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RESEARCH ARTICLE

A General Equilibrium Analysis of Predefined-Time Control and Energy Consumption for Neural Networks With Time-Varying Delays

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ABSTRACT This paper mainly focuses on the equilibrium problem of predefined-time stability and control energy consumption in nonlinear neural networks with time-varying delays. A new criterion for one global composite switching controller to assure predefined-time stability is provided by employing inequality technologies and Lyapunov stability theorem. Under the constructed controller, it is proved that the system is predefined-time stable when the initial conditions are inside and outside the unit sphere. Then, the energy consumption required for the system to reach the control target is estimated, which is related to the preset control time. Moreover, the equilibrium problem of the control energy consumption and the settling time is investigated by constructing an evaluation index function, and the optimal preset control time is obtained. The results show that a suitable preset control time can better balance the energy consumed by the controller, which has practical implications. Finally, a simulation example has clearly verified the theoretical results.

INDEX TERMS Equilibrium analysis, delayed neural networks, predefined-time stability, energy consumption.

I. INTRODUCTION

In past 20 years, neural networks dynamics has caused extensive concern due to its broad application in the area of nonlinear dynamic systems, including machine learning, biological, engineering, and thus generates a group of typical theoretical results and applications [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. As an important research topic, several stability concepts of neural networks have been proposed, for example exponential stability and asymptotic stability. It should be noted that the control time of asymptotic stability or exponential stability is infinite. In actual applications, it is envisaged to hasten the stabilization of the system. In consideration of this, various conclusions about finite-time stability(FTS) have been presented [13], [14], [15], [16], [17]. FTS demonstrates faster convergence and improved disturbance rejection properties [18]. The main issue of FTS is that the settling time function depends on the initial conditions and it is often an unbounded function. In order to solve this problem, an improved form of stability called the fixed-time stability(FxTS) is proposed [19], [34], [38], [39], in which the settling time function is independent of the system's initial conditions. FxTS improves the classical finite-time stability in a sense, but it is generally difficult to estimate the settling time function, because the relationship between the tuning parameters and convergence

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time is not specific. Many estimations of the upper bound of the fixed stability time are often much larger than the actual true convergence time. For some problems, it will be very convenient if the upper bound of settling time function can be determined in advance, such as state estimation and dynamic optimization [21]. To accomplish this, a new class of finite-time stability notion known as predefined-time stability has been developed. References [20] and [40], in which the settling time is a predefined constant and explicitly set as a function of system's parameters.

When control a differential system, an important and unavoidable issue is the control cost. In order to achieve the control goal, the controller needs to consume a certain amount of energy [22]. For example, in order to control an electronic or mechanical network, some energy must be consumed to drive some components. If the system's stabilization time is finite but the control energy consumption is infinite, application in practice is not possible. Therefore, it is necessary to evaluate the energy consumption in the process of system control. In general, shorter control time means more energy consumed by the system. Therefore, how to coordinate the control time and control energy consumption is a very meaningful topic. In [22], the expression of control energy consumption was given. In two time scales, the different scaling behaviors of control time of general neural networks were analyzed. On this basis, a closed-loop control framework for complex networks to ensure FTS of the system was developed, and a trade-off between time and energy was investigated [23]. Inspired by this, the method was extended to neural networks [24]. A composite switching controller was developed to ensure the FxTS of a class of nonlinear neural networks without delays, and the effect of modifying parameters on the stability time and energy was thoroughly investigated. The specific control parameter to guarantee trade-off between them was given [25].

It should be noted that the above conclusions on control energy consumption is for the system without time delays. Time delays are often one of those factors that must be considered in neural networks. For example, there are time delays in information transmission and signal conversion. It is therefore a challenge to establish the criteria for FxTS of delayed systems, which motivates our present work. Analysing the balance between stabilization time and energy cost of delayed system naturally becomes a topic of research. To handle the effect of time delays, two compound switching controllers $u(t) = -kx(t) - p\text{sign}(x(t))$ and $u(t) = -k\text{sign}(x(t))^{\alpha} - p\text{sign}(x(t))$ have been developed to ensure FTS of nonlinear delayed system, and the switching controller's control energy consumption was estimated [35]. For general nonlinear differential systems, the control energy consumption of proposed controller in FTS is estimated with and without delays [26].

The settling time's upper bound in predefined-time stability can be chosen arbitrarily in advance, but the energy consumption is associated with the initial states and is dependent on the parameters of the system, the control parameters and the settling time. This is different from the cases of FTS and FxTS. Up to now, the research on the equilibrium between predefined-time control and control energy consumption of delayed system has not been discovered. Overall, the fundamental goal for this research is to analyze the relationship of predefined-time control and control energy consumption of delayed system. To facilitate readers, the main contributions and innovations are summarized as follows:

1. To handle the effect of time delays, a novel composite switching controller is constructed to assure predefined-time stability of delayed neural networks. In practical applications, such a controller design would have more selectivity and flexibility. A sufficient condition has been introduced based on the constructed Lyapunov function.

2. The specific formula of control energy consumption is present. By analyzing the equilibrium problem of the settling time function and control energy consumption, the optimal settling time function is given, which will facilitate the application of the conclusion in practice. The remainder of this paper is organized as follows. Section II will present some assumptions, definitions, and lemmas. Section III describes the main outcomes of our research. A numerical example is provided in the fourth part to validate our theoretical results. Finally, the thesis is outlined in Section V.

Notations: Throughout this article, let $n > 0$ denote an integer, $\text{sig}(\cdot)^a = |\cdot|^a\text{sign}(\cdot)$ with signum function $\text{sign}(\cdot)$. $C([a, b], R)$ symbolizes the continuous function family from interval $[a, b]$ to real number set R, $R^+ = \{x|x > 0\}$. The notation x^T denotes the transpose of x. The L_2 norm of x is denoted by $\|x\| = \sqrt{x^T x}$, and $|x(t)| = [|x_1(t)|, \cdots, |x_n(t)|]^T$. For $f(t) = [f_1(t), \cdots, f_n(t)]^T$, $f_i(t) \in C([a, b], R)$, $\|f(t)\|_c = \sup_{t\in[a,b]}\sqrt{f^T(t)f(t)}$.

II. PROBLEM FORMULATION

Consider a class of nonlinear delayed neural networks described by:

$$\dot{x}_i(t) = -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau(t))) + u_i(t), \quad (1)$$

where $x(t) = [x_1(t), \cdots, x_n(t)]^T \in R^n$ is the system state, $D = \text{diag}(d_1, \cdots, d_n) \in R^{n\times n}$, $d_i \geq 0$ and $A = (a_{ij}) \in R^{n\times n}$ are connection strength matrices. $B = (b_{ij}) \in R^{n\times n}$ symbolizes constant connection weight matrices at $t - \tau(t)$. $\tau(t)$ symbolizes the time-varying delay, meeting $0 \leq \tau(t) \leq \tau$, where the constant τ is known. $f_j(x_j(t))$, $g_j(x_j(t - \tau(t)))$ represent the activation functions at t and $t - \tau(t)$. $u_i(t)$ is the controller we will design later. The initial values related to DNNs (1) are given by $x_i(0) = \varphi_i(s)$, where $\varphi_i(s) \in C([-\tau, 0], R)$, $\varphi(s) = [\varphi_1(s), \cdots, \varphi_n(s)]^T$.

To get the main results, we present some basic lemmas and assumptions.

Assumption 1: For f_j and g_j, there are two constants $L_j > 0$, $M_j > 0$ such that $|f_j(y)| \leq L_j|(y)|$, $|g_j(y)| \leq L_j|(y)|$, $|f_j(\cdot)| \leq M_j$, $|g_j(\cdot)| \leq M_j, \forall y \in R$. Additionally, $f_j(0) = g_j(0) = 0$.

Definition 1 [20]: Given a constant $T_c > 0$ in advance, system (1) is said to be predefined-time stable if it is FxTS and the settling time function $T_0(\varphi)$ is such that $T_0(\varphi) \leq T_c, \forall\varphi \in \Omega$, of which the open set $\Omega \subseteq C([-\tau, 0], R^n)$ contains 0. In this case, T_c is called a predefined time.

Lemma 1 [27]: Assume $\zeta_1, \zeta_2, \cdots, \zeta_n \geq 0$ and $0 < p_1 \leq 1$, $p_2 > 1$, then

$$\sum_{i=1}^{m} \zeta_i^{p_1} \geq (\sum_{i=1}^{m} \zeta_i)^{p_1}, \qquad \sum_{i=1}^{m} \zeta_i^{p_2} \geq m^{1-p_2}(\sum_{i=1}^{m} \zeta_i)^{p_2}.$$

Lemma 2 [28]: Assume that there exist two numbers $k_1, k_2 \in R^+$, such that $\forall 0 < a_1 \leq a_2$, $k_1 \|.\|_{a_1} \leq \|.\|_{a_2} \leq k_2 \|.\|_{a_1}$, where $\|.\|_{a_1}$ is the L_{a_1} norm for R^n, and $\|.\|_{a_2}$ denotes the L_{a_2} norm for R^n. In particular, $k_1 = 1$ and $k_2 = n^{\frac{1}{a_1} - \frac{1}{a_2}}$.

In previous literatures [30], [31], the switching controller design has been well used. In addition, one of our goals in this paper is to design a delay-independent controller. The usual approach is to restrict the activation function [32], [33]. Inspired by this, a global composite switching controller is constructed as follows:

$$u_i(t) = \begin{cases} u_i^{(1)}(t), & \|x(0)\|_c \geq 1, \\ u_i^{(2)}(t), & \|x(0)\|_c < 1. \end{cases}$$

where $u_i^{(1)}(t) = -kx_i(t) - \frac{2n^{\frac{\beta-1}{2}}}{(\beta-1)T_c}\text{sig}(x_i(t))^\beta - h_i\text{sig}(x_i(t))$, $u_i^{(2)}(t) = -kx_i(t) - \frac{2}{(1-\alpha)T_c}\text{sig}(x_i(t))^\alpha - h_i\text{sig}(x_i(t))$, $\beta > 1, 0 < \alpha < 1$, $u(t) = [u_1(t), \cdots, u_n(t)]^T$, $u^{(1)}(t) = [u_1^{(1)}(t), \cdots, u_n^{(1)}(t)]^T$, $u^{(2)}(t) = [u_1^{(2)}(t), \cdots, u_n^{(2)}(t)]^T$, $H = [h_1, \cdots, h_n]^T$, and $k > 0$, $h_i > 0$, T_c is a predefined positive constant.

Obviously the controller $u_i(t)$ is discontinuous, leading to the discontinuity on the right hand side of system (1). Therefore, we consider the solutions of system in Filippov sence [36], [37].

Remark 1: Compared with the method in existing studies [34], It is more convenient to estimate settling time and energy consumption using the global composite switching controller above. we only choose the term of $\alpha < 1$ inside the unit ball $\ominus = \{\|x\| \leq 1\}$ and only choose the term of $\beta > 1$ outside the unit ball. Different controllers are selected inside and outside the unit ball $\ominus = \{\|x\| \leq 1\}$, and the control time can be estimated separately. It should be noted that we can also choose other controllers in complementary regions above to achieve the same control target, which means the design of the controller has more selectivity and flexibility.

III. MAIN RESULTS

In this section, we will prove that DNNs (1) is predefined-time stable and give the specific formula of control energy consumption of the designed controller.

A. PREDEFINED-TIME STABILIZATION OF SYSTEM

In this part, we give the sufficient condition of predefined-time stabilization of DNNs (1) under global control protocol $u(t)$.

Theorem 1: Assume the assumption 1 is satisfied, the control strength k and matrices A, B, D satisfy the inequality $k > L\lambda_A - \lambda_D$ and $h_i \geq \|B\|M_i$, the DNNs (1) is predefined-time stable under $u(t)$, where $L = \max\{L_j|j = 1, \cdots n\}$, λ_D,λ_A denote the smallest and largest eigenvalue of matrices D, A separately, and $\|B\| = \sqrt{r(B^TB)}$, $r(B^TB) = \max\{|\lambda(B^TB)|\}$. T_c is the predefined control time.

Proof: We formulate the Lyapunov function $V(x(t)) = x^T(t)x(t)$. Combining definition of the switching controller, we consider the initial conditions in two cases: $\|x(0)\|_c \geq 1$ and $\|x(0)\|_c < 1$.

Case A: When $\|x(0)\|_c \geq 1$.

Step 1: we first calculate the settling time before trajectories enter the unit ball. Before the trajectories enter the interior of the unit ball $\ominus$, the controller $u_i^{(1)}(t) = -kx_i(t) - \frac{2n^{\frac{\beta-1}{2}}}{(\beta-1)T_c}\text{sig}(x_i(t))^\beta - h_i\text{sig}(x_i(t))$ works. The differentiation of $V(x(t))$ along the solution of the DNN (1) leads to

$$\begin{aligned} \frac{dV(x(t))}{dt} &= (\sum_{i=1}^{n} x_i^2(t))' \\ &= -2x^T(t)Dx(t) + 2x^T(t)Af(x(t)) \\ &\quad + 2x^T(t)Bg(x(t-\tau(t))) - 2|x(t)|^TH \\ &\quad - \frac{2n^{\frac{\beta-1}{2}}}{(\beta-1)T_c}x^T(t)\text{sig}(x(t))^\beta \\ &\quad - 2kx^T(t)x(t). \end{aligned} \tag{2}$$

In accordance with the lemma 1, we have

$$x^T(t)\text{sig}(x(t))^\beta \geq \qquad n^{\frac{1-\beta}{2}}V^{\frac{\beta+1}{2}}(x(t)).$$

Since $k > L\lambda_A - \lambda_D$, $\|B\|M_i - h_i \leq 0$, (2) can be further reduced to

$$\begin{aligned} \frac{dV(x(t))}{dt} &\leq -2(k + \lambda_D - L\lambda_A)V(x(t)) \\ &\quad - \frac{4}{(\beta-1)T_c}V^{\frac{\beta+1}{2}}(x(t)) \\ &\leq -\frac{4}{(\beta-1)T_c}V^{\frac{\beta+1}{2}}(x(t)). \end{aligned} \tag{3}$$

Obviously, from (3), we can get a constant t^* such that $\|x(t^*)\| = 1$. Simplifying (3) and integrating it from 0 to t, one can have

$$\int_0^t \frac{dV}{V^{\frac{\beta+1}{2}}} \leq \int_0^t -\frac{4}{(\beta-1)T_c}dt,$$

Solving this inequality, we can obtain $t \leq \frac{T_C}{2}V^{\frac{1-\beta}{2}}(x(t))$. Further, since $V^{\frac{1-\beta}{2}}(x(t)) \leq 1$, we can get the upper bound of t^*, that is $t^* \leq \frac{T_C}{2}$. Next, using the method of contradiction, we will prove $\|x(t)\| < 1$ $\forall t \in (t^*, +\infty)$. Suppose the trajectories of the system cross the unit sphere again, which means there is at least a constant satisfying $\|x(t)\| = 1$ on the interval $(t^*, +\infty)$. We record the smallest moment when the trajectories cross the sphere again as

$$t' = \inf\{t \in [\hat{t}, t_1) |\, \|x(t)\| = 1\},$$

where $t^* < t < \hat{t} < t_1 < +\infty$ and $\|x(t)\| < 1$. All the constants in inequality can be obtained because $x(t)$ is continuous. For $t \in [\hat{t}, t')$, differentiating $V(x(t))$ along (1) yields

$$\begin{aligned}\frac{dV(x(t))}{dt} &= -2x^T(t)Dx(t) + 2x^T(t)Af(x(t)) \\ &\quad + 2x^T(t)Bg(x(t-\tau(t))) - 2|x(t)|^T H \\ &\quad - \frac{4}{(1-\alpha)T_c}x^T(t)\text{sig}(x(t))^\alpha - 2kx^T(t)x(t) \\ &\le -2(k + \lambda_D - L\lambda_A)V(x(t)) \\ &\quad - \frac{4}{(1-\alpha)T_c}V^{\frac{\alpha+1}{2}}(x(t)),\end{aligned} \tag{4}$$

where

$$x^T(t)\text{sig}(x(t))^\alpha \ge V^{\frac{\alpha+1}{2}}(x(t)),$$

Apparently, $V(x(t))$ is a monotonically decreasing when $\hat{t} \le t < t'$. Then, $1 = V(x(t')) \le V(x(\hat{t})) < 1$ can be gotten. Clearly, this is untenable. Therefore, $\|x(t)\| \le 1, \forall t \in (t^*, t_1)$, and we can extend the open interval (t^*, t_1) to $(t^*, +\infty)$.

Step 2: Estimate control time when trajectories enter sphere. Clearly, $V(x(t)) \le V^{\frac{\alpha+1}{2}}(x(t))$ holds when $t \in (t^*, +\infty)$. From (4), we have

$$\frac{dV(x(t))}{dt} \le -\frac{4}{(1-\alpha)T_c}V^{\frac{\alpha+1}{2}}(x(t)). \tag{5}$$

Based on lemma 3 [29], [35], we have $J(x(t)) \ge V(x(t))$ when $t \in (t^*, +\infty)$, where $\frac{dJ(x(t))}{dt} = -\frac{4}{(1-\alpha)T_c}J(x(t))^{\frac{\alpha+1}{2}}$, $J(x(t^*)) = V(x(t^*))$. Integrating it, one has $\frac{1}{1-\alpha}J^{\frac{1-\alpha}{2}}(x(t)) = -\frac{4}{(1-\alpha)T_c}t + c_0, t > t^*$, where $c_0 = \frac{4}{(1-\alpha)T_c}t^* + \frac{2}{1-\alpha}$. Thus, we can get $V(x(t)) \le J(x(t)) = [(1-\alpha)(-\frac{2}{(1-\alpha)T_c}t + \frac{c_0}{2})]^{\frac{2}{1-\alpha}}$. Taking $J(x(t)) = 0$, we have

$$T_f \le t^* + \frac{T_c}{2} \le T_c.$$

Case B: When $\|x(0)\|_c < 1$.

In this case, $u_i^{(2)}(t) = -kx_i(t) - \frac{2}{(1-\alpha)T_c}\text{sig}(x_i(t))^\alpha - h_i\text{sig}(x_i(t))$ works. Similar to the proof in step 2 above,we have

$$T_f \le \frac{T_c}{2} \le T_c.$$

To sum up, for the two cases above, the stabilization time is less than the predefined constant T_c. According to definition 1, the DNNs (1) is predefined-time stable.

This is all proof.

Remark 2: From the proof process, it can be found that once trajectories of the system enter the unit ball, they will remain in the ball, and the controller will not switch repeatedly. The system considered is predefined-time stable inside and outside the unit ball under the controller $u(t)$. Furthermore, if $\|x(0)\|_c < 1$, the control time is only half the preset time T_c. We will still use $\frac{T_c}{2}$ when calculating the energy.

B. ESTIMATION OF ENERGY

Based on the result in [22], the control energy consumption was defined as $\Xi_c = \int_0^{T_f} \|u(t)\|^2\, dt$. For convenience, we will also denote Ξ_c as the upper energy bound. Then, the control energy cost is given as follows.

Theorem 2: For DNNs (1), the upper bound of the energy cost Ξ_c can be estimated as

$$\Xi_c = \begin{cases} 3k^2T_c\|\varphi(s)\|_c^2 + \frac{12n^{\frac{2\beta-1}{2}}\|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)T_c} + \frac{6n^{1-\alpha}}{(1-\alpha^2)T_c} \\ \quad + \frac{3k^2(1-\alpha)T_c}{2(3-\alpha)} + \frac{6n^{1-\alpha}}{(1-\alpha^2)T_c} \\ \quad + 3\|H\|^2T_c, \qquad \|x(0)\|_c \ge 1, \\ \quad \frac{3k^2(1-\alpha)T_c\|\varphi(s)\|_c^{3-\alpha}}{2(3-\alpha)} + \frac{6n^{1-\alpha}\|\varphi(s)\|_c^{1+\alpha}}{T_c(1-\alpha^2)} \\ \quad + \frac{3}{2}\|H\|^2T_c, \qquad \|x(0)\|_c < 1. \end{cases} \tag{6}$$

where these parameters k, α, β are the same as in theorem 1. T_c is the predefined stabilization time. $\|H\| = \max\{h_1, \ldots, h_n\}$.

Proof: Corresponding to theorem 1, we still prove theorem 2 in two scenarios.

Case A: When $\|x(0)\|_c \ge 1$

In view of the definition of the switch controller $u(t)$, $u^{(1)}(t)$ works when $t < t^*$, while it is $u^{(2)}(t)$ when $t \in (t^*, T_f)$. The control energy consumption Ξ_c can be calculated as follows

$$\begin{aligned}\Xi_c &= \int_0^{T_f}\|u(t)\|^2\,dt \\ &= \int_0^{t^*}\left\|u^{(1)}(t)\right\|^2 dt + \int_{t^*}^{T_f}\left\|u^{(2)}(t)\right\|^2 dt,\end{aligned} \tag{7}$$

when $t < t^*$, we have

$$\begin{aligned}&\left\|u^{(1)}(t)\right\|^2 \\ &\le \left(\|kx(t)\| + \left\|\frac{2n^{\frac{\beta-1}{2}}}{(\beta-1)T_c}sig(x(t))^\beta\right\| \right. \\ &\quad \left. + \|H\text{sig}(x(t))\|\right)^2 \\ &\le 3k^2\|x(t)\|^2 + \frac{12n^{\beta-1}}{(\beta-1)^2T_c^2}\|x(t)\|_{2\beta}^{2\beta} + 3\|H\|^2.\end{aligned}$$

In addition, from the inequality in lemma 2, it can be deduced that $\|x(t)\|_{2\beta}^{2\beta} \le \|x(t)\|_1^{2\beta} \le n^{\frac{1}{2}}\|x(t)\|^{2\beta} = n^{\frac{1}{2}}V(x(t))^\beta$.

So we can get

$$\begin{aligned}&\int_0^{t^*}\left\|u^{(1)}(t)\right\|^2 dt \\ &\le 3k^2\int_0^{t^*}V(x(t))dt + \frac{12n^{\frac{2\beta-1}{2}}}{(\beta-1)^2T_c^2}\int_0^{t^*}V(x(t))^\beta dt \\ &\quad + 3\int_0^{t^*}\|H\|^2\,dt\end{aligned}$$

$$\leq 3k^2 \int_0^{t^*} [V^{\frac{1-\beta}{2}}(x(0)) + \frac{1}{T_c} t]^{\frac{2}{1-\beta}} dt + \frac{12n^{\frac{2\beta-1}{2}}}{(\beta-1)^2 T_c^2} \int_0^{t^*} [V^{\frac{1-\beta}{2}}(x(0)) + \frac{1}{T_c} t]^{\frac{2\beta}{1-\beta}} + 3\|H\|^2 t^* \leq 3k^2 T_c \|\varphi(s)\|_c^2 + \frac{12n^{\frac{2\beta-1}{2}} \|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)T_c} + \frac{3}{2}\|H\|^2 T_c. \tag{8}$$

Similar to the above method, we can get

$$\left\|u^{(2)}(t)\right\|^2 \leq \left(\|kx(t)\| + \left\| \frac{2}{(1-\alpha)T_c} \mathrm{sig}(x(t))^\alpha \right\| + \|H \mathrm{sig}(x(t))\| \right)^2 \leq 3k^2 \|x(t)\|^2 + \frac{12}{(1-\alpha)^2 T_c^2} \|x(t)\|_{2\alpha}^{2\alpha} + 3\|H\|^2.$$

In addition, based on the inequality in lemma 2, it can be deduced that $\|x(t)\|_{2\alpha}^{2\alpha} \leq \xi \|x(t)\|^{2\alpha} = \xi V^\alpha(x(t))$, where $\xi = (\xi_2)^{2\alpha} = [n^{\frac{1}{2\alpha}-\frac{1}{2}}]^{2\alpha} = n^{1-\alpha}$. Thus, when $t \in [t^*, T_f)$, the control energy consumption is estimated as.

$$\int_{t^*}^{T_f} \left\|u^{(2)}(t)\right\|^2 dt \leq 3k^2 \int_{t^*}^{T_f} \|x(t)\|^2 dt + \frac{12}{(1-\alpha)^2 T_c^2} \int_{t^*}^{T_f} \|x(t)\|_{2\alpha}^{2\alpha} dt + 3\int_{t^*}^{T_f} \|H\|^2 dt \leq 3k^2 \int_{t^*}^{T_f} V(x(t))dt + \frac{12n^{1-\alpha}}{(1-\alpha)^2 T_c^2} \int_{t^*}^{T_f} V^\alpha x(t) dt + 3\|H\|^2 (T_f - t^*) \leq 3k^2 \int_{t^*}^{T_f} \left[(1-\alpha)(-\frac{2}{(1-\alpha)T_c} t + c_0)\right]^{\frac{2}{1-\alpha}} dt + \frac{12n^{1-\alpha}}{(1-\alpha)^2 T_c^2} \int_{t^*}^{T_f} \left[(1-\alpha)(-\frac{2}{(1-\alpha)T_c} t + c_0)\right]^{\frac{2\alpha}{1-\alpha}} dt + 3\|H\|^2 (T_f - t^*) \leq \frac{3k^2(1-\alpha)T_c}{2(3-\alpha)} + \frac{6n^{1-\alpha}}{(1-\alpha^2)T_c} + \frac{3\|H\|^2 T_c}{2}, \tag{9}$$

where $c_0 = \frac{2}{T_c(1-\alpha)} t^* + \frac{1}{1-\alpha}$. Based on (8) and (9), the upper bound of energy can be obtained

$$\Xi_c = 3k^2 T_c \|\varphi(s)\|_c^2 + \frac{12n^{\frac{2\beta-1}{2}} \|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)T_c} + 3\|H\|^2 T_c + \frac{3k^2(1-\alpha)T_c}{2(3-\alpha)} + \frac{6n^{1-\alpha}}{(1-\alpha^2)T_c}. \tag{10}$$

Case B: When $\|x(0)\|_c < 1$. In accordance with theorem 1, the control time T_f satisfies $T_f \leq \frac{T_c}{2}$. Thus, we have

$$\int_0^{\frac{T_c}{2}} \left\|u^{(2)}(t)\right\|^2 dt \leq 3k^2 \int_0^{\frac{T_c}{2}} \|x(t)\|^2 dt + \frac{12}{(1-\alpha)^2 T_c^2} \int_0^{\frac{T_c}{2}} \|x(t)\|_{2\alpha}^{2\alpha} dt + 3\int_0^{\frac{T_c}{2}} \|H\|^2 dt \leq 3k^2 \int_0^{\frac{T_c}{2}} V(x(t))dt + \frac{12n^{1-\alpha}}{(1-\alpha)^2 T_c^2} \int_0^{\frac{T_c}{2}} V^\alpha(x(t))dt + \frac{3}{2}\|H\|^2 T_c \leq 3k^2 \int_0^{\frac{T_c}{2}} \left[(1-\alpha)(-\frac{2}{(1-\alpha)T_c} t + c_0)\right]^{\frac{2}{1-\alpha}} dt + \frac{12n^{1-\alpha}}{(1-\alpha)^2 T_c^2} \int_0^{\frac{T_c}{2}} \left[(1-\alpha)(-\frac{2}{(1-\alpha)T_c} t + c_0)\right]^{\frac{2\alpha}{1-\alpha}} dt + \frac{3}{2}\|H\|^2 T_c \leq \frac{3k^2(1-\alpha)T_c \|\varphi(s)\|_c^{3-\alpha}}{2(3-\alpha)} + \frac{6n^{1-\alpha} \|\varphi(s)\|_c^{1+\alpha}}{T_c(1-\alpha^2)} + \frac{3}{2}\|H\|^2 T_c,$$

where $c_0 = \frac{V^{\frac{1-\alpha}{2}} x(0)}{1-\alpha}$. Finally, we can obtain the upper bound Ξ_c as

$$\Xi_c = \frac{3k^2(1-\alpha)T_c \|\varphi(s)\|_c^{3-\alpha}}{2(3-\alpha)} + \frac{6n^{1-\alpha} \|\varphi(s)\|_c^{1+\alpha}}{T_c(1-\alpha^2)} + \frac{3}{2}\|H\|^2 T_c. \tag{11}$$

Therefore, the upper bound Ξ_c is summarized as

$$\Xi_c = \begin{cases} 3k^2 T_c \|\varphi(s)\|_c^2 + \frac{12n^{\frac{2\beta-1}{2}} \|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)T_c} + \frac{6n^{1-\alpha}}{(1-\alpha^2)T_c} \\ \quad + \frac{3k^2(1-\alpha)T_c}{2(3-\alpha)} + 3\|H\|^2 T_c, \; \|x(0)\|_c \geq 1, \\ \frac{3k^2(1-\alpha)T_c \|\varphi(s)\|_c^{3-\alpha}}{2(3-\alpha)} + \frac{6n^{1-\alpha} \|\varphi(s)\|_c^{1+\alpha}}{T_c(1-\alpha^2)} \\ \quad + \frac{3}{2}\|H\|^2 T_c, \qquad \|x(0)\|_c < 1. \end{cases} \tag{12}$$

Remark 3: Compared with corresponding conclusions in [24], [35], the predefined control time can be arbitrarily chosen in advance, which is different from the finite/fixed-time stability, the energy cost is related to the initial conditions of system, and is associated with control parameters and control time. This means that we can attempt to find the best preset time with minimum control energy consumption.

C. EQUILIBRIUM ANALYSIS

According to the results of the first two sections, it can be found that although the control time is set in advance, the control energy consumption is not only relevant to the control parameters, such as exponents α, β and the control intensity k, but also relevant to the control time. This means that different control parameters or control time will affect the control energy consumption. In general, we expect the system to achieve stability as soon as possible, while the controller consumes as little energy as possible. How to balance control time and control energy consumption will be discussed below. Next, supposing the parameters α, β, and the initial condition $\varphi(s)$ are set, How does changing parameter k and preset control time T_c affect control energy consumption is studied, and the optimal k and T_c that minimize control energy consumption are found. Specifically, we discuss two problems. First, when the control time T_c is fixed, we try to find suitable control strength k to minimize the control energy consumption Ξ_c. Second, when the control time T_c is adjustable within one range, we try to discuss the equilibrium between the stabilization time and energy.

First, when the control time T_c is fixed, we try to find suitable control parameters to minimize the control energy consumption Ξ_c. According to the formula of Ξ_c in section III-B, with respect to the parameter k, the control energy consumption Ξ_c increases monotonically. According to theorem 1, we can let $k = L\lambda_A - \lambda_D$, and the minimum energy Ξ_c can be gotten.

Next we will have a look at the equilibrium between control time and control energy consumption. When the control time T_c is adjustable within one range, the control energy consumption is a binary function of k and T_c. It is expected that the stabilization time and energy are both as small as possible. Therefore, we study this bi-objective optimization problem: min T_f, min Ξ_c. Apparently, the control energy consumption Ξ_c is a monotonically increasing function of the parameter k, the control time T_c is independent of the parameter k. Thus, with respect to the parameter k, the following evaluation index function is also monotonically increasing. We only need to discuss the influence of changing the predefined constant T_c. We discuss the evaluation index function

$$\Upsilon_{\gamma_1,\gamma_2}(T_c) = \gamma_1\Gamma[T_f] + \gamma_2\Gamma[\Xi_c],$$

where γ_1, γ_2 are the adjustable weights, and $\gamma_1 + \gamma_2 = 1$. $\Gamma[.]$ is a normalization function. For the sake of discussion, we select linear normalization

$$\Upsilon_{\gamma_1,\gamma_2}(T_c) = \gamma_1 T_f + \gamma_2\Xi_c, \tag{13}$$

and we still use γ_1, γ_2 to represent the weights of the objective function. It is necessary to add that the specific method of linear normalization is not unique in practical applications. Depending on the situation, some non-linear normalisation functions can be selected.

Next, we have a separate discussion of the minimum value of (14) in two different cases.

Case A: When $\|x(0)\|_c < 1$.

$$\begin{aligned}&\Upsilon_{\gamma_1,\gamma_2}(T_c)\\ &= \gamma_1 T_f + \gamma_2\Xi_c\\ &= \gamma_1 T_c + \gamma_2\Bigg(\frac{3k^2(1-\alpha)\,\|\varphi(s)\|_c^{3-\alpha}\,T_c}{2(3-\alpha)}\\ &\quad + \frac{6n^{1-\alpha}\,\|\varphi(s)\|_c^{1+\alpha}}{(1-\alpha^2)T_c} + \frac{3\,\|H\|^2\,T_c}{2}\Bigg),\end{aligned} \tag{14}$$

Differentiating $\Upsilon_{\gamma_1,\gamma_2}(T_c)$ with respect to T_c, we have

$$\begin{aligned}\frac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} &= \gamma_1 + \frac{3\gamma_2 k^2(1-\alpha)\,\|\varphi(s)\|_c^{3-\alpha}}{2(3-\alpha)} + \frac{3\gamma_2\,\|H\|^2}{2}\\ &\quad + \frac{6\gamma_2 n^{1-\alpha}\,\|\varphi(s)\|_c^{1+\alpha}}{(1-\alpha^2)T_c^2}.\end{aligned}$$

Taking $\frac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} = 0$ and noting that $T_c > 0$, we have

$$T_c = \sqrt{\frac{12\gamma_2(3-\alpha)n^{1-\alpha}\,\|\varphi(s)\|_c^{1+\alpha}}{(1-\alpha^2)\bar{\delta}}} \triangleq \check{T}_1 > 0,$$

where $\bar{\delta} = (6-2\alpha)\gamma_1 + 3\gamma_2 k^2(1-\alpha)\,\|\varphi(s)\|_c^{3-\alpha} + 3\gamma_2\,\|H\|^2$, then

$$\begin{cases}\dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} < 0, & T_c \in (0, \check{T}_1),\\ \dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} = 0, & T_c = \check{T}_1,\\ \dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} > 0, & T_c \in (\check{T}_1, +\infty).\end{cases} \tag{15}$$

Obviously, when $T_c = \check{T}_1$, objective function $\Upsilon_{\gamma_1,\gamma_2}(T_c)$ reaches the minimum, $\check{T}_1$ is optimal choice to reach equilibrium between the stabilization time and energy.

Case B: When $\|x(0)\|_c \geq 1$.

$$\begin{aligned}\Upsilon_{\gamma_1,\gamma_2}(T_c) =& \gamma_1 T_f + \gamma_2\Xi_c\\ =& \gamma_1 T_c + \gamma_2\Bigg((3k^2\,\|\varphi(s)\|_c^2 + \frac{3k^2(1-\alpha)}{6-2\alpha}\\ &+ 3\,\|H\|^2)T_c + (\frac{12n^{\frac{2\beta-1}{2}}\,\|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)}\\ &+ \frac{6n^{1-\alpha}}{1-\alpha^2})\frac{1}{T_c}\Bigg)\\ \triangleq& \theta_1 T_c + \theta_2\frac{1}{T_c},\end{aligned}$$

where $\theta_1 = \gamma_1 + \gamma_2(3k^2\,\|\varphi(s)\|_c^2 + \frac{3k^2(1-\alpha)}{6-2\alpha} + 3\,\|H\|^2)$, $\theta_2 = \gamma_2(\frac{12n^{\frac{2\beta-1}{2}}\|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)} + \frac{6n^{1-\alpha}}{1-\alpha^2})$.

Taking $\frac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} = 0$ and noting that $T_c > 0$, we have

$$T_c = \sqrt{\frac{\theta_2}{\theta_1}} \triangleq \check{T}_2 > 0.$$

then

$$\begin{cases}\dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} < 0, & T_c \in (0, \check{T}_2),\\ \dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} = 0, & T_c = \check{T}_2,\\ \dfrac{d\Upsilon_{\gamma_1,\gamma_2}(T_c)}{dT_c} > 0, & T_c \in (\check{T}_2, +\infty).\end{cases} \tag{16}$$

Obviously, when $T_c = \check{T}_2$, $\Upsilon_{\gamma_1,\gamma_2}(T_c)$ achieves the minimum value, $T_c = \check{T}_2$ is the optimal choice to reach equilibrium between the stabilization time and energy.

Theorem 3: To summarise, for the linear evaluated indexed function (13), the predefined settling time T_c ensuring the equilibrium between the stabilization time and energy satisfies

$$\begin{cases} T_c = \check{T}_1, & \|x(0)\|_c < 1, \\ T_c = \check{T}_2, & \|x(0)\|_c \geq 1, \end{cases} \tag{17}$$

where $\check{T}_1 = \sqrt{\frac{12\gamma_2(3-\alpha)n^{1-\alpha}\|\varphi(s)\|_c^{1+\alpha}}{(1-\alpha^2)\bar{\delta}}}$, $\check{T}_2 = \sqrt{\frac{\theta_2}{\theta_1}}$,

$\bar{\delta} = (6-2\alpha)\gamma_1 + 3\gamma_2 k^2(1-\alpha)\|\varphi(s)\|_c^{3-\alpha} + 3\gamma_2\|H\|^2$,

$\theta_1 = \gamma_1 + \gamma_2(3k^2\|\varphi(s)\|_c^2 + \frac{3k^2(1-\alpha)}{6-2\alpha} + 3\|H\|^2)$,

$\theta_2 = \gamma_2(\frac{12n^{\frac{2\beta-1}{2}}\|\varphi(s)\|_c^{2\beta}}{(\beta^2-1)(\beta-1)} + \frac{6n^{1-\alpha}}{1-\alpha^2})$.

Remark 4: In predefined-time stability, although the control time can be arbitrarily preset as a constant, a smaller control time often means more energy consumption in control process. As analysed above, we can find an optimal preset time by studying the equilibrium between the stabilization time and energy cost, which is meaningful in practical applications. In addition, since the control time in the predefined-time stability is a preset constant and has no direct relationship with the control parameters, it is not possible to discuss the equilibrium of these two indicators about the control parameters, which is different from the finite/fixed-time stability.

IV. SIMULATION EXAMPLES

In this section, we will have a simulation example as an illustration of our theoretical results.

Consider this delayed system with two nodes:

$$\dot{x}(t) = -Dx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + u(t), \tag{18}$$

where $x(t) = [x_1(t), x_2(t)]^T$, $u(t) = [u_1(t), u_2(t)]^T$ $D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; $A = \begin{bmatrix} 2 & 1 \\ 2 & -0.5 \end{bmatrix}$; $B = \begin{bmatrix} -0.2 & -0.8 \\ 0.1 & -0.05 \end{bmatrix}$.

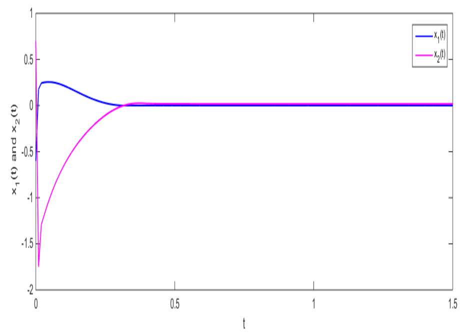

FIGURE 1. Trajectories of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-0.6, 0.7]$, $\forall s \in [-1, 0)$.

According to the above values, we have $\lambda_D = 2$. We choose $g_j(x) = f_j(x) = \frac{|x+1|-|x-1|}{2}$, $j = 1, 2$ which satisfy lemma 1. $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t))]^T$, $g(x(t-\tau(t))) = [g_1(x_1(t-\tau(t))), g_2(x_2(t-\tau(t)))]^T$, $\tau(t) = \frac{e^t}{1+e^t}$. then we can take $L = \text{diag}[1, 1]$, $0 < \tau(t) < 1$, $M = [1, 1]^T$. By simple computation, one obtains $\|B\|M = [0.84, 0.84]^T$,$h_i = 0.84$.

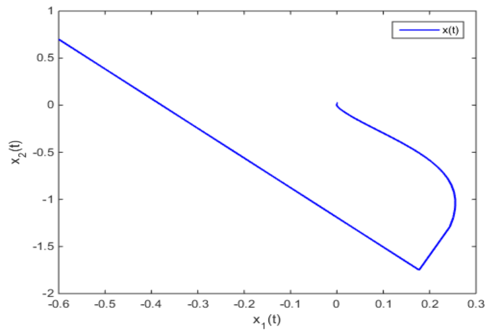

FIGURE 2. Phase portrait of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-0.6, 0.7]$, $\forall s \in [-1, 0)$.

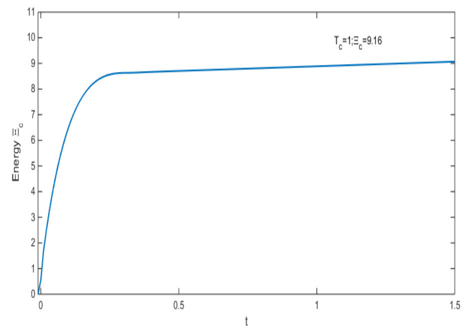

FIGURE 3. The energy consumption curve of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-0.6, 0.7]$, $\forall s \in [-1, 0)$.

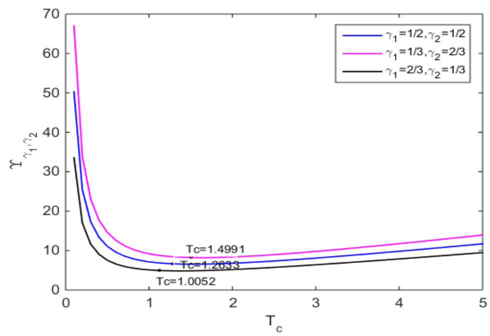

FIGURE 4. The curve between the function $\Upsilon_{\gamma_1,\gamma_2}$ and control time T_c.

Select $k = 3$, $\alpha = 0.5$, $\beta = 2$ in controller $u(t)$. Next, consider two initial conditions.

Case A: Let the initial condition be $\varphi_1(s) = -0.6$, $\varphi_2(s) = 0.7$, $\forall s \in [-1, 0)$. For a preset control time $T_c = 1$, we can obtain $\Xi_c = 13.3$ from theorem 2. Trajectories of DNN (18) are shown in Figure 1, where it is clear that state is converging to zero within 1. This shows that our conclusion is accurate. The phase portrait of the DNNs (18) is shown in Figure 2. The simulation results of corresponding energy consumption by controller is shown in Figure 3. When the control time is 1, the required energy cost is $9.16 < \Xi_c = 13.3$ from Figure 3, which is a test of the validity of Theorem 2.

Based on the formula (14), Figure 4 shows the curve between the function $\Upsilon_{\gamma_1,\gamma_2}$ and control time T_c with different weights, where $T_c \in [0.1, 5]$. From (15) we calculate the

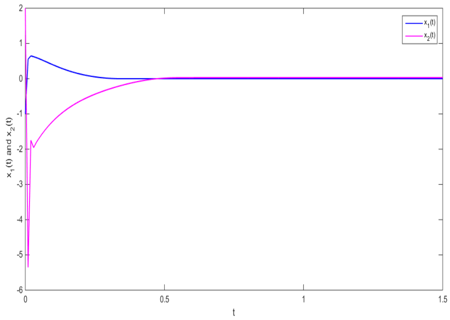

FIGURE 5. Trajectories of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-1, 2]$, $\forall s \in [-1, 0)$.

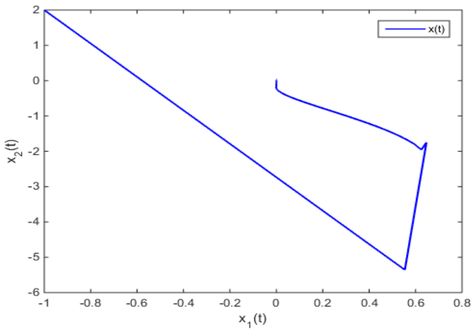

FIGURE 6. Phase portrait of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-1, 2]$, $\forall s \in [-1, 0)$.

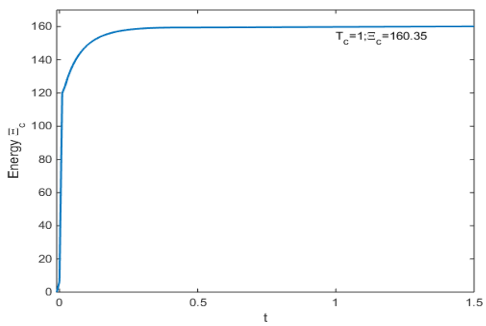

FIGURE 7. The energy consumption curve of the DNNs (18) with $[\varphi_1(s), \varphi_2(s)] = [-1, 2]$, $\forall s \in [-1, 0)$.

values of $T_c = \check{T}_1$ as

$$\check{T}_1 = \begin{cases} 1.2633, & \gamma_1 = 1/2,\ \gamma_2 = 1/2, \\ 1.4991, & \gamma_1 = 1/3,\ \gamma_2 = 2/3, \\ 1.0052, & \gamma_1 = 2/3,\ \gamma_2 = 1/3. \end{cases}$$

They are consistent with the corresponding values of T_c when $\Upsilon_{\gamma_1,\gamma_2}$ takes the minimum value in Figure 4.

Case B: In this case, let the initial condition be $\varphi_1(s) = -1, \varphi_2(s) = 2, \forall s \in [-1, 0)$. According to theorems 2, for a preset control time $T_c = 1$, we can have $\Xi_c = 433.97$. Trajectories of DNN (18) are shown in Figure 5, where it is clear that state is converging to zero within 1. This shows that our conclusion is accurate. The phase diagram of the DNNs (18) is shown in Figure 6. The simulation results of corresponding energy consumption by controller is shown in Figure 7. When the control time is 1, the required energy consumption is $160.35 < \Xi_c = 433.97$ from Figure 7, which is a test of the validity of Theorem 2.

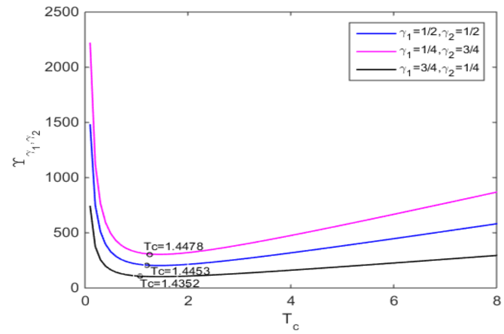

FIGURE 8. The curve between the function $\Upsilon_{\gamma_1,\gamma_2}$ and control time T_c.

Based on the formula (13), Figure 8 shows the curve between the function $\Upsilon_{\gamma_1,\gamma_2}$ and control time T_c with different weights, where $T_c \in [0.1, 8]$. From (17) we calculate the values of $T_c = \check{T}_2$ as

$$\check{T}_2 = \begin{cases} 1.4453, & \gamma_1 = 1/2,\ \gamma_2 = 1/2, \\ 1.4487, & \gamma_1 = 1/4,\ \gamma_2 = 3/4, \\ 1.4352, & \gamma_1 = 3/4,\ \gamma_2 = 1/4. \end{cases}$$

They are consistent with the corresponding values of T_c when $\Upsilon_{\gamma_1,\gamma_2}$ takes the minimum value in Figure 8.

V. CONCLUSION

This paper mainly focuses on the equilibrium problem of predefined-time stability and control energy consumption in nonlinear neural networks with delays. A new criterion for one global composite switching controller to assure predefined-time stability is provided by employing inequality technologies and Lyapunov stability theorem. Under the constructed controller, it is proved that the system is predefined time-stable when the initial conditions are inside and outside the unit sphere. Then, the energy consumption required for the system to reach the control target is estimated, which is related to the preset control time. Moreover, the equilibrium problem of the control energy consumption and the settling time is investigated by constructing an evaluation index function, and the optimal preset control time is obtained. The results show that a suitable preset control time can better balance the energy consumed by the system, which has practical implications. Finally, a simulation example has clearly verified the theoretical results.

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