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RESEARCH ARTICLE

Function Perturbation Impact on Robust Stability and Stabilization of Boolean Networks With Disturbances

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ABSTRACT This article concentrates on the function perturbation impact on robust stability and robust stabilization of Boolean networks with disturbance inputs (DBNs). First, using the semi-tensor product (STP) of matrices, the algebraic representation of DBNs with function perturbation is given. Second, a state set is determined to detect the robust stability of DBNs subject to function perturbation. The result shows that the robust stability of DBNs remains unchanged if and only if the perturbed point is not in the constructed state set. Third, DBNs with control inputs (DBCNs) are considered, and several criteria to verify whether DBCNs with function perturbation can still maintain robust stabilization under a given state feedback stabilizer are presented. Finally, two examples are provided to illustrate the validity of the theoretical results.

INDEX TERMS Boolean network, disturbance inputs, function perturbation, semi-tensor product of matrices, robust stability.

I. INTRODUCTION

With the great interest of human genome engineering, gene regulatory networks (GRNs) have become hot topic and research front in systems biology [1], [2], [3]. In order to study GRNs, many models have been constructed, such as Boolean network models (BNs) [4], [5], linear models [6], [7] and Markovian models [8], [9]. Among these models, BNs are more suitable for GRNs because they are parameter-free and can be applied to quantifying the large-scale GRNs. In BNs, each node has two states: ON or OFF (1 or 0). State evolution of each node is related to a pre-assigned Boolean function consisting of its neighboring nodes, itself and some basic logical operators. Although BNs are simple models, it is quite difficult to characterize the dynamics of BNs because of lacking effective mathematical tools to handle logical dynamic systems. In the last decade, an algebraic state space representation (ASSR) method has been provided for analysing BNs via the STP of matrices. The ASSR method was introduced in [10], based on which BNs can be expressed as discrete-time systems. Using the ASSR method, numerous landmark results about BNs have been achieved, such as observability and controllability [11], [12], output tracking [13], [14], disturbance decoupling [15], optimal control [16], [17], and other issues [18], [19], [20].

It is known that there exist some uncertain factors in the process of modeling a GRN, such as environmental changes, external interference, and experimental noises, which may have an obstructive effect on the effectiveness of control strategies. For instance, in [21], cancer was regarded as failures in the robustness against genetic uncertainties. Hence, as an appropriate model of GRNs, modeling BNs with disturbances (DBNs) is necessary. The robust stability and robust stabilization of DBNs are two basic concepts of the modern control theory. And many typical results on robust stability and robust stabilization of DBNs have been obtained by the ASSR approach. Zhong et al. [22] presented the concept of robust stability to a limit cycle for DBNs, and determined the corresponding stability criteria. The robust stabilization of DBNs was discussed in [23], and state feedback stabilizers

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were developed by constructing the robust reachable set. In addition to robust stability and robust stabilization, many other issues on DBNs have also been well studied [24], [25].

Gene mutation is a change in DNA sequence, which can occur spontaneously or be induced by environmental factors such as ultraviolet rays or ionizing radiation. These mutations can have a wide range of effects on organisms. For example, gene mutation in hemoglobin is a key factor leading to sickle cell anemia [26]. For BNs, authors in [27] introduced function perturbation to depicted gene mutation, and analyzed the changes of the topological structure in BNs after function perturbation. The function perturbation influence on the dynamical behaviours of BNs became a hot topic of research in recent years, especially based on the ASSR approach [28], [29], [30], [31]. Li et al. discussed the stochastic function perturbation influence on the stability and stabilization of BNs [32]. References [33] and [34] studied asymptotical stability and output tracking issues of probabilistic Boolean networks (PBNs) with function perturbation, respectively. It is worth noting that the existing literature on function perturbation mainly focused on BNs and PBNs. However, BNs are often subject to disturbances, and the existing results are not directly applicable to DBNs. Therefore, it is necessary to study the function perturbation influence on the robust stability and robust stabilization for BNs under arbitrary disturbance. To our best knowledge, there exist few results on the robust stability and robust stabilization problems of DBNs with function perturbation at present.

In this article, utilizing the ASSR approach, we investigate the robust stability and robust stabilization issues of DBNs with function perturbation. The main contributions can be concluded in following aspects: (1) The robust stability and robust stabilization of DBNs with function perturbation are studied for the first time. By constructing a state set, several criteria are provided to guarantee the robust stability and robust stabilization of DBNs after function perturbation. (2) Our results can be regarded as a generalization of references [28]. When there is no disturbance in system, our results will degenerate to the results of global robust stability in reference [28]. However, the conditions obtained in this paper are easier to understand and verify than previous methods.

We organize the remainder of this article as follows. Section II provides some necessary notations and results. Section III describes the problem of function perturbation in DBNs. In Section IV, we discuss the impact of function perturbation on robust stability and robust stabilization of DBNs. Section V presents two illustrative examples to support the theoretical results. Section VI is a brief conclusion.

II. PRELIMINARIES

In this section, some useful notations and definitions related to the STP of matrices are given.

$\mathbb{R}$ and $\mathbb{Z}_+$ denote the set of real numbers and positive integers, respectively. Symbol $[a, b]$ denotes the set of integers λ with $a \leq \lambda \leq b$. $\mathbb{R}_{n\times s}$ denotes the set of all $n \times s$ real matrices. $\mathcal{D} := \{0, 1\}$, $\mathcal{D}^n := \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{n}$. $\mathrm{Col}_i(L)$ is the ith column of matrix L. The set of columns of L is denoted by $\mathrm{Col}(L)$. $\Delta_n := \{\delta_n^i \mid i \in [1, n]\}$, where $\delta_n^i = \mathrm{Col}_i(I_n)$. For compactness, $\Delta := \Delta_2$. A matrix $L \in \mathbb{R}_{n\times s}$ is called a logical matrix, if $\mathrm{Col}(L) \subseteq \Delta_n$. Denote by $\mathcal{L}_{n\times s}$ the set of all $n \times s$ logical matrices. If $L \in \mathcal{L}_{n\times s}$, denote L briefly by $L = \delta_n[i_1\ i_2\ \cdots\ i_s]$. $[P]_{i,j}$ denotes the (i, j)-element of matrix P.

Definition 1: ([10]) For matrix $\mathcal{P} \in \mathbb{R}_{m\times n}$ and matrix $\mathcal{Q} \in \mathbb{R}_{s\times t}$, let λ be the least common multiple of n and s. Then the STP of $\mathcal{P}$ and $\mathcal{Q}$ is defined as

$$\mathcal{P} \ltimes \mathcal{Q} = (\mathcal{P} \otimes I_{\frac{\lambda}{n}})(\mathcal{Q} \otimes I_{\frac{\lambda}{s}}), \tag{1}$$

where $\otimes$ denotes the Kronecker product of matrices.

Remark 1: Note that $\mathcal{P} \ltimes \mathcal{Q} = \mathcal{P}\mathcal{Q}$ if $n = s$. It follows that the STP can be regarded as a generalization of the ordinary matrix product. Hence, the symbol "$\ltimes$" will be omitted for convenience.

Lemma 1: ([10]) Let $x \in \Delta_{2^n}$. Then $x^2 = M_{2^n}^r x$, where $M_{2^n}^r = diag\{\delta_{2^n}^1, \delta_{2^n}^2, \ldots, \delta_{2^n}^{2^n}\}$ is the power-reducing matrix.

Let $1 \sim \delta_2^1$, $0 \sim \delta_2^2$, then $\mathcal{D} \sim \Delta$, where "$\sim$" stands for the equivalence relation. For $x \in \mathcal{D}$, we have the vector form $x = \delta_2^1$ if $x = 1$, and the vector form $x = \delta_2^2$ if $x = 0$.

Lemma 2: ([10]) Given a logical function $f : \mathcal{D}^n \to \mathcal{D}$. Then, there exists a unique matrix $F \in \mathcal{L}_{2\times 2^n}$ such that

$$f(x_1, x_2, \cdots, x_n) = F \ltimes x_1 \ltimes \cdots \ltimes x_n,$$

where $x_i \in \Delta$, $i \in [1, n]$, $\mathrm{Col}_j(F) = f(\delta_{2^n}^j)$ and $j \in [1, 2^n]$. Here, F is called the structure matrix of function f.

III. PROBLEM FORMULATION

Consider the following DBN:

$$X_i(t + 1) = f_i(X(t), \Xi(t)), \quad i \in [1, n] \tag{2}$$

with the state $X(t) = (X_1(t), \cdots, X_n(t)) \in \mathcal{D}^n$ and disturbance $\Xi(t) = (\Xi_1(t), \cdots, \Xi_q(t)) \in \mathcal{D}^q$. $f_i : \mathcal{D}^n \times \mathcal{D}^q \to \mathcal{D}$, $i \in [1, n]$ are Boolean functions.

Using the vector form of logical variables and ASSR method, we obtain the algebraic representation of DBN (2)

$$x(t + 1) = L\xi(t)x(t), \tag{3}$$

where $x(t) = \ltimes_{i=1}^n x_i(t) \in \Delta_{2^n}$ and $\xi(t) = \ltimes_{i=1}^q \xi_i(t) \in \Delta_{2^q}$. The state transition matrix of DBN (2) is

$$L := [L_1\ L_2\ \cdots\ L_{2^q}] \in \mathcal{L}_{2^n\times 2^{q+n}},$$

where $L_k := \delta_{2^n}[\alpha_{k,1}\ \alpha_{k,2}\ \cdots\ \alpha_{k,j}\ \cdots\ \alpha_{k,2^n}] \in \mathcal{L}_{2^n\times 2^n}$, $k \in [1, 2^q]$. Given $s \in \mathbb{Z}_+$, the trajectory of DBN (3) from $x(0) \in \Delta_{2^n}$ under the disturbance sequence $\{\xi(0), \cdots, \xi(s-1)\} \subseteq \Delta_{2^q}$ can be expressed by

$$\begin{aligned} x(s; x(0), \xi) &= L\xi(s-1)x(s-1) \\ &= \cdots = \ltimes_{t=s-1}^{0}(L\xi(t))x(0), \end{aligned} \tag{4}$$

TABLE 1. Truth table of DBN (5).

X_1	X_2	Ξ	f_1	f_2
1	1	1	0	1
1	0	1	1	1
0	1	1	0	0
0	0	1	0	0
1	1	0	0	1
1	0	0	0	1
0	1	0	0	0
0	0	0	0	0

where $\ltimes_{t=s-1}^{0}(L\xi(t)) = L\xi(s-1) \ltimes \cdots \ltimes L\xi(0)$. Notice that the trajectories of DBN (3) are not unique with various disturbance sequences.

Based on the algebraic representation (3), we review the definitions of robust reachability and robust stability of DBNs [22].

Definition 2: For DBN (3), x_e is robustly reachable from initial state $x(0) \in \Delta_{2^n}$, if there exists $s \in \mathbb{Z}_+$, such that $x(s; x(0), \xi) = x_e$, $\forall \{\xi(t) : t \in \mathbb{N}\} \subseteq \Delta_{2^q}$.

Definition 3: DBN (3) is robustly stable at x_e, if for any initial state $x(0) \in \Delta_{2^n}$, there exists $\tau \in \mathbb{Z}_+$, such that $x(t; x(0), \xi) = x_e$, $\forall t \geq \tau$ and $\{\xi(t) : t \in \mathbb{N}\} \subseteq \Delta_{2^q}$.

Since gene mutation is a general phenomenon in GRNs, function perturbation may be occurred in DBNs, that is, some truth values of f are flipped in the truth table of DBN (2). Correspondingly, some columns of L in (3) are changed. Here, we give an example to illustrate this phenomenon.

Example 1: Consider a BN consisting of two nodes and one disturbance as

$$X_i(t+1) = f_i(X(t), \Xi(t)), \quad i \in [1, 2], \tag{5}$$

where $f_1 = X_1 \wedge \neg X_2 \wedge \Xi$ and $f_2 = X_1$. Table 1 shows the truth table of DBN (5).

Using ASSR technique, the algebraic representation of DBN (5) is $x(t+1) = L\xi(t)x(t)$, where $L = [L_1, L_2]$ with $L_1 = \delta_4[3, 1, 4, 4]$ and $L_2 = \delta_4[3, 3, 4, 4]$. This article only discusses an one-bit perturbation. For instance, $f_1(0, 1, 1)$ is flipped from 0 to 1 (see the red text in Table 1). Then, the value $\mathrm{Col}_3(L_1)$ is changed from δ_4^4 to δ_4^2, and L is changed to $\delta_4[3, 1, 2, 4, 3, 3, 4, 4]$.

Before function perturbation, DBN (5) is robustly stable at δ_4^4. Is DBN (5) still robustly stable at δ_4^4 after function perturbation? In order to solve this problem, we establish some criteria to detect whether DBN (5) can be still robustly stable at x_e after function perturbation.

IV. MAIN RESULTS

Robust stability and robust stabilization of DBNs under function perturbation are investigated.

A. ROBUST STABILITY OF DBNs WITH FUNCTION PERTURBATION

In the following, two assumptions about the function perturbation of DBN (3) are given.

Assumption 1: Before function perturbation, DBN (3) is robustly stable at $x_e = \delta_{2^n}^{\gamma}$, $\gamma \in [1, 2^n]$.

Assumption 2: After function perturbation, the j^*-th column of L_{k^*} is perturbed from $\delta_{2^n}^{\alpha_{k^*,j^*}}$ to $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$, where $j^* \neq \gamma$ and $\delta_{2^n}^{\alpha_{k^*,j^*}} \neq \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$.

Remark 2: One-bit perturbation is only considered in Assumption 1 for DBN (3), and the state $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ is called the perturbed point of DBN (3).

After function perturbation, matrix L changes to

$$L' = [L_1' \ L_2' \ \cdots \ L_{2^q}'],$$

where $L_k' = \delta_{2^n}[\beta_{k,1} \ \beta_{k,2} \ \cdots \ \beta_{k,j} \ \cdots \ \beta_{k,2^n}]$, $k \in [1, 2^q]$ with

$$\beta_{k,j} = \begin{cases} \hat{\alpha}_{k^*,j^*}, & \text{if } (k,j) = (k^*, j^*), \\ \alpha_{k,j}, & \text{otherwise}. \end{cases} \tag{6}$$

Then, the algebraic representation of DBN (3) changes to

$$x(t+1) = L'\xi(t)x(t). \tag{7}$$

Thus, for any $x = \delta_{2^n}^j$ and any $\xi = \delta_{2^q}^k$, if $j \neq j^*$, we have

$$L\xi x = \delta_{2^n}^{\alpha_{k,j}} = \delta_{2^n}^{\beta_{k,j}} = L'\xi x. \tag{8}$$

If $j = j^*$ and $k = k^*$, then

$$L\xi x = \delta_{2^n}^{\alpha_{k^*,j^*}} \neq \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} = L'\xi x. \tag{9}$$

Next, we analyze how function perturbation affect the robust stability of DBN (3). Construct the state set:

$$\Gamma = \{\delta_{2^n}^j \in \Delta_{2^n} : [P]_{j^*,j} > 0\} \cup \{\delta_{2^n}^{j^*}\}, \tag{10}$$

where $P := \sum_{s=1}^{2^n} Q^s$ and $Q = \sum_{k=1}^{2^q} L_k$. Then, the set Γ contains all the states which can reach $\delta_{2^n}^{j^*}$ under a sequence of disturbance inputs, including $\delta_{2^n}^{j^*}$ itself. If $\delta_{2^n}^j \in \Gamma \setminus \{\delta_{2^n}^{j^*}\}$, there must exist one path from $\delta_{2^n}^j$ to $\delta_{2^n}^{j^*}$ for DBN (3) before function perturbation.

Utilizing the set Γ, we present the following result for the robust stability of DBN (3) with function perturbation.

Theorem 1: Under Assumption 1, DBN (3) with function perturbation in Assumption 2 is still robustly stable at x_e, if and only if $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \notin \Gamma$.

Proof. (Necessity) We prove it by contradiction. Assume that DBN (3) is still robustly stable at x_e after function perturbation, and $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \in \Gamma$. From Assumption 1, we know that $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ can robustly reach x_e. Since $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \in \Gamma$, there exists at least one path from $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ to x_e containing $\delta_{2^n}^{j^*}$. One of these paths is assumed as

$$\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \xrightarrow{\xi(0)} \cdots \rightarrow \delta_{2^n}^{j^*} \rightarrow \cdots \xrightarrow{\xi(\tau-1)} x_e, \tag{11}$$

where the disturbance sequence is $\{\xi(0), \cdots, \xi(\tau-1)\} \subseteq \Delta_{2^q}$, and τ is the number of steps from $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ to x_e.

Under Assumption 2, the j^*-th column of L_{k^*} is perturbed from $\delta_{2^n}^{\alpha_{k^*,j^*}}$ to $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$, that is, after function perturbation, $\delta_{2^n}^{j^*}$

can reach $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ in one step under disturbance $\xi = \delta_{2^q}^{k^*}$. Combining with the path (11), we obtain that

$$\delta_{2^n}^{j^*} \to \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \to \cdots \to \delta_{2^n}^{j^*}. \tag{12}$$

Thereby a cycle (12) is formed for DBN (3) with function perturbation. This contradicts with the hypothesis that DBN (3) is still robustly stable at x_e after function perturbation. Thus, $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \notin \Gamma$.

(Sufficiency) Before function perturbation, DBN (3) is robustly stable at x_e. Therefore, for every state $\delta_{2^n}^{\eta} \in \Delta_{2^n}$, the paths from $\delta_{2^n}^{\eta}$ to x_e can be divided into two cases as follows.

- Case 1: $\delta_{2^n}^{\eta} \notin \Gamma$, that is, $\delta_{2^n}^{\eta}$ can robustly reach x_e, meanwhile there is no path from $\delta_{2^n}^{\eta}$ to x_e containing $\delta_{2^n}^{j^*}$.
- Case 2: $\delta_{2^n}^{\eta} \in \Gamma$, that is, $\delta_{2^n}^{\eta}$ can robustly reach x_e, meanwhile there is at least one path from $\delta_{2^n}^{\eta}$ to x_e containing $\delta_{2^n}^{j^*}$.

For Case 1, one path from $\delta_{2^n}^{\eta}$ to x_e is arbitrarily chosen and assumed as

$$\delta_{2^n}^{\eta} \to \cdots \to x(t) \to \cdots \to x_e, \tag{13}$$

where the disturbance sequence is $\{\xi(0), \cdots, \xi(\tau-1)\} \subseteq \Delta_{2^q}$, τ is the number of steps from $\delta_{2^n}^{\eta}$ to x_e, and $\{x(1), \cdots, x(\tau-1)\}$ denotes a series of states in the path from $\delta_{2^n}^{\eta}$ to x_e. Obviously, $x(t) \neq \delta_{2^n}^{j^*}$, $t \in [1, \tau-1]$.

After function perturbation, we can know from (8) that

$$\begin{aligned} x(\tau; \delta_{2^n}^{\eta}, \xi) &= L'\xi(\tau-1)x(\tau-1) \\ &= L'\xi(\tau-1)L'\xi(\tau-2)x(\tau-2) \\ &= \cdots \\ &= \ltimes_{t=\tau-1}^{0}(L'\xi(t))\delta_{2^n}^{\eta} \\ &= L\xi(\tau-1)x(\tau-1) \\ &= L\xi(\tau-1)L\xi(\tau-2)x(\tau-2) \\ &= \cdots \\ &= \ltimes_{t=\tau-1}^{0}(L\xi(t))\delta_{2^n}^{\eta} \\ &= x_e. \end{aligned}$$

Hence, the path (13) is not affected by function perturbation, which together with Assumption 1 shows that x_e is still robustly reachable from any state $\delta_{2^n}^{\eta} \in \Delta_{2^n}$.

For Case 2, we take an arbitrary path from $\delta_{2^n}^{\eta}$ to x_e as

$$\begin{aligned} &\delta_{2^n}^{\eta} \to \cdots \to x(t_1) \to \cdots \to \delta_{2^n}^{j^*} \to \delta_{2^n}^{\alpha} \to \\ &\cdots \to x(t_2) \to \cdots \to x_e, \end{aligned} \tag{14}$$

where the corresponding disturbance sequence is $\{\xi(t) = \delta_{2^q}^{i_t} : t = 0, \ldots, \tau_1 + \tau_2 - 1\} \subseteq \Delta_{2^q}$, $\tau_1 + \tau_2$ is the number of time steps from $\delta_{2^n}^{\eta}$ to x_e, and $\{x(t_1) : t_1 = 1, \ldots, \tau_1 - 1\}$ denotes the states in the path from $\delta_{2^n}^{\eta}$ to $\delta_{2^n}^{\alpha}$, $\{x(t_2) : t_2 = \tau_1 + 1, \ldots, \tau_1 + \tau_2 - 1\}$ represents the states in the path from $\delta_{2^n}^{\alpha}$ to x_e. Obviously, $x(\tau_1 - 1) = \delta_{2^n}^{j^*}$, $x(\tau_1) = \delta_{2^n}^{\alpha}$ and $x(t) \neq \delta_{2^n}^{j^*}$, $t \in \{1, \ldots, \tau_1 + \tau_2 - 1\} \backslash \{\tau_1 - 1\}$.

There are two cases for $\delta_{2^n}^{\alpha}$ in path (14), that is, $\alpha \neq \alpha_{k^*,j^*}$ or $\alpha = \alpha_{k^*,j^*}$. If $\alpha \neq \alpha_{k^*,j^*}$, similar to the analysis of path (13), path (14) is not affected by function perturbation, and $\delta_{2^n}^{\eta}$ can still robustly reach to x_e. If $\alpha = \alpha_{k^*,j^*}$, after function perturbation, we obtain that

$$\begin{cases} x(\tau_1 - 1; \delta_{2^n}^{\eta}, \xi) & = \ltimes_{t=\tau_1-2}^{0}(L'\delta_{2^q}^{i_t})\delta_{2^n}^{\eta} \\ & = \ltimes_{t=\tau_1-2}^{0}(L\delta_{2^q}^{i_t})\delta_{2^n}^{\eta} = \delta_{2^n}^{j^*}, \\ x(1; \delta_{2^n}^{j^*}, \xi) & = L'\delta_{2^q}^{k^*}\delta_{2^n}^{j^*} = \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}. \end{cases} \tag{15}$$

Considering $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \notin \Gamma$, we have

$$\begin{aligned} x(\tau_3 - 1; \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}, \xi) &= \ltimes_{t=\tau_3-1}^{0}(L'\delta_{2^q}^{j_t})\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \\ &= \ltimes_{t=\tau_3-1}^{0}(L\delta_{2^q}^{j_t})\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \\ &= x_e, \end{aligned} \tag{16}$$

where τ_3 is the number of time steps from $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ to x_e, and the corresponding disturbance sequence is $\{\xi(t) = \delta_{2^q}^{j_t} : t = 0, \ldots, \tau_3 - 1\} \subseteq \Delta_{2^q}$.

Combining (15) with (16), we have

$$\begin{aligned} &x(\tau_1 + \tau_3; \delta_{2^n}^{\eta}, \xi) \\ &\quad = \ltimes_{t=\tau_3-1}^{0}(L'\delta_{2^q}^{j_t})L'\delta_{2^q}^{k^*} \ltimes_{t=\tau_1-2}^{0} (L'\delta_{2^q}^{i_t})\delta_{2^n}^{\eta} \\ &\quad = \ltimes_{t=\tau_1+\tau_3-1}^{0}(L'\xi(t))\delta_{2^n}^{\eta} \\ &\quad = x_e, \end{aligned}$$

which implies that path (14) changes to

$$\begin{aligned} &\delta_{2^n}^{\eta} \to \cdots \to x(t_1) \to \cdots \\ &\quad \to \delta_{2^n}^{j^*} \not\to \delta_{2^n}^{\alpha_{k^*,j^*}} \to \cdots \to x(t_2) \to \cdots \to x_e \\ &\quad \downarrow \\ &\quad \delta_{2^n}^{\hat{\alpha}_{k^*,j^*}} \to \cdots \to x(t_3) \to \cdots \to x_e, \end{aligned} \tag{17}$$

where the corresponding disturbance sequence is $\{\xi(t) = \delta_{2^q}^{i_t} : t = 0, \ldots, \tau_1 - 2\} \cup \{\xi(t) = \delta_{2^q}^{k^*} : t = \tau_1 - 1\} \cup \{\xi(t) = \delta_{2^q}^{j_{t-\tau_1}} : t = \tau_1, \ldots, \tau_1 + \tau_3 - 1\}$, $\tau_1 + \tau_3$ is the number of time steps from $\delta_{2^n}^{\eta}$ to x_e, and $\{x(t_3) : t_3 = \tau_1 + 1, \ldots, \tau_1 + \tau_3 - 1\}$ denots a series of states in the path from $\delta_{2^n}^{\hat{\alpha}_{k^*,j^*}}$ to x_e. Then, $\delta_{2^n}^{\eta}$ can still reach to x_e when $\alpha = \alpha_{k^*,j^*}$. Therefore, based on the above discussion, for case 2, one knows that x_e is still robustly reachable from any state $\delta_{2^n}^{\eta} \in \Delta_{2^n}$ after function perturbation.

By Definition 3, DBN (3) is still robustly stable at x_e after function perturbation of Assumption 2. ■

B. ROBUST STABILIZATION OF DBCNs WITH FUNCTION PERTURBATION

Now, consider the following DBCN:

$$x_i(t+1) = f_i(X(t), U(t), \Xi(t)), \quad i \in [1, n], \tag{18}$$

with control input $U(t) = (u_1(t), u_2(t), \cdots, u_m(t)) \in \mathcal{D}^m$. $f_i : \mathcal{D}^{m+n+q} \to \mathcal{D}$ $(i \in [1, n])$ is Boolean function. Then, the concept of robust stabilization of DBCNs is recalled as follows [22].

Definition 4: DBCN (18) is robustly feedback stabilizable to X_e, if for any initial state $X(0) \in \mathcal{D}^n$, there exist the state *feedback controller $U(t) = g(X(t))$ and $\tau \in \mathbb{Z}_+$, such that $X(t; X(0), U_g, \Xi) = X_e$, $\forall t \geq \tau$ and $\{\Xi(t) : t \in \mathbb{N}\} \subseteq \mathcal{D}^q$, where $g : \mathcal{D}^n \to \mathcal{D}^m$, and U_g is the control sequence generated by g.*

Setting u_i be the vector form of U_i and $u(t) = \ltimes_{i=1}^m u_i(t) \in \Delta_{2^m}$, based on Lemma 2, the ASSR of DBCN (18) is expressed as

$$x(t+1) = L\xi(t)u(t)x(t), \tag{19}$$

where $L \in \mathcal{L}_{2^n \times 2^{m+n+q}}$ is the state transition matrix of (18). Divide $L \in \mathcal{L}_{2^n \times 2^{m+n+q}}$ into 2^q equal blocks as $L = [L_1 \;\; L_2 \;\; \cdots \;\; L_{2^q}]$, where $L_k \in \mathcal{L}_{2^n \times 2^{m+n}}$, $k \in [1, 2^q]$. Denote $L_k := [L_{k,1} \;\; L_{k,2} \;\; \cdots \;\; L_{k,2^m}]$, where $L_{k,l} := \delta_{2^n}[\theta_{k,l}^1 \;\; \theta_{k,l}^2 \;\; \cdots \;\; \theta_{k,l}^{2^n}]$, $l = 1, \cdots, 2^m$.

Analogously, the controller $U(t) = g(X(t))$ is converted into

$$u(t) = Gx(t), \tag{20}$$

where $G := \delta_{2^m}[v_1 \;\; v_2 \;\; \cdots \;\; v_{2^n}]$ is the state feedback gain matrix, which can be derived through the stabilizer design method presented in [23].

Next, we discuss whether DBCNs can still maintain robust stabilization under the given state feedback stabilizer after function perturbation.

Two natural assumptions are given in the following.

Assumption 3: Before function perturbation, DBCN (19) is robustly stabilizable at $x_e = \delta_{2^n}^{\gamma}$ under the state feedback control (20).

Assumption 4: After function perturbation, $\delta_{2^n}^{\theta_{k^*,l^*}^{j^*}}$ changes to $\delta_{2^n}^{\hat{\theta}_{k^*,l^*}^{j^*}}$, where $k^* \in [1, 2^q]$, $l^* \in [1, 2^m]$, $j^* \in [1, 2^n]$, $j^* \neq \gamma$ and $\delta_{2^n}^{\theta_{k^*,l^*}^{j^*}} \neq \delta_{2^n}^{\hat{\theta}_{k^*,l^*}^{j^*}}$.

Plugging (20) into (19), the closed-loop system can be given as

$$\begin{aligned} x(t+1) &= L\xi(t)Gx(t)x(t) \\ &= L(I_{2^q} \otimes GM_{2^n}^r)\xi(t)x(t) \\ &:= \widetilde{L}\xi(t)x(t). \end{aligned} \tag{21}$$

Here $\widetilde{L} = L(I_{2^q} \otimes GM_{2^n}^r)$. Divide $\widetilde{L}$ into 2^q equal blocks as

$$\widetilde{L} = [\widetilde{L}_1 \;\; \widetilde{L}_2 \;\; \ldots \;\; \widetilde{L}_{2^q}],$$

with $\widetilde{L}_k = L_k(I_{2^q} \otimes GM_{2^n}^r)$, $k \in [1, 2^q]$. A simple calculation gives

$$\widetilde{L}_k = \delta_{2^n}[\theta_{k,v_1}^1 \;\; \theta_{k,v_2}^2 \;\; \cdots \;\; \theta_{k,v_{2^n}}^{2^n}], \; k \in [1, 2^q].$$

In the following, we discuss the changes in the structure matrix L of DBCN (19) with function perturbation.

Under Assumption 4, only one column in some block of L is changed. Assume that matrix L in DBCN (19) changes to

$$L' = [L_1' \;\; L_2' \;\; \cdots \;\; L_{2^q}'],$$

where $L_k' = [L_{k,1}' \;\; L_{k,2}' \;\; \cdots \;\; L_{k,2^m}']$, $k = 1, \cdots, 2^q$. We know that

$$L_k' = \begin{cases} L_k, & \text{if } k \neq k^*; \\ [L_{k^*,1}' \;\; L_{k^*,2}' \;\; \cdots \;\; L_{k^*,2^m}'], & \text{if } k = k^*. \end{cases} \tag{22}$$

When $l \neq l^*$, $L_{k^*,l}$ is unchanged, that is $L_{k^*,l}' = L_{k^*,l}$. When $l = l^*$, we have

$$L_{k^*,l^*}' = \delta_{2^n}[\theta_{k^*,l^*}^1 \cdots \theta_{k^*,l^*}^{j^*-1} \;\; \hat{\theta}_{k^*,l^*}^{j^*} \;\; \theta_{k^*,l^*}^{j^*+1} \;\; \cdots \theta_{k^*,l^*}^{2^n}].$$

According to the above description, we further study the changes of matrix $\widetilde{L}$ in closed-loop system (21).

When function perturbation of Assumption 4 occurs, if $k \neq k^*$, $\widetilde{L}_k$ is unchanged, $k \in \{1, \ldots, 2^q\} \backslash \{k^*\}$; if $k = k^*$ and $v_{j^*} \neq l^*$, since $\theta_{k^*,v_{j^*}}^{j^*}$ does not change, there are still no change in $\widetilde{L}_{k^*}$; if $k = k^*$ and $v_{j^*} = l^*$, since $\theta_{k^*,v_{j^*}}^{j^*}$ changes to $\hat{\theta}_{k^*,v_{j^*}}^{j^*}$, one knows that $\widetilde{L}_{k^*}$ changes to $\widetilde{L}_{k^*}'$, where

$$\widetilde{L}_{k^*}' = \delta_{2^n}[\theta_{k^*,v_1}^1 \;\; \cdots \;\; \theta_{k^*,v_{j^*-1}}^{j^*-1} \;\; \hat{\theta}_{k^*,l^*}^{j^*} \;\; \cdots \;\; \theta_{k^*,v_{2^n}}^{2^n}]. \tag{23}$$

Theorem 2: Under Assumption 3, if $v_{j^*} \neq l^*$, then DBCN (19) with function perturbation in Assumption 4 is still robustly stabilizable at x_e under the controller (20).

Proof. Since $v_{j^*} \neq l^*$, $\widetilde{L}$ will not change, which implies that the function perturbation in Assumption 4 has no effect on closed-loop system (21). Based on Assumption 3, the conclusion holds. ■

For the case $v_{j^*} = l^*$, it is easy to see from (23) that system (21) is affected by function perturbation. Denote the state set as

$$\widetilde{\Gamma} = \{\delta_{2^n}^j \in \Delta_{2^n} : [\widetilde{P}]_{j^*,j} > 0\} \cup \{\delta_{2^n}^{j^*}\}, \tag{24}$$

where $\widetilde{P} := \sum_{j=1}^{2^n} \widetilde{Q}^j$ and $\widetilde{Q} = \sum_{k=1}^{2^q} \widetilde{L}_k$. Similar to the analyses of DBNs with function perturbation, the following result can be established.

Theorem 3: Under Assumption 3, if $v_{j^*} = l^*$, then DBCN (19) with function perturbation in Assumption 4 is still robustly stabilizable at x_e under the controller (20), if and only if $\delta_{2^n}^{\hat{\theta}_{k^*,l^*}^{j^*}} \notin \widetilde{\Gamma}$.

Proof. This proof is similar to Theorem 1, so we omitted it. ■

V. ILLUSTRATIVE EXAMPLES

In the following, we given two examples to verify the obtained results.

Example 2: Recall DBN (5) in Example 1. A direct calculation shows that DBN (5) is robustly stable at $x_e = \delta_4^4$ before function perturbation. After function perturbation, the value $\mathrm{Col}_3(L_1)$ is changed from δ_4^4 to δ_4^2. Then, under Assumption 2, we have $x_e = \delta_4^4$, $j^* = 3$, $k^* = 1$, $\alpha_{1,3} = 4$ and $\hat{\alpha}_{1,3} = 2$. By calculations, the following matrices are obtained:

$$Q = L_1 + L_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

and

$$P = Q^1 + Q^2 + Q^3 + Q^4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 28 & 26 & 30 & 30 \end{bmatrix}.$$

We know from $[P]_{3,1} > 0$ and $[P]_{3,2} > 0$ that $\Gamma = \{\delta_4^1, \delta_4^2, \delta_4^3\}$. Since the perturbed point $\delta_4^2 \in \Gamma$, by Theorem 1, we obtain that DBN (5) cannot robustly stable at $x_e = \delta_4^4$ after function perturbation. We can explain it by Figure 1. Figure 1(a) shows that DBN (5) is robustly stable to δ_4^4 before function perturbation. After function perturbation, a cycle (see Figure 1(b)) of length 2 is formed as

$$\delta_4^2 \xrightarrow{\xi(0)=\delta_2^2} \delta_4^3 \xrightarrow{\xi(1)=\delta_2^1} \delta_4^2, \tag{25}$$

which means that DBN (5) cannot converge to δ_4^4 under arbitrary disturbance.

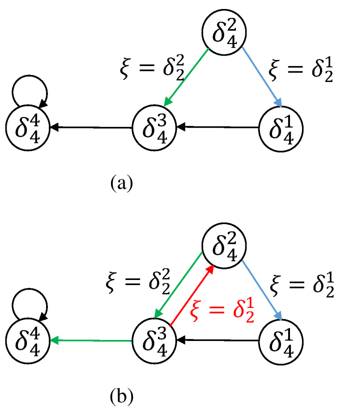


FIGURE 1. State transfer diagrams of DBN (5): (a) before function perturbation; (b) after function perturbation.

In Figure 1, the blue and green arrows represent the state transfer under $\xi = \delta_2^1$ and $\xi = \delta_2^2$, respectively. The black arrows represent the state transfer under arbitrary disturbance input (δ_2^1 and δ_2^2), and the red arrow represents the state transfer of perturbed point.

Example 3: Consider a reduced E. coli lactose operon network with disturbance [35]:

$$\begin{cases} x_1(t+1) &= \neg u_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) &= \neg u_1(t) \wedge u_2(t) \wedge x_1(t) \wedge \xi(t), \\ x_3(t+1) &= \neg u_1(t) \wedge (u_2(t) \vee (u_3(t) \wedge x_1(t))). \end{cases} \tag{26}$$

where state variables x_1, x_2 and x_3, respectively, denote the *lac* mRNA, the high-concentration lactose, and medium-concentration lactose; input variables u_1, u_2 and u_3 stand for the extracellular glucose, the high exolactose, and the medium exolactose, respectively; ξ indicates an artificial disturbance.

Using the ASSR method, the algebraic representation for DBCN (26) can be denoted by

$$x(t+1) = L\xi(t)u(t)x(t), \tag{27}$$

where $L = [L_1, L_2]$ with

$$\begin{aligned} L_1 = \delta_8[\, & 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 1\ 1\ 1\ 5\ 3\ 3\ 3\ 7\ 1\ 1\ 1\ 5\ 3\ 3\ 3\ 7 \\ & 3\ 3\ 3\ 7\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\,], \end{aligned}$$

and

$$\begin{aligned} L_2 = \delta_8[\, & 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8\ 8 \\ & 3\ 3\ 3\ 7\ 3\ 3\ 3\ 7\ 3\ 3\ 3\ 7\ 3\ 3\ 3\ 7 \\ & 3\ 3\ 3\ 7\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\ 4\ 4\ 4\ 8\,]. \end{aligned}$$

According to the control design approach in [23], 48 state feedback stabilizers can be obtained to robustly stabilize DBCN (27) to $x_e = \delta_8^3$, and one of state feedback stabilizers is proposed below

$$u(t) = \delta_8[7\ 7\ 7\ 5\ 5\ 5\ 5\ 5]x(t). \tag{28}$$

Plugging control (28) into DBCN (27), the closed-loop system can be obtained as

$$x(t+1) = \widetilde{L}\xi(t)x(t), \tag{29}$$

where $\widetilde{L} = [\widetilde{L}_1, \widetilde{L}_2]$ with $\widetilde{L}_1 = \delta_8[3\ 3\ 3\ 5\ 3\ 3\ 3\ 7]$, and $\widetilde{L}_2 = \delta_8[3\ 3\ 3\ 7\ 3\ 3\ 3\ 7]$. The state transfer diagram of DBCN (27) under control (28) is shown in Figure 2.

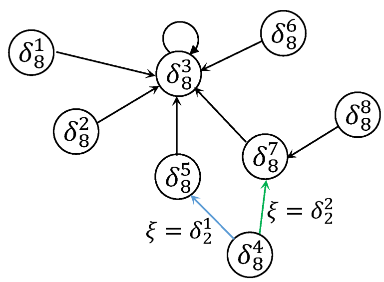
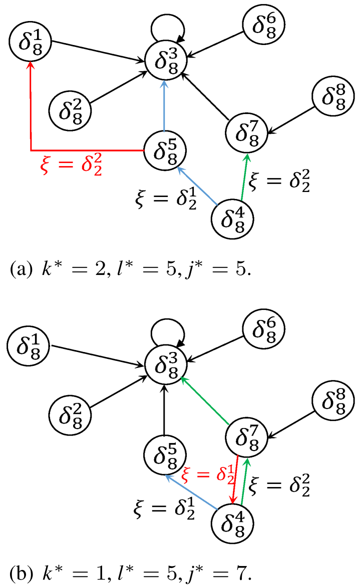


FIGURE 2. State transfer diagram of DBCN (27) under control (28). The blue and green arrows represent the state transfer under $\xi = \delta_2^1$ and $\xi = \delta_2^2$, respectively. The black arrows represent the state transfer under arbitrary disturbance input (δ_2^1 and δ_2^2).

Now, we study whether the given control (28) can make DBCN (27) maintain robust stabilization after the following three types of function perturbations.

1) After function perturbation, the value $\mathrm{Col}_9(L_2)$ is changed from δ_8^8 to δ_8^3, that is, $k^* = 2$, $l^* = 2$ and $j^* = 1$. It is easy to see from (28) that $v_{j^*} = v_1 = 7$. Then, we obtain $v_{j^*} \neq l^*$. By Theorem 2, DBCN (26) is still robustly stabilizable to x_e under control (28).

2) After function perturbation, the value $\mathrm{Col}_{37}(L_2)$ is changed from δ_8^3 to δ_8^1, that is, $k^* = 2$, $l^* = 5$ and $j^* = 5$, thus, $v_{j^*} = v_5 = 5$ and $\widetilde{L}'_1 = \delta_8[3\ 3\ 3\ 5\ 3\ 3\ 3\ 7]$, $\widetilde{L}'_2 = \delta_8[3\ 3\ 3\ 7\ 1\ 3\ 3\ 7]$. We know from (24) that $\widetilde{\Gamma} = \{\delta_8^4, \delta_8^5\}$. Then, $v_{j^*} = l^*$ and $\delta_8^1 \notin \widetilde{\Gamma}$. According to Theorem 3, DBCN



(a) $k^* = 2, l^* = 5, j^* = 5$.

(b) $k^* = 1, l^* = 5, j^* = 7$.

FIGURE 3. State transfer diagrams of DBCN (27) under control (28) after function perturbation: (a) $k^* = 2, l^* = 5, j^* = 5$; (b)$k^* = 1, l^* = 5, j^* = 7$. The red arrow represents the state transfer of perturbed point.

(26) is still robustly stabilizable to x_e under control (28) (see Figure 3(a)).

3) After function perturbation, the value $\mathrm{Col}_{39}(L_1)$ is changed from δ_8^3 to δ_8^4, that is, $k^* = 1$, $l^* = 5$ and $j^* = 7$, thus, $v_{j^*} = v_7 = 5$ and $\widetilde{L}'_1 = \delta_8[3\ 3\ 3\ 5\ 3\ 3\ 4\ 7]$, $\widetilde{L}'_2 = \delta_8[3\ 3\ 3\ 7\ 3\ 3\ 3\ 7]$. By a simple calculation, we obtain $\widetilde{\Gamma} = \{\delta_8^4, \delta_8^7\}$. Then, $v_{j^*} = l^*$ and $\delta_8^4 \in \widetilde{\Gamma}$. According to Theorem 3, DBCN (26) is not robustly stabilizable to x_e under control (28) (see Figure 3(b)).

VI. CONCLUSION

In this article, we have studied the robust stability and robust stabilization issues of DBNs with function perturbation. On one hand, we have constructed a state set to detect whether the robust stability of DBNs remains unchanged after function perturbation. On the other hand, we have derived several criteria to verify whether DBCNs with function perturbation can still maintain robust stabilization under a given state feedback stabilizer.

Future works contain the following contents.

1) This article only considers one-bit function perturbation in DBNs. Future work can study multi-bit function perturbations in DBNs.
2) One can generalize the obtained results to robust set stabilization and robust output tracking of DBCNs subject to function perturbations.
3) Gene mutation often occurs in a stochastic manner in practical genetic regulatory networks. Hence, we will focus on the impact of multi-bit stochastic function perturbations on the behavior of DBNs.

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