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RESEARCH ARTICLE

Detectability of Timed Discrete Event Systems

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ABSTRACT This paper introduces T-detectability, which extends detectability to timed discrete event systems within the context of communication networks. We propose network T-detectability to address the challenges posed by observation delays and losses in network environments. Our approach involves the construction of an augmented automaton to model delays and losses and an extended automaton to calculate state estimates. We present a method for checking network T-detectability and T-detectability. These findings contribute to a better understanding of system behavior in complex scenarios and provide analytical techniques for assessing detection properties in timed discrete event systems operating within communication networks.

INDEX TERMS Communication networks, detectability, observation delays, state estimation, observation losses, timed discrete event systems.

I. INTRODUCTION

Discrete event systems theory is a vibrant research field, with active efforts underway to tackle new challenges posed by complex, large-scale, and distributed systems, and to explore emerging applications in areas such as manufacturing, transportation, communication networks, and robotics [1], [2], [3], [4].

In manufacturing systems, such as fast assembly test and pack-out process, determining the state of each produced unit is crucial for tracking the overall manufacturing process. In the event of failures, it becomes essential to quickly locate and fix the problems to minimize downtime and ensure the prompt resumption of normal operations. Such scenarios can be effectively modeled as discrete event systems, characterized by discrete states and event-driven behavior. Consequently, the problem at hand can be framed as the state estimation problem of discrete event systems. The study of state estimation in discrete event systems has been a topic of research for several decades, with early work conducted in [5] and subsequent systematic works in [6] and [7]. Detectability, defined as the ability to estimate the current state and subsequent states of a system based on observations, plays a critical role in state estimation. It refers to the ability to detect the occurrence of faults or abnormal events in a system and involves determining whether a fault can be observed or identified when it occurs. A system is considered detectable if all faults or events occurring within the system can be detected.

In the context of manufacturing systems, detectability assumes great importance in identifying and promptly resolving problems. By ensuring timely detection of faults or failures, manufacturing processes can be effectively monitored. Additionally, high detectability facilitates the diagnosability process, which involves identifying the specific fault or failure responsible for the observed abnormality. Diagnosability, however, relies not only on detectability but also on additional analysis, reasoning, and historical knowledge or models of the system. Therefore, in manufacturing systems, the relationship between detectability and diagnosability becomes crucial. While detectability is a prerequisite for diagnosability, diagnosability encompasses more than just fault detection, requiring identification of the fault location and potentially the fault type or nature. By considering both detectability and diagnosability, manufacturing systems can optimize their fault management and troubleshooting processes, thereby minimizing downtime and ensuring efficient operations.

It is worth noting that in the field of discrete event systems, early research in detectability and subsequent systematic works have contributed to advancing the understanding of state estimation and fault management. The literature, such

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as the work done in [5] and the systematic works in [6] and [7], provides valuable insights into the principles and techniques involved in detectability and diagnosability of discrete event systems. By leveraging the knowledge and methodologies developed in these studies, manufacturing systems can enhance their ability to estimate the current and subsequent states of the system based on observations. This, in turn, facilitates effective fault detection, localization, and resolution, ultimately improving the overall performance and productivity of the manufacturing processes.

With the increasing use of networks in practical systems, the impact of communication delays, communication losses, and their effect on discrete event system observation and diagnosis has become a significant research area. For example, communication delays and losses have a significant impact on the observability and state estimation of discrete event system operating in communication networks. These delays can occur when events are transmitted from the controlled system to the supervisor or when observations are relayed back to the system. They can result from network congestion, transmission delays, and packet losses, among other factors. As a consequence, the observed event sequence may deviate from the actual event sequence, leading to potential errors in state estimation and detectability analysis. To tackle these challenges, recent research has focused on developing optimized estimation techniques for networked systems. Notably, the work presented in [8] proposes an advanced estimation approach that considers communication delays and losses in the system model. Researches in [9], [10], [11], and [12] have also investigated how to handle communication delays and losses in fault diagnosis of discrete event systems. Furthermore, the state estimation of networked discrete event systems with communication delays and losses between supervisors and plants has been systematically explored in studies conducted in [13], [14], [15], and [16]. In these works, delays are typically measured by the number of events that occur, often resulting in conservative results.

In this paper, we address communication delays and losses using a timed automaton proposed in [17]. This automaton introduces a special event, called *tick*, which represents the passage of time. It serves as a timing mechanism within the timed discrete event system model and is not subject to delays. By measuring communication delays using the number of *tick*, we provide a more accurate representation, to capture the effects of communication delays on the system's behavior and provide accurate state estimation. We assume that communication delays are random but upper-bounded. To handle observation delays and losses, we analyze their mechanism and propose an augmented automaton technique. Based on this technique, we construct an augmented automaton that incorporates all possible observations and a corresponding extended automaton. These automata enable us to calculate state estimates and determine the detectability of timed discrete event systems under communication delays and losses.

Our work presents a significant and novel approach compared to [18]. Our computational complexity is linear with respect to the upper bound N_o on observation delays and the number of states in the system. In compare with the delayed detectability studies in [7], our focus in this paper is to extend it to timed discrete event systems in the context of communication networks. We aim to provide a comprehensive framework that captures the effects of both observation delays and losses on the detectability of timed discrete event systems. To the best of our knowledge, no previous work has investigated the detectability of timed discrete event systems under observation delays and losses.

In our work, we primarily focus on addressing the challenges associated with first-in-first-out (FIFO) systems, as they are prevalent in various domains such as manufacturing, transportation, and communication networks. By assuming a FIFO order for delayed events, we aim to provide an applicable solution for a significant number of real-world systems. For systems observed from different channels where FIFO may not be satisfied, the addition of a timestamp is necessary to recover the order of event occurrences.

The remainder of the paper is organized as follows: Section II reviews timed discrete event systems. Section III extends the definition of detectability to the timed case and introduces T-detectability for timed discrete event systems. In Section IV, we discuss communication delays and losses and propose network T-detectability. We establish that T-detectability is a special case of network T-detectability, with our focus primarily on network T-detectability. Section V presents a method to construct an augmented automaton and an extended automaton for calculating state estimates. We also propose a method to check network T-detectability and T-detectability. Finally, we conclude the paper in Section VI. Due to page limitations, some proofs are omitted but are available from the author.

II. TIMED DISCRETE EVENT SYSTEMS

A timed discrete event system can be represented by its activity transition graph (ATG) or its timed transition graph (TTG). Here, we adopt the TTG $\widetilde{G}$ to describe a timed discrete event system as in [17]:

$$\widetilde{G} = (Q, \widetilde{\Sigma}, \rho, q_0),$$

where Q is the set of states, $\widetilde{\Sigma} = \Sigma \cup t$ is the set of events including *tick*, denoted by t, representing the elapse of a unit of time. Σ is the set of activity events. $\rho : Q \times \widetilde{\Sigma} \to Q$ is the (partial) state transition function, and q_0 is the initial state. For a given timed discrete event system, the TTG can be constructed from its ATG model. For more details, readers are referred to [17] and [3, Ch. 9].

In the usual way, we extend the transition function to $\rho : Q \times \widetilde{\Sigma}^* \to Q$. Additionally, we use $\rho(q, s)!$ to indicate that the transition $\rho(q, s)$ is defined. We assume that some activity events in Σ are observable, while the other activity events are unobservable. The *tick* event t is always observable. We use Σ_o and Σ_{uo} to denote the set of observable activity events and the set of unobservable activity events, respectively. For a given timed discrete event system, the set of observable events is denoted by $\widetilde{\Sigma}_o = \Sigma_o \cup t$, and the set of unobservable events is denoted by $\widetilde{\Sigma}_{uo} = \widetilde{\Sigma} - \widetilde{\Sigma}_o = \Sigma_{uo}$. The observation is defined by the natural projection $P : \widetilde{\Sigma}^* \rightarrow \widetilde{\Sigma}_o^*$ as

$$P(\varepsilon) = \varepsilon, P(s\sigma) = \begin{cases} P(s)\sigma & \sigma \in \widetilde{\Sigma}_o \\ P(s) & \sigma \notin \widetilde{\Sigma}_o. \end{cases}$$

The behavior of a timed discrete event system $\widetilde{G}$ is described by the language generated by $\widetilde{G}$. This language is defined as

$$L(\widetilde{G}) = \{s \in \widetilde{\Sigma}^* : \rho(q_0, s)!\}.$$

For an event sequence s, we use $Pr(s)$ to denote its prefix set, $|s|$ to denote its length, and $|t|(s)$ to denote the number of *tick* events t in s. For a set of states Q', we use $|Q'|$ to denote the number of elements in Q'. The prefix closure of a language is the set of prefixes of event sequences in that language. A language is considered (prefix) closed if it is equal to its prefix closure. In this paper, we focus only on closed languages.

For a given system, a possible trajectory is represented by an infinite sequence of events that the system may generate. The set of all trajectories is denoted as the ω language. Given a timed discrete event system $\widetilde{G}$, the set of trajectories is defined on the set of events $\widetilde{\Sigma}$ in $\widetilde{G}$ and is denoted as $L^{\omega}(\widetilde{G})$. Specifically, $L^{\omega}(\widetilde{G})$ is defined as

$$L^{\omega}(\widetilde{G}) = \{s \in L(\widetilde{G}) : |s| = \infty\}.$$

In an automaton, we refer to the transitions between any two states as paths. A path contains information about the states and events involved, and it is denoted as

$$q_0\sigma_1 q_1 \cdots \sigma_i q_i \cdots \sigma_n q_n,$$

where $q_i (i = 1, 2, \cdots, n)$ is a state transferred from state q_{i-1} by transitions labeled with event σ_i.

We use an example to show a timed discrete event system and how the above mentioned notations and operations work.

Example 1: Given a timed discrete event system modeled as TTG, denoted by $\widetilde{G}$, which is shown as an automaton in Fig. 1.

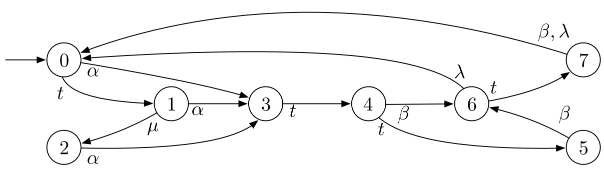

FIGURE 1. TTG model for a timed discrete event system $\widetilde{G}$.

The set of states is given as $Q = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and its cardinality is $|Q| = 8$. The set of events is denoted as $\widetilde{\Sigma} = \{\alpha, \beta, \mu, \lambda, t\}$. The initial state is $q_0 = 0$. Assuming that event β is unobservable, we have the set of observable events as $\widetilde{\Sigma}_o = \{\alpha, \mu, \lambda, t\}$, and the set of unobservable events as $\widetilde{\Sigma}_{uo} = \{\beta\}$. Starting from the initial state 0, the event α can occur, causing a transition from state 0 to state 3. Therefore, the transition $\rho(0, \alpha)$ is defined in $\widetilde{G}$, and $\rho(0, \alpha) = 3$.

We use $L(\widetilde{G})$ to denote the language generated by $\widetilde{G}$. It is evident that the event sequence $s = t\alpha t\beta \in L(\widetilde{G})$, and $\rho(q_0, s) = 6$. The prefix set of s is $Pr(s) = \{\varepsilon, t, t\alpha, t\alpha t, t\alpha t\beta\}$, where ε denotes the empty event sequence. The length of s is $|s| = 4$, and the number of *tick* events in s is $|t|(s) = 2$. The observation of s can be calculated as $P(s) = t\alpha t$. One of the trajectories generated by $\widetilde{G}$ is $(\alpha t\beta\lambda)^* = \alpha t\beta\lambda\alpha t\beta\lambda\cdots$. Additionally, the path $0t\ 1\alpha 3$ is a path in $\widetilde{G}$.

As discussed in [17], for timed discrete event systems, we have the following two assumptions.

Assumption 1: Only a finite number of events can occur in one unit of time, that is, $\widetilde{G}$ is Σ-loop free:

$$(\forall q \in Q)(\forall s \in \Sigma^* \setminus \{\varepsilon\})\rho(q, s) \neq q.$$

Assumption 2: $\widetilde{G}$ is deadlock free, that is,

$$(\forall q \in Q)(\exists \sigma \in \widetilde{\Sigma})\rho(q, \sigma)!$$

Note that Assumption 1 excludes the physically unrealistic possibility of infinite occurrences of activity events within one unit of time. Assumption 2 assumes that at any given state, there are either defined transitions with activity events or at least the t transition is defined, as the flow of time cannot be halted.

III. T-DETECTABILITY

In [6], the authors investigate the state estimation problem of discrete event systems and introduce the concept of detectability. A discrete event system is considered detectable if it is possible to determine the current state and subsequent states after a finite number of observations for all trajectories. In this section, we extend the definition of detectability to the timed case and further investigate the state estimation problem.

For a timed discrete event system $\widetilde{G}$ with an initial state denoted as q_0, when an event sequence $w \in P(L(\widetilde{G}))$ is observed, we denote the set of states in which $\widetilde{G}$ may reside as

$$E(w) = \{q \in Q : (\exists s \in \widetilde{\Sigma}^*)P(s) = w \wedge q \in \rho(q_0, s)\}.$$

We call the states set $E(w)$ the current state estimates. When observing an event sequence w, we can determine the current state of a timed discrete event system if the state estimate set contains only one state. In other words,

$$|E(w)| = 1.$$

We then formally define the T-detectability of timed discrete event systems as follows:

Definition 1 (T-Detectability): A timed discrete event system $\widetilde{G}$ is T-detectable if for all trajectories $L^{\omega}(\widetilde{G})$, after a finite of number observation $w \in P(Pr(s))(\in \widetilde{\Sigma}_o^*)$, we can always determine the current state and the subsequent states of the system, that is,

$$(\forall s \in L^{\omega}(\widetilde{G}))(\forall w \in P(Pr(s)))(\exists n \in \mathbb{N}^{+})$$
$$|w| \geq n \Rightarrow |E(w)| = 1.$$

We then use an example to show how we estimate the current state for a given timed discrete event system.

Example 2: We continue with the timed discrete event system shown in Fig. 1. If we assume that all events in $\widetilde{G}$ are observable, then regardless of the observation, we can always determine the current state. For instance, when observing αt, we can conclude that the system is in state 4. This holds true for all other observations as well. In this case, according to the definition of T-detectability, the timed discrete event system $\widetilde{G}$ is T-detectable.

However, if we assume that not all events are observable, specifically, the event β can never be observed, the situation changes. If an event sequence from $(\alpha t\lambda)^*$ is observed, we can still determine that the system is in state 0. However, if an event sequence from$(\alpha t\lambda\alpha t)^*$ is observed, the actual sequence of events that occurred could be from $(\alpha t\beta\lambda\alpha t)^*$ or $(\alpha t\beta\lambda\alpha t)^*\beta$.. Since we cannot determine whether the event β occurred or not after the last t, the state estimates become nondeterministic. The system could be in state 4 or state 6. Therefore, under the assumptions of this case, $\widetilde{G}$ is not T-detectable.

IV. NETWORK T-DETECTABILITY

In practical systems, the information exchange between the controlled system and the supervisor often takes place through communication networks. As a result, network delays and packet losses are inevitable. When considering these delays and losses, the state estimation problem becomes more complex.

We first focus on observation delays. In a timed discrete event system with communication delays, we need to consider two types of delays. (1) When an event is enabled, it may not occur instantly but with some delay. These delays, known as occurrence delays, have been studied in [17], and they are already included in the model of timed discrete event systems. (2) After an event occurs, it may not be observed immediately but with some delay. These delays are referred to as communication/observation delays. It is important to note that communication/observation delays are distinct from occurrence delays. In this paper, we specifically address communication/observation delays.

We use the *tick* event t to measure observation delays and assume that these delays are random but upper-bounded by N_o.[1] It is worth mentioning that the *tick* event t will always be observed without any delay. This implies that when an activity event occurs, it will be observed within a maximum of N_o units of time. To illustrate this concept, we provide the following example, which presents one possible observation for a given sequence of events that occurred.

Example 3: We continue with the timed discrete event system in Fig. 1. Suppose all events are observable and currently, the event sequence $\alpha tt\beta\lambda$ occurs. The occurrence time of each activity event as well as the *tick* event t is shown in Fig. 2.

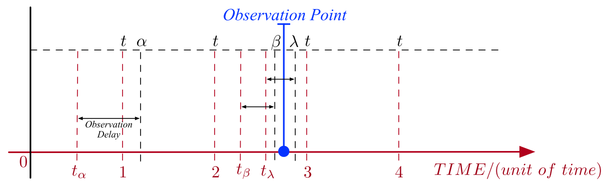

FIGURE 2. Occurrence and one possible observation of event sequence $\alpha tt\beta\lambda$ with $N_o = 1$.

In Fig. 2, t occurs at every $1, 2, 3, \cdots$ units of time. Event α, β and λ occur at t_α, t_β and t_λ, respectively. When there are no observation delays, the occurrences of event α, β and λ will be observed instantly, the observation will be $\alpha tt\beta\lambda$.

We then assume that random observation delays exist and the delays are upper bounded by 1. In this example, event α is observed with delay. Specifically, α occurs before the first *tick* but is actually observed after the first *tick* (before the upper bound of its observation delay). Event β and λ are also observed with delay. Before the shown observation point in Fig. 2, β is observed whereas λ has not been observed yet. The length of observation delays for each event is denoted in Fig. 2. For this case, at the shown observation point, the actual event sequence that is observed is $t\alpha t\beta$. In other words, when observation delay is upper bounded by 1, $t\alpha t\beta$ is one the possible observations of $\alpha tt\beta\lambda$. Note that we assume that the observation channel is FIFO, so α will always be observed before β and β will always be observed before λ. Following this strategy, we can enumerate the set of all the possible observations as

$$\{\alpha tt\beta\lambda, \alpha tt\beta, \alpha tt, t\alpha t\beta\lambda, t\alpha t\beta, t\alpha t\}.$$

Now let us consider the package losses in the observation. To model communication losses in observation, we denote the observation mapping under communication losses by Φ_L. We assume that only observable events may be lost. With a slight abuse of notations, we use ρ to denote the set of all possible transitions as $\rho = \{(q, \sigma, q') : \rho(q, \sigma) = q'\}$. The set of observable transitions is denoted by $\rho_o = \{(q, \sigma, q') : \rho(q, \sigma) = q' \wedge \sigma \in \Sigma_o\}$. Let $\rho_{loss} \subseteq \rho_o$ denotes the set of transitions that may be lost in communication. Note that transitions labeled with t will never be lost. For example, given an event sequence $t\alpha tt\beta\lambda \in L(\widetilde{G})$, assume that transitions labeled with event β may be lost in the observation channel. That means once event β occurs, it may be observed if the event is not lost in the observation channel. Another possible case is that it may not be observed because the event is lost in the observation channel. Assume no observation delays

[1] When we mention that delays are upper-bounded or bounded by N_o, it means that they are limited by N_o units of time. In this paper, it corresponds to N_o ticks.

exist in this case, the set of all possible observations after the occurrence of α and β will be $\{t\alpha tt\beta\lambda, t\alpha tt\lambda\}$.

If we consider both observation delays and losses, for event sequence $t\alpha tt\beta\lambda \in L(\widetilde{G})$, assume that observation delay is upper-bounded by $N_o = 1$ and all transitions labeled with β may be lost. The set of all possible event sequences will be

$$\{\alpha tt\beta\lambda, \alpha tt\lambda, \alpha tt\beta, \alpha tt, t\alpha t\beta\lambda, t\alpha t\beta, t\alpha t\}.$$

We then show how to model the observation delays and losses. We first model observation delays. We use $\Theta^i(s)$ to denote the set of observations with respect to i *ticks* fixed delays when all events are observable.

For a given event sequence $s \in L(\widetilde{G})$, we re-write it as

$$s = u_1tu_2t\cdots u_ltu_{l+1},$$

where $u_k \in \Sigma^*(k = 1, 2, \cdots, l+1)$ does not include any *ticks*. Note that u_k may be empty, that is, $u_k = \varepsilon$.

Assume there exists one *tick* observation delay. For any $u_k = \sigma_1\sigma_2\cdots\sigma_n$, any event in u_k may be delayed for 1 *tick* and the observation, denoted as $\theta(u_kt)$, is defined as

$$\theta(u_kt) = \{\sigma_1\cdots\sigma_nt, \sigma_1\cdots t\sigma_n, \cdots, t\sigma_1\cdots\sigma_n\}.$$

Particularly, if $u_k = \varepsilon$, we have $\theta(u_kt) = \{t\}$. The operation θ has the following property:

$$\theta(\sigma_1\ldots\sigma_jt) = \theta(\sigma_1\ldots\sigma_{j-1}t)\sigma_j \cup \{\sigma_1\ldots\sigma_jt\}.$$

We then define the set of observations with respect one *tick* delay as

$$DELAY^1(s) = \theta(u_1t)\theta(u_2t)\cdots\theta(u_lt)u_{l+1}.$$

The set of observations with respect to j *tick* delays can be obtained from $DELAY^1(s)$ recursively as

$$DELAY^j(s) = DELAY^1(DELAY^{j-1}(s)).$$

Finally, for any $s \in L(\widetilde{G})$, we have

$$\Theta^i(s) = DELAY^i(s)/\Sigma^*.$$

Note that, due to observation delays, activity events at the end of $DELAY^i(s)$ (after the last t) may not be observed yet.

For language $L(\widetilde{G})$, assuming that observation delays are upper-bounded by N_o. The set of all possible observations is given by

$$\Theta^{N_o}(L(\widetilde{G})) = \cup_{s\in L(\widetilde{G})}\Theta^{N_o}(s).$$

Now we extend observation to a general case in which some events are unobservable by natural projection P. We use $\Phi_D^{N_o}(s)$ to denote the observation set with partial observation. We then have

$$\Phi_D^{N_o}(s) = P(\Theta^{N_o}(s))(= \Phi_D(s)),$$

where $(= \Phi_D(s))$ means that, if N_o is understood, we use the simplified notation Φ_D for $\Phi_D^{N_o}$, and similarly for other notations to be introduced later.

We further extend the definition $\Phi_D^{N_o}(\cdot)$ from event sequences to language $L(\widetilde{G})$ as

$$\Phi_D^{N_o}(L(\widetilde{G})) = \cup_{s\in L(\widetilde{G})}\Phi_D^{N_o}(s)(= \Phi_D(L(\widetilde{G}))).$$

The inverse mapping of $\Phi_D^{N_o}$ is denoted as $(\Phi_D^{N_o})^{-1}$. For an observation w, $(\Phi_D^{N_o})^{-1}(w)$ is defined as

$$(\Phi_D^{N_o})^{-1}(w) = \{s \in L(\widetilde{G}) : w \in \Phi_D^{N_o}(s)\}(= \Phi_D{}^{-1}(w)).$$

To model communication losses in observation, let $s = \sigma_1\sigma_2t\sigma_3\cdots\sigma_it\cdots\sigma_k$, $(\sigma_i \in \Sigma)$. $\Phi_L(s)$ is obtained by replacing σ_i with ε or σ_i for all events of which the transitions may be lost. We extend $\Phi_L(\cdot)$ from event sequence to language as

$$\Phi_L(L(\widetilde{G})) = \cup_{s\in L(\widetilde{G})}\Phi_L(s).$$

For an event sequence s, to consider both observation delays and losses, we use $\Phi_{DL}^{N_o}$ to denote the set of all possible observations under communication delays and losses, the observation delays are bounded by N_o. For an event sequence $s \in L(\widetilde{G})$, $\Phi_{DL}^{N_o}(s)$ is obtained by the composition of two mappings Φ_D and Φ_L:

$$\Phi_{DL}^{N_o}(s) = \Phi_D^{N_o}(\Phi_L(s))(= \Phi_{DL}(s)).$$

Therefore, observation delays and losses are completely captured by observation mapping $\Phi_{DL}^{N_o}$. We extend $\Phi_{DL}^{N_o}$ from an event sequence to a language in the same way.

The inverse mapping of $\Phi_{DL}^{N_o}$ is denoted as $(\Phi_{DL}^{N_o})^{-1}$. For an observation w under observation delays and losses, $(\Phi_{DL}^{N_o})^{-1}(w)$ is defined as

$$(\Phi_{DL}^{N_o})^{-1}(w) = \{s \in L(\widetilde{G}) : w \in \Phi_{DL}^{N_o}(s)\} \; (= \Phi_{DL}{}^{-1}(w)).$$

We use an example to illustrate the results.

Example 4: We still consider the timed discrete event system shown in Fig. 1. Assume that observation delay is bounded by $N_o = 1$. We first assume that all events in $\widetilde{G}$ are observable and no events may be lost. Given an event sequence $s = \alpha tt\beta\lambda$, we calculate $\Theta_D(s)$ as follows:

Firstly, we re-write s as $s = u_1tu_2tu_3$, where $u_1 = \alpha$, $u_2 = \varepsilon$, $u_3 = \beta\lambda$. Secondly, we calculate $\theta(u_1t) = \{\alpha t, t\alpha\}$, $\theta(u_2t) = \{t\}$. Thirdly, we calculate $DELAY^{N_o}(s) = \{\alpha tt\beta\lambda, t\alpha t\beta\lambda\}$. Finally, we calculate $\Theta_D(s)$ as

$$\Theta_D = \{\alpha tt\beta\lambda, \alpha tt\beta, \alpha tt, t\alpha t\beta\lambda, t\alpha t\beta, t\alpha t\}.$$

We then assume all events are observable, but all transitions labeled with β may be lost in the observation channel. Following the above calculation steps, we calculate $\Phi_{DL}(s)$ as:

$$\Phi_{DL}(s) = \{\alpha tt\beta\lambda, \alpha tt\lambda, \alpha tt\beta, \alpha tt, t\alpha t\beta\lambda, t\alpha t\beta, t\alpha t\}.$$

The result is the same as what we obtained in Example 3.

We then consider the state estimation and detectability problem under observation delays and losses. Given a timed discrete event system $\widetilde{G}$, the initial state is q_0. Assume that there exist observation delays and losses in the observation channel. The observation delays are upper-bounded by N_o. After observing event sequence w, the set of the states that $\widetilde{G}$ may be in is denoted as

$$TE_{DL}^{N_o}(w) = \{q \in Q : (\exists s \in L(\widetilde{G}))\Phi_{DL}^{N_o}(s) = w \wedge q \in \rho(q_0, s)\}(= TE(w)).$$

We call the states set the current state estimates of timed discrete event systems under observation delays and losses.

Similarly, for an observation w, if the state estimate set $TE(w)$ has only one state, we can determine the current state of the system. That is

$$|TE(w)| = 1.$$

Under communication networks, with the consideration of the influences of both observation delays and losses, we further extend T-detectability to networked cases and define network T-detectability as follows:

Definition 2 (Network T-Detectability): Give a timed discrete event system $\widetilde{G}$ under observation delays and losses, the observation delays are upper-bounded by N_o and the observation losses are denoted by ρ_{loss}. $\widetilde{G}$ is networked T-detectable if we can determine the current state and the subsequent states after a finite number of observations for all trajectories, that is

$$(\forall s \in L^{\omega}(\widetilde{G}))(\forall w \in \Phi_{DL}^{N_o}(Pr(s)))(\exists n \in \mathbb{N}^{+})|w| \geq n \Rightarrow |TE(w)| = 1.$$

In other words, for any trajectory s in the language $L^{\omega}(\widetilde{G})$ and any observation sequence w in the set $\Phi_{DL}^{N_o}(Pr(s))$, there exists a positive integer n such that if the length of w is greater than or equal to n, then the number of reachable states from the set of timed events $TE(w)$ is equal to one, indicating a unique determination of the system's state. This definition captures the notion of network T-detectability, emphasizing the requirement for a finite number of observations to yield a unique determination of the system's state, even in the presence of observation delays and losses.

Remark 1: When there are no communication delays and losses in the observation channel, for any occurred event sequence $s \in L(\widetilde{G})$, the observed event sequence will be $P(s)$. Then Definition 2 will be reduced to Definition 1. Therefore, T-Detectability is a special case of network T-Detectability. We will focus on network T-Detectability.

Remark 2: Network T-detectability differs from the traditional notions of weak detectability and strong detectability. While weak and strong detectability primarily address the general ability to detect faults or events in a system, network T-detectability specifically considers the impact of observation delays and losses in a networked environment.

V. CHECKING NETWORK T-DETECTABILITY

A. AUGMENTED AUTOMATON

In order to check network T-detectability, we need to calculate the current state estimate of the timed discrete event system considering observation delays and losses. Assuming that the currently observed event sequence is denoted as w, let's discuss the procedure for determining the current state estimate $TE(w)$ in the presence of observation delays and losses.

We firstly construct an augmented automaton as a tool to calculate the state estimates. The augmented automaton $\widetilde{G}^{N_o}$ is constructed as

$$\widetilde{G}^{N_o} = (Z, \widetilde{\Sigma}, \rho^{N_o}, z_0),$$

where $z = Q \times \{0, 1, 2, \cdots, N_o\}$. Every state $z = (q, n)$ in Z is a pair, of which the first element is the state that the system is in, the second element is the number of *ticks* which denotes the current specific observation delays. The initial state is $z_0 = (q_0, 0)$. The transitions of the augmented automaton are defined as follows.

1. For any state $(q, n) \in Z$, $n \leq N_o$, for any event $\sigma \in \widetilde{\Sigma}$, if $\rho(q, \sigma) = q'$, the transition $((q, n), \sigma, (q', n))$ is then defined. Hence, we have

$$\rho_1^{N_o} = \{((q, n), \sigma, (q', n)) : (q, n) \in Z \wedge n \leq N_o \wedge \rho(q, \sigma) = q'\}. \quad (1)$$

It describes the case in which the current delays do not change.

2. For any state $(q, n) \in Z$, one more *tick* delay can occur if $n < N_o$. Hence, we have

$$\rho_2^{N_o} = \{((q, n), t, (q, n+1)) : (q, n) \in Z \wedge n < N_o\}. \quad (2)$$

It describes the case in which the current delays increase.

3. For any state $(q, n) \in Z$, if $n > 0$ and $\rho(q, t)!$, events at state $\rho(q, t)$ may be observed with delays reduced by one *tick*, that is,

$$\rho_3^{N_o} = \{((q, n), \varepsilon, (q', n-1)) : (q, n) \in Z \wedge 0 < n \leq N_o \wedge \rho(q, t) = q'\}. \quad (3)$$

It describes the case in which the current delays decrease.

We then formally define the transitions ρ^{N_o} as

$$\rho^{N_o} = \rho_1^{N_o} \cup \rho_2^{N_o} \cup \rho_3^{N_o}.$$

Note that $\widetilde{G}^{N_o}$ is nondeterministic.

Property 1: For any path tr generated by $\widetilde{G}^{N_o}$ such that

$$tr = (q_0, 0)\sigma_1(q_1, n_1)\sigma_2 \cdots \sigma_m(q_m, n_m),$$

there exists an event sequence $s \in L(\widetilde{G})$ such that

$$\sigma_1\sigma_2 \cdots \sigma_m \in DELAY^{N_o}(st^{n_m}) \wedge q_m = \rho(q_0, s).$$

Property 2: For any event sequence $s \in L(\widetilde{G})$, any $n_m \leq N_o$ and any observation $w = \sigma_1\sigma_2 \cdots \sigma_m \in DELAY^{N_o}(st^{n_m})$, there exists a path tr generated by $\widetilde{G}^{N_o}$ such that

$$tr = (q_0, 0)\sigma_1(q_1, n_1) \cdots \sigma_m(q_m, n_m) \wedge q_m = \rho(q_0, s).$$

Theorem 1: The language generated by $\widetilde{G}^{N_o}$ equals the language obtained by performing the operation Θ^{N_o} on $L(\widetilde{G})$, that is,

$$L(\widetilde{G}^{N_o}) = \Theta^{N_o}(L(\widetilde{G})).$$

Proof: In this proof, we aim to demonstrate the equivalence between the event sequences in $L(\widetilde{G}^{N_o})$ and $\Theta^{N_o}(L(\widetilde{G}))$. To establish this equivalence, we split the proof into two parts: the "if" part and the "only if" part, to prove that $w \in L(\widetilde{G}^{N_o})$ if and only if $w \in \Theta^{N_o}(L(\widetilde{G}))$.

(If part) Suppose there exists an event sequence $w = \sigma_1\sigma_2 \cdots \sigma_m \in L(\widetilde{G}^{N_o})$, and $(q_m, n_m) \in \rho^{N_o}((q_0, 0), w)$, we prove that $w \in \Theta^{N_o}(L(\widetilde{G}))$ as follows.

$$
\begin{aligned}
& w \in L(\widetilde{G}^{N_o}) \\
& \Rightarrow (\exists tr \in Tr(w)) tr = (q_0, 0)\sigma_1 \cdots \sigma_m (q_m, n_m) \\
& \Rightarrow (\exists s \in L(\widetilde{G})) w \in DELAY^{N_o}(st^{n_m}) \wedge q_m = \rho(q_0, s) \\
& \quad \text{(By Property 1)} \\
& \Rightarrow (\exists s \in L(\widetilde{G})) w \in DELAY^{N_o}(st^{n_m}) \\
& \wedge (q_m, n_m) \in \rho^{N_o}((q_0, 0), w) \\
& \quad \text{(By Equation (1) and Equation (2))} \\
& \Rightarrow (\exists s \in L(\widetilde{G}))(\exists s' \in \widetilde{\Sigma}^*)|t|(s') = n_m \wedge ss' \in L(\widetilde{G}) \\
& \wedge w \in DELAY^{N_o}(st^{n_m}) \wedge (q_m, n_m) \in \rho^{N_o}((q_0, 0), w) \\
& \quad \text{(By Assumption 2)} \\
& \Rightarrow (\exists s \in L(\widetilde{G}))(\exists s' \in \widetilde{\Sigma}^*)(\exists q \in Q)|t|(s') = n_m \\
& \wedge w \in DELAY^{N_o}(st^{n_m}) \\
& \wedge (q_m, n_m) \in \rho^{N_o}((q_0, 0), w) \wedge \rho(q_m, s') = q \\
& \Rightarrow (\exists s \in L(\widetilde{G}))(\exists s' = u_1tu_2t \cdots u_{n_m}tu_{n_m+1}) \\
& (\exists q \in Q) w \in DELAY^{N_o}(st^{n_m}) \\
& \wedge (q_m, n_m) \in \rho^{N_o}((q_0, 0), w) \\
& \wedge \rho(q_m, u_1tu_2t \cdots u_{n_m}tu_{n_m+1}) = q \\
& \Rightarrow (\exists s \in L(\widetilde{G}))(\exists s' = u_1tu_2t \cdots u_{n_m}tu_{n_m+1}) \\
& (\exists q \in Q) w \in DELAY^{N_o}(st^{n_m}) \\
& \wedge (q, 0) \in \rho^{N_o}((q_0, 0), wu_1u_2 \cdots u_{n_m}u_{n_m+1}) \\
& \quad \text{(By Equation (1) and Equation (3))} \\
& \Rightarrow (\exists s'' \in L(\widetilde{G})) s'' = ss' = su_1tu_2t \cdots u_{n_m}tu_{n_m+1} \\
& \wedge wu_1u_2 \cdots u_{n_m}u_{n_m+1} \in DELAY^{N_o}(s'') \\
& \wedge q = \rho(q_0, s'') \\
& \quad \text{(By Property 1 and } n_m = 0) \\
& \Rightarrow (\exists s'' \in L(\widetilde{G})) w \in DELAY^{N_o}(s'')/\Sigma^* \\
& \Rightarrow (\exists s'' \in L(\widetilde{G})) w \in \Theta^{N_o}(s'') \\
& \Rightarrow w \in \Theta^{N_o}(L(\widetilde{G})).
\end{aligned}
$$

(Only if part) Suppose there exists an event sequence $w \in \Theta^{N_o}(L(\widetilde{G}))$, we prove that $w \in L(\widetilde{G}^{N_o})$ as follows.

$$
\begin{aligned}
& w \in \Theta^{N_o}(L(\widetilde{G})) \\
& \Rightarrow (\exists s \in L(\widetilde{G})) w \in \Theta^{N_o}(s) \\
& \Rightarrow (\exists s \in L(\widetilde{G})) w \in DELAY^{N_o}(s)/\Sigma^* \\
& \Rightarrow (\exists s \in L(\widetilde{G}))(\exists u \in \Sigma^*) wu \in DELAY^{N_o}(s) \\
& \Rightarrow (\exists u \in \Sigma^*) wu \in L(\widetilde{G}^{N_o}) \\
& \quad \text{(By Property 2)} \\
& \Rightarrow w \in L(\widetilde{G}^{N_o}).
\end{aligned}
$$

■

Remark 3: Although automaton $\widetilde{G}$ is deterministic, the proposed method to construct an augmented automaton is also applicable to a nondeterministic automaton. In addition, if there are no observation delays and losses, the augmented automaton will be the same as $\widetilde{G}$.

We now demonstrate how to handle partial observations and observation losses using the augmented automaton method. To accommodate observations under observation delays and losses, we define the set of all observations as $\Phi_{DL}(L(\widetilde{G}))$. Our objective is to construct an automaton $\widetilde{G}_{DL}$ that can represent all observations in $\Phi_{DL}(L(\widetilde{G}))$. The concept is illustrated in Fig. 3.

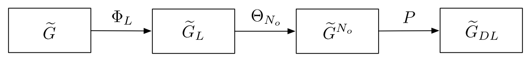

FIGURE 3. The process for calculating $\Phi_{DL}(L(\widetilde{G}))$.

In detail, for a given timed discrete event system $\widetilde{G} = \{Q, \widetilde{\Sigma}, \rho, q_0\}$, assume that delays upper bounded by N_o and there exist observation losses described by ρ_{loss}. We construct $\widetilde{G}_{DL}$ to describe all the observations under observation delays and losses as the following steps.

For any transition $(q, \sigma, q') \in \rho_{loss}$, we add another $\varepsilon-$transition from q to q' as (q, ε, q'). We then construct the automaton $\widetilde{G}_L$, which is denoted as:

$$\widetilde{G}_L = (Q, \widetilde{G}, \rho_L, q_0),$$

where $\rho_L = \rho \cup \{(q, \varepsilon, q') :\in \rho_{loss}\}$.

In the next step, we construct the augmented automaton for $\widetilde{G}_L$ according to our proposed methods and get the automaton $(\widetilde{G}_L)^{N_o}$. The $(\widetilde{G}_L)^{N_o}$ is denoted as

$$(\widetilde{G}_L)^{N_o} = (Z, \widetilde{G}, \rho_L^{N_o}, z_0).$$

We then consider and partial observation and add the natural project to construct the automaton $\widetilde{G}_{DL}$ from $(\widetilde{G}_L)^{N_o}$. We complete this step by replacing any event σ in $(\widetilde{G}_L)^{N_o}$ with their natural project $P(\sigma)$. The automaton $\widetilde{G}_{DL}$ is denoted as

$$\widetilde{G}_{DL} = (Z, \widetilde{G}, \rho_{DL}^{N_o}, z_0),$$

where $\rho_{DL}^{N_o} = \{(z, P(\sigma), z') : (z, \sigma, z') \in \rho_L^{N_o}\}$.

Consolidating all the constructing processes together, we can get the relations between constructed automatons and operations as

$$
\begin{aligned}
L(\widetilde{G}_L) &= \Phi_L(L(\widetilde{G})), \\
L((\widetilde{G}_L)^{N_o}) &= \Theta^{N_o}(L(\widetilde{G}_L)), \\
L(\widetilde{G}_{DL}) &= P(L((\widetilde{G}_L)^{N_o})).
\end{aligned}
$$

Remark 4: When there are no observation delays and losses, the $\widetilde{G}_{DL}$ will be obtained by simply applying the natural project on $\widetilde{G}$.

Corollary 1: For any path tr generated by $(\widetilde{G}_L)^{N_o}$ such that

$$tr = (q_0, 0)\sigma_1(q_1, n_1)\sigma_2 \cdots \sigma_m(q_m, n_m),$$

there exists an event sequence $s \in L(\widetilde{G}_L)$ such that

$$\sigma_1\sigma_2\cdots\sigma_m \in DELAY^{N_o}(st^{n_m}) \wedge q_m = \rho_L(q_0, s).$$

Corollary 2: For any event sequence $s \in L(\widetilde{G}_L)$, any $n_m \leq N_o$ and any observation $w = \sigma_1\sigma_2\cdots\sigma_m \in DELAY^{N_o}(st^{n_m})$, there exists a path tr generated by $(\widetilde{G}_L)^{N_o}$ such that

$$tr = (q_0, 0)\sigma_1(q_1, n_1)\cdots\sigma_m(q_m, n_m) \wedge q_m = \rho_L(q_0, s).$$

The following theorem shows automaton $\widetilde{G}_{DL}$ generates all the observations in $\Phi_{DL}(L(\widetilde{G}))$.

Theorem 2: The language generated by $\widetilde{G}_{DL}$ is equal to $\Phi_{DL}(L(\widetilde{G}))$, that is,

$$L(\widetilde{G}_{DL}) = \Phi_{DL}(L(\widetilde{G})).$$

Proof: To demonstrate this equality, we start by observing that $L(\widetilde{G}_{DL})$ can be expressed as $P(L((\widetilde{G}_L)^{N_o}))$. This follows from the definition of $\widetilde{G}_{DL}$, which represents the timed system under observation delays and losses. It is clear that

$$\begin{aligned} L(\widetilde{G}_{DL}) &= P(L((\widetilde{G}_L)^{N_o})) \\ &= P(\Theta^{N_o}(L(\widetilde{G}_L))) \\ &\quad \text{(By Theorem 1)} \\ &= \Phi_D(L(\widetilde{G}_L)) \\ &\quad \text{(By the definition of } \Phi_D) \\ &= \Phi_D(\Phi_L(L(\widetilde{G}))) \\ &= \Phi_{DL}(L(\widetilde{G})). \\ &\quad \text{(By the definition of } \Phi_{DL}) \end{aligned}$$

■

Remark 5: Let us consider the computational complexity of constructing $\widetilde{G}_{DL}$. The number of states in $\widetilde{G}_{DL}$ equals the number of states in $(\widetilde{G}_L)^{N_o}$. It equals $|Q| \times |N_o + 1|$. Hence, the computational complexity of constructing $\widetilde{G}_{DL}$ is linear with respect to the number of states in $\widetilde{G}$ and the upper bound of N_o.

We use an example to illustrate the result.

Example 5: We still consider the timed discrete event system shown in Fig. 1. Assume that observation delays are bounded by $N_o = 1$ and all events are observable. All the transitions labeled with β may be lost. We then construct the automaton $\widetilde{G}_{DL}$ as in Fig. 4.

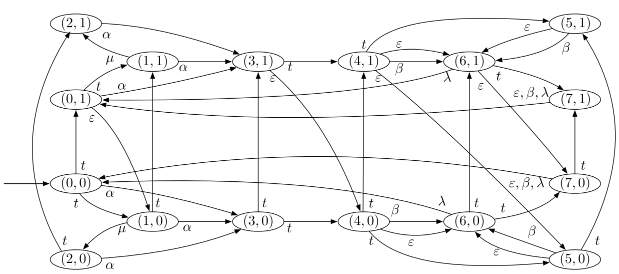

FIGURE 4. The automaton $\widetilde{G}_{DL}$.

With the augmented automaton $\widetilde{G}_{DL}$ at our disposal, we can effectively track the observations and estimate the states of the timed discrete event system. For example, if the event sequence $tt\alpha$ is observed, from the initial state, following the path in $\widetilde{G}_{DL}$, the augmented automaton is transferred to state (3, 1), which means the current observation has one *tick* delay. The actual state that the system may be in is the states that can be reached after one *tick* from state 3, following the path in Fig. 1, they are the states 4, 6, 0 and 3.

Note that if $N_o > 1$, we can still apply the same methodology by iterating the process for each unit of observation delay. Instead of directly considering a single observation delay of N_o, we can divide the overall delay into multiple units of observation delay, each of which corresponds to a delay of one time unit. We will discuss about more details in the next section.

B. METHOD TO CHECK NETWORK T-DETECTABILITY

We first show how to calculate the state estimates after observing $w \in \Phi_{DL}(L(\widetilde{G}))$. Since automaton $\widetilde{G}_{DL}$ is nondeterministic, we convert it into an equivalent deterministic automaton by constructing the observer $\widetilde{G}_{DL,obs}$ as

$$\widetilde{G}_{DL,obs} = (X, \widetilde{\Sigma}_o, \xi, x_0) = Ac(2^Z, \widetilde{\Sigma}_o, \xi, UR(z_0)),$$

where the set of marked states contains all the singleton states in the observer. $Ac(\cdot)$ denotes the accessible part, and UR denotes the unobservable reach, defined for $x \subseteq Z$ as

$$UR(x) = \{z' \in Z : (\exists z \in x)\rho_{DL}(z, \varepsilon) = z'\}.$$

The transition function ξ is defined for $x \in X$ and $\sigma \in \widetilde{\Sigma}_o$ as

$$\xi(x, \sigma) = UR(\{z \in Z : (\exists z' \in x)\rho_{DL}(z', \sigma) = z\}).$$

For an observed event sequence w that lead to state (q, n), there may exist observation delays of n *ticks*. So the state that system may be in after observing w is not state q, but one of these states reached by event sequences including n *ticks* from state q. We define an operation DR on state $z = (q, n)$ as

$$DR(z) = \{q' \in Q : (\exists s \in \widetilde{\Sigma}^*)|t|(s) = n \wedge \rho(q, s) = q'\}.$$

We extend the operation DR to a set of states. For a set of states $Z_i \subseteq Z$, the set of states reachable from Z_i with respect to the corresponding current observation delays, is defined as

$$DR(Z_i) = \cup_{z \in Z_i} DR(z).$$

To consider current observation delays, we extend every state set $Z_i \in X$ to $DR(Z_i)$ and use y to denote the resulting states set. Furthermore, we use Y to represent all the extensions. In other words, if $X = \{x_0, x_q, \cdots, x_m\}$, we always have $Y = \{y_0, y_q, \cdots, y_m\}$. For every x_i, the corresponding y_i is obtained by $y_i = DR(x_i)$. We then set all state $y_i \in Y$ which have only one state element as marked states and define the new extended automaton $\widetilde{G}_{exd}$ as

$$\widetilde{G}_{exd} = (Y, \widetilde{\Sigma}_o, \zeta, y_0, Y_m),$$

where $Y \in 2^Q$. The transition function is defined for $y_i, y_j \in Y$ and $\sigma \in \widetilde{\Sigma}_o$ as $\zeta = \{(y_i, \sigma, y_j) : (x_i, \sigma, x_j) \in \xi\}$.

Remark 6: When there are no observation delays and losses, then we can firstly follow Remark 4 to construct $\widetilde{G}_{DL}$, then the extended automaton $\widetilde{G}_{exd}$ is actually the observer of $\widetilde{G}_{DL}$.

Given an extended automaton $\widetilde{G}_{exd}$, we use Y_{loop} to denote all the states in loops, which is denoted as

$$Y_{loop} = \{y \in Y : (\exists s \in \widetilde{\Sigma}^*) s \neq \varepsilon \wedge \zeta(y, s) = y\}.$$

For any state y of an extended automaton $\widetilde{G}_{exd}$, we use $QR(y)$ to denote the set of states that can be reached from y after number of events. $QR(y)$ is denoted as follows.

$$QR(y) = \{y' \in Y : (\exists s \in \widetilde{\Sigma}_o^*)\zeta(y, s)!\}.$$

$QR(\cdot)$ can also be extended from state to state set. We use Y to denote a states set, $QR(Y)$ is then defined as

$$QR(Y) = \cup_{y \in Y} QR(y).$$

We then have the following theorem to show that under network environment where observation losses exist and observation delays are upper-bounded by N_o, the extend automaton $\widetilde{G}_{exd}$ can be used to calculate the state estimate for any given observation.

Theorem 3: Given a timed discrete event system $\widetilde{G}$ with observation losses described by ρ_{loss} and observation delays upper-bounded by N_o, for any given observation w, the state estimate $TE(w)$ can be obtained by the extend automaton $\widetilde{G}_{exd}$ as

$$TE(w) = \zeta(y_0, w). \tag{4}$$

Proof: In this proof, we aim to establish a connection between the extended observer $\widetilde{G}_{exd}$ and the transition behavior of the observation sequence w. We first prove that for any given observation w, the transition $\zeta(y_0, w)$ is defined in the extend observer $\widetilde{G}_{exd}$. Actually, for any $w \in \widetilde{\Sigma}_o^*$,

$$\begin{aligned} & w \in \Phi_{DL}(L(\widetilde{G})) \\ \Leftrightarrow & w \in L(\widetilde{G}_{DL}) \\ & \quad \text{(By Theorem 2.)} \\ \Leftrightarrow & w \in L(\widetilde{G}_{DL,obs}) \\ & \quad \text{(Since } \widetilde{G}_{DL,obs} \text{ is the observer of } \widetilde{G}_{DL}) \\ \Leftrightarrow & w \in L(\widetilde{G}_{exd}) \\ \Leftrightarrow & \zeta(y_0, w)!. \end{aligned}$$

Next, we proceed to prove the theorem by showing that for any state $q \in Q$, $q \in TE(w)$ if and only if $q \in \zeta(y_0, w)$. We go through a series of equivalences to establish this result. On the one hand,

$$\begin{aligned} & q \in \zeta(y_0, w) \\ & \Leftrightarrow q \in Q \wedge (\exists (q', n) \in \xi(x_0, w))(\exists s' \in \widetilde{\Sigma}^*)|t|(s') = n \\ & \wedge \rho(q', s') = q \\ & \Leftrightarrow q \in Q \wedge (\exists (q', n) \in \rho_{DL}(z_0, w))(\exists s' \in \widetilde{\Sigma}^*) \\ & |t|(s') = n \wedge \rho(q', s') = q \\ & \quad \text{(By the definition of } \widetilde{G}_{DL,obs}) \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(q', n) \in \rho_L^{N_o}(z_0, s) \\ & \wedge (\exists s' \in \widetilde{\Sigma}^*)|t|(s') = n \wedge \rho(q', s') = q \\ & \quad \text{(By the definition of } \rho_{DL}) \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(q', n) \in \rho_L^{N_o}(z_0, s) \\ & \wedge (\exists s' = u_1 t u_2 t \cdots u_n t u_{n+1} \in \widetilde{\Sigma}^*) \\ & \rho(q', u_1 t u_2 t \cdots u_n t u_{n+1}) = q \\ & \quad \text{(Because } |t|(s') = n) \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(\exists u = u_i u_2 \cdots u_n u_{n+1} \in \Sigma^*) \\ & (q', n) \in \rho_L^{N_o}(z_0, s) \\ & \wedge (q, 0) \in \rho_L^{N_o}((q', n), u_1 u_2 \cdots u_n u_{n+1}) \\ & \quad \text{(By Equation (1) and Equation (3))} \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(\exists u \in \Sigma^*) \\ & (q', n) \in \rho_L^{N_o}(z_0, s) \wedge (q, 0) \in \rho_L^{N_o}((q', n), u) \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(\exists u \in \Sigma^*) \\ & (q, 0) \in \rho_L^{N_o}(z_0, su). \end{aligned}$$

On the other hand,

$$\begin{aligned} & q \in TE(w) \\ & \Leftrightarrow q \in Q \wedge (\exists s' \in L(\widetilde{G}))w \in \Phi_{DL}(s') \wedge q = \rho(q_0, s') \\ & \Leftrightarrow q \in Q \wedge (\exists s' \in L(\widetilde{G}))w \in \Phi_D(\Phi_L(s')) \wedge q = \rho(q_0, s') \\ & \Leftrightarrow q \in Q \wedge (\exists s' \in L(\widetilde{G}))w \in P(\Theta^{N_o}(\Phi_L(s'))) \\ & \wedge q = \rho(q_0, s') \\ & \Leftrightarrow q \in Q \wedge (\exists s' \in L(\widetilde{G}))(\exists s \in P^{-1}(w))(\exists s'' \in \Phi_L(s')) \\ & \wedge s \in \Theta^{N_o}(s'') \wedge q = \rho_L(q_0, s'') \\ & \Leftrightarrow q \in Q \wedge (\exists s' \in L(\widetilde{G}))(\exists s \in P^{-1}(w))(\exists s'' \in \Phi_L(s')) \\ & s \in DELAY^{N_o}(s'')/\widetilde{\Sigma}^* \wedge q = \rho_L(q_0, s'') \\ & \Leftrightarrow q \in Q \wedge (\exists s \in P^{-1}(w))(\exists s'' \in L(\widetilde{G}_L)) \\ & s \in DELAY^{N_o}(s'')/\widetilde{\Sigma}^* \wedge q = \rho_L(q_0, s''). \\ & \quad \text{(By the definition of } \widetilde{G}_L) \end{aligned}$$

Hence, it remains to prove

$$\begin{aligned} & (\exists u \in \Sigma^*)(q, 0) \in \rho_L^{N_o}(z_0, su) \\ \Leftrightarrow & (\exists s'' \in L(\widetilde{G}_L))s \in DELAY^{N_o}(s'')/\widetilde{\Sigma}^* \\ & \wedge q = \rho_L(q_0, s'') \end{aligned}$$

as follows.

On one hand, under both communication delays and losses, suppose there exists an event sequence $su = \sigma_1'\sigma_2' \cdots \sigma_n' u \in L((\widetilde{G}_L)^{N_o})$, and $(q, 0) \in \rho_L^{N_o}((q_0, 0), su)$, we then have

$$\begin{aligned} & (\exists u \in \Sigma^*)(q, 0) \in \rho_L^{N_o}(z_0, su) \\ \Rightarrow & (\exists u \in \Sigma^*)(\exists tr' \in Tr(su))tr' = (q_0, 0)\sigma_1' \cdots \sigma_n' u(q, 0) \\ \Rightarrow & (\exists u \in \Sigma^*)(\exists s'' \in L(\widetilde{G}_L))su \in DELAY^{N_o}(s'') \\ & \wedge q = \rho_L(q_0, s'') \end{aligned}$$

$$\text{(By Corollary 1 with } n_m = 0)$$
$$\Rightarrow (\exists s'' \in L(\widetilde{G}_L)) s \in DELAY^{N_o}(s'')/\Sigma^* \wedge q = \rho_L(q_0, s'')$$

On the other hand,

$$(\exists s'' \in L(\widetilde{G}_L)) s \in DELAY^{N_o}(s'')/\widetilde{\Sigma}^* \wedge q = \rho_L(q_0, s'')$$
$$\Rightarrow (\exists s'' \in L(\widetilde{G}_L))(\exists u \in \Sigma^*) su \in DELAY^{N_o}(s'') \wedge q = \rho_L(q_0, s'')$$
$$\Rightarrow (\exists u \in \Sigma^*)(\exists tr' \in Tr(su)) tr' = (q_0, 0) su (q, 0) \wedge su \in L((\widetilde{G}_L)^{N_o})$$
$$\text{(By Corollary 2 with } n_m = 0)$$
$$\Rightarrow (\exists u \in \Sigma^*) su \in L((\widetilde{G}_L)^{N_o}) \wedge (q, 0) \in \rho_L^{N_o}(z_0, su)$$
$$\text{(Because } tr' \text{ is a path of } (\widetilde{G}_L)^{N_o})$$
$$\Rightarrow (\exists u \in \Sigma^*)(q, 0) \in \rho_L^{N_o}(z_0, su).$$

■

With the state estimation result, we then use the following theorem to show how to check network T-detectability.

Theorem 4: For a timed discrete event system $\widetilde{G} = \{Q, \widetilde{\Sigma}, \rho, q_0\}$, considering observation delays upper-bounded by N_o and observation losses described by ρ_{loss}. $\widetilde{G}$ is network T-detectable if and only if all loops in $\widetilde{G}_{exd}$ and the reachable states from all loops are entirely within Y_m.

Proof: To prove the network T-detectability of the timed discrete event system $\widetilde{G}$, the proof is divided into two parts: the "if" part and the "only if" part.

If part. Assume that all loops in $\widetilde{G}_{exd}$ and the reachable states from all loops are entirely within Y_m. Then the following equation holds.

$$(\forall y \in Y_{loop}) y \in Y_m \wedge QR(y) \subseteq Y_m$$

Meanwhile,

$$(\forall s \in L^\omega(\widetilde{G}))(\forall w \in \Phi_{DL}^{N_o}(Pr(s)))(\exists n \in \mathbb{N}^+)|w| \geq n$$
$$\Rightarrow \zeta(y_0, w) \in Y_{loop} \cup QR(Y_{loop})$$
$$\Rightarrow TE(w) \in Y_{loop} \cup QR(Y_{loop})$$
$$\text{(By Equation (4))}$$
$$\Rightarrow TE(w) \in Y_m$$
$$\text{(Because } (\forall y \in Y_{loop}) y \in Y_m \wedge QR(y) \subseteq Y_m)$$
$$\Rightarrow |TE(w)| = 1$$

Therefore, the current states and the subsequent states can always be determined after finite observations, which means that the system is network T-detectable.

Only if part. We prove this by contradiction. Assume that the given timed discrete event system $\widetilde{G}$ is network T-detectable and there exists at least one state in the loop, that state is not marked. Therefore,

$$(\exists y \in Y_{loop} \cup QR(Y_{loop})) y \notin Y_m$$
$$\Rightarrow (\exists s \in L^\omega(\widetilde{G}))(\exists w \in \Phi_{DL}^{N_o}(Pr(s)))(\forall n \in \mathbb{N}^+) |w| \geq n \Rightarrow \zeta(y_0, w) \notin Y_m$$
$$\Rightarrow (\exists s \in L^\omega(\widetilde{G}))(\exists w \in \Phi_{DL}^{N_o}(Pr(s)))(\forall n \in \mathbb{N}^+) |w| \geq n \Rightarrow TE(w) \notin Y_m$$
$$\text{(By Equation (4))}$$
$$\Rightarrow (\exists s \in L^\omega(\widetilde{G}))(\exists w \in \Phi_{DL}^{N_o}(Pr(s)))(\forall n \in \mathbb{N}^+) |w| \geq n \Rightarrow |TE(w)| > 1$$

This contradicts the definition of network T-detectability. ■

Remark 7: In Theorem 4, for a given automaton $\widetilde{G}_{exd}$, the complexity of traversing and checking the reachable states from all loops is polynomial in the number of states in $\widetilde{G}_{exd}$. However, to obtain the extended automaton $\widetilde{G}_{exd}$, we actually need to construct both the augmented automaton $\widetilde{G}_{DL}$ and its observer $\widetilde{G}_{DL,obs}$ from $\widetilde{G}$. As mentioned in Remark 5, the complexity of constructing $\widetilde{G}_{DL}$ is linear in the number of states in $\widetilde{G}$. On the other hand, when constructing the observer, the number of states increases from $|Z|$ to $|2^Z|$, resulting in an exponential complexity. Subsequently, the number of states changes from $|X|$ to $|Y|$ during the construction of the extended automaton, which has only linear complexity. Therefore, the complexity of checking T-detectability is exponential.

We use an example to illustrate the results.

Example 6: We use a simplified model to illustrate our results. Given another timed discrete event system as shown in Fig. 5. Assume that all events are observable. The observation delay is bounded by $N_o = 1$.

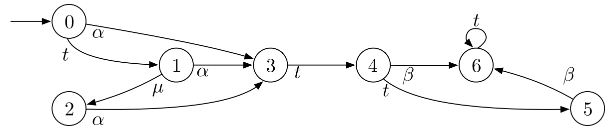

FIGURE 5. A simplified timed discrete event system $\widetilde{G}'$.

We first construct the observer $\widetilde{G}'_{DL,obs}$, which is shown in Fig. 6.

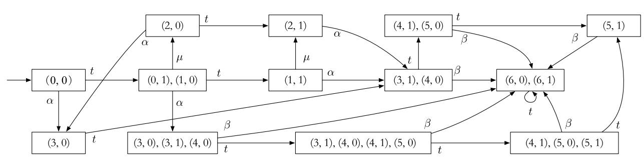

FIGURE 6. The observer $\widetilde{G}'_{DL,obs}$.

We construct its corresponding extended automaton $\widetilde{G}'_{exd}$ as shown in Fig. 7. Assume the observed event sequence $w = t\alpha t$, we calculate the state estimate $TE(w)$ by Theorem 3 as

$$TE(w) = \zeta(y_0, w) = \{4, 5, 6\}.$$

Therefore, once $w = t\alpha t$, under observation delay bounded by 1. The system actually stays in state 4, 5 or state 6.

We proceed to check whether the system depicted in Fig. 5 is network T-detectable. By examining the extended automaton $\widetilde{G}'_{exd}$ illustrated in Fig. 7, we observe that all loops and

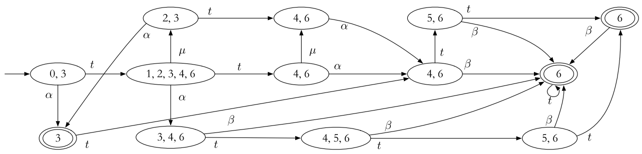

FIGURE 7. The extended automaton $\widetilde{G}'_{exd}$ of $\widetilde{G}'$.

the reachable states from these loops are confined within Y_m. To be precise, all loops and the reachable states from these loops are entirely contained within state 6. As a result, we can conclude that the system in Fig. 5 is network T-detectable.

VI. CONCLUSION

This paper focused on the theoretical investigation of detectability in timed discrete event systems under observation delays and losses. The main contributions of this work can be summarized as follows: (1) We provide a precise description of observations in the presence of communication delays and losses. (2) We introduce the concepts of T-detectability and network T-detectability. We demonstrate that T-detectability can be viewed as a specific case of network T-detectability. (3) We propose an augmented automaton that incorporates the effects of observation delays in timed discrete event systems, providing a structured representation of the system's behavior under varying observation conditions. (4) We develop an extended automaton which facilitates the estimation of the system's state for each observation, accounting for the specific delays and losses incurred. (5) We present a method for checking network T-detectability, enabling the evaluation of whether a timed discrete event system remains detectable even in the presence of observation delays and losses.

There are several avenues for future research in this area. Some potential directions for further investigation include: (1) Expanding the scope to multi-channel observation systems. (2) Experimental validation and industrial case studies. (3) Incorporating more complex communication delay and loss models. By exploring these future research directions, we can further enhance our understanding of detectability and its implications in timed discrete event systems under observation delays and losses. These investigations will contribute to the development of more robust fault management and diagnosis techniques for practical applications.

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