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RESEARCH ARTICLE

System Decomposition Method-Based Exponential Stability of Clifford-Valued BAM Delayed Neural Networks

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ABSTRACT This study explores new theoretical results for the global exponential stability of bidirectional associative memory delayed neural networks in the Clifford domain. By considering time-varying delays, a general class of Clifford-valued bidirectional associative memory neural networks is formulated, which encompasses real-, complex-, and quaternion-valued neural network models as special cases. To analyze the global exponential stability, we first decompose the considered n-dimensional Clifford-valued networks into $2^m n$-dimensional real-valued networks, which avoids the inconvenience caused by the non-commutativity of the multiplication of Clifford numbers. Subsequently, we establish new sufficient conditions to guarantee the existence, uniqueness, and global exponential stability of equilibrium points for the considered networks by constructing a new Lyapunov functional and applying homeomorphism theory. Finally, we provide a numerical example accompanied by simulation results to illustrate the validity of the obtained theoretical results. The present results remain valid even when the considered neural networks degenerate into real-, complex-, and quaternion-valued networks.

INDEX TERMS Bidirectional associative memory neural networks, Clifford-valued neural networks, global exponential stability, Lyapunov functions, time-varying delays.

I. INTRODUCTION

Over the past few decades, neural networks (NNs) have attracted significant attention in the field of artificial intelligence [1], [2], [3], [4], [5], [6], [7], [8]. Compared with other NNs, bidirectional associative memory neural networks (BAMNNs) have been attracting more attention from researchers because of their widespread applications in many scientific and engineering fields such as image processing, associative memories, pattern recognition, automatic control, secure communication, signal processing, optimization problems, and other practical applications [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

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Also, using the BAMNNs perfectly solves the known bitwise XOR problem, compared to the existing NNs, which makes it possible to store the data, encoded to binary form, in its memory. In such practical applications, it is important to consider the dynamics of the designed NNs, particularly their stability. As such, recent studies have investigated various stability methods for BAMNNs and published significant and interesting results [18], [19], [20], [21], [22], [23], [24], [25].

Recently, Clifford-valued NNs have attracted increasing attention from researchers owing to their generalized form of quaternion-valued, complex-valued, and real-valued NNs [26], [27]. Moreover, Clifford-valued NNs are powerful and effective models for representing and solving geometrical and engineering problems. In Clifford-valued NNs, the state variables, connection weights, and external inputs are

Clifford numbers, and they have more complex algebraic algorithms. It has been used in a wide range of applications, including medical imaging, robotics, and natural language processing [27], [28]. Moreover, Clifford-valued NNs offer several advantages in that they process data that are not solved by real-, complex-, and quaternion-valued NNs with high accuracy, and they can also handle multidimensional data. Consequently, Clifford-valued NNs have emerged as an important research field. In [29], the problem of global asymptotic stability in Clifford-valued NNs was examined using a decomposition method. In [32], the authors derived a globally asymptotic almost automorphic synchronization of Clifford-valued recurrent NNs networks with delays. In [34], the authors considered a class of Clifford-valued neutral high-order Hopfield NNs with leakage delays and studied their existence and global stability analysis. Using the Lyapunov function method, the global exponential stability of an anti-periodic solution for Clifford-valued inertial Cohen-Grossberg NNs was investigated in [37]. Other results of Clifford-valued NNs have been reported in earlier works [35], [36], [38], [39].

In reality, time delays inevitably exist in biological and artificial NNs and cannot be neglected [5], [10], [17], [21], [40], [41]. On the other hand, the appearance of discrete time delays in NNs are usual because of the limited switching speed of neurons and amplifiers. Moreover, NNs are typically spatial in nature and the propagation velocity distribution along these paths results in a delayed propagation distribution owing to the presence of significant parallel paths with different axon sizes and lengths [25], [31], [33]. Numerous studies have demonstrated that the existence of time delays has an impact on NNs and can result in complex dynamic behaviors such as oscillations, divergence, or instability of NNs, all of which can be detrimental to the system [42], [43], [44], [45]. Consequently, incorporating mixed delays in network modeling boosts the value of the considered NNs in theory and practice.

To the best of our knowledge, no studies have been published on the existence, uniqueness, and global exponential stability of BAMNNs with time-varying delays in the Clifford domain. To fill this gap, we aim to investigate the global exponential stability of Clifford-valued BAMNNs using the system decomposition method. In recent years, a number of studies have been published on the stability of Clifford-valued NNs; however, Clifford-valued BAMNNs have not been fully explored, which motivated us to investigate this topic. The main aspects of this paper can be summarized as follows: (1) A general form of Clifford-valued BAMNNs with time-varying delays was presented to derive more realistic Clifford-valued NNs dynamics. (2) The system decomposition method was used to investigate the global exponential stability of Clifford-valued delayed BAMNNs. (3) Lyapunov stability theory, homeomorphism theory, and inequality techniques were applied to Clifford-valued delayed BAMNNs to determine the enhanced stability conditions. (4) The effectiveness of the main results was illustrated using a numerical example and simulation results.

The remainder of this paper is structured as follows: Section II provides the basic concepts of Clifford algebra, problem model, definitions, and useful lemmas. The main results of this study are presented in Section III, Theorem (3.1) presents sufficient criteria for the existence of the equilibrium point and the global exponential stability of the considered NNs. In Corollary (3.2), the results of the stability criteria are discussed for a special case. Section IV presents a numerical case study that demonstrates the feasibility of the derived results. Section V presents the conclusions of this study.

II. MATHEMATICAL FORMULATION AND PROBLEM DEFINITION

A. NOTATIONS

In the remainder of this paper, the n-space real vectors, n-space real Clifford vectors, set of all $n \times m$ real matrices, and set of all $n \times m$ real Clifford matrices are denoted by $\mathbb{R}^n$, $\mathcal{A}^n$, $\mathbb{R}^{n\times m}$, and $\mathcal{A}^{n\times m}$, respectively. The transposition and involution transposition of the matrices are denoted by T and $*$, respectively. The Clifford algebra with m generators over a real number is denoted by $\mathcal{A}$. The norm of $\mathbb{R}^n$ is defined as $\|z\| = \sum_{p=1}^{n} |z_p|$. For $z = \sum_{A\in\Xi} z^A e_A \in \mathcal{A}$ denote $\|z\|_{\mathcal{A}} = \sum_{A\in\Xi} |z^A|$. For all $\overline{1,n} = 1, 2, \ldots, n$ and $\overline{1,m} = 1, 2, \ldots, m$.

B. CLIFFORD ALGEBRA

The Clifford real algebra over $\mathbb{R}^m$ is given by

$$\mathcal{A} = \left\{ \sum_{A\subseteq\{\overline{1,m}\}} a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $e_A = e_{w_1} e_{w_2} \ldots e_{w_\eta}$ with $A = w_1, w_2, \ldots, w_\eta$, $1 \leq w_1 < w_2 < \ldots < \underline{w_\eta} \leq m$. Furthermore, $e_\emptyset = e_0 = 1$ and e_w, $w = \overline{1,m}$ denote the Clifford generators that satisfy the following conditions: (i) $e_i e_j + e_j e_i = 0$, $i \neq j$, $i,j = \overline{1,m}$, (ii) $e_i^2 = -1$, $i = \overline{1,m}$. For convenience, an element is defined as the product of many Clifford generators $e_1 e_2 e_3 e_4 = e_{1234}$. Define $\Xi = \{\emptyset, 1, 2, \ldots, A, \ldots, 12\ldots m\}$, we get

$$\mathcal{A} = \left\{ \sum_A a^A e_A,\ a^A \in \mathbb{R} \right\},$$

where $\sum_A$ is the short form of $\sum_{A\in\Xi}$ and $\mathcal{A}$ is isomorphic to $\mathbb{R}^{2^m}$. For any Clifford number $z = \sum_{A\in\Xi} z^A e_A \in \mathcal{A}$, the involution of z is denoted by $\bar{z} = \sum_{A\in\Xi} z^A \bar{e}_A$, where $\bar{e}_A = (-1)^{\frac{\varrho[A](\varrho[A]+1)}{2}} e_A$, and $\varrho[A] = 0$ if $A = \emptyset$ and $\varrho[A] = \eta$ if $A = w_1 w_2 \ldots w_\eta \in \Xi$. From this definition, we obtain $e_A \bar{e}_A = \bar{e}_A e_A = 1$, and $z = \sum_{A\in\Xi} z^A e_A : \mathbb{R} \to \mathcal{A}$, where $z^A : \mathbb{R} \to \mathbb{R}$, $A \in \Xi$, and $\dot{z}(t) = \sum_{A\in\Xi} \dot{z}^A(t) e_A$. We refer the reader to [26], [27], [28], [29], [30], [31], [32], and [33] for more information on Clifford algebra.

Remark 2.1: Since $e_B\bar{e}_A = (-1)^{\frac{\varrho[A](\varrho[A]+1)}{2}} e_B e_A$, then $e_B\bar{e}_A = e_C$ or $e_B\bar{e}_A = -e_C$, e_C is the basis for Clifford algebra $\mathcal{A}$. Define $\varrho[B.\bar{A}] = 0$ if $e_B\bar{e}_A = e_C$ and $\varrho[B.\bar{A}] = 1$ if $e_B\bar{e}_A = -e_C$ then, $e_B\bar{e}_A = (-1)^{\varrho[B.\bar{A}]}e_C$. This shows that there exists a unique $\mathcal{S}^C$ for any $\mathcal{S}^C \in \mathcal{A}$ that satisfies $\mathcal{S}^{B.\bar{A}} = (-1)^{\varrho[B.\bar{A}]}\mathcal{S}^C$ for $e_B\bar{e}_A = (-1)^{\varrho[B.\bar{A}]}e_C$. Hence, $\mathcal{S}^{B.\bar{A}}e_B\bar{e}_A = \mathcal{S}^{B.\bar{A}}(-1)^{\varrho[B.\bar{A}]}e_C = (-1)^{\varrho[B.\bar{A}]}\mathcal{S}^C(-1)^{\varrho[B.\bar{A}]}e_C = \mathcal{S}^C e_C$, and $\mathcal{S} = \sum_C \mathcal{S}^C e_C \in \mathcal{A}$.

C. PROBLEM FORMULATION

In this section, we consider a class of Clifford-valued BAMNNs with time-varying delays as follows:

$$\begin{cases} \dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{m} a_{qp} g_q(y_q(t-\sigma_{qp}(t))) + u_p, \\ \dot{y}_q(t) = -c_q y_q(t) + \sum_{p=1}^{n} b_{pq} f_p(x_p(t-\tau_{pq}(t))) + v_q, \end{cases} \tag{1}$$

where for all $t \geq 0$; $p = \overline{1,n}$, $q = \overline{1,m}$; $x_p(t) \in \mathcal{A}$ and $y_q(t) \in \mathcal{A}$ denote the state variables; $0 < d_p \in \mathbb{R}^+$ and $0 < c_q \in \mathbb{R}^+$ denote the self-feedback connection weights; $a_{qp} \in \mathcal{A}$, $b_{pq} \in \mathcal{A}$ denote the interconnection weights; $f_p(\cdot) : \mathcal{A} \to \mathcal{A}$ and $g_q(\cdot) : \mathcal{A} \to \mathcal{A}$ denote the neuron activation functions; $u_p \in \mathcal{A}$ and $v_q \in \mathcal{A}$ denote the external inputs; $\tau_{pq}(t) \in \mathbb{R}^+$ and $\sigma_{qp}(t) \in \mathbb{R}^+$ denote the transmission delays.

The initial conditions of NNs (1) are given by

$$\begin{cases} x_p(t) = \varphi_p(t),\ t \in [-\sigma, 0],\ \sigma = \max_{1\leq p\leq n}\max_{1\leq q\leq m}\{\sigma_{qp}\}, \\ y_q(t) = \psi_q(t),\ t \in [-\tau, 0],\ \tau = \max_{1\leq q\leq m}\max_{1\leq p\leq n}\{\tau_{pq}\}, \end{cases} \tag{2}$$

where $\varphi_p \in \mathcal{C}([-\sigma, 0], \mathcal{A})$ and $\psi_q \in \mathcal{C}([-\tau, 0], \mathcal{A})$ are continuous functions.

The transmission delays $\sigma_{qp}(t)$, $\tau_{pq}(t)$ and the signal functions $f_p(\cdot)$, $g_q(\cdot)$ are assumed to satisfy the following hypothesis:

(H1) Delays $\sigma_{qp}(t)$, $\tau_{pq}(t)$ ($p = \overline{1,n}$; $q = \overline{1,m}$) are differentiable and satisfy

$$\begin{cases} 0 \leq \sigma_{qp}(t) \leq \sigma_{qp},\ \dot{\sigma}_{qp}(t) \leq \mu_1 < 1, \\ 0 \leq \tau_{pq}(t) \leq \tau_{pq},\ \dot{\tau}_{pq}(t) \leq \mu_2 < 1, \end{cases} \tag{3}$$

where $\sigma_{qp} \in \mathbb{R}^+$, $\tau_{pq} \in \mathbb{R}^+$, $\mu_1 \in \mathbb{R}^+$ and $\mu_2 \in \mathbb{R}^+$ are constants.

(H2) Functions $f_p(\cdot)$, $g_q(\cdot) : \mathcal{A} \to \mathcal{A}$ ($p = \overline{1,n}$; $q = \overline{1,m}$) are Lipschitz continuous. Then there exist constants $\mathcal{L}_q^g \in \mathbb{R}^+$ and $\mathcal{K}_p^f \in \mathbb{R}^+$ such that

$$\begin{cases} |g_q(x) - g_q(y)|_{\mathcal{A}} \leq \mathcal{L}_q^g|x-y|_{\mathcal{A}},\ q = \overline{1,m}, \\ |f_p(x) - f_p(y)|_{\mathcal{A}} \leq \mathcal{K}_p^f|x-y|_{\mathcal{A}},\ p = \overline{1,n}, \end{cases} \tag{4}$$

for all $x, y \in \mathcal{A}$ and $g_q(0) = 0, f_p(0) = 0$.

For any solution $(x, y)^T = \big(x_1(t), x_2(t), \ldots, x_n(t), y_1(t), y_2(t), \ldots, y_m(t)\big)^T$ and the equilibrium point $(x^*, y^*)^T = \big(x_1^*(t), x_2^*(t), \ldots, x_n^*(t), y_1^*(t), y_2^*(t), \ldots, y_m^*(t)\big)^T$ of NNs (1), we denote $\|(\varphi, \psi)^T - (x^*, y^*)^T\|$ as follows:

$$\|(\varphi, \psi)^T - (x^*, y^*)^T\| = \sup_{-\sigma\leq t\leq 0}\sum_{p=1}^{n}|\varphi_p(t) - x_p^*|^r + \sup_{-\tau\leq t\leq 0}\sum_{q=1}^{m}|\psi_q(t) - y_q^*|^r, \tag{5}$$

where $r > 1$ is a constant.

To prove the main results of this paper, the following definitions and lemmas will be used:

Definition 2.2: [42] The point of equilibrium $(x^*, y^*)^T$ of NNs (1) is globally exponentially stable, if there exist scalars $\alpha \geq 1$ and $\lambda \in \mathbb{R}^+$, then

$$\sum_{p=1}^{n}|x_p(t) - x_p^*|^r + \sum_{q=1}^{m}|y_q(t) - y_q^*|^r \leq \alpha\|(\varphi, \psi)^T - (x^*, y^*)^T\|e^{-\lambda t},\ t \geq 0.$$

Definition 2.3: [43] Let $f : \mathbb{R}^{2^m n} \to \mathbb{R}^{2^m n}$ be continuous, and the upper-right Dini derivative $D^+(f)$ of f is expressed as

$$D^+(f(t)) = \lim_{h\to 0^+}\sup\frac{f(t+h) - f(t)}{h}.$$

Lemma 2.4: [44] Assume that $a \in \mathbb{R}^+$, $b \in \mathbb{R}^+$, $1 < p$, and $\frac{1}{p} + \frac{1}{q} = 1$, then the following condition holds:

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q.$$

Lemma 2.5: [45] Let $\mathcal{H}(x, y) : \mathbb{R}^{2^m(n+m)} \to \mathbb{R}^{2^m(n+m)}$ continuous. If $\mathcal{H}(x, y)$ satisfies the following conditions:
(1) $\mathcal{H}(x, y)$ is injective on $\mathbb{R}^{2^m(n+m)}$,
(2) $\|\mathcal{H}(x, y)\| \to \infty$ as $\|x, y\| \to \infty$.
Then $\mathcal{H}(x, y)$ is a homeomorphism of $\mathbb{R}^{2^m(n+m)}$.

III. MAIN RESULTS

This section presents the delay-independent criteria for the existence, uniqueness, and global exponential stability of the equilibrium point for NNs (1) using homeomorphism theory and Lyapunov functions.

Based on the previous discussion about Clifford algebra, we can decompose the Clifford-valued function into the real-valued function. For example, for the second term in NNs (1), we have

$$\sum_{q=1}^{m} a_{qp} g_q(y_q(t-\sigma_{qp}(t))) = \sum_{q=1}^{m}\sum_{C\in\Xi} a_{qp}^C e_C \sum_{B\in\Xi} g_q^B(y_q(t-\sigma_{qp}(t)))e_B$$

$$= \sum_{q=1}^{m}\sum_{A\in\Xi}\sum_{B\in\Xi}(-1)^{\varrho[A.\bar{B}]}a_{qp}^{A.\bar{B}}(-1)^{\varrho[A.\bar{B}]}e_A\bar{e}_B \times g_q^B(y_q(t-\sigma_{qp}(t)))e_B$$
$$= \sum_{q=1}^{m}(-1)^{2\varrho[A.\bar{B}]}\sum_{A\in\Xi}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}g_q^B(y_q(t-\sigma_{qp}(t)))e_A\bar{e}_Be_B$$
$$= \sum_{q=1}^{m}\sum_{A\in\Xi}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}g_q^B(y_q(t-\sigma_{qp}(t)))e_A,\ p=\overline{1,n}. \quad (6)$$

According to the above discussion, Clifford-valued NNs (1) can be decomposed into equivalent real-valued NNs (7) to overcome the non-commutativity problem:

$$\begin{cases} \dot{x}_p^A(t) = -d_px_p^A(t) + \sum_{q=1}^{m}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}g_q^B(y_q^A(t-\sigma_{qp}(t))) + u_p^A, \\ \dot{y}_q^A(t) = -c_qy_q^A(t) + \sum_{p=1}^{n}\sum_{B\in\Xi}b_{pq}^{A.\bar{B}}f_p^B(x_p^A(t-\tau_{pq}(t))) + v_q^A. \end{cases} \quad (7)$$

The initial conditions of NNs (7) are given by

$$\begin{cases} x_p^A(t) = \varphi_p^A(t),\ t\in[-\sigma,0],\ p=\overline{1,n}, \\ y_q^A(t) = \psi_q^A(t),\ t\in[-\tau,0],\ q=\overline{1,m}, \end{cases} \quad (8)$$

where

$$\begin{cases} x^A(t) = (x_1^A(t), x_2^A(t), \ldots, x_n^A(t))^T, \\ y^A(t) = (y_1^A(t), y_2^A(t), \ldots, y_n^A(t))^T, \\ x(t) = \sum_{A\in\Xi}x^A(t)e_A,\ y(t) = \sum_{A\in\Xi}y^A(t)e_A, \\ u^A = (u_1^A(t), u_2^A(t), \ldots, u_n^A(t))^T,\ u = \sum_{A\in\Xi}u^Ae_A, \\ v^A = (v_1^A(t), v_2^A(t), \ldots, v_n^A(t))^T,\ v = \sum_{A\in\Xi}v^Ae_A, \\ a_{qp} = \sum_{C\in\Xi}a_{qp}^Ce_C,\ a_{qp}^{A.\bar{B}} = (-1)^{\varrho[A.\bar{B}]}a_{qp}^C, \\ b_{pq} = \sum_{C\in\Lambda}b_{pq}^Ce_C,\ b_{pq}^{A.\bar{B}} = (-1)^{\varrho[A.\bar{B}]}b_{pq}^C, \\ e_A\bar{e}_B = (-1)^{\varrho[A.\bar{B}]}e_C, \\ g^B\big(y^A(t-\sigma_{qp}(t))\big) \\ = g_1^B\big(y_1^{A_1}(t-\sigma_{qp}(t)), \ldots, y_1^{A_{2^m}}(t-\sigma_{qp}(t))\big), \\ g_2^B\big(y_2^{A_1}(t-\sigma_{qp}(t)), \ldots, y_2^{A_{2^m}}(t-\sigma_{qp}(t))\big), \ldots, \\ g_n^B\big(y_n^{A_1}(t-\sigma_{qp}(t)), \ldots, y_n^{A_{2^m}}(t-\sigma_{qp}(t))\big)^T, \\ f^B\big(x^A(t-\tau_{pq}(t))\big) \\ = f_1^B\big(x_1^{A_1}(t-\tau_{pq}(t)), \ldots, x_1^{A_{2^m}}(t-\tau_{pq}(t))\big), \\ f_2^B\big(x_2^{A_1}(t-\tau_{pq}(t)), \ldots, x_2^{A_{2^m}}(t-\tau_{pq}(t))\big), \ldots, \\ f_n^B\big(x_n^{A_1}(t-\tau_{pq}(t)), \ldots, x_n^{A_{2^m}}(t-\tau_{pq}(t))\big)^T. \end{cases} \quad (9)$$

(H3) By hypothesis **(H2)**, we have

$$\begin{cases} \sum_{B\in\Xi}\big|g_q^B(y_q^{A_1}, y_q^{A_2}, \ldots, y_q^{A_{2^m}}) - g_q^B(\hat{y}_q^{A_1}, \hat{y}_q^{A_2}, \ldots, \hat{y}_q^{A_{2^m}})\big| \\ \le \mathcal{L}_q^g\sum_{A\in\Xi}\big|y_q^A - \hat{y}_q^A\big|,\ q=\overline{1,m}, \\ \sum_{B\in\Xi}\big|f_p^B(x_p^{A_1}, x_p^{A_2}, \ldots, x_p^{A_{2^m}}) - f_p^B(\hat{x}_p^{A_1}, \hat{x}_p^{A_2}, \ldots, \hat{x}_p^{A_{2^m}})\big| \\ \le \mathcal{K}_p^f\sum_{A\in\Xi}\big|x_p^A - \hat{x}_p^A\big|,\ p=\overline{1,n}. \end{cases} \quad (10)$$

The equilibrium point of NNs (1) is also the equilibrium point of NNs (7), and the stability of NNs (1) is the same as that of NNs (7). Thus, we investigate NNs (7) to obtain sufficient criteria to establish that the equilibrium point of NNs (7) is globally exponentially stable.

Theorem 3.1: Let **(H1)–(H3)** hold, there exist constants $1 < r$, $\lambda_p \in \mathbb{R}^+$, $\lambda_{n+q} \in \mathbb{R}^+$ $(p=\overline{1,n};\ q=\overline{1,m})$ such that

$$\begin{cases} -r\lambda_pd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\big|a_{qp}^{A.\bar{B}}\big|\mathcal{L}_q^g \\ + \sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\big|b_{pq}^{A.\bar{B}}\big|\mathcal{K}_p^f < 0,\ p=\overline{1,n}, \\ -r\lambda_{n+q}c_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_{n+q}\big|b_{pq}^{A.\bar{B}}\big|\mathcal{K}_p^f \\ + \sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_p\big|a_{qp}^{A.\bar{B}}\big|\mathcal{L}_q^g < 0,\ q=\overline{1,m}, \end{cases} \quad (11)$$

then the equilibrium point of NNs (7) is globally exponentially stable.

Proof: The proof of this theorem involves two steps: The first step is to prove the existence and uniqueness of the equilibrium point.

The point $(x^*, y^*)^T$ is an equilibrium point of NNs (7) if and only if it is a solution of the following equation:

$$\begin{cases} -d_px_p^* + \sum_{q=1}^{m}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}g_q^B(y_q^*) + u_p^A = 0,\ p=\overline{1,n}, \\ -c_qy_q^* + \sum_{p=1}^{n}\sum_{B\in\Xi}b_{pq}^{A.\bar{B}}f_p^B(x_p^*) + v_q^A = 0,\ q=\overline{1,m}. \end{cases} \quad (12)$$

Define a map

$$\Upsilon(x^A, y^A) = \big(\Upsilon_1(x^A, y^A), \ldots, \Upsilon_n(x^A, y^A), \Upsilon_{n+1}(x^A, y^A), \ldots, \Upsilon_{n+m}(x^A, y^A)\big)^T, \quad (13)$$

where

$$\begin{cases} \Upsilon_p(x^A, y^A) = -d_px_p^A + \sum_{q=1}^{m}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}g_q^B(y_q^A) + u_p^A, \\ \Upsilon_{n+q}(x^A, y^A) = -c_qy_q^A + \sum_{p=1}^{n}\sum_{B\in\Xi}b_{pq}^{A.\bar{B}}f_p^B(x_p^A) + v_q^A. \end{cases}$$

for all $p=\overline{1,n}$ and $q=\overline{1,m}$.

The following proves $\Upsilon(x^A, y^A)$ is a homeomorphism.

We claim that in the first step $\Upsilon(x^A, y^A)$ is injective to $\mathbb{R}^{2^m(n+m)}$. If there exist $(x^A, y^A)^T, (\hat{x}^A, \hat{y}^A)^T \in \mathbb{R}^{2^m(n+m)}$ with $(x^A, y^A)^T \neq (\hat{x}^A, \hat{y}^A)^T$, such that $\Upsilon(x^A, y^A) = \Upsilon(\hat{x}^A, \hat{y}^A)$, using Lemma (2.4), we get for all $A \in \Xi$

$$\begin{aligned}
0 =& \sum_{p=1}^{n} r\lambda_p |x_p^A - \hat{x}_p^A|^{r-1} sgn(x_p^A - \hat{x}_p^A)(\Upsilon_p(x^A, y^A) - \Upsilon_p(\hat{x}^A, \hat{y}^A)) \\
=& \sum_{p=1}^{n} r\lambda_p |x_p^A - \hat{x}_p^A|^{r-1} sgn(x_p^A - \hat{x}_p^A)\Big(- d_p(x_p^A - \hat{x}_p^A) \\
&+ \sum_{q=1}^{m}\sum_{B\in\Xi} a_{qp}^{A.\bar{B}}(g_q^B(y_q^A) - g_q^B(\hat{y}_q^A))\Big) \\
\leq& - \sum_{p=1}^{n} r\lambda_p d_p |x_p^A - \hat{x}_p^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p |a_{qp}^{A.\bar{B}}| \\
&\times \mathfrak{L}_q^g |x_p^A - \hat{x}_p^A|^{r-1} |y_q^A - \hat{y}_q^A| \\
\leq& - \sum_{p=1}^{n} r\lambda_p d_p |x_p^A - \hat{x}_p^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p |a_{qp}^{A.\bar{B}}| \\
&\times \mathfrak{L}_q^g \Big(\frac{r-1}{r}|x_p^A - \hat{x}_p^A|^r + \frac{1}{r}|y_q^A - \hat{y}_q^A|^r\Big) \\
=& \sum_{p=1}^{n} \lambda_p \Big(- rd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi} |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g\Big)|x_p^A - \hat{x}_p^A|^r \\
&+ \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} \lambda_p |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g |y_q^A - \hat{y}_q^A|^r. \qquad (14)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
0 =& \sum_{q=1}^{m} r\lambda_{n+q} |y_q^A - \hat{y}_q^A|^{r-1} sgn(y_q^A - \hat{y}_q^A) \\
&\times \Big(\Upsilon_{n+q}(x^A, y^A) - \Upsilon_{n+q}(\hat{x}^A, \hat{y}^A)\Big) \\
=& \sum_{q=1}^{m} r\lambda_{n+p} |y_q^A - \hat{y}_q^A|^{r-1} sgn(y_q^A - \hat{y}_q^A) \\
&\times \Big(- c_q(y_q^A - \hat{y}_q^A) + \sum_{p=1}^{n}\sum_{B\in\Xi} b_{pq}^{A.\bar{B}}(f_p^B(x_p^A) - f_p^B(\hat{x}_p^A))\Big) \\
\leq& - \sum_{q=1}^{m} r\lambda_{n+q} c_q |y_q^A - \hat{y}_q^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q} |b_{pq}^{A.\bar{B}}| \\
&\times \mathcal{K}_p^f |y_q^A - \hat{y}_q^A|^{r-1} |x_p^A - \hat{x}_p^A| \\
\leq& - \sum_{q=1}^{m} r\lambda_{n+q} c_q |y_q^A - \hat{y}_q^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q} |b_{pq}^{A.\bar{B}}| \\
&\times \mathcal{K}_p^f \Big(\frac{r-1}{r}|y_q^A - \hat{y}_q^A|^r + \frac{1}{r}|x_p^A - \hat{x}_p^A|^r\Big) \\
=& \sum_{q=1}^{m} \lambda_{n+q} \Big(- rc_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi} |b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big)|y_q^A - \hat{y}_q^A|^r \\
&+ \sum_{q=1}^{m}\sum_{p=1}^{n}\sum_{B\in\Xi} \lambda_{n+q} |b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f |x_p^A - \hat{x}_p^A|^r. \qquad (15)
\end{aligned}$$

Using (14), (15) and (11), we obtain for all $p = \overline{1, n},\ q = \overline{1, m}$

$$\begin{aligned}
0 \leq& \sum_{p=1}^{n} \Big(\lambda_p\Big(- rd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi} |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g\Big) \\
&+ \sum_{q=1}^{m}\sum_{B\in\Xi} \lambda_{n+q} |b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big)|x_p^A - \hat{x}_p^A|^r \\
&+ \sum_{q=1}^{m} \Big(\lambda_{n+q}\Big(- rc_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi} |b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big) \\
&+ \sum_{p=1}^{n}\sum_{B\in\Xi} \lambda_p |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g\Big)|y_q^A - \hat{y}_q^A|^r < 0, \qquad (16)
\end{aligned}$$

which is a contradiction. Therefore, $\Upsilon(x^A, y^A)$ is injective on $\mathbb{R}^{2^m(n+m)}$.

Second, we show that $\|\Upsilon(x^A, y^A)\| \to \infty$ as $\|(x^A, y^A)\| \to \infty$. By Lemma (2.5), we have

$$\begin{aligned}
&\sum_{p=1}^{n} r\lambda_p |x_p^A|^{r-1} sgn(x_p^A)\Big(\Upsilon_p(x^A, y^A) - \Upsilon_p(0, 0)\Big) \\
&= \sum_{p=1}^{n} r\lambda_p |x_p^A|^{r-1} sgn(x_p^A) \\
&\quad \times \Big(- d_p x_p^A + \sum_{q=1}^{m}\sum_{B\in\Xi} a_{qp}^{A.\bar{B}}(g_q^B(y_q^A) - g_q^A(0))\Big) \\
&\leq - \sum_{p=1}^{n} r\lambda_p d_p |x_p^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g |x_p^A|^{r-1}|y_q^A| \\
&\leq - \sum_{p=1}^{n} r\lambda_p d_p |x_p^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p |a_{qp}^{A.\bar{B}}| \\
&\quad \times \mathfrak{L}_q^g \Big(\frac{r-1}{r}|x_p^A|^r + \frac{1}{r}|y_q^A|^r\Big) \\
&= \sum_{p=1}^{n} \lambda_p \Big(- rd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi} |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g\Big)|x_p^A|^r \\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} \lambda_p |a_{qp}^{A.\bar{B}}|\mathfrak{L}_q^g |y_q^A|^r. \qquad (17)
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\sum_{q=1}^{m} r\lambda_{n+q} |y_q^A|^{r-1} sgn(y_q^A)(\Upsilon_{n+q}(x^A, y^A) - \Upsilon_{n+q}(0, 0)) \\
&= \sum_{q=1}^{m} r\lambda_{n+q} |y_q^A|^{r-1} sgn(y_q^A) \\
&\Big(- c_q y_q^A + \sum_{p=1}^{n}\sum_{B\in\Xi} b_{pq}^{A.\bar{B}}(f_p^B(x_p^A) - f_p^B(0))\Big)
\end{aligned}$$

$$\leq -\sum_{q=1}^{m} r\lambda_{n+q}c_q|y_q^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q}|b_{pq}^{A.\bar{B}}|$$
$$\times \mathcal{K}_p^f|y_q^A|^{r-1}|x_p^A|$$
$$\leq -\sum_{q=1}^{m} r\lambda_{n+q}c_q|y_q^A|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q}|b_{pq}^{A.\bar{B}}|$$
$$\times \mathcal{K}_p^f\Big(\frac{r-1}{r}|y_q^A|^r + \frac{1}{r}|x_p^A|^r\Big)$$
$$= \sum_{q=1}^{m}\lambda_{n+q}\Big(-rc_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big)|y_q^A|^r$$
$$+\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f|x_p^A|^r. \tag{18}$$

Using (17) and (18), we can get

$$\sum_{p=1}^{n} r\lambda_p|x_p^A|^{r-1}sgn(x_p^A)(\Upsilon_p(x^A,y^A)-\Upsilon_p(0,0))$$
$$+\sum_{q=1}^{m} r\lambda_{n+q}|y_q^A|^{r-1}sgn(y_q^A)(\Upsilon_{n+q}(x^A,y^A)-\Upsilon_{n+q}(0,0))$$
$$\leq \sum_{p=1}^{n}\Big[\lambda_p\Big(-rd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\Big)$$
$$+\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big]|x_p^A|^r$$
$$+\sum_{q=1}^{m}\Big[\lambda_{n+q}\Big(-rc_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big)$$
$$+\sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\Big]|y_q^A|^r$$
$$\leq -\alpha\Big(\sum_{p=1}^{n}|x_p^A|^r + \sum_{q=1}^{m}|y_q^A|^r\Big). \tag{19}$$

where $\alpha = min\{\alpha_1, \alpha_2\}$,

$$\begin{cases}\alpha_1 = \min_{1\leq p\leq n}\Big\{\lambda_p\Big(-rd_p + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\Big) \\ \quad +\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big\} > 0, \\ \alpha_2 = \min_{1\leq q\leq m}\Big\{\lambda_{n+q}\Big(-rc_q + (r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\Big) \\ \quad +\sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\Big\} > 0.\end{cases} \tag{20}$$

Thus, using Hölder inequality, we get

$$\alpha\Big(\sum_{p=1}^{n}|x_p^A|^r + \sum_{q=1}^{m}|y_q^A|^r\Big)$$
$$\leq -\sum_{p=1}^{n} r\lambda_p|x_p^A|^{r-1}sgn(x_p^A)(\Upsilon_p(x^A,y^A)-\Upsilon_p(0,0))$$
$$-\sum_{q=1}^{m} r\lambda_{n+q}|y_q^A|^{r-1}sgn(y_q^A)(\Upsilon_{n+q}(x^A,y^A)-\Upsilon_{n+q}(0,0))$$
$$\leq r\max_{1\leq s\leq n+m}\{\lambda_s\}\Big[\sum_{p=1}^{n}|x_p|^{r-1}|\Upsilon_p(x^A,y^A)-\Upsilon_p(0,0)|$$
$$+\sum_{q=1}^{m}|y_q^A|^{r-1}|\Upsilon_{n+q}(x^A,y^A)-\Upsilon_{n+q}(0,0)|\Big]$$
$$\leq r\max_{1\leq s\leq n+m}\{\lambda_s\}\Big(\sum_{p=1}^{n}|x_p^A|^r + \sum_{q=1}^{m}|y_q^A|^r\Big)^{\frac{r-1}{r}}$$
$$\times\Big(\sum_{p=1}^{n}\big|\Upsilon_p(x^A,y^A)-\Upsilon_p(0,0)\big|^r$$
$$+\sum_{q=1}^{m}\big|\Upsilon_{n+q}(x^A,y^A)-\Upsilon_{n+q}(0,0)\big|^r\Big)^{\frac{1}{r}}, \tag{21}$$

that is,

$$\|(x^A,y^A)\|^T \leq \left(\frac{r\max_{1\leq s\leq n+m}\{\lambda_s\}}{\alpha}\right)^r\|\Upsilon(x^A,y^A)-\Upsilon(0,0)\|. \tag{22}$$

Thus, $\|\Upsilon(x^A,y^A)\| \to \infty$ as $\|(x^A,y^A)^T\| \to \infty$.

According to Lemma (2.5), $\Upsilon(x^A,y^A)$ are homeomorphisms of $\mathbb{R}^{2^m(n+m)}$. Thus, NNs (7) has a unique solution $(x^*,y^*)^T$ which is the unique equilibrium point.

The second step is to show that the equilibrium point of the NNs (7) is globally exponentially stable. Let $\bar{x}_p^A(t) = x_p^A(t) - x_p^*, \bar{y}_q^A(t) = y_q^A(t) - y_q^*, \bar{f}_p^B(\bar{x}_p^A(t-\tau_{pq}(t))) = f_p^B(x_p^A(t-\tau_{pq}(t)) + x_p^*) - f_p^B(x_p^*), \bar{g}_q^B(\bar{y}_q^A(t-\sigma_{qp}(t))) = g_q^B(y_q^A(t-\sigma_{qp}(t)) + y_q^*) - g_q^B(y_q^*)$. Then, NNs (7) can be reduced to the following model

$$\begin{cases}\dot{\bar{x}}_p^A(t) = -d_p\bar{x}_p^A(t) + \sum_{q=1}^{m}\sum_{B\in\Xi}a_{qp}^{A.\bar{B}}\bar{g}_q^B(\bar{y}_q^A(t-\sigma_{qp}(t))), \\ \dot{\bar{y}}_q^A(t) = -c_q\bar{y}_q^A(t) + \sum_{p=1}^{n}\sum_{B\in\Xi}b_{pq}^{A.\bar{B}}\bar{f}_p^B(\bar{x}_p^A(t-\sigma_{pq}(t))).\end{cases} \tag{23}$$

Choose the Lyapunov functional $\mathcal{V}(t)$ which is given as follows:

$$\mathcal{V}(t) = \sum_{p=1}^{n}\lambda_p\Big[e^{r\epsilon t}|\bar{x}_p^A(t)|^r + \frac{1}{1-\mu_1}\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|$$
$$\times\mathcal{L}_q^g\int_{t-\sigma_{qp}(t)}^{t}|\bar{y}_q^A(s)|^r e^{r\epsilon(s+\sigma)}ds\Big],$$
$$+\sum_{q=1}^{m}\lambda_{n+q}\Big[e^{r\epsilon t}|\bar{y}_q^A(t)|^r + \frac{1}{1-\mu_2}\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|$$

$$\times \mathcal{K}_p^f \int_{t-\tau_{pq}(t)}^{t} \left|\bar{x}_p^A(s)\right|^r e^{r\epsilon(s+\tau)} ds\Big]. \tag{24}$$

Based on **(H1)** and $e^{r\epsilon(t-\sigma_{qp}(t)+\sigma)} = e^{r\epsilon(t-\tau_{pq}(t)+\tau)} \geq e^{r\epsilon t}$, computing $D^+\mathcal{V}(t)$ along the solutions of NNs (23), we obtain

$$\begin{aligned}
&D^+\mathcal{V}(t)\\
&j \leq \sum_{p=1}^{n} \lambda_p r e^{r\sigma t}\left|\bar{x}_p^A(t))\right|^{r-1} sgn(\bar{x}_p^A(t))\Big(-d_p\bar{x}_p^A(t)\\
&\quad + \sum_{q=1}^{m}\sum_{B\in\Xi}\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right|\Big)\\
&\quad + \sum_{p=1}^{n}\lambda_p r\epsilon e^{r\epsilon t}\left|\bar{x}_p^A(t)\right|^r + \frac{1}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{L}_q^g\left|\bar{y}_q^A(t)\right|^r e^{r\epsilon(t+\sigma)} - \frac{1}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right|^r e^{r\epsilon(t-\sigma_{qp}(t)+\sigma)}(1-\dot{\sigma}_{qp}(t))\\
&\quad + \sum_{q=1}^{m}\lambda_{n+q} r e^{r\tau t}\left|\bar{y}_q^A(t))\right|^{r-1} sgn(\bar{y}_q^A(t))\Big(-c_q\bar{y}_q^A(t)\\
&\quad + \sum_{p=1}^{n}\sum_{B\in\Xi}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|\Big)\\
&\quad + \sum_{q=1}^{m}\lambda_{n+q} r\epsilon e^{r\epsilon t}\left|\bar{y}_q^A(t)\right|^r + \frac{1}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{K}_p^f\left|\bar{x}_p^A(t)\right|^r e^{r\epsilon(t+\tau)} - \frac{1}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|^r e^{r\epsilon(t-\tau_{pq}(t)+\tau)}(1-\dot{\tau}_{pq}(t))\\
&\leq e^{r\epsilon t}\Big[r\sum_{p=1}^{n}\lambda_p(\epsilon-d_p)\left|\bar{x}_p^A(t)\right|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p\left|\bar{x}_p^A(t)\right|^{r-1}\\
&\quad \times \left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right| + \frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{L}_q^g\left|\bar{y}_q^A(t)\right|^r - \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right|^r\Big]\\
&\quad + e^{r\epsilon t}\Big[r\sum_{q=1}^{m}\lambda_{n+q}(\epsilon-c_q)\left|\bar{y}_q^A(t)\right|^r + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q}\\
&\quad \times \left|\bar{y}_q^A(t)\right|^{r-1}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|\\
&\quad + \frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t)\right|^r\\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|^r\Big].
\end{aligned} \tag{25}$$

By the Lemma (2.4) it follows

$$\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_p\left|\bar{x}_p^A(t)\right|^{r-1}\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right|\\
&\quad \leq (r-1)\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{x}_p^A(t)\right|^r\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q(t-\sigma_{qp}(t))\right|^r,
\end{aligned} \tag{26}$$

$$\begin{aligned}
&\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi} r\lambda_{n+q}\left|\bar{y}_q^A(t)\right|^{r-1}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|\\
&\quad \leq (r-1)\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{y}_q^A(t)\right|^r\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p(t-\tau_{pq}(t))\right|^r.
\end{aligned} \tag{27}$$

From (25)-(27), we get

$$\begin{aligned}
D^+\mathcal{V}(t) &\leq e^{r\epsilon t}\Big[r\sum_{p=1}^{n}\lambda_p(\epsilon-d_p)\left|\bar{x}_p^A(t)\right|^r\\
&\quad + (r-1)\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{x}_p^A(t)\right|^r\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q(t-\sigma_{qp}(t))\right|^r\\
&\quad + \frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t)\right|^r\\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t-\sigma_{qp}(t))\right|^r\Big]\\
&\quad + e^{r\epsilon t}\Big[r\sum_{q=1}^{m}\lambda_{n+q}(\epsilon-c_q)\left|\bar{y}_q^A(t)\right|^r\\
&\quad + (r-1)\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{y}_q^A(t)\right|^r\\
&\quad + \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p(t-\tau_{pq}(t))\right|^r\\
&\quad + \frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t)\right|^r\\
&\quad - \sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}\left|b_{pq}^{A.\bar{B}}\right|\mathcal{K}_p^f\left|\bar{x}_p^A(t-\tau_{pq}(t))\right|^r\Big].
\end{aligned} \tag{28}$$

Thus, we have

$$\begin{aligned}
&D^+\mathcal{V}(t)\\
&\leq e^{r\epsilon t}\Big[\sum_{p=1}^{n}\lambda_p\Big(r(\epsilon-d_p) + (r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}\left|a_{qp}^{A.\bar{B}}\right|\\
&\quad \times \mathcal{L}_q^g\Big)\left|\bar{x}_p^A(t)\right|^r + \frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\lambda_p\left|a_{qp}^{A.\bar{B}}\right|\mathcal{L}_q^g\left|\bar{y}_q^A(t)\right|^r\Big]
\end{aligned}$$

$$+e^{r\epsilon t}\Bigg[\sum_{q=1}^{m}\lambda_{n+q}\bigg(r(\epsilon-c_q)+(r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg)\times|\bar{y}_q^A(t)|^r+\frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f|\bar{x}_p^A(t)|^r\Bigg]. \tag{29}$$

Based on condition (11), we can select a small constant $\epsilon>0$, such that

$$\begin{cases}\lambda_p\bigg(r(\epsilon-d_p)+(r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\bigg)\\ +\frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f<0,\\ \lambda_{n+q}\bigg(r(\epsilon-c_q)+(r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg)\\ +\frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g<0.\end{cases} \tag{30}$$

Therefore, we have

$$\begin{aligned}D^+\mathcal{V}(t)&\le e^{r\epsilon t}\Bigg[\sum_{p=1}^{n}\bigg(\lambda_p\bigg(r(\epsilon-d_p)+(r-1)\sum_{q=1}^{m}\sum_{B\in\Xi}\times|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\bigg)+\frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg)|\bar{x}_p^A(t)|^r\\&+\sum_{q=1}^{m}\bigg(\lambda_{n+q}\bigg(r(\epsilon-c_q)+(r-1)\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg)\\&+\frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{B\in\Xi}\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g\bigg)|\bar{y}_q(t)|^r\Bigg]<0.\end{aligned} \tag{31}$$

Furthermore, $\mathcal{V}(t)\le\mathcal{V}(0)$ for $t\ge0$. Hence

$$\mathcal{V}(t)\ge\min_{1\le s\le n+m}\{\lambda_s\}e^{r\epsilon t}\Bigg\{\sum_{p=1}^{n}|\bar{x}_p^A(t)|^r+\sum_{q=1}^{m}|\bar{y}_q^A(t)|^r\Bigg\}. \tag{32}$$

Moreover, from (24), we have

$$\begin{aligned}\mathcal{V}(0)&\le\sum_{p=1}^{n}\lambda_p\bigg[|\bar{x}_p(0)|^r+\frac{1}{1-\mu_1}\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\times\mathcal{L}_q^g\int_{-\sigma_{qp}(0)}^{0}|\bar{y}_q(s)|^r e^{r\epsilon(s+\sigma)}ds\bigg]\\&+\sum_{q=1}^{m}\lambda_{n+q}\bigg[|\bar{y}_q(0)|^r+\frac{1}{1-\mu_2}\sum_{p=1}^{n}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\times\mathcal{K}_p^f\int_{-\tau_{pq}(0)}^{0}|\bar{x}_p(s)|^r e^{r\epsilon(s+\tau)}ds\bigg]\\&\le\sum_{p=1}^{n}\lambda_p|\bar{x}_p(0)|^r+\frac{e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_p|a_{qp}^{A.\bar{B}}|\times\mathcal{L}_q^g\int_{-\sigma_{qp}}^{0}|\bar{y}_q(s)|^r e^{r\epsilon s}ds+\sum_{q=1}^{m}\lambda_{n+q}|\bar{y}_q(0)|^r\\&+\frac{e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\times\int_{-\tau_{pq}}^{0}|\bar{x}_p(s)|^r e^{r\epsilon s}ds\\&\le\max_{1\le s\le n+m}\{\lambda_s\}\bigg[1+\frac{\sigma e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g+\frac{\tau e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg]\\&\times\|(\varphi^A,\psi^A)^T-(x^*,y^*)^T\|\\&=\mathcal{M}^*\|(\varphi^A,\psi^A)^T-(x^*,y^*)^T\|,\end{aligned} \tag{33}$$

where

$$\mathcal{M}^*=\bigg[1+\frac{\sigma e^{r\epsilon\sigma}}{1-\mu_1}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g+\frac{\tau e^{r\epsilon\tau}}{1-\mu_2}\sum_{p=1}^{n}\sum_{q=1}^{m}\sum_{B\in\Xi}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f\bigg].$$

Setting $\mathcal{M}=\frac{\max_{1\le s\le n+m}\{\lambda_s\}}{\min_{1\le s\le n+m}\{\lambda_s\}}\mathcal{M}^*>1$. It is inferred from (32) and (33) that for all $t\ge0$

$$\sum_{p=1}^{n}|\bar{x}_p^A(t)|^r+\sum_{q=1}^{m}|\bar{y}_q^A(t)|^r\le\mathcal{M}\|(\varphi^A,\psi^A)^T-(x^*,y^*)^T\|e^{-r\epsilon t}. \tag{34}$$

This means that the equilibrium point of NNs (7) is globally exponentially stable. This completes this proof.

When we consider $r=2$, Corollary (3.2) can be derived by using Theorem (3.1).

Corollary 3.2: Let **(H1)–(H3)** hold, there exist constants $\lambda_p\in\mathbb{R}^+$, $\lambda_{n+q}\in\mathbb{R}^+$ ($p=\overline{1,n};\ q=\overline{1,m}$) such that

$$\begin{cases}-\lambda_p d_p+\sum_{q=1}^{m}\sum_{B\in\Xi}\frac{\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g}{2}+\sum_{q=1}^{m}\sum_{B\in\Xi}\frac{\lambda_{n+q}|b_{pq}^{A.\bar{B}}|\mathcal{K}_p^f}{2}<0,\\ -\lambda_{n+q}c_q+\sum_{p=1}^{n}\sum_{B\in\Xi}\frac{\lambda_{n+q}|b_{qp}^{A.\bar{B}}|\mathcal{K}_p^f}{2}+\sum_{p=1}^{n}\sum_{B\in\Xi}\frac{\lambda_p|a_{qp}^{A.\bar{B}}|\mathcal{L}_q^g}{2}<0,\end{cases} \tag{35}$$

then the equilibrium point of NNs (7) is globally exponentially stable.

Remark 3.3: Theorem (3.1) examines the global exponential stability criteria for Clifford-valued NNs by dividing the original Clifford-valued NNs into multidimensional real-valued NNs. It should be noted that the main results of this study are related to Clifford-valued NNs.

Remark 3.4: In [30], the authors investigated the S^p-almost periodic solutions of a fuzzy Clifford-valued cellular NN model with time-varying delays. In [31], the authors examined asymptotic almost automorphic synchronization criteria for neutral type fuzzy cellular NNs Clifford-valued recurrent NNs with time delays. In [38], the authors analyzed the global asymptotic stability criteria for Clifford-valued NNs incorporating impulsive effects and time-varying delays. However, no studies have investigated the stability of Clifford-valued BAMNNs with time delays using the decomposition method. Therefore, we investigated the global exponential stability of Clifford-valued BAMNNs with time delays using Lyapunov stability and system decomposition method. In addition, the results proposed in this study are new and differ from those in the existing literature [30], [31], [32], [33], [34], [35], [36], [37], [38], [39].

Remark 3.5: In this study, the proposed NNs (1) is more general than those presented in previous studies; therefore, there are significant differences between them. For example, by setting the Clifford generators m as 0, 1, and 2, the NNs (1) becomes a real-, complex-, and quaternion-valued NNs, respectively.

IV. NUMERICAL EXAMPLES

This section presents an example of the effectiveness and feasibility of the proposed method.

Example 1: For $m = 2$, the two neurons Clifford-valued BAMNNs are considered as follows:

$$\begin{cases} \dot{x}_1(t) = -d_1x_1(t) + a_{11}g_1(y_1(t-\sigma_{11}(t))) \\ \qquad +a_{21}g_2(y_2(t-\sigma_{21}(t))) + u_1, \\ \dot{x}_2(t) = -d_2x_2(t) + a_{12}g_1(y_1(t-\sigma_{12}(t))) \\ \qquad +a_{22}g_2(y_2(t-\sigma_{22}(t))) + u_2, \\ \dot{y}_1(t) = -c_1y_1(t) + b_{11}f_1(x_1(t-\tau_{11}(t))) \\ \qquad +b_{21}f_2(x_2(t-\tau_{21}(t))) + v_1, \\ \dot{y}_2(t) = -c_2y_2(t) + b_{12}f_1(x_1(t-\tau_{12}(t))) \\ \qquad +b_{22}f_2(x_2(t-\tau_{22}(t))) + v_2. \end{cases} \tag{36}$$

The Clifford generators are: $e_1^2 = e_2^2 = e_{12}^2 = e_1e_2e_{12} = -1$, $e_1e_2 = -e_2e_1 = e_{12}$, $e_1e_{12} = -e_{12}e_1 = -e_2$, $e_2e_{12} = -e_{12}e_2 = e_1$, $x_1 = x_1^0e_0 + x_1^1e_1 + x_1^2e_2 + x_1^{12}e_{12}$, $x_2 = x_2^0e_0 + x_2^1e_1 + x_2^2e_2 + x_2^{12}e_{12}$, $y_1 = y_1^0e_0 + y_1^1e_1 + y_1^2e_2 + y_1^{12}e_{12}$, $y_2 = y_2^0e_0 + y_2^1e_1 + y_2^2e_2 + y_2^{12}e_{12}$.

We also considered the following parameters

$$\begin{aligned} d_1 &= 2, \ d_2 = 2, \\ c_1 &= 3, \ c_2 = 3, \\ a_{11} &= 0.2e_0 + e_1, \\ a_{12} &= 0.1e_0 + 0.3e_2 - 0.6e_{12}, \\ a_{21} &= 0.05e_0 - 0.2e_2 + 0.4e_{12}, \\ a_{22} &= 0.1e_0 + 0.1e_1 + 0.05e_{12}, \\ b_{11} &= 0.3e_0 + 0.01e_1, \\ b_{12} &= 0.1e_0 + 0.1e_2 + 0.02e_2 - 0.3e_{12}, \\ b_{21} &= 0.05e_0 - 0.3e_2 + 0.05e_{12}, \\ b_{22} &= 0.2e_0 + 0.2e_1 + 0.05e_{12}, \\ u_1 &= 0.3e_0 + 0.1e_1 - 0.2e_{12}, \end{aligned}$$

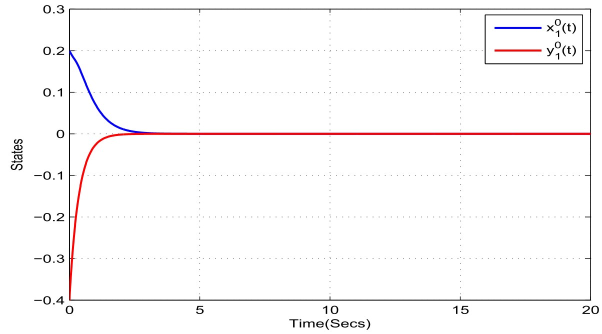

FIGURE 1. Time responses of the states $x_1^0(t)$, $y_1^0(t)$ of the NNs (36).

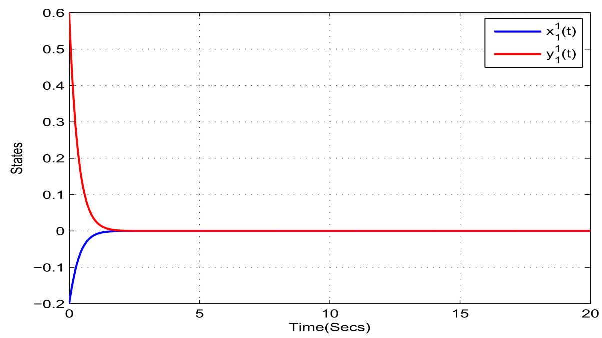

FIGURE 2. Time responses of the states $x_1^1(t)$, $y_1^1(t)$ of the NNs (36).

$$\begin{aligned} u_2 &= 0.1e_0 - 0.3e_1 + 0.1e_2 - 0.4e_{12}, \\ v_1 &= 0.3e_0 + 0.1e_1 - 0.1e_{12}, \\ v_2 &= 0.2e_0 + 0.1e_1 + 0.1e_2 - 0.2e_{12}. \end{aligned}$$

Let $\sigma_{11}(t) = \sigma_{21}(t) = 0.5 + 0.2sint$, $\sigma_{12}(t) = \sigma_{22}(t) = 0.4 + 0.3sint$, $\tau_{11}(t) = \tau_{21}(t) = 0.6 + 0.1sint$, and $\tau_{12}(t) = \tau_{22}(t) = 0.4 + 0.3sint$. It is clear that $0 \leq \sigma_{11} = \sigma_{21} \leq 0.7$, $0 \leq \sigma_{12} = \sigma_{22} \leq 0.7$, $0 \leq \tau_{11} = \tau_{21} \leq 0.7$, $0 \leq \tau_{12} = \sigma_{22} \leq 0.7$, and the time derivative can be obtained as $\dot{\sigma}_{11}(t) = \dot{\sigma}_{21}(t) = 0.2\ cost$, $\dot{\sigma}_{12}(t) = \dot{\sigma}_{22}(t) = 0.3\ cost$, $\dot{\tau}_{11}(t) = \dot{\tau}_{21}(t) = 0.1\ cost$, and $\dot{\tau}_{12}(t) = \dot{\tau}_{22}(t) = 0.3\ cost$. Moreover, the activation functions are chosen as $g_1(y_1) = g_2(y_2) = 0.5\ tanh(y^0)e_0 + 0.5\ tanh(y^1)e_1 + 0.5\ tanh(y^2)e_2 + 0.5\ tanh(y^{12})e_{12}$, $f_1(x_1) = f_2(x_2) = 0.5\ tanh(x^0)e_0 + 0.5\ tanh(x^1)e_1 + 0.5\ tanh(x^2)e_2 + 0.5\ tanh(x^{12})e_{12}$. It is obvious that It seems that the activation functions $g_q(\cdot)$ and $f_p(\cdot)$ are satisfying hypothesis **(H2)** with $\mathcal{L}_1^g = \mathcal{L}_2^g = 0.5$ and $\mathcal{K}_1^f = \mathcal{K}_2^f = 0.5$, respectively.

Further, it is easy to obtain $a_{11}^{A.\bar{B}} = 1.2$, $a_{12}^{A.\bar{B}} = -0.2$, $a_{21}^{A.\bar{B}} = 0.25$, $a_{22}^{A.\bar{B}} = 0.25$, $b_{11}^{A.\bar{B}} = 0.31$, $b_{12}^{A.\bar{B}} = -0.8$, $b_{21}^{A.\bar{B}} = 0.2$, $b_{22}^{A.\bar{B}} = 0.45$ and take $r = 2$, $\lambda_n = 1$, $(n = 1, 2)$, $\lambda_{2+q} = 1$, $(q = 1, 2)$. Using a simple calculation, we obtain

$$\begin{aligned} -r\lambda_1 d_1 + (r-1)\sum_{q=1}^{2}\sum_{B\in\Xi}\lambda_1|a_{q1}^{A.\bar{B}}|\mathcal{L}_q^g \\ + \sum_{q=1}^{2}\sum_{B\in\Xi}\lambda_{2+q}|b_{1q}^{A.\bar{B}}|\mathcal{K}_1^f < 0, \ q = 1, 2, \end{aligned} \tag{37}$$

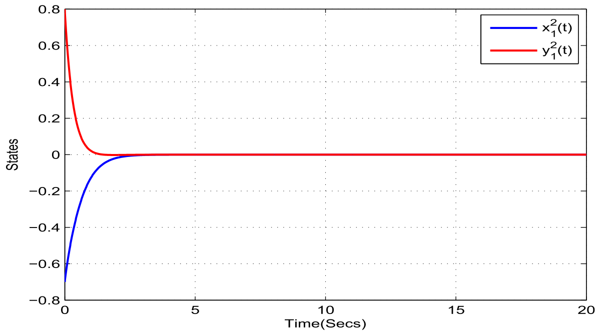

FIGURE 3. Time responses of the states $x_1^2(t)$, $y_1^2(t)$ of the NNs (36).

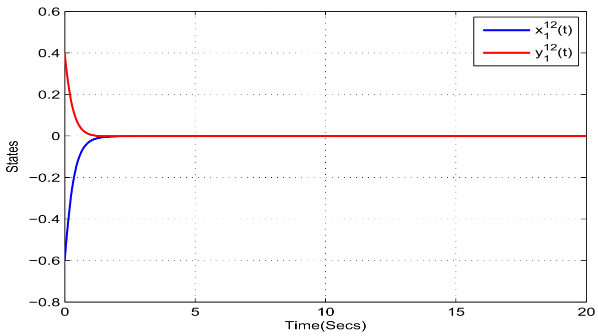

FIGURE 4. Time responses of the states $x_1^{12}(t)$, $y_1^{12}(t)$ of the NNs (36).

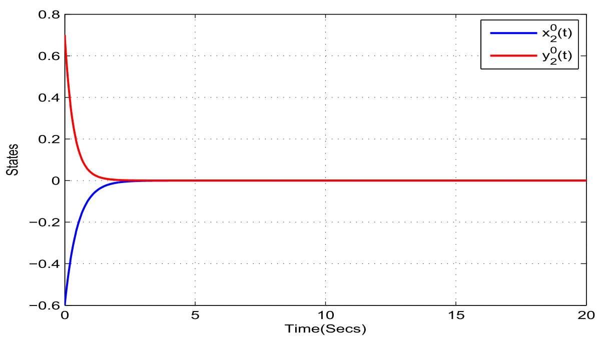

FIGURE 5. Time responses of the states $x_2^0(t)$, $y_2^0(t)$ of the NNs (36).

$$- r\lambda_2 d_2 + (r-1)\sum_{q=1}^{2}\sum_{B\in\Xi}\lambda_2|a_{q2}^{A.\bar{B}}|\mathcal{L}_q^g + \sum_{q=1}^{2}\sum_{B\in\Xi}\lambda_{2+q}|b_{2q}^{A.\bar{B}}|\mathcal{K}_2^f < 0,\ q=1,2, \tag{38}$$

$$- r\lambda_3 c_1 + (r-1)\sum_{p=1}^{2}\sum_{B\in\Xi}\lambda_3|b_{p1}^{A.\bar{B}}|\mathcal{K}_p^f + \sum_{p=1}^{2}\sum_{B\in\Xi}\lambda_p|a_{1p}^{A.\bar{B}}|\mathcal{L}_1^g < 0,\ p=1,2, \tag{39}$$

$$- r\lambda_4 c_2 + (r-1)\sum_{p=1}^{2}\sum_{B\in\Xi}\lambda_4|b_{p2}^{A.\bar{B}}|\mathcal{K}_p^f + \sum_{p=1}^{2}\sum_{B\in\Xi}\lambda_p|a_{2p}^{A.\bar{B}}|\mathcal{L}_2^g < 0,\ p=1,2. \tag{40}$$

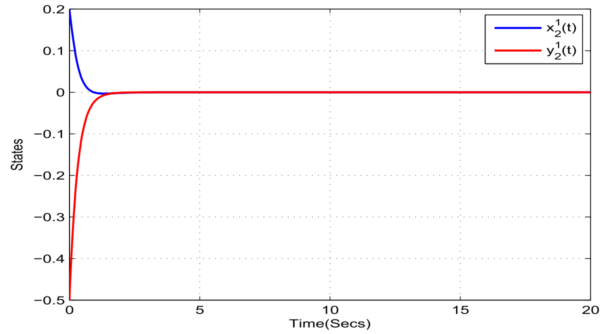

FIGURE 6. Time responses of the states $x_2^1(t)$, $y_2^1(t)$ of the NNs (36).

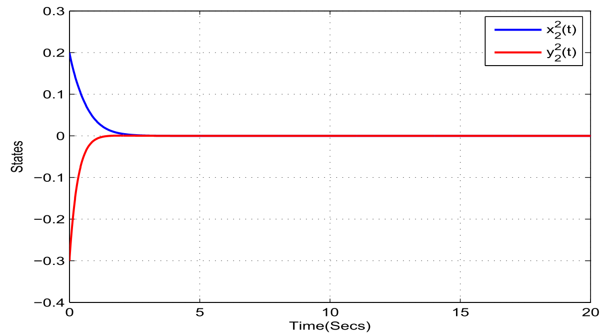

FIGURE 7. Time responses of the states $x_2^2(t)$, $y_2^2(t)$ of the NNs (36).

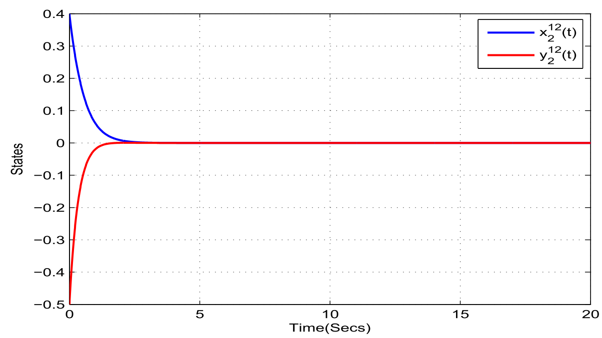

FIGURE 8. Time responses of the states $x_2^{12}(t)$, $y_2^{12}(t)$ of the NNs (36).

Under the initial conditions $\varphi_1(t) = 0.2e_0 - 0.2e_1 - 0.7e_2 - 0.6e_{12}$, $\varphi_2(t) = -0.6e_0 + 0.2e_1 + 0.2e_2 + 0.4e_{12}$, $\psi_1(t) = -0.4e_0 + 0.5e_1 + 0.8e_2 + 0.4e_{12}$, and $\psi_2(t) = 0.7e_0 - 0.5e_1 - 0.3e_2 - 0.5e_{12}$, the time responses of the states of the NNs (36) were obtained using MATLAB, as shown in Figures (1)-(8). From these figures (1)-(8), it can be seen that the time response of the states of the NNs (36) converge to the equilibrium point over time. This shows that all conditions in Theorem (3.1) are satisfied; therefore, the NNs considered in (36) have a unique equilibrium point that is globally exponentially stable.

V. CONCLUSION

This study investigated the global exponential stability problem for a class of Clifford-valued BAMNNs with time-varying delays. We first decomposed the n-dimensional Clifford-valued NNs into $2^m n$-dimensional real-valued NNs to avoid the inconvenience caused by the non-commutativity of Clifford number multiplication. We then established new sufficient conditions for the existence, uniqueness, and global exponential stability of the equilibrium points for the considered networks using Lyapunov functions, homeomorphism theory, and inequality techniques. Finally, the results presented in this paper are illustrated using a numerical example accompanied by the simulation results.

The results of this study can be used to explore various dynamics of Clifford-valued BAMNNs, including finite-time stability, state estimation, and synchronization. Thus, we will examine finite-time stability for the following Clifford-valued BAMNNs with impulsive effects

$$\begin{cases} \dot{x}_p(t) = -d_p x_p(t) + \sum_{q=1}^{m} a_{qp} g_q(y_q(t-\sigma_{qp}(t))) \\ \qquad +u_p,\ t \geq 0,\ t \neq t_k,\ p = \overline{1,n}, \\ \triangle x_p(t_k) = \alpha_k(x_p(t_k)),\ t = t_k,\ k = 1,2,\ldots,\ p = \overline{1,n}, \\ \dot{y}_q(t) = -c_q y_q(t) + \sum_{p=1}^{n} b_{pq} f_p(x_p(t-\tau_{pq}(t))) \\ \qquad +v_q,\ t \geq 0,\ t \neq t_k,\ q = \overline{1,m}, \\ \triangle y_q(t_k) = \beta_k(y_q(t_k)),\ t = t_k,\ k = 1,2,\ldots,\ q = \overline{1,m}. \end{cases}$$

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