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RESEARCH ARTICLE

Some Novel Theories of Triangular Intuitionistic Fuzzy Numbers and Its Application in Two-Sided Matching

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ABSTRACT Some existing theories of triangular intuitionistic fuzzy numbers (TIFNs) and triangular intuitionistic fuzzy sets (TIFSs) are proved in this paper. In addition, this paper also provides some novel theorems and related proofs on TIFNs and TIFSs. Moreover, the presented theories are applied in the field of two-sided matching decision-making. A numerical case is adopted for demonstrating the effectiveness of the proposed theories and method. This work will effectively complement the theories of TIFNs and TIFSs, and popularize their application scope.

INDEX TERMS Triangular intuitionistic fuzzy number, triangular intuitionistic fuzzy set, novel theory, two-sided matching.

I. INTRODUCTION

In 1965, Zadeh put forward the concept of fuzzy set [1]. On this basis, Atanassov proposed the intuitionistic fuzzy set [2], which provided a better data expression tool for dealing with uncertainty. After that, some extended concepts of fuzzy sets were put forward successively, such as interval intuitionistic fuzzy set [3], Pythagorean fuzzy set [4], interval hesitation fuzzy set [5]. As a tool to collect and convey uncertainty, the research of intuitionistic fuzzy sets has attracted extensive attention of scholars.

Many scholars have deeply explored the decision-making theory and method that the preference of decision-makers was all kinds of intuitionistic fuzzy sets. However, considering the complexity of real decision-making problems, the information asymmetry and fuzziness of decision-makers, the preference of decision-makers may be in the form of triangular intuitionistic fuzzy set (TIFS) after mathematical statistics. TIFS theory has been widely used in multi-attribute decision-making. For example, Zhang and Liu [6] used triangular fuzzy numbers to represent the membership and nonmembership of intuitionistic fuzzy sets, and proposed the concept of triangular intuitionistic fuzzy number (TIFN). Then, the algorithm, weighted aggregation operator, score function and exact function of TIFNs were defined and applied to the field of fuzzy decision-making. De et al. [7] and Shu et al. [8] gave the distance measure of intuitionistic fuzzy set and interval-valued intuitionistic fuzzy set based on Hausdorff measure, which could be regarded as the generalization of Hamming distance and Euclidean distance. On the basis of extending the cut set of TIFNs based on triangular norm, Yi and Li [9] proposed the probability mean standard deviation ratio of TIFNs, and Furthermore, put forward a multi-attribute decision-making method. Based on the concept of intuitionistic triangular fuzzy preference relations, Zhang and Meng [10] proposed an intuitionistic triangular fuzzy group decision-making method based on preference relations. Wan [11] and Wang et al. [12] defined the arithmetic operation of TIFNs. Babatunde et al. [13] improved the triangular intuitionistic fuzzy clustering and ranking function (TIFARF) model, and identified the best ''scrapping'' treatment scheme of batteries in the energy industry. Xu et al. [14] proposed a decision-making method based on zero sum game for the multi-attribute decision-making problem with triangular intuitionistic fuzzy number and unknown attribute weight.

In recent years, the research depth of triangular fuzzy numbers has been continuously expanded and fruitful research results have been achieved. For example, Xu and Yager [3]

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realized the combination of intuitionistic fuzzy concept and linear programming, and considered all parameters and variables in the form of TIFNs to establish a triangular intuitionistic fuzzy linear programming model. Nayagam and Murugan [15] analyzed the weighted triangular approximation process of new intuitionistic fuzzy numbers, and discussed some important properties of triangular approximation of intuitionistic fuzzy numbers. Garg and Rani [16] evaluated the weight of each criterion and the rating of alternatives in the context of TIFNs using language terms, and introduced Euclidean distance between two TIFNs to calculate the interval between the two alternatives. Hashemi et al. [17] proposed a multi type scheduling problem with TIFNs, and gave algorithms to solve the optimal scheduling, minimum total running time and machine idle time. Khan et al. [18] introduced a new concept of triangular linear Diophantine fuzzy numbers (TLDFNs) in a generic way and the arithmetic operations on these numbers. Yang and Xu [19] solved the linear programming problem of TIFN by simplex algorithm. By combining the probability and possibility of triangular fuzzy numbers (TFNs), the preference degree of TFN was defined. Wang [20] designed a TFN ranking algorithm based on preference degree, and proposed triangular fuzzy MAGDM method. Al-Qudaimi [21] constructed an inter region valued triangular fuzzy regression model (IVTFRM) whose regression component is interval valued triangular fuzzy number, and proposed a method to find IVTFRM parameters by using mathematical linear optimization problem, so as to minimize the difference between the given model and the predicted value of dependent variable. Ding et al. [22] proposed a group decision-making method in which the decision-making value is triangular fuzzy number and the decision-maker gives the pairwise comparison of scheme and evaluation matrix. Wan et al. [23] proposed a new fuzzy best worst method for multi criteria decision-making based on triangular fuzzy numbers, and proposed the concepts of fuzzy consistency index and fuzzy consistency ratio. Dong et al. [24] proposed a new method to compare TIFNs and a method to solve the type-2 triangular intuitionistic fuzzy transportation problem. Karmakar and De [25] studied various fuzzy triangular norms in fuzzy sets and applied them to the economic order quantity model. Yue et al. [26] developed a novel TIFN theory including score function, distance measure, aggregation operator, and proposed a two-sided matching (TSM) method based on the TIFN preference information.

Although TIFSs have achieved fruitful results in the field of fuzzy decision-making, there are still some challenges.

(1) In the existing literature on the algorithm and aggregation operator of TIFN, it is basically required that the TIFN and coefficient are positive, and the calculation of the highest membership degree and the lowest membership degree in the algorithm is not completely unified. However, it is obvious that sometimes there are operation and aggregation of negative TIFNs. Therefore, it is necessary to study the algorithms and aggregation operators in this case.

(2) In the existing literature on the cut set, score and distance measure of TIFNs, most of them only consider the degree of membership and non membership, but ignore the degree of hesitation. In intuitionistic fuzzy sets, hesitation is a very important concept. Therefore, this paper will study it combined with hesitation.

(3) In the research proposed by Yue et al. [26], some theories of TIFSs are proposed, which have not been proved effectively. At the same time, the practicability and validity of these theorems have not been verified by numerical cases. Therefore, it would be meaning to prove and extended the theories proposed by Yue et al. [26] and verify its validity by a numerical example.

Based on this, this paper proposes some theories of TIFSs and TIFNs. Aiming at the new theoretical research of TIFSs, this paper retrospects the algorithm and generalized aggregation operator of TIFNs, the expectation function and distance measure of TIFNs, and the distance measures of TIFNs and TIFSs. At the same time, the rationality of these theories of TIFSs is proved. Then, some proposed theories of TIFNs are applied in the field of TSM decision-making. A case of ERP software supply-demand matching is offered to illustrate the effectiveness of the proposed theory of TIFNs.

The rest of this paper is as follows: In Section II, some relevant definitions and theories of TIFN and TIFS are proposed and proved. Section III presents a decision-making method for solving the TSM problem with positive/negative TIFNs. A numerical case is given in Section IV. Section V expounds the conclusion of this paper.

II. SOME BASIS CONCEPTS AND THEORIES

A. TIFN

Definition 1 [12], [13]: Let $\hat{t} = <(\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}>$, $\underset{\sim}{t} < t < \tilde{t}$, if the membership $l_{\hat{t}}(x)$ and the non membership $f_{\hat{t}}(x)$ are expressed as

$$l_{\hat{t}}(x) = \begin{cases} 0, & x < \underset{\sim}{t} \\ \dfrac{(x-\underset{\sim}{t})l_{\hat{t}}^{\max}}{t-\underset{\sim}{t}}, & \underset{\sim}{t} \le x < t \\ l_{\hat{t}}^{\max}, & x = t \\ \dfrac{(\tilde{t}-x)l_{\hat{t}}^{\max}}{\tilde{t}-t}, & t < x \le \tilde{t} \\ 0, & x > \tilde{t} \end{cases} \tag{1}$$

$$f_{\hat{t}}(x) = \begin{cases} 1, & x < \underset{\sim}{t} \\ \dfrac{t-x+f_{\hat{t}}^{\min}(x-\underset{\sim}{t})}{t-\underset{\sim}{t}}, & \underset{\sim}{t} \le x < t \\ f_{\hat{t}}^{\min}, & x = t \\ \dfrac{x-t+f_{\hat{t}}^{\min}(\tilde{t}-x)}{\tilde{t}-t}, & t < x \le \tilde{t} \\ 1, & x > \tilde{t} \end{cases} \tag{2}$$

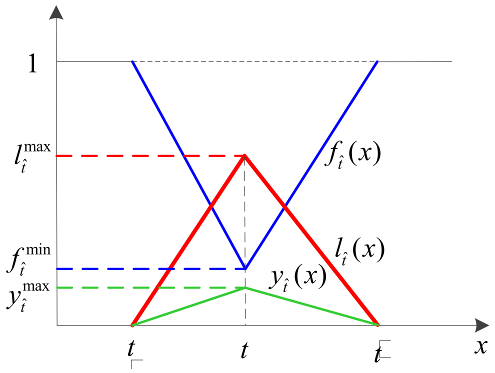

FIGURE 1. TIFN $\hat{t}$.

then $\hat{t}$ is called a TIFN on the set of real numbers. Hereinto, $l_{\hat{t}}^{\max}$ and $f_{\hat{t}}^{\min}$ is the maximum membership degree and minimum non membership degree of $\hat{t}$, which meet the $l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}$, $l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0, 1]$.

Remark 1 [26]: Let

$$y_{\hat{t}}(x) = 1 - l_{\hat{t}}(x) - f_{\hat{t}}(x) \tag{3}$$

At this point, $y_{\hat{t}}(x)$ is called the degree of hesitation of $\hat{t}$. Furthermore, according to Eqs. (1)-(3), it can be calculated, i.e.,

$$y_{\hat{t}}(x) = \begin{cases} 0, & x < \underset{\sim}{t} \\ \dfrac{(1 - l_{\hat{t}}^{\max} - f_{\hat{t}}^{\min})(x - \underset{\sim}{t})}{t - \underset{\sim}{t}}, & \underset{\sim}{t} \le x < t \\ 1 - l_{\hat{t}}^{\max} - f_{\hat{t}}^{\min}, & x = t \\ \dfrac{(1 - l_{\hat{t}}^{\max} - f_{\hat{t}}^{\min})(\tilde{t} - x)}{\tilde{t} - t}, & t < x \le \tilde{t} \\ 0, & x > \tilde{t} \end{cases} \tag{4}$$

In Eq. (4), let $y_{\hat{t}}^{\max} = 1 - l_{\hat{t}}^{\max} - f_{\hat{t}}^{\min}$; at this time, TIFN $\hat{t}$ can be represented by Figure 1.

Remark 2: TIFN $\hat{t} =< (\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min} >$ can be expressed as a concept of fuzzy quantity, that is, the fuzzy quantity of "approximate to t". Different from the fuzzy number, TIFN $\hat{t} =< (\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min} >$ uses $l_{\hat{t}}, f_{\hat{t}}$ and $y_{\hat{t}}$ to describe the degrees of membership, non membership and hesitation of any number in the interval $[\underset{\sim}{t}, \tilde{t}]$ for the fuzzy number "similar to".

According to Definition 1, $y_{\hat{t}} \in [0, 1], \forall x$; If $y_{\hat{t}}$=0, $\forall x$, then $\hat{t}$ is degenerated into a fuzzy number.

B. GENERALIZED ALGORITHM OF TIFNs

Definition 2 [26]: Let $\hat{t} =< (\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min} >, \hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min} >$ and $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min} >$ be three TIFNs. At this time, the generalized algorithm of $\hat{t}$, $\hat{t}_1$ and $\hat{t}_2$ is expressed as (5)–(11), shown at the bottom of the next page.

In Definition 2, "mid" is the operator taking the intermediate value; "odd/even" represents the odd/even number, which meaning is the same as that in Theorem 8, 9, and others.

Theorem 1 [26]: Let $\hat{t} =< (\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min} >, \hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min} >$ and $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min} >$ be any three TIFNs. At this time, the result calculated by Eqs. (5)-(11) is a TIFN.

C. AGGREGATION OPERATOR AND GENERALIZED AGGREGATION OPERATOR OF TIFNs

According to the generalized algorithm, the aggregation operator and generalized aggregation operator of TIFN are proposed.

Definition 3: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min} >, \hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l_{\hat{t}_n}^{\max}, f_{\hat{t}_n}^{\min} >$ is a set of TIFNs. At this time, the triangular intuitionistic fuzzy weighted arithmetic average (TIFWAA) operator is expressed as

$$\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) = \sum_{s=1}^{n} w_s \hat{t}_s \tag{12}$$

In Eq. (12), $W = (w_1, w_2, \ldots, w_n)$ is the weight vector, where w_s is the weight of $\hat{t}_s$, and satisfies $w_s \in [0, 1], s = 1, \ldots, n, \sum_{s=1}^{n} w_s = 1$.

In Definition 3, if $w_s = 1/n$, then the TIFWAA operator is degenerated into a triangular intuitionistic fuzzy arithmetic average (TIFAA) operator, i.e.,

$$\text{TIFAA}_{(1/n, 1/n, \ldots, 1/n)}(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) = \sum_{s=1}^{n} \frac{1}{n} \hat{t}_s \tag{13}$$

Theorem 2: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min} >, \hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l_{\hat{t}_n}^{\max}, f_{\hat{t}_n}^{\min} >$ be a set of TIFNs; $W = (w_1, w_2, \ldots, w_n)$ is a weight vector, where $w_s \in [0, 1], \forall s = 1, 2, \ldots, n$. Then the result calculated by the TIFWAA operator is a TIFN, and

$$\begin{aligned} &\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) \\ &= < (\sum_{s=1}^{n} w_s \underset{\sim}{t}_s, \sum_{s=1}^{n} w_s t_s, \sum_{s=1}^{n} w_s \tilde{t}_s); \\ &\quad 1 - \prod_{s=1}^{n} (1 - l_{\hat{t}_s}^{\max})^{w_s}, \prod_{s=1}^{n} (f_{\hat{t}_s}^{\min})^{w_s} > \end{aligned} \tag{14}$$

Definition 4: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min} >, \hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l_{\hat{t}_n}^{\max}, f_{\hat{t}_n}^{\min} >$ be a set of TIFNs. At this time, the triangular intuitionistic fuzzy extend weighted arithmetic average (TIFEWAA) operator is expressed as

$$\text{TIFEWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) = \sum_{s=1}^{n} w_s \hat{t}_s \tag{15}$$

In Eq. (15), $W = (w_1, w_2, \ldots, w_n)$ is the weight vector, and w_s is the weight coefficient of $\hat{t}_s$, $w_s \neq 0, \forall s = 1, 2, \ldots, n$.

Remark 3: In Eq. (15), w_s can be greater than or less than zero. Therefore, the TIFEWAA operator is a generalization of TIFWAA operator.

Theorem 3: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of TIFNs; $W = (w_1, w_2, \ldots, w_n)$ is a weight vector, where $w_s < 0, \forall s = 1, 2, \ldots, n$. Then, the result calculated by TIFEWAA operator is still TIFN, and

$$\begin{aligned} &\mathrm{TIFEWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) \\ &= <(\sum_{s=1}^{n} w_s \tilde{t}_s, \sum_{s=1}^{n} w_s t_s, \sum_{s=1}^{n} w_s \underset{\sim}{t}_s); \\ &\quad 1 - \prod_{s=1}^{n} \frac{1}{(1 - l^{\max}_{\hat{t}_s})^{w_s}}, \prod_{s=1}^{n} \frac{1}{(f^{\min}_{\hat{t}_s})^{w_s}} > \end{aligned} \tag{16}$$

$$\hat{t}_1 + \hat{t}_2 =< (\underset{\sim}{t}_1 + \underset{\sim}{t}_2, t_1 + t_2, \tilde{t}_1 + \tilde{t}_2); l^{\max}_{\hat{t}_1} + l^{\max}_{\hat{t}_2} - l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} > \tag{5}$$

$$k\hat{t} = \begin{cases} < (k\underset{\sim}{t}, kt, k\tilde{t}); 1 - (1 - l^{\max}_{\hat{t}})^k, (f^{\min}_{\hat{t}})^k >, & k > 0 \\ < (k\tilde{t}, kt, k\underset{\sim}{t}); 1 - \dfrac{1}{(1 - l^{\max}_{\hat{t}})^k}, \dfrac{1}{(f^{\min}_{\hat{t}})^k} >, & k < 0 \end{cases} \tag{6}$$

$$\hat{t}_1 - \hat{t}_2 =< (\underset{\sim}{t}_1 - \tilde{t}_2, t_1 - t_2, \tilde{t}_1 - \underset{\sim}{t}_2); l^{\max}_{\hat{t}_1} + l^{\max}_{\hat{t}_2} - l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} > \tag{7}$$

$$\hat{t}_1 \times \hat{t}_2 = \begin{cases} < (\underset{\sim}{t}_1 \underset{\sim}{t}_2, t_1 t_2, \tilde{t}_1 \tilde{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 \geq 0,\ \hat{t}_2 \geq 0 \\ < (\underset{\sim}{t}_1 \tilde{t}_2, t_1 t_2,\ \tilde{t}_1 \underset{\sim}{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 < 0,\ \hat{t}_2 \geq 0 \\ < (\tilde{t}_1 \tilde{t}_2, t_1 t_2, \underset{\sim}{t}_1 \underset{\sim}{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 < 0,\ \hat{t}_2 < 0 \\ < (\min\{\underset{\sim}{t}_1 \underset{\sim}{t}_2, \underset{\sim}{t}_1 \tilde{t}_2, \tilde{t}_1 \underset{\sim}{t}_2, \tilde{t}_1 \tilde{t}_2\}, t_1 t_2, \max\{\underset{\sim}{t}_1 \underset{\sim}{t}_2, \underset{\sim}{t}_1 \tilde{t}_2, \tilde{t}_1 \underset{\sim}{t}_2, \tilde{t}_1 \tilde{t}_2\}); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \text{else} \end{cases} \tag{8}$$

$$\hat{t}^{-1} = \begin{cases} < (1/\tilde{t}, 1/t, 1/\underset{\sim}{t}); l^{\max}_{\hat{t}}, f^{\min}_{\hat{t}} >, & \hat{t} > 0, \hat{t} < 0 \\ < (\min\{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\}, \mathrm{mid}\{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\}, \max\{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\}); l^{\max}_{\hat{t}}, f^{\min}_{\hat{t}} >, & \text{else} \end{cases} \tag{9}$$

$$\hat{t}_1/\hat{t}_2 = \begin{cases} < (\underset{\sim}{t}_1/\tilde{t}_2, t_1/t_2, \tilde{t}_1/\underset{\sim}{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 \geq 0,\ \hat{t}_2 > 0 \\ < (\underset{\sim}{t}_1/\underset{\sim}{t}_2, t_1/t_2, \tilde{t}_1/\tilde{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 < 0,\ \hat{t}_2 > 0 \\ < (\tilde{t}_1/\underset{\sim}{t}_2, t_1/t_2, \underset{\sim}{t}_1/\tilde{t}_2); l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \hat{t}_1 < 0,\ \hat{t}_2 < 0 \\ < (\min\{\underset{\sim}{t}_1/\tilde{t}_2, \underset{\sim}{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\tilde{t}_2\}, t_1/t_2, \max\{\underset{\sim}{t}_1/\tilde{t}_2, \underset{\sim}{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\tilde{t}_2\}); & \\ \quad l^{\max}_{\hat{t}_1} l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_1} + f^{\min}_{\hat{t}_2} - f^{\min}_{\hat{t}_1} f^{\min}_{\hat{t}_2} >, & \text{else} \end{cases} \tag{10}$$

$$\hat{t}^k = \begin{cases} < (\underset{\sim}{t}^k, t^k, \tilde{t}^k); (l^{\max}_{\hat{t}})^k, 1 - (1 - f^{\min}_{\hat{t}})^k >, & \hat{t} \geq 0,\ k > 0 \\ < (\underset{\sim}{t}^k, t^k, \tilde{t}^k); (l^{\max}_{\hat{t}})^k, 1 - (1 - f^{\min}_{\hat{t}})^k >, & \hat{t} < 0, k > 0 \text{ and } k = \dfrac{\text{odd}}{\text{odd}} \\ < (\tilde{t}^k, t^k, \underset{\sim}{t}^k); (l^{\max}_{\hat{t}})^k, 1 - (1 - f^{\min}_{\hat{t}})^k >, & \hat{t} < 0, k > 0 \text{ and } k = \dfrac{\text{even}}{\text{odd}} \\ < (\min\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}, \mathrm{mid}\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}, \max\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}); & \\ \quad (l^{\max}_{\hat{t}})^k, 1 - (1 - f^{\min}_{\hat{t}})^k >, & \hat{t} \text{ is else},\ k > 0 \\ < (\tilde{t}^k, t^k, \underset{\sim}{t}^k); \dfrac{1}{(l^{\max}_{\hat{t}})^k}, 1 - \dfrac{1}{(1 - f^{\min}_{\hat{t}})^k} >, & \hat{t} \geq 0,\ k < 0 \\ < (\tilde{t}^k, t^k, \underset{\sim}{t}^k); \dfrac{1}{(l^{\max}_{\hat{t}})^k}, 1 - \dfrac{1}{(1 - f^{\min}_{\hat{t}})^k} >, & \hat{t} < 0,\ k < 0 \text{ and } k = \dfrac{\text{even}}{\text{odd}} \\ < (\underset{\sim}{t}^k, t^k, \tilde{t}^k); \dfrac{1}{(l^{\max}_{\hat{t}})^k}, 1 - \dfrac{1}{(1 - f^{\min}_{\hat{t}})^k} >, & \hat{t} < 0,\ k < 0 \text{ and } k = \dfrac{\text{even}}{\text{odd}} \\ < (\min\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}, \mathrm{mid}\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}, \max\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}); & \\ \quad \dfrac{1}{(l^{\max}_{\hat{t}})^k}, 1 - \dfrac{1}{(1 - f^{\min}_{\hat{t}})^k} >, & \hat{t} \text{ is else},\ k < 0 \end{cases} \tag{11}$$

Theorem 4: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of TIFNs; $W = (w_1, w_2, \ldots, w_n)$is a weight vector, where $w_s > 0, s = 1, 2, \ldots, g$ and $w_s < 0, s = g+1, g+2, \ldots, n$. Then the result calculated by TIFEWAA operator is a TIFN, and

$$\begin{aligned}
&\text{TIFEWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)\\
&=< (\sum_{s=1}^{g} w_s \underset{\sim}{t}_s + \sum_{s=g+1}^{n} w_s \tilde{t}_s, \sum_{s=1}^{n} w_s t_s, \sum_{s=1}^{g} w_s \tilde{t}_s + \sum_{s=g+1}^{n} w_s \underset{\sim}{t}_s);\\
&\quad 1 - \prod_{s=1}^{g} (1 - l^{\max}_{\hat{t}_s})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1 - l^{\max}_{\hat{t}_s})^{w_s}},\\
&\quad \prod_{s=1}^{g} (f^{\min}_{\hat{t}_s})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(f^{\min}_{\hat{t}_s})^{w_s}} >
\end{aligned} \tag{17}$$

Definition 5: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of TIFNs. At this time, the triangular intuitionistic fuzzy weighted geometric average (TIFWGA) operator is expressed as

$$\text{TIFWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) = \prod_{s=1}^{n} (\hat{t}_s)^{w_s} \tag{18}$$

In Eq. (18), $W = (w_1, w_2, \ldots, w_n)$ is the weight vector, where w_s is the weight of $\hat{t}_s$, and satisfies $w_s \in [0, 1], s = 1, \ldots, n, \sum_{s=1}^{n} w_s = 1$.

In Definition 5, if $w_s = 1/n$, then the TIFWGA operator is degenerated into the triangular intuitionistic fuzzy geometric average (TIFAA) operator, i.e.,

$$\text{TIFGA}_{(1/n, 1/n, \ldots, 1/n)}(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n) = \prod_{s=1}^{n} (\hat{t}_s)^{\frac{1}{n}} \tag{19}$$

Theorem 5: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of positive TIFNs; $W = (w_1, w_2, \ldots, w_n)$ be a weight vector, where $w_s \in [0, 1], \forall s = 1, 2, \ldots, n$. Then the result calculated by TIFWGA operator is a positive TIFN, and

$$\begin{aligned}
&\text{TIFWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)\\
&=< (\prod_{s=1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{n} (\tilde{t}_s)^{w_s});\\
&\quad \prod_{s=1}^{n} (l^{\max}_{\hat{t}_s})^{w_s}, 1 - \prod_{s=1}^{n} (1 - f^{\min}_{\hat{t}_s})^{w_s} >
\end{aligned} \tag{20}$$

Definition 6: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ is a set of TIFNs. At this time, the triangular intuitionistic fuzzy extend weighted geometric average operator (TIFEWGA) is expressed as

$$\text{TIFEWGA}_w(\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_n) = \prod_{s=1}^{n} (\hat{t}_s)^{w_s} \tag{21}$$

In Eq. (21), $W = (w_1, w_2, \ldots, w_n)$ is the weight vector, and w_s is the weight coefficient of $\hat{t}_s$, $w_s \neq 0, \forall s = 1, 2, \ldots, n$.

Remark 4: In Eq. (21), w_s can be greater than or less than zero. Therefore, the TIFEWGA operator is a generalization of TIFWGA operator.

Theorem 6: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of positive TIFNs; $W = (w_1, w_2, \ldots, w_n)$ is a weight vector, where $w_s < 0, \forall s = 1, 2, \ldots, n$. Then the result calculated by TIFEWGA operator is a positive TIFN, and

$$\begin{aligned}
&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)\\
&=< (\prod_{s=1}^{n} (\tilde{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{n} (\underset{\sim}{t}_s)^{w_s});\\
&\quad \prod_{s=1}^{n} \frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}}, 1 - \prod_{s=1}^{n} \frac{1}{(1 - f^{\min}_{\hat{t}_s})^{w_s}} >
\end{aligned} \tag{22}$$

Theorem 7: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of positive TIFNs; $W = (w_1, w_2, \ldots, w_n)$ is a weight vector, where $w_s > 0, s = 1, 2, \ldots, g, w_s < 0, s = g+1, g+2, \ldots, n$. Then the result calculated by TIFEWGA operator is a TIFN, and

$$\begin{aligned}
&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)\\
&=< (\prod_{s=1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s},\\
&\quad \prod_{s=1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s});\\
&\quad \prod_{s=1}^{g} (l^{\max}_{\hat{t}_s})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},\\
&\quad 1 - \prod_{s=1}^{g} (1 - f^{\min}_{\hat{t}_s})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1 - f^{\min}_{\hat{t}_s})^{w_s}} >
\end{aligned} \tag{23}$$

Theorem 8: Let $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$, $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >, \ldots, \hat{t}_n =< (\underset{\sim}{t}_n, t_n, \tilde{t}_n); l^{\max}_{\hat{t}_n}, f^{\min}_{\hat{t}_n} >$ be a set of negative TIFN; $W = (w_1, w_2, \ldots, w_n)$ is a weight vector, then the following conclusions are true:

(i) If $w_s = \frac{\text{odd}}{\text{odd}} > 0, \forall s = 1, 2, \ldots, n$, then the result calculated by TIFEWGA operator is a TIFN, and

$$\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)$$

$$=\begin{cases} <(\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\tilde{t}_s)^{w_s});\prod_{s=1}^{n}(l^{\max}_{\hat{t}_s})^{w_s},\\ \quad 1-\prod_{s=1}^{n}(1-f^{\min}_{\hat{t}_s})^{w_s}>, \quad n \text{ is odd}\\ <(\prod_{s=1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{n}(l^{\max}_{\hat{t}_s})^{w_s},\\ \quad 1-\prod_{s=1}^{n}(1-f^{\min}_{\hat{t}_s})^{w_s}>, \quad n \text{ is even} \end{cases} \tag{24}$$

(ii) If $w_s=\frac{\text{even}}{\text{odd}}>0, \forall s=1,2,\ldots,n$, then the result calculated by TIFEWGA operator is a positive TIFN, and

$$\begin{aligned} &\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\ &=<(\prod_{s=1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s});\\ &\prod_{s=1}^{n}(l^{\max}_{\hat{t}_s})^{w_s},1-\prod_{s=1}^{n}(1-f^{\min}_{\hat{t}_s})^{w_s}> \end{aligned} \tag{25}$$

(iii) If $w_s=\frac{\text{odd}}{\text{odd}}<0, \forall s=1,2,\ldots,n$, then the result calculated by TIFEWGA operator is a TIFN, and

$$\begin{aligned} &\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\ &=\begin{cases} <(\prod_{s=1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{n}\frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},\\ \quad 1-\prod_{s=1}^{n}\frac{1}{(1-f^{\min}_{\hat{t}_s})^{w_s}}>, \quad n \text{ is odd}\\ <(\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\tilde{t}_s)^{w_s});\prod_{s=1}^{n}\frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},\\ \quad 1-\prod_{s=1}^{n}\frac{1}{(1-f^{\min}_{\hat{t}_s})^{w_s}}>, \quad n \text{ is even} \end{cases} \end{aligned} \tag{26}$$

(iv) If $w_s=\frac{\text{even}}{\text{odd}}<0, \forall s=1,2,\ldots,n$, then the result calculated by TIFEWGA operator is still a positive TIFN, and

$$\begin{aligned} &\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\ &=<(\prod_{s=1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{n}(\tilde{t}_s)^{w_s});\\ &\prod_{s=1}^{n}\frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},1-\prod_{s=1}^{n}\frac{1}{(1-f^{\min}_{\hat{t}_s})^{w_s}}> \end{aligned} \tag{27}$$

(v) If $w_s=\frac{\text{odd}}{\text{odd}}>0, s=1,\ldots,b$, $w_s=\frac{\text{even}}{\text{odd}}>0, s=b+1,\ldots,g$, $w_s=\frac{\text{odd}}{\text{odd}}<0, s=g+1,\ldots,c$, $w_s=\frac{\text{even}}{\text{odd}}<0, s=c+1,\ldots,n$, then the result calculated by TIFEWGA operator is still TIFN, and

$$\begin{aligned} &\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\ &=\begin{cases} <(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\\ \quad \prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s});\\ \quad \prod_{s=1}^{g}(l^{\max}_{\hat{t}_s})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},\\ \quad 1-\prod_{s=1}^{g}(1-f^{\min}_{\hat{t}_s})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f^{\min}_{\hat{t}_s})^{w_s}}>,\\ \text{if } b \text{ and } c-g \text{ are both odd and even numbers}\\ <(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\\ \quad \prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\ \quad \prod_{s=1}^{g}(l^{\max}_{\hat{t}_s})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l^{\max}_{\hat{t}_s})^{w_s}},\\ \quad 1-\prod_{s=1}^{g}(1-f^{\min}_{\hat{t}_s})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f^{\min}_{\hat{t}_s})^{w_s}}>,\\ \text{if the parity of } b \text{ and } c-g \text{ are opposite} \end{cases} \end{aligned} \tag{28}$$

Theorem 9: Let $\hat{t}_1=<(\underset{\sim}{t}_1,t_1,\tilde{t}_1);l^{\max}_{\hat{t}_1},f^{\min}_{\hat{t}_1}>$, $\hat{t}_2=<(\underset{\sim}{t}_2,t_2,\tilde{t}_2);l^{\max}_{\hat{t}_2},f^{\min}_{\hat{t}_2}>,\ldots,\hat{t}_n=<(\underset{\sim}{t}_n,t_n,\tilde{t}_n);l^{\max}_{\hat{t}_n},f^{\min}_{\hat{t}_n}>$ be a set of TIFNs, where $\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_a>0,\hat{t}_{a+1},\hat{t}_{a+2},\ldots,\hat{t}_n<0$; $W=(w_1,w_2,\ldots,w_n)$ is the weight vector, where $w_s>0, s=1,2,\ldots,r$, $w_s<0, s=r+1,r+2,\ldots,a$, $w_s=\frac{\text{odd}}{\text{odd}}>0, s=a+1,\ldots,b$, $w_s=\frac{\text{even}}{\text{odd}}>0, s=b+1,\ldots,g$, $w_s=\frac{\text{odd}}{\text{odd}}<0, s=g+1,\ldots,c$, $w_s=\frac{\text{even}}{\text{odd}}<0, s=c+1,\ldots,n$. Then the result calculated by TIFEWGA operator is a TIFN, and (29), as shown at the bottom of the next page.

Remark 5: If the values of TIFNs $\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n$ are other cases and the weight coefficient w_s is positive or negative, a series of theorems can also be deduced according to Eq. (11) and Theorems 5-9. To save the space, it is omitted here.

D. CUT SETS OF TIFNs

Definition 7 [26]: Let $\hat{t}=<(\underset{\sim}{t},t,\tilde{t});l^{\max}_{\hat{t}},f^{\min}_{\hat{t}}>$ be a TIFN, $\alpha\in[0,l^{\max}_{\hat{t}}]$, $\beta\in[f^{\min}_{\hat{t}},1]$, $\gamma\in[0,1-l^{\max}_{\hat{t}}-f^{\min}_{\hat{t}}]$, then the α cut set expression of TIFN $\hat{t}$ is $\hat{t}_\alpha=[L_{\hat{t}}(\alpha),R_{\hat{t}}(\alpha)]=\{x|\,l_{\hat{t}}(x)\geq\alpha\}$, the β cut set expression is $\hat{t}_\beta=[L_{\hat{t}}(\beta),R_{\hat{t}}(\beta)]=\{x|f_{\hat{t}}(x)\leq\beta\}$, and the γ cut set expression is $\hat{t}_\gamma=[L_{\hat{t}}(\gamma),R_{\hat{t}}(\gamma)]=\{x|\,y_{\hat{t}}(x)\leq\gamma\}$.

Remark 6: According to Eqs. (1), (2) and (4), cut sets, $\hat{t}_\alpha=[L_{\hat{t}}(\alpha),R_{\hat{t}}(\alpha)]=\left(\underset{\sim}{t}+\frac{t-\underset{\sim}{t}}{l^{\max}_{\hat{t}}}\alpha,\tilde{t}-\frac{\tilde{t}-t}{l^{\max}_{\hat{t}}}\alpha\right)$ and $\hat{t}_\alpha=[L_{\hat{t}}(\alpha),R_{\hat{t}}(\alpha)]=\left(\underset{\sim}{t}+\frac{t-\underset{\sim}{t}}{l^{\max}_{\hat{t}}}\alpha,\tilde{t}-\frac{\tilde{t}-t}{l^{\max}_{\hat{t}}}\alpha\right)$ can be calculated respectively. i.e.,

$$\hat{t}_\alpha=[L_{\hat{t}}(\alpha),R_{\hat{t}}(\alpha)]=\left(\underset{\sim}{t}+\frac{t-\underset{\sim}{t}}{l^{\max}_{\hat{t}}}\alpha,\tilde{t}-\frac{\tilde{t}-t}{l^{\max}_{\hat{t}}}\alpha\right) \tag{30}$$

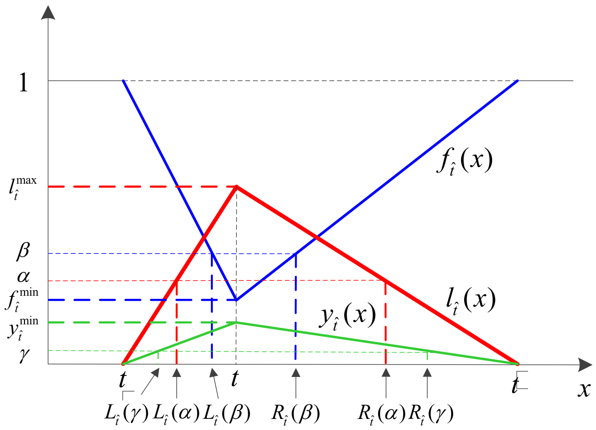

FIGURE 2. Cut sets $\hat t_\alpha = [L_{\hat t}(\alpha), R_{\hat t}(\alpha)]$, $\hat t_\beta = [L_{\hat t}(\beta), R_{\hat t}(\beta)]$ and $\hat t_\gamma = [L_{\hat t}(\gamma), R_{\hat t}(\gamma)]$.

$$\hat t_\beta = [L_{\hat t}(\beta), R_{\hat t}(\beta)] = \left(\begin{array}{c}\dfrac{(1-\beta)t+(\beta-f_{\hat t}^{\min})\underset{\sim}{t}}{1-f_{\hat t}^{\min}},\\ \dfrac{(1-\beta)t+(\beta-f_{\hat t}^{\min})\tilde t}{1-f_{\hat t}^{\min}}\end{array}\right) \quad (31)$$

$$\hat t_\gamma = [L_{\hat t}(\gamma), R_{\hat t}(\gamma)] = \left(\begin{array}{c}\underset{\sim}{t}+\gamma\dfrac{t-\underset{\sim}{t}}{1-l_{\hat t}^{\max}-f_{\hat t}^{\min}},\\ \tilde t-\gamma\dfrac{\tilde t-t}{1-l_{\hat t}^{\max}-f_{\hat t}^{\min}}\end{array}\right) \quad (32)$$

The α-cut set, β-cut set and γ-cut set of TIFN $\hat t$ can be represented by Figure 2.

E. EXPECTED SCORE OF TIFN

Definition 8 [26]: Let $\hat t = <(\underset{\sim}{t}, t, \tilde t); l_{\hat t}^{\max}, f_{\hat t}^{\min}>$ be a TIFN, $\alpha \in [0, l_{\hat t}^{\max}]$, $\beta \in [f_{\hat t}^{\min}, 1]$, $\gamma \in [0, 1-l_{\hat t}^{\max}-f_{\hat t}^{\min}]$. At this time, the expected score (ES) of TIFN $\hat t$ is expressed as:

$$\mathrm{ES}(\hat t) = \left(\mathrm{ES}(\hat t^l) + \omega^l \mathrm{ES}(\hat t^\gamma)\right) - \left(\mathrm{ES}(\hat t^f) + \omega^f \mathrm{ES}(\hat t^\gamma)\right) - \left(\omega^y \mathrm{ES}(\hat t^\gamma)\right) \quad (33)$$

In Eq. (33), $\omega^l, \omega^f, \omega^y$ are the support ratio of the hesitation degree for $l_{\hat t}(x)$, $f_{\hat t}(x)$ and $y_{\hat t}(x)$; $\mathrm{ES}(\hat t^l)$, $\mathrm{ES}(\hat t^f)$, $\mathrm{ES}(\hat t^\gamma)$ are expressed as

$$\begin{cases}\mathrm{ES}(\hat t^l) = \int_0^{l_{\hat t}^{\max}} \left(\omega_R R_{\hat t}(\alpha) - \omega_L L_{\hat t}(\alpha)\right) d\alpha \\ \mathrm{ES}(\hat t^f) = \int_{f_{\hat t}^{\min}}^{1} \left(\omega_R R_{\hat t}(\beta) - \omega_L L_{\hat t}(\beta)\right) d\beta \\ \mathrm{ES}(\hat t^\gamma) = \int_0^{1-l_{\hat t}^{\max}-f_{\hat t}^{\min}} \left(\omega_R R_{\hat t}(\gamma) - \omega_L L_{\hat t}(\gamma)\right) d\gamma\end{cases} \quad (34)$$

where ω_R and ω_L represent the optimistic and pessimistic coefficient, satisfying $\omega_R + \omega_L = 1$.

Remark 7: In Definition 8, $\omega^l, \omega^f, \omega^y$ can be determined by the preference of TIFS; ω_R and ω_L can be determined by the decision-maker.

F. GENERALIZED DISTANCE OF TIFNs

Definition 9 [26]: Let $\hat t_1 = <(\underset{\sim}{t}_1, t_1, \tilde t_1); l_{\hat t_1}^{\max}, f_{\hat t_1}^{\min}>$ and $\hat t_2 = <(\underset{\sim}{t}_2, t_2, \tilde t_2); l_{\hat t_2}^{\max}, f_{\hat t_2}^{\min}>$ be any two TIFNs. At this time, the triangular intuitionistic fuzzy weighted Minkowski distance (TIFWMD) is expressed as (35), shown at the bottom of the next page.

In Eq. (35), the parameter $\lambda \in [1, +\infty)$, and $\omega = (\omega_1, \omega_2, \ldots, \omega_6)$ is a weight vector, which satisfies $\omega_k \geq 0$, $\sum_{k=1}^{6} \omega_k = 1$.

$$\mathrm{TIFEWGA}_W(\hat t_1, \hat t_2, \ldots, \hat t_n) = \begin{cases} <(\prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s} \prod_{s=r+1}^{g} (\tilde t_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{r} (\tilde t_s)^{w_s} \prod_{s=r+1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde t_s)^{w_s}); \\ \prod_{s=1}^{r} (l_{\hat t_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat t_s}^{\max})^{w_s}} \prod_{s=a+1}^{g} (l_{\hat t_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat t_s}^{\max})^{w_s}}, \\ 1-\prod_{s=1}^{r} (1-f_{\hat t_s}^{\min})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(1-f_{\hat t_s}^{\min})^{w_s}} \prod_{s=a+1}^{g} (1-f_{\hat t_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat t_s}^{\min})^{w_s}} >, \\ \text{if } b-a \text{ and } c-g \text{ are both odd and even numbers} \\ <(\prod_{s=1}^{r} (\tilde t_s)^{w_s} \prod_{s=r+1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde t_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s} \prod_{s=r+1}^{g} (\tilde t_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}); \\ \prod_{s=1}^{r} (l_{\hat t_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat t_s}^{\max})^{w_s}} \prod_{s=a+1}^{g} (l_{\hat t_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat t_s}^{\max})^{w_s}}, \\ 1-\prod_{s=1}^{r} (l_{\hat t_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat t_s}^{\max})^{w_s}} \prod_{s=a+1}^{g} (1-f_{\hat t_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat t_s}^{\min})^{w_s}} >, \\ \text{if the parity of } b-a \text{ and } c-g \text{ are opposite} \end{cases} \quad (29)$$

Remark 8: Parameter λ has the same meaning as p-norm. If $\lambda = 1$, TIFWMD is degenerated into the triangular intuitionistic fuzzy weighted Hamming distance (TIFWHD), i.e.,

$$\begin{aligned}
&\mathrm{TIFWHD}(\hat{t}_1,\hat{t}_2)\\
&\quad=\mathrm{TIFWMD}_1(\hat{t}_1,\hat{t}_2)\\
&\quad=\omega_1\left|\int_0^{l_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\alpha)d\alpha\right|\\
&\qquad+\omega_2\left|\int_0^{l_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\alpha)d\alpha\right|\\
&\qquad+\omega_3\left|\int_{f_{\hat{t}_1}^{\min}}^1 L_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1 L_{\hat{t}_2}(\beta)d\beta\right|\\
&\qquad+\omega_4\left|\int_{f_{\hat{t}_1}^{\min}}^1 R_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1 R_{\hat{t}_2}(\beta)d\beta\right|\\
&\qquad+\omega_5\left|\int_0^{y_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\gamma)d\gamma\right|\\
&\qquad+\omega_6\left|\int_0^{y_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\gamma)d\gamma\right|
\end{aligned}\tag{36}$$

If $\lambda = 2$, TIFWMD is degenerated into the triangular intuitionistic fuzzy weighted Euclidean distance (TIFWED), i.e., (37), as shown at the bottom of the next page.

If $\lambda \to \infty$, TIFWMD is degenerated into the triangular intuitionistic fuzzy weighted Chebyshev distance (TIFWCD), i.e.,

$$\begin{aligned}
&\mathrm{TIFWCD}(\hat{t}_1,\hat{t}_2)\\
&\quad=\mathrm{TIFWMD}_{\to\infty}(\hat{t}_1,\hat{t}_2)\\
&\quad=\max\Bigg\{\omega_1\left|\int_0^{l_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\alpha)d\alpha\right|,\\
&\qquad\omega_2\left|\int_0^{l_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\alpha)d\alpha\right|,\\
&\qquad\omega_3\left|\int_{f_{\hat{t}_1}^{\min}}^1 L_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1 L_{\hat{t}_2}(\beta)d\beta\right|,\\
&\qquad\omega_4\left|\int_{f_{\hat{t}_1}^{\min}}^1 R_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1 R_{\hat{t}_2}(\beta)d\beta\right|,\\
&\qquad\omega_5\left|\int_0^{y_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\gamma)d\gamma\right|,\\
&\qquad\omega_6\left|\int_0^{y_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\gamma)d\gamma\right|\Bigg\}
\end{aligned}\tag{38}$$

Remark 9: Eq. (35) is further simplified as (39), shown at the bottom of the next page, where $A_L^l(\hat{t}_1,\hat{t}_2)$, $A_R^l(\hat{t}_1,\hat{t}_2)$, $A_L^f(\hat{t}_1,\hat{t}_2)$, $A_R^f(\hat{t}_1,\hat{t}_2)$, $A_L^y(\hat{t}_1,\hat{t}_2)$, $A_R^y(\hat{t}_1,\hat{t}_2)$ can be calculated from Eqs. (30)-(32) and expressed as

$$\begin{aligned}
A_L^l(\hat{t}_1,\hat{t}_2)&=\left|\int_0^{l_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\alpha)d\alpha\right|\\
&=\left|\frac{(t_1+\underset{\sim}{t}_1)l_{\hat{t}_1}^{\max}}{2}-\frac{(t_2+\underset{\sim}{t}_2)l_{\hat{t}_2}^{\max}}{2}\right|
\end{aligned}\tag{40}$$

$$\begin{aligned}
A_R^l(\hat{t}_1,\hat{t}_2)&=\left|\int_0^{l_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\alpha)d\alpha\right|\\
&=\left|\frac{(t_1+\tilde{t}_1)l_{\hat{t}_1}^{\max}}{2}-\frac{(t_2+\tilde{t}_2)l_{\hat{t}_2}^{\max}}{2}\right|
\end{aligned}\tag{41}$$

$$\begin{aligned}
A_L^f(\hat{t}_1,\hat{t}_2)&=\left|\int_{f_{\hat{t}_1}^{\min}}^1 L_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1 L_{\hat{t}_2}(\beta)d\beta\right|\\
&=\left|\frac{(t_1+\underset{\sim}{t}_1)(1-f_{\hat{t}_1}^{\min})}{2}-\frac{(t_2+\underset{\sim}{t}_2)(1-f_{\hat{t}_2}^{\min})}{2}\right|
\end{aligned}\tag{42}$$

$$\begin{aligned}
&\mathrm{TIFWMD}_\lambda(\hat{t}_1,\hat{t}_2)\\
&=\sqrt[\lambda]{\begin{aligned}&\omega_1\left|\int_0^{\max\{l_{\hat{t}_1}^{\max},l_{\hat{t}_2}^{\max}\}}[L_{\hat{t}_1}(\alpha)-L_{\hat{t}_2}(\alpha)]d\alpha\right|^\lambda+\omega_2\left|\int_0^{\max\{l_{\hat{t}_1}^{\max},l_{\hat{t}_2}^{\max}\}}[R_{\hat{t}_1}(\alpha)-R_{\hat{t}_2}(\alpha)]d\alpha\right|^\lambda+\omega_3\left|\int_{\min\{f_{\hat{t}_1}^{\min},f_{\hat{t}_2}^{\min}\}}^1[L_{\hat{t}_1}(\beta)-L_{\hat{t}_2}(\beta)]d\beta\right|^\lambda\\&+\omega_4\left|\int_{\min\{f_{\hat{t}_1}^{\min},f_{\hat{t}_2}^{\min}\}}^1[R_{\hat{t}_1}(\beta)-R_{\hat{t}_2}(\beta)]d\beta\right|^\lambda+\omega_5\left|\int_0^{\max\{y_{\hat{t}_1}^{\max},y_{\hat{t}_2}^{\max}\}}[L_{\hat{t}_1}(\gamma)-L_{\hat{t}_2}(\gamma)]d\gamma\right|^\lambda+\omega_6\left|\int_0^{\max\{y_{\hat{t}_1}^{\max},y_{\hat{t}_2}^{\max}\}}[R_{\hat{t}_1}(\gamma)-R_{\hat{t}_2}(\gamma)]d\gamma\right|^\lambda\end{aligned}}\\
&=\sqrt[\lambda]{\begin{aligned}&\omega_1\left|\int_0^{l_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\alpha)d\alpha\right|^\lambda+\omega_2\left|\int_0^{l_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\alpha)d\alpha-\int_0^{l_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\alpha)d\alpha\right|^\lambda+\omega_3\left|\int_{f_{\hat{t}_1}^{\min}}^1L_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1L_{\hat{t}_2}(\beta)d\beta\right|^\lambda\\&+\omega_4\left|\int_{f_{\hat{t}_1}^{\min}}^1R_{\hat{t}_1}(\beta)d\beta-\int_{f_{\hat{t}_2}^{\min}}^1R_{\hat{t}_2}(\beta)d\beta\right|^\lambda+\omega_5\left|\int_0^{y_{\hat{t}_1}^{\max}}L_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}L_{\hat{t}_2}(\gamma)d\gamma\right|^\lambda+\omega_6\left|\int_0^{y_{\hat{t}_1}^{\max}}R_{\hat{t}_1}(\gamma)d\gamma-\int_0^{y_{\hat{t}_2}^{\max}}R_{\hat{t}_2}(\gamma)d\gamma\right|^\lambda\end{aligned}}
\end{aligned}\tag{35}$$

$$A_R^f(\hat{t}_1, \hat{t}_2) = \left| \int_{f_{\hat{t}_1}^{\min}}^{1} R_{\hat{t}_1}(\beta) d\beta - \int_{f_{\hat{t}_2}^{\min}}^{1} R_{\hat{t}_2}(\beta) d\beta \right| = \left| \frac{(t_1 + \tilde{t}_1)(1 - f_{\hat{t}_1}^{\min})}{2} - \frac{(t_2 + \tilde{t}_2)(1 - f_{\hat{t}_2}^{\min})}{2} \right| \tag{43}$$

$$A_L^y(\hat{t}_1, \hat{t}_2) = \left| \int_0^{y_{\hat{t}_1}^{\max}} L_{\hat{t}_1}(\gamma) d\gamma - \int_0^{y_{\hat{t}_2}^{\max}} L_{\hat{t}_2}(\gamma) d\gamma \right| = \left| \frac{\left(t_1 + \underset{\sim}{t}_1\right) y_{\hat{t}_1}^{\max}}{2} - \frac{\left(t_2 + \underset{\sim}{t}_2\right) y_{\hat{t}_2}^{\max}}{2} \right| \tag{44}$$

$$A_R^y(\hat{t}_1, \hat{t}_2) = \left| \int_0^{y_{\hat{t}_1}^{\max}} R_{\hat{t}_1}(\gamma) d\gamma - \int_0^{y_{\hat{t}_2}^{\max}} R_{\hat{t}_2}(\gamma) d\gamma \right| = \left| \frac{\left(\tilde{t}_1 + t_1\right) y_{\hat{t}_1}^{\max}}{2} - \frac{\left(\tilde{t}_2 + t_2\right) y_{\hat{t}_2}^{\max}}{2} \right| \tag{45}$$

Theorem 10: $\mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2)$ satisfies:

(i) Non-negativity: $\mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2) \geq 0$;

(ii) Symmetry: $\mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2) = \mathrm{TIFWMD}_\lambda(\hat{t}_2, \hat{t}_1)$;

(iii) Trigonometric inequality: $\mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2) + \mathrm{TIFWMD}_\lambda(\hat{t}_2, \hat{t}_3) \geq \mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_3)$.

Remark 10: According to Eqs. (39)-(45), the geometric meaning of TIFWMD can be expressed by the weighted combination of TIFNs $A_L^l(\hat{t}_1, \hat{t}_2)$, $A_R^l(\hat{t}_1, \hat{t}_2)$, $A_L^f(\hat{t}_1, \hat{t}_2)$, $A_R^f(\hat{t}_1, \hat{t}_2)$, $A_L^y(\hat{t}_1, \hat{t}_2)$, $A_R^y(\hat{t}_1, \hat{t}_2)$ and $A_L^l(\hat{t}_1, \hat{t}_2)$, $A_R^l(\hat{t}_1, \hat{t}_2)$, $A_L^f(\hat{t}_1, \hat{t}_2)$, $A_R^f(\hat{t}_1, \hat{t}_2)$, $A_L^y(\hat{t}_1, \hat{t}_2)$, $A_R^y(\hat{t}_1, \hat{t}_2)$ and the absolute value of the difference between the area enclosed in the coordinate axis, as shown in Figure 3. The geometric meaning of $A_L^l(\hat{t}_1, \hat{t}_2)$, $A_R^l(\hat{t}_1, \hat{t}_2)$, $A_L^f(\hat{t}_1, \hat{t}_2)$, $A_R^f(\hat{t}_1, \hat{t}_2)$, $A_L^y(\hat{t}_1, \hat{t}_2)$, $A_R^y(\hat{t}_1, \hat{t}_2)$ is the absolute value of the difference between the areas of two shadow trapezoids, which can be further represented by Figure 4-Figure 9.

G. TIFSs

Definition 10: Let G be the universe set, then G is usually a finite object set ($G = \{G_1, G_2, \ldots, G_n\}$) or finite interval ($G = [t_{\text{low}}, t^{\text{up}}]$, $t_{\text{low}}, t^{\text{up}} \in R$). At this time, the TIFS is expressed as $\hat{T} = \{< x, \hat{t}(x) > | x \in G\}$, where $\hat{t}(x)$ is the TIFN of element x that belongs to $\hat{t}$.

Definition 11 [26]: Let G_n be the universe set of finite objects, that is $G_n = \{G_1, G_2, \ldots, G_n\}$. At this time, the TIFS on G_n is expressed as $\hat{T}_{G_n} = \{< x_s, \hat{t}_s(x) > | x_s \in G_n, s = 1, 2, \ldots, n\}$, where $\hat{t}_s(x)$ is the TIFN of $x = G_s$ that belongs to $\hat{t}$.

This paper discusses the TIFS on the domain $\hat{T} = \hat{T}_{G_n} = \{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n\}$ in Definition 11. For convenience, note $\hat{T} = \hat{T}_{G_n} = \{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n\}$.

H. DISTANCE OF TIFSs

Definition 12 [26]: Suppose $G_n = \{G_1, G_2, \ldots, G_n\}$; let $\hat{T}^1 = \{\hat{t}_1^1, \hat{t}_2^1, \ldots, \hat{t}_n^1\}$ and $\hat{T}^2 = \{\hat{t}_1^2, \hat{t}_2^2, \ldots, \hat{t}_n^2\}$ are two TIFSs on domain G_n; in this case, the new TIFS weighted Minkowski distance (TIFSWMD) is expressed as

$$\mathrm{TIFSWMD}_\lambda(\hat{T}^1, \hat{T}^2) = \sqrt[\lambda]{\frac{1}{n} \sum_{j=1}^{n} \left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^1, \hat{t}_j^2)\right)^\lambda} \tag{46}$$

In Eq. (46), $\mathrm{TIFWMD}_\lambda(\hat{t}_j^1, \hat{t}_j^2)$ can be calculated from Eq. (39).

Theorem 11 $\mathrm{TIFSWMD}(\hat{T}_1, \hat{T}_2)$ satisfies:

(i) Non-negativity: $\mathrm{TIFSWMD}(\hat{T}_1, \hat{T}_2) \geq 0$;

(ii)Symmetry: $\mathrm{TIFSWMD}(\hat{T}_1, \hat{T}_2) = \mathrm{TIFSWMD}(\hat{T}_2, \hat{T}_1)$;

(iii) Trigonometric inequalities: $\mathrm{TIFSWMD}(\hat{T}_1, \hat{T}_2) + \mathrm{TIFSWMD}(\hat{T}_2, \hat{T}_3) \geq \mathrm{TIFSWMD}(\hat{T}_1, \hat{T}_3)$.

I. TWO-SIDED MATCHING

Given two agent sets $A = \{A_1, A_2, \ldots, A_m\}$ and $B = \{B_1, B_2, \ldots, B_n\}$, where A_p and B_q is the pth and the qth

$$\begin{aligned} &\mathrm{TIFWED}(\hat{t}_1, \hat{t}_2) \\ &= \mathrm{TIFWMD}_2(\hat{t}_1, \hat{t}_2) \\ &= \sqrt{\begin{aligned} &\omega_1 \left| \int_0^{l_{\hat{t}_1}^{\max}} L_{\hat{t}_1}(\alpha) d\alpha - \int_0^{l_{\hat{t}_2}^{\max}} L_{\hat{t}_2}(\alpha) d\alpha \right|^2 + \omega_2 \left| \int_0^{l_{\hat{t}_1}^{\max}} R_{\hat{t}_1}(\alpha) d\alpha - \int_0^{l_{\hat{t}_2}^{\max}} R_{\hat{t}_2}(\alpha) d\alpha \right|^2 + \omega_3 \left| \int_{f_{\hat{t}_1}^{\min}}^{1} L_{\hat{t}_1}(\beta) d\beta - \int_{f_{\hat{t}_2}^{\min}}^{1} L_{\hat{t}_2}(\beta) d\beta \right|^2 \\ &+\omega_4 \left| \int_{f_{\hat{t}_1}^{\min}}^{1} R_{\hat{t}_1}(\beta) d\beta - \int_{f_{\hat{t}_2}^{\min}}^{1} R_{\hat{t}_2}(\beta) d\beta \right|^2 + \omega_5 \left| \int_0^{y_{\hat{t}_1}^{\max}} L_{\hat{t}_1}(\gamma) d\gamma - \int_0^{y_{\hat{t}_2}^{\max}} L_{\hat{t}_2}(\gamma) d\gamma \right|^2 + \omega_6 \left| \int_0^{y_{\hat{t}_1}^{\max}} R_{\hat{t}_1}(\gamma) d\gamma - \int_0^{y_{\hat{t}_2}^{\max}} R_{\hat{t}_2}(\gamma) d\gamma \right|^2 \end{aligned}} \end{aligned} \tag{37}$$

$$\begin{aligned} &\mathrm{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2) \\ &= \sqrt[\lambda]{\omega_1 \left(A_L^l(\hat{t}_1, \hat{t}_2)\right)^\lambda + \omega_2 \left(A_R^l(\hat{t}_1, \hat{t}_2)\right)^\lambda + \omega_3 \left(A_L^f(\hat{t}_1, \hat{t}_2)\right)^\lambda + \omega_4 \left(A_R^f(\hat{t}_1, \hat{t}_2)\right)^\lambda + \omega_5 \left(A_L^y(\hat{t}_1, \hat{t}_2)\right)^\lambda + \omega_6 \left(A_R^y(\hat{t}_1, \hat{t}_2)\right)^\lambda} \end{aligned} \tag{39}$$

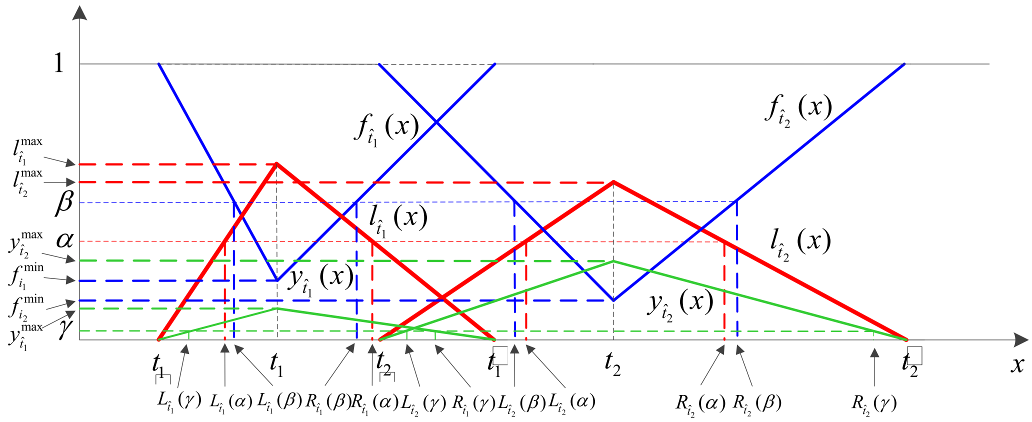

FIGURE 3. α cut set, β cut set and γ cut set of TIFNs $\hat{t}_1$ and $\hat{t}_2$.

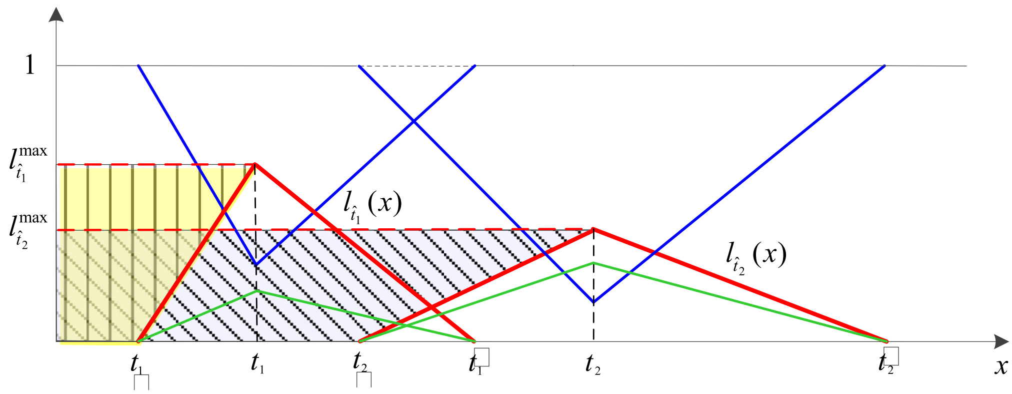

FIGURE 4. Geometric meaning of $A_L^l\left(\hat{t}_1, \hat{t}_2\right)$.

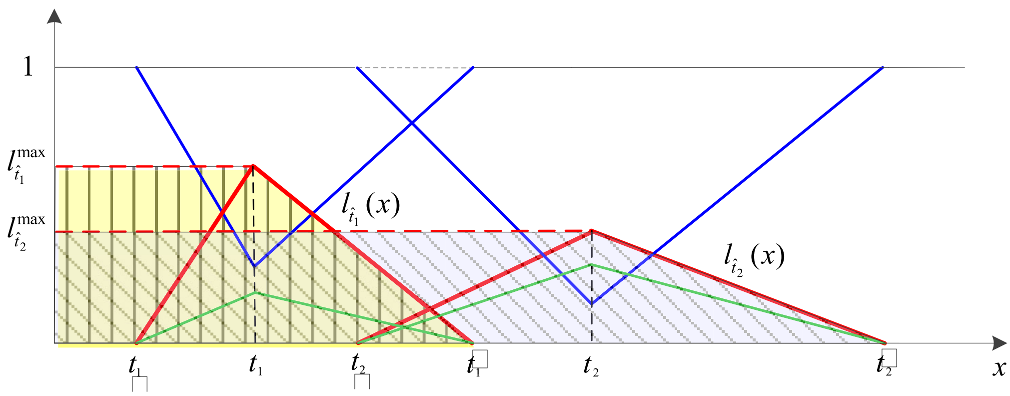

FIGURE 5. Geometric meaning of $A_R^l\left(\hat{t}_1, \hat{t}_2\right)$.

agent of sides A and B respectively. For convenience, suppose $n \geq m > 2$.

Definition 13 [26], [27], [28]: Assume that $\Lambda : A \cup B \to A \cup B$ is a one-to-one mapping, if $\forall A_p \in A$, $\forall B_q \in B$ meet the following conditions: (i) $\Lambda(A_p) \in B$, (ii) $\Lambda(B_q) \in A \cup \{B_q\}$, (iii) $\Lambda(A_p) = B_q$ if and only if $\Lambda(B_q) = A_p$, then Λ is called a TSM.

In Definition 13, (A_p, B_q) denotes that A_p and B_q is matched successfully in TSM Λ; (A_p, A_p) and (B_q, B_q) denote that A_p and B_q is not matched in in TSM Λ.

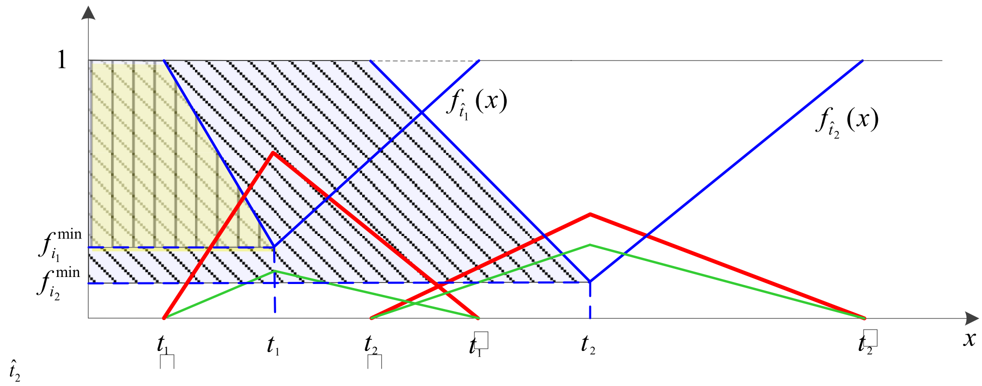

FIGURE 6. Geometric meaning of $A_L^f(\hat{t}_1, \hat{t}_2)$.

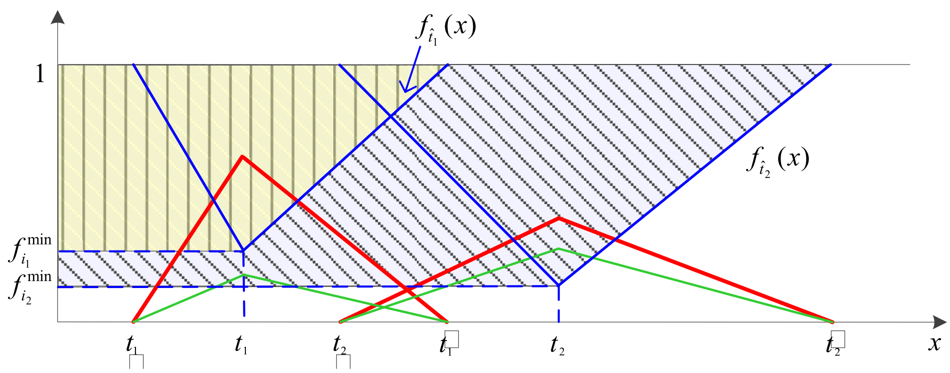

FIGURE 7. Geometric meaning of $A_R^f(\hat{t}_1, \hat{t}_2)$.

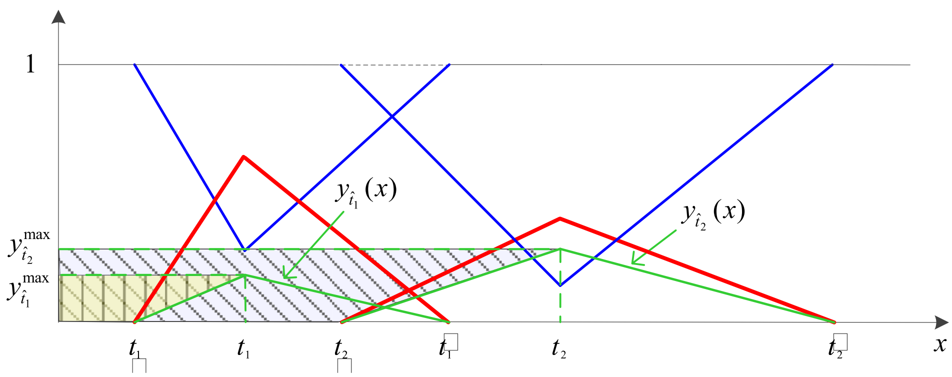

FIGURE 8. Geometric meaning of $A_L^y(\hat{t}_1, \hat{t}_2)$.

III. TWO-SIDED MATCHING DECISION-MAKING WITH POSITIVE/NEGATIVE TIFNs

A. TWO-SIDED MATCHING PROBLEM WITH TIFNs

For the TIFN two-sided matching problem, assume that $\hat{T}_A = [\hat{t}_{pq}^A]_{m\times n}$ is the TIFN information matrix given by side A to side B, where $\hat{t}_{pq}^A$ refers to the TIFN information given by A_p to B_q. Assume that $\hat{T}_B = [\hat{t}_{pq}^B]_{m\times n}$ is the TIFN information matrix by side B to side A, where $\hat{t}_{pq}^B$ refers to the TIFN information given by B_q to A_p.

This paper considers solving this TSM problem according to the TIFN preference matrices $\hat{T}_A = [\hat{t}_{pq}^A]_{m\times n}$ and $\hat{T}_B = [\hat{t}_{pq}^B]_{m\times n}$ provided by matching agents $A = \{A_1, A_2, \ldots, A_m\}$ and $B = \{B_1, B_2, \ldots, B_n\}$.

Remark 11: In the considered problem, TIFNs $\hat{t}_{pq}^A$ and $\hat{t}_{pq}^B$ can be negative, which are ignored in most related studies.

B. SOLUTION PROCESS OF TWO-SIDED MATCHING DECISION-MAKING

The decision-making process of the TSM method with TIFN preference is summarized as follows:

Step 1: Transform TIFN matrices $\hat{T}_A = [\hat{t}_{pq}^A]_{m\times n}$ and $\hat{T}_B = [\hat{t}_{pq}^B]_{m\times n}$ into ES matrices $\hat{S}_A = [ES_{pq}^A]_{m\times n}$ and $\hat{S}_B = [ES_{pq}^B]_{m\times n}$ by Eqs. (33)-(34).

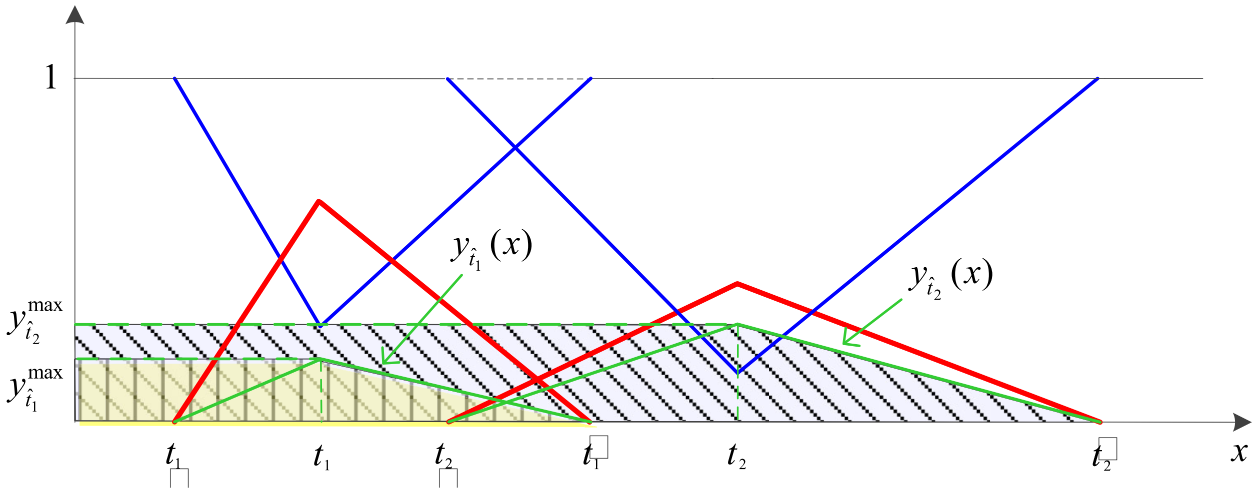

FIGURE 9. Geometric meaning of $A_R^y(\hat{t}_1, \hat{t}_2)$.

Step 2: Construct a TSM Model (M-1) according to ES matrices $\hat{S}_A = [ES_{pq}^A]_{m\times n}$ and $\hat{S}_B = [ES_{pq}^B]_{m\times n}$ considering the constrains of one to one matching as follows:

$$(\text{M-1})\begin{cases} \text{Max } Z_1 = \sum_{p=1}^{m}\sum_{q=1}^{n} ES_{pq}^A x_{pq} \\ \text{Max } Z_2 = \sum_{p=1}^{m}\sum_{q=1}^{n} ES_{pq}^B x_{pq} \\ \text{s.t. } \sum_{p=1}^{m} x_{pq} \le 1, q = 1, 2, \ldots, n; \\ \sum_{q=1}^{n} x_{pq} \le 1, p = 1, 2, \ldots, m; \\ x_{pq} + \sum_{k:ES_{pk}^A > ES_{pq}^A} x_{pk} + \sum_{l:ES_{lq}^A > ES_{pq}^A} x_{lq} \ge 1 \\ x_{pq} \in \{0, 1\}, \quad p = 1, 2, , , m; \ q = 1, 2, \ldots, n \end{cases}$$

Step 3: Transform model (M-1) into model (M-2) by using the linear weighted method, i.e.,

$$(\text{M-2})\begin{cases} \text{Max} Z = \sum_{p=1}^{m}\sum_{q=1}^{n} ES_{pq} x_{pq} \\ \text{s.t. } \sum_{p=1}^{m} x_{pq} \le 1, q = 1, 2, \ldots, n; \\ \sum_{q=1}^{n} x_{pq} \le 1, p = 1, 2, \ldots, m; \\ x_{pq} + \sum_{k:ES_{pk}^A > ES_{pq}^A} x_{pk} + \sum_{l:ES_{lq}^A > ES_{pq}^A} x_{lq} \ge 1 \\ x_{pq} \in \{0, 1\}, \quad p = 1, 2, \ldots, m; \ q = 1, 2, \ldots, n \end{cases}$$

In Model (M-2), $ES_{pq} = w_A ES_{pq}^A + w_B ES_{pq}^B$, where w_A and w_B refer to the weights of side A and side B respectively.

Step 4: Gain the best TSM scheme by solving model (M-2).

IV. CASE ANALYSIS

A. INSTRUCTION OF CASE

With the approach of intelligent era, the business environment and production process of manufacturing enterprises have changed greatly. Therefore, manufacturing enterprises need to take some ways to make themselves in the future wave of not be eliminated and merged. Adopting advanced ERP software is one of the important ways for manufacturing enterprises to store competitive capital in the new era. On the one hand, ERP software can help enterprises to inherit the important resources and production factors, such as customers, projects, sales and quotations, in a set of systems, and minimize the manual participation, so that enterprises can use advanced information systems to operate, and thus improve the efficiency of enterprise management. On the other hand, ERP software eliminates various management loopholes from the source and optimizes the business process of the enterprise. For manufacturing enterprises, whether to find suitable ERP software developers is an important factor to achieve the successful transformation of information technology. ERP software developers also need to find the right customers in order to facilitate long-term cooperation. However, due to the complex market environment and diversified information, both parties need to find the suitable TSM scheme through third-party intermediaries.

On this basis, the TSM problem between manufacturing enterprises and ERP software developers is provided to demonstrate the effectiveness of the proposed theories and method.

B. THE PROBLEM OF ERP SOFTWARE SUPPLY-DEMAND MATCHING WITH TIFN PREFERENCES

There are three construction machinery manufacturing enterprises $A = \{A_1, A_2, A_3\}$ in Changsha. To save the cost and improve the production management level of the enterprises, the three construction machinery manufacturing enterprises need to find relevant ERP software enterprises to order ERP software with the help of the third-party intermediary. At the same time, four ERP software enterprises are looking for long-term cooperation with manufacturing customers. Both parties submit their own information to a third-party intermediary. On this basis, machinery manufacturing enterprises evaluated the four ERP software enterprises $B = \{B_1, B_2, B_3, B_4\}$ from four indices, including the functionality, reliability, ease of use and maintainability. ERP software enterprises $B = \{B_1, B_2, B_3, B_4\}$ evaluated the three construction machinery manufacturing enterprises $A = \{A_1, A_2, A_3\}$ from four indices of the enterprise scale, management level, information level and financial condition. Due to the complexity of the market environment, the evaluation information given by both parties is the TIFS. At the same time, the evaluation method adopts the scoring system. If the match object reaches the decision-maker's ideal level, the score is greater than 0, otherwise, the score is less than 0. The range of fuzzy numbers in TIFN is $[-10, 10]$, where -10/10 is the lower/upper limit of the score. On this basis, the decision-maker gives its upper limit, lower limit and possible value. The TIFN information provided by three construction machinery manufacturing enterprises $A = \{A_1, A_2, A_3\}$ is marked as $\hat{T}_A = [\hat{t}^A_{pq}]_{3\times 4}$, as displayed in Table 1. The TIFN information provided by four ERP software enterprises $B = \{B_1, B_2, B_3, B_4\}$ is marked as $\hat{T}_B = [\hat{t}^B_{pq}]_{3\times 4}$, as displayed in Table 2.

C. TSM DECISION-MAKING PROCESS

The main decision-making steps of the proposed approach for solving the TSM problem are displayed as follows.

Step 1: Transform TIFN matrices $\hat{T}_A = [\hat{t}^A_{pq}]_{m\times n}$ and $\hat{T}_B = [\hat{t}^B_{pq}]_{m\times n}$ into ES matrices $\hat{S}_A = [ES^A_{pq}]_{m\times n}$ and $\hat{S}_B = [ES^B_{pq}]_{m\times n}$ by Eqs. (33)-(34), as displayed in Table 3 and Table 4, where $\omega_\alpha = \omega_\beta = \omega_\gamma = \frac{1}{3}, \omega_R = \omega_L = \frac{1}{2}$.

Step 2: Construct a TSM Model (M-1) according to ES matrices $\hat{S}_A = [ES^A_{pq}]_{m\times n}$ and $\hat{S}_B = [ES^B_{pq}]_{m\times n}$ considering stable one to one matching constrains as follows:

$$
\text{(M-1)}\begin{cases}
\text{Max } Z_1 = \sum_{p=1}^{m}\sum_{q=1}^{n} ES^A_{pq}x_{pq} \\
\text{Max } Z_2 = \sum_{p=1}^{m}\sum_{q=1}^{n} ES^B_{pq}x_{pq} \\
\text{s.t. } \sum_{p=1}^{m} x_{pq} \le 1, q = 1, 2, 3, 4; \\
\sum_{q=1}^{n} x_{pq} \le 1, p = 1, 2, 3; \\
x_{pq} + \sum_{k:ES^A_{pk} > ES^A_{pq}} x_{pk} + \sum_{l:ES^A_{lq} > ES^A_{pq}} x_{lq} \ge 1 \\
x_{pq} \in \{0, 1\}, \quad p = 1, 2, 3, \ q = 1, 2, 3, 4
\end{cases}
$$

Step 3: Transform model (M-1) into the following model (M-2) by using the linear weighted method, i.e.,

$$
\text{(M-2)}\begin{cases}
\text{Max } Z = \sum_{p=1}^{m}\sum_{q=1}^{n} ES_{pq}x_{pq} \\
\text{s.t. } \sum_{p=1}^{m} x_{pq} \le 1, \quad q = 1, 2, 3, 4; \\
\sum_{q=1}^{n} x_{pq} \le 1, \quad p = 1, 2, 3; \\
x_{pq} + \sum_{k:ES^A_{pk} > ES^A_{pq}} x_{pk} + \sum_{l:ES^A_{lq} > ES^A_{pq}} x_{lq} \ge 1 \\
x_{pq} \in \{0, 1\}, \quad p = 1, 2, 3, \ q = 1, 2, 3, 4
\end{cases}
$$

where $w_A = w_B = 0.5$.

Step 4: Gain the best TSM variable x_{pq} by solving Model (M-2), as displayed in Table 5.

Therefore, the best TSM scheme is (A_1, B_2), (A_2, B_1), (A_3, B_4). In other words, construction machinery manufacturing enterprise A_1 is matched with ERP software enterprise B_2, construction machinery manufacturing enterprise A_2 is matched with ERP software enterprise B_1, construction machinery manufacturing enterprise A_3 is matched with ERP software enterprise B_4.

D. COMPARATIVE ANALYSIS

To better illustrate the stable matching constrains in the formation of matching schemes, we reconstruct the Model (M-3) without stable constrains, i.e.,

$$
\text{(M-3)}\begin{cases}
\text{Max } Z = \sum_{p=1}^{m}\sum_{q=1}^{n} ES_{pq}x_{pq} \\
\text{s.t. } \sum_{p=1}^{m} x_{pq} \le 1, \quad q = 1, 2, 3, 4; \\
\sum_{q=1}^{n} x_{pq} \le 1, \quad p = 1, 2, 3; \\
x_{pq} \in \{0, 1\}, \quad p = 1, 2, 3, \ q = 1, 2, 3, 4
\end{cases}
$$

On this basis, the comparison of TSM scheme based on two different methods is displayed in Table 6.

As we can see from Table 6, there are slight differences between the two TSM schemes. Only one of the three pairs of TSM schemes is the same, while the other two pairs of TSM schemes are totally different. Therefore, it can be inferred that the stable matching conditions effectively guarantee the formation of stable TSM schemes.

Furthermore, we analyze the impact of weights $w_A = 1, w_B = 0$ and $w_A = 0, w_B = 1$ on the TSM scheme and the objective function value. The sensitive analysis on weights $w_A = 1, w_B = 0$ and $w_A = 0, w_B = 1$ are displayed in Table 7.

The variation of objective function value Z from cases 1-11 are shown in Figure 10. The variation of TSM variable x_{pq} from cases 1-11 are shown in Figure 11. The variation of

TABLE 1. TIFN matrix $\hat{T}_A = [\hat{t}^A_{pq}]_{3\times4}$.

	B_1	B_2	B_3	B_4
A_1	$<(2,5,6);0.4,0.1>$	$<(1,3,4);0.5,0.4>$	$<(4,7,9);0.2,0.1>$	$<(1,3,8);0.5,0.3>$
A_2	$<(1,4,7);0.7,0.2>$	$<(2,6,7);0.3,0.2>$	$<(-4,5,6);0.3,0.1>$	$<(2,4,7);0.4,0.2>$
A_3	$<(3,4,8);0.3,0.1>$	$<(1,2,8);0.8,0.1>$	$<(-8,2,4);0.6,0.3>$	$<(2,4,8);0.2,0.7>$

TABLE 2. TIFN matrix $\hat{T}_B = [\hat{t}^B_{pq}]_{3\times4}$.

	B_1	B_2	B_3	B_4
A_1	$<(1,2,7);0.3,0.4>$	$<(1,4,7);0.4,0.1>$	$<(3,4,8);0.6,0.2>$	$<(-4,3,4);0.3,0.2>$
A_2	$<(1,8,9);0.2,0.7>$	$<(1,2,6);0.3,0.2>$	$<(1,2,9);0.7,0.1>$	$<(4,8,9);0.1,0.3>$
A_3	$<(-1,3,4);0.2,0.4>$	$<(4,7,8);0.8,0.1>$	$<(-2,1,6);0.3,0.2>$	$<(-2,1,6);0.5,0.1>$

TABLE 3. ES matrix $\hat{S}_A = [ES^A_{pq}]_{m\times n}$.

	B_1	B_2	B_3	B_4
A_1	-0.6667	-0.01	-1.1667	-0.4667
A_2	-0.2000	-0.8333	-0.2	-0.5
A_3	-1	-0.2333	-0.4	-0.2

TABLE 4. ES matrix $\hat{S}_B = [ES^B_{pq}]_{m\times n}$.

	B_1	B_2	B_3	B_4
A_1	-0.6	-0.5733	-0.3333	-1.3333
A_2	-0.1667	-0.8333	-0.5333	-1
A_3	-0.6667	-0.1333	-1.3333	-1.0667

TABLE 5. Best matching variable x_{pq}.

	B_1	B_2	B_3	B_4
A_1	0	1	0	0
A_2	1	0	0	0
A_3	0	0	0	1

TSM variable x_{pq} and objective function value Z from cases 1-11 are shown in Figure 12.

From Figure 10, it can be concluded that objective function value x_{pq} decline constantly form cases 1-6 when weight values changes. However, the objective function value x_{pq} rises gradually from cases 6-11. Hence, we can see that

TABLE 6. Comparison of two different methods.

Method	Matching scheme
TSM method considering stable constrains	$(A_1,B_2),(A_2,B_3),(A_3,B_4)$
TSM method without stable constrains	$(A_1,B_2),(A_2,B_1),(A_3,B_4)$

the weight values have a significant role in the change of objective function value Z.

As we can see from Figure 11, along with the change in weight values, there is no significant change for the TSM variable x_{pq}. Therefore, it could be concluded that the weight values play little important role in the formation of TSM scheme.

From Figure 12, it can be concluded that the weight values have a significant role in the change of objective function value Z. However, the weight values have little important role in the formation of TSM scheme.

V. CONCLUSION

This paper further refined the new theory of TIFNs and TIFSs proposed by Yue et al. [26], where the proofs of theorems of the generalized algorithm of TIFNs are given. This paper also presented some new theorems and relative proofs on the aggregation operator and distance measure of TIFNs and TIFSs. Moreover, the novel theories and method of TIFNs and TIFSs are adopted to solve the TSM problem under a positive/negative TIFN environment. These results can provide a theoretical foundation for the development of TIFNs and TIFSs.

TABLE 7. Sensitive analysis.

Cases	Different weight values	Matching scheme	Objective function values
1	$w_A = 1, w_B = 0$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.6714
2	$w_A = 0.9, w_B = 0.1$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.5871
3	$w_A = 0.8, w_B = 0.2$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.5027
4	$w_A = 0.7, w_B = 0.3$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.4184
5	$w_A = 0.6, w_B = 0.4$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.3339
6	$w_A = 0.5, w_B = 0.5$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.2768
7	$w_A = 0.4, w_B = 0.6$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.3404
8	$w_A = 0.3, w_B = 0.7$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.4566
9	$w_A = 0.2, w_B = 0.8$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.573
10	$w_A = 0.1, w_B = 0.9$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.2768
11	$w_A = 0, w_B = 1$	$(A_4, B_1), (A_2, B_1), (A_3, B_2)$	2.8055

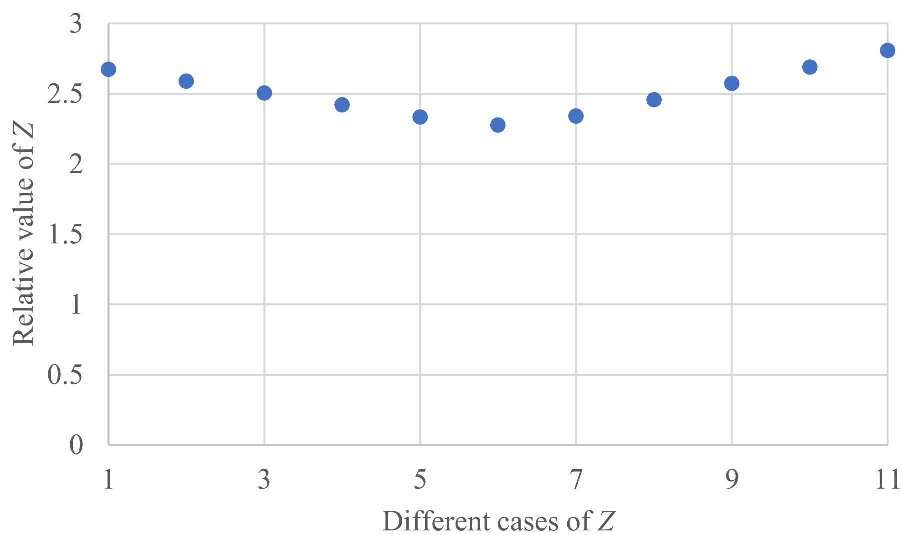

FIGURE 10. Objective function value *Z* from cases 1-11.

Overall, on the one hand, the new theory of TIFN and TIFS proposed in this paper could further refine the relevant theoretical system of TIFN and TIFS. On the other hand, the proposed method could provide a novel prospective for the research on TSM problems under a positive/negative TIFN environment.

However, the proposed TSM decision-making with TIFNs cannot be applied in some more complex decision-making scenarios, such as dynamic decision-making scenarios. Therefore, it would be practically to develop a novel TIFN theory which could be applied in dynamic decision-making scenarios. Also, in continuation of the efforts in Section II, aggregation of temporal TIFSs and its utilization in decision-making can be a line for future research.

APPENDIX

PROOF OF THEOREM 1

(i) Prove that the result calculated by Eq. (5) is a TIFN.

Because $\hat{t}_1 =< (\underset{\sim}{t}_1, t_1, \tilde{t}_1); l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1} >$ and $\hat{t}_2 =< (\underset{\sim}{t}_2, t_2, \tilde{t}_2); l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2} >$ are TIFNs, $\underset{\sim}{t}_1 \leq t_1 \leq \tilde{t}_1$, $\underset{\sim}{t}_2 \leq t_2 \leq \tilde{t}_2$, $l^{\max}_{\hat{t}_1}, f^{\min}_{\hat{t}_1}, l^{\max}_{\hat{t}_1} + f^{\min}_{\hat{t}_1} \in [0, 1]$ and $l^{\max}_{\hat{t}_2}, f^{\min}_{\hat{t}_2}, l^{\max}_{\hat{t}_2} +$

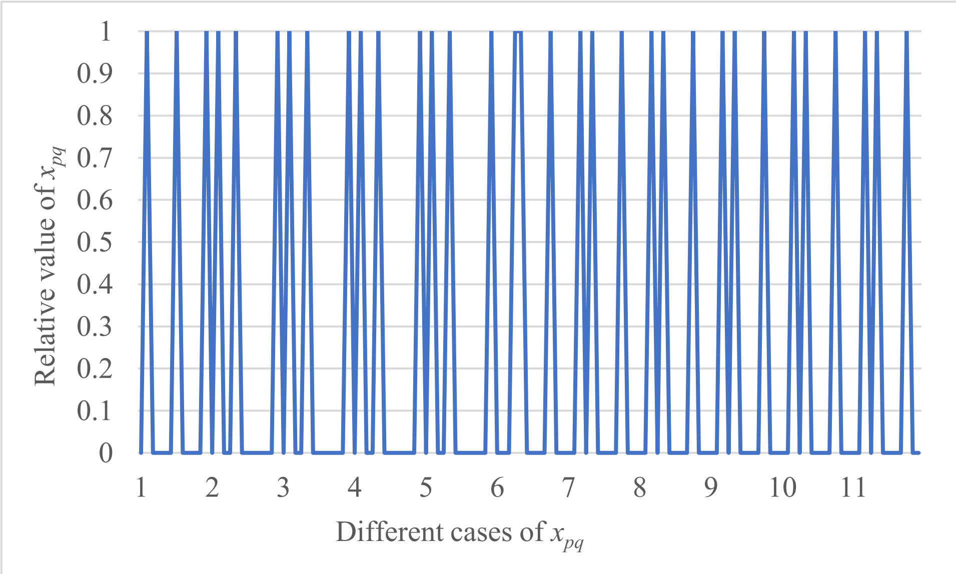

FIGURE 11. TSM variable x_{pq} from cases 1-11.

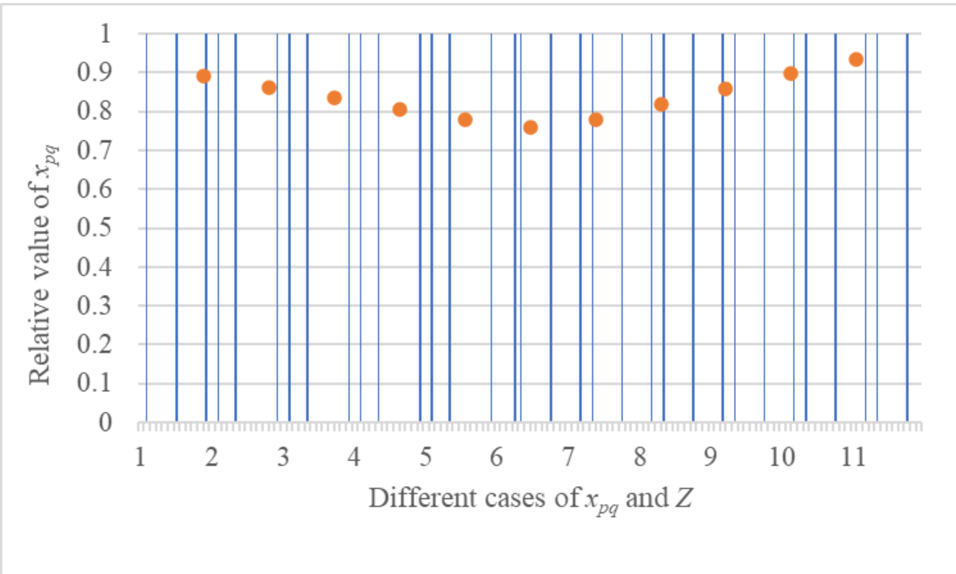

FIGURE 12. TSM variable x_{pq} and objective function value Z from cases 1-11.

$f_{\hat{t}_2}^{\min} \in [0, 1]$ can be obtained. Therefore we get $\underset{\sim}{t}_1 + \underset{\sim}{t}_2 \leq t_1 + t_2 \leq \tilde{t}_1 + \tilde{t}_2$ and $f_{\hat{t}_1}^{\min} f_{\hat{t}_2}^{\min} \in [0, 1]$.

Because $l_{\hat{t}_1}^{\max} + l_{\hat{t}_2}^{\max} - l_{\hat{t}_1}^{\max} l_{\hat{t}_2}^{\max} = 1 - (1 - l_{\hat{t}_1}^{\max})(1 - l_{\hat{t}_2}^{\max})$ and $l_{\hat{t}_1}^{\max}, l_{\hat{t}_2}^{\max} \in [0, 1]$, $l_{\hat{t}_1}^{\max} + l_{\hat{t}_2}^{\max} - l_{\hat{t}_1}^{\max} l_{\hat{t}_2}^{\max} \in [0, 1]$ can be obtained.

Because $l_{\hat{t}_1}^{\max} + f_{\hat{t}_1}^{\min} \in [0, 1]$ and $l_{\hat{t}_2}^{\max} + f_{\hat{t}_2}^{\min} \in [0, 1]$, $0 \leq f_{\hat{t}_1}^{\min} \leq 1 - l_{\hat{t}_1}^{\max}$ and $0 \leq f_{\hat{t}_2}^{\min} \leq 1 - l_{\hat{t}_2}^{\max}$ can be obtained. Therefore we get $0 \leq l_{\hat{t}_1}^{\max} + l_{\hat{t}_2}^{\max} - l_{\hat{t}_1}^{\max} l_{\hat{t}_2}^{\max} + f_{\hat{t}_1}^{\min} f_{\hat{t}_2}^{\min} \leq l_{\hat{t}_1}^{\max} + l_{\hat{t}_2}^{\max} - l_{\hat{t}_1}^{\max} l_{\hat{t}_2}^{\max} + (1 - l_{\hat{t}_1}^{\max})(1 - l_{\hat{t}_2}^{\max}) = 1$. Therefore, $\hat{t}_1 + \hat{t}_2 =< (\underset{\sim}{t}_1 + \underset{\sim}{t}_2, t_1 + t_2, \tilde{t}_1 + \tilde{t}_2); l_{\hat{t}_1}^{\max} + l_{\hat{t}_2}^{\max} - l_{\hat{t}_1}^{\max} l_{\hat{t}_2}^{\max}, f_{\hat{t}_1}^{\min} f_{\hat{t}_2}^{\min} >$ is still a TIFN.

(ii) Prove that the result calculated by Eq. (6) is a TIFN.

Since $\hat{t} =< (\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min} >$ is a TIFN, $\underset{\sim}{t} \leq t \leq \tilde{t}$ and $l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}, l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0, 1]$ can be obtained. The following two cases are respectively proved.

① When $k > 0$, $k\underset{\sim}{t} \leq tk \leq k\tilde{t}$ can be obtained because $\underset{\sim}{t} \leq t \leq \tilde{t}$.

Because $0 \leq l_{\hat{t}}^{\max} \leq 1$, $0 \leq 1 - l_{\hat{t}}^{\max} \leq 1$ can be obtained. Therefore we get $0 \leq 1 - (1 - l_{\hat{t}}^{\max})^k \leq 1$. Since $f_{\hat{t}}^{\min} \in [0, 1]$, $0 \leq (f_{\hat{t}}^{\min})^k \leq 1$ can be obtained.

Because $l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0, 1]$, $f_{\hat{t}}^{\min} \leq 1 - l_{\hat{t}}^{\max}$ can be obtained. Therefore, we get $(f_{\hat{t}}^{\min})^k \leq (1 - l_{\hat{t}}^{\max})^k$ and $0 \leq 1 - (1 - l_{\hat{t}}^{\max})^k + (f_{\hat{t}}^{\min})^k \leq 1 - (1 - l_{\hat{t}}^{\max})^k + (1 - l_{\hat{t}}^{\max})^k \leq 1$.

Hence, when $k > 0$, $<(k\underset{\sim}{t}, kt, k\tilde{t}); 1-(1-l_{\hat{t}}^{\max})^k, (f_{\hat{t}}^{\min})^k>$ is still a TIFN.

② When $k<0$, $k\tilde{t} \le tk \le k\underset{\sim}{t}$ can be obtained because $\underset{\sim}{t} \le t \le \tilde{t}$.

Because $0 \le l_{\hat{t}}^{\max} \le 1$, $0 \le 1-l_{\hat{t}}^{\max} \le 1$ can be obtained. Therefore, we get $0 \le \frac{1}{(1-l_{\hat{t}}^{\max})^k} \le 1$. Furthermore, we obtain $0 \le 1-\frac{1}{(1-l_{\hat{t}}^{\max})^k} \le 1$. Since $f_{\hat{t}}^{\min} \in [0,1]$, $0 < \frac{1}{(f_{\hat{t}}^{\min})^k} < 1$ can be obtained.

Because $l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0,1]$, $f_{\hat{t}}^{\min} \le 1-l_{\hat{t}}^{\max}$ can be obtained. Therefore we get $\frac{1}{(f_{\hat{t}}^{\min})^k} \le \frac{1}{(1-l_{\hat{t}}^{\max})^k}$ and $0 \le 1-\frac{1}{(1-l_{\hat{t}}^{\max})^k}+\frac{1}{(f_{\hat{t}}^{\min})^k} \le 1-\frac{1}{(1-l_{\hat{t}}^{\max})^k}+\frac{1}{(1-l_{\hat{t}}^{\max})^k} \le 1$.

Hence, when $k<0$, $<(k\tilde{t}, kt, k\underset{\sim}{t}); 1-\frac{1}{(1-l_{\hat{t}}^{\max})^k}, \frac{1}{(f_{\hat{t}}^{\min})^k}>$ is still TIFN.

To sum up, the result calculated by Eq. (6) is still a TIFN.

(iii) Prove that the result calculated by Eq. (7) is a TIFN.

Since $\hat{t}_1 =<(\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}>$ and $\hat{t}_2 =<(\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}>$ are TIFNs, $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1$, $\underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$, $l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}, l_{\hat{t}_1}^{\max}+f_{\hat{t}_1}^{\min} \in [0,1]$ and $l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}, l_{\hat{t}_2}^{\max}+f_{\hat{t}_2}^{\min} \in [0,1]$ can be obtained. Therefore, we get $\underset{\sim}{t}_1-\tilde{t}_2 \le t_1-t_2 \le \tilde{t}_1-\underset{\sim}{t}_2$.

Because the maximum membership degree and minimum non membership degree calculated by Eq. (7) and Eq. (5) are the same, the proof process is omitted.

To sum up, $\hat{t}_1-\hat{t}_2 =<(\underset{\sim}{t}_1-\tilde{t}_2, t_1-t_2, \tilde{t}_1-\underset{\sim}{t}_2); l_{\hat{t}_1}^{\max}+l_{\hat{t}_2}^{\max}-l_{\hat{t}_1}^{\max}l_{\hat{t}_2}^{\max}, f_{\hat{t}_1}^{\min}f_{\hat{t}_2}^{\min}>$ is still a TIFN.

(iv) Prove that the result calculated by Eq. (8) is a TIFN.

Since $\hat{t}_1 =<(\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}>$ and $\hat{t}_2 =<(\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}>$ are TIFNs, $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1$, $\underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$, $l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}, l_{\hat{t}_1}^{\max}+f_{\hat{t}_1}^{\min} \in [0,1]$ and $l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}, l_{\hat{t}_2}^{\max}+f_{\hat{t}_2}^{\min} \in [0,1]$ can be obtained.

Firstly, prove the rationality of the TIFN calculated by Eq. (8) from the following four cases.

① When $\hat{t}_1 \ge 0, \hat{t}_2 \ge 0$, we obtain $0 \le \underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1$ and $0 \le \underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$. Hence $\underset{\sim}{t}_1\underset{\sim}{t}_2 \le t_1t_2 \le \tilde{t}_1\tilde{t}_2$.

② When $\hat{t}_1 < 0, \hat{t}_2 \ge 0$, we obtain $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1 < 0$ and $0 \le \underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$. Hence $\underset{\sim}{t}_1\tilde{t}_2 \le t_1t_2 \le \tilde{t}_1\underset{\sim}{t}_2$.

③ When $\hat{t}_1 < 0, \hat{t}_2 < 0$, we obtain $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1 < 0$ and $\underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2 < 0$. Hence $\tilde{t}_1\tilde{t}_2 \le t_1t_2 \le \underset{\sim}{t}_1\underset{\sim}{t}_2$.

④ When $\hat{t}_1$ and $\hat{t}_2$ belong to other situations, it is obvious that $\min\{\underset{\sim}{t}_1\underset{\sim}{t}_2, \underset{\sim}{t}_1\tilde{t}_2, \tilde{t}_1\underset{\sim}{t}_2, \tilde{t}_1\tilde{t}_2\} < t_1t_2 < \max\{\underset{\sim}{t}_1\underset{\sim}{t}_2, \underset{\sim}{t}_1\tilde{t}_2, \tilde{t}_1\underset{\sim}{t}_2, \tilde{t}_1\tilde{t}_2\}$.

Then, prove the rationality of the membership degree and non membership degree calculated by Eq. (8).

Since $l_{\hat{t}_1}^{\max} \in [0,1]$ and $l_{\hat{t}_2}^{\max} \in [0,1]$, $l_{\hat{t}_1}^{\max}l_{\hat{t}_2}^{\max} \in [0,1]$ can be obtained. Because $f_{\hat{t}_1}^{\min}+f_{\hat{t}_2}^{\min}-f_{\hat{t}_1}^{\min}f_{\hat{t}_2}^{\min} = 1-(1-f_{\hat{t}_1}^{\min})(1-f_{\hat{t}_2}^{\min})$ and $f_{\hat{t}_1}^{\min}, f_{\hat{t}_2}^{\min} \in [0,1]$, $f_{\hat{t}_1}^{\min}+f_{\hat{t}_2}^{\min}-f_{\hat{t}_1}^{\min}f_{\hat{t}_2}^{\min} \in [0,1]$ can be obtained.

Because $l_{\hat{t}_1}^{\max}+f_{\hat{t}_1}^{\min} \in [0,1]$ and $l_{\hat{t}_2}^{\max}+f_{\hat{t}_2}^{\min} \in [0,1]$, $0 \le l_{\hat{t}_1}^{\max} \le 1-f_{\hat{t}_1}^{\min}$ and $0 \le l_{\hat{t}_2}^{\max} \le 1-f_{\hat{t}_2}^{\min}$ can be obtained. Therefore $0 \le f_{\hat{t}_1}^{\min}+f_{\hat{t}_2}^{\min}-f_{\hat{t}_1}^{\min}f_{\hat{t}_2}^{\min}+l_{\hat{t}_1}^{\max}l_{\hat{t}_2}^{\max} \le f_{\hat{t}_1}^{\min}+f_{\hat{t}_2}^{\min}-f_{\hat{t}_1}^{\min}f_{\hat{t}_2}^{\min}+(1-f_{\hat{t}_1}^{\min})(1-f_{\hat{t}_2}^{\min}) = 1$ can be obtained.

To sum up, the result calculated by Eq. (8) is still a TIFN.

(v) Proved that the result calculated by Eq. (9) is a TIFN.

Since $\hat{t} =<(\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}>$ is a TIFN, $\underset{\sim}{t} \le t \le \tilde{t}$ and $l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}, l_{\hat{t}}^{\max}+f_{\hat{t}}^{\min} \in [0,1]$ can be obtained. Therefore, when $\hat{t}>0$ (i.e., $0<\underset{\sim}{t} \le t \le \tilde{t}$) or $\hat{t}<0$ (i.e., $\underset{\sim}{t} \le t \le \tilde{t} < 0$), there is $1/\tilde{t} \le 1/t \le 1/\underset{\sim}{t}$. When $\hat{t}_1$ and $\hat{t}_2$ belong to other situations, there is $\min\{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\} <| \{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\} < \max\{1/\tilde{t}, 1/\tilde{t}, 1/\underset{\sim}{t}\}$.

To sum up, the result calculated by Eq. (9) is a TIFN.

(vi) Prove that the result calculated by Eq. (10) is a TIFN.

Since $\hat{t}_1 =<(\underset{\sim}{t}_1, t_1, \tilde{t}_1); l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}>$ and $\hat{t}_2 =<(\underset{\sim}{t}_2, t_2, \tilde{t}_2); l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}>$ are TIFNs, $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1$, $\underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$, $l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}, l_{\hat{t}_1}^{\max}+f_{\hat{t}_1}^{\min} \in [0,1]$ and $l_{\hat{t}_2}^{\max}, f_{\hat{t}_2}^{\min}, l_{\hat{t}_2}^{\max}+f_{\hat{t}_2}^{\min} \in [0,1]$ can be obtained.

When $\hat{t}_1 \ge 0, \hat{t}_2 > 0$, we obtain $0 \le \underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1$ and $0 < \underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$; then we have $\underset{\sim}{t}_1/\tilde{t}_2 \le t_1/t_2 \le \tilde{t}_1/\underset{\sim}{t}_2$. When $\hat{t}_1 < 0, \hat{t}_2 > 0$, we obtain $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1 < 0$ and $0 < \underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2$; then we have $\underset{\sim}{t}_1/\underset{\sim}{t}_2 \le t_1/t_2 \le \tilde{t}_1/\tilde{t}_2$. When $\hat{t}_1 < 0, \hat{t}_2 < 0$, we obtain $\underset{\sim}{t}_1 \le t_1 \le \tilde{t}_1 < 0$ and $\underset{\sim}{t}_2 \le t_2 \le \tilde{t}_2 < 0$; then we have $\tilde{t}_1/\underset{\sim}{t}_2 \le t_1/t_2 \le \underset{\sim}{t}_1/\tilde{t}_2$. When $\hat{t}_1$ and $\hat{t}_2$ belong to other situations, we have $\min\{\underset{\sim}{t}_1/\tilde{t}_2, \underset{\sim}{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\tilde{t}_2\} \le t_1/t_2 \le \max\{\underset{\sim}{t}_1/\tilde{t}_2, \underset{\sim}{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\underset{\sim}{t}_2, \tilde{t}_1/\tilde{t}_2\}$.

Since the maximum membership degree and minimum non membership degree calculated by Eq. (10) and Eq. (8) are the same, the proof process is omitted.

To sum up, the result calculated by Eq. (10) is a TIFN.

(vii) Finally, prove that the result calculated by Eq. (11) is a TIFN.

Since $\hat{t} =<(\underset{\sim}{t}, t, \tilde{t}); l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}>$ is a TIFN, $\underset{\sim}{t} \le t \le \tilde{t}$ and $l_{\hat{t}}^{\max}, f_{\hat{t}}^{\min}, l_{\hat{t}}^{\max}+f_{\hat{t}}^{\min} \in [0,1]$ can be obtained.

First, the rationality of the TFN of the calculation result is proved from eight different cases.

① If $\hat{t} \ge 0$ (i.e., $0 \le \underset{\sim}{t} \le t \le \tilde{t}$) and $k>0$, then $\underset{\sim}{t}^k < t^k < \tilde{t}^k$.

② If $\hat{t} < 0$ (i.e., $\underset{\sim}{t} \le t \le \tilde{t} < 0$) and $k>0$, $k=\frac{\text{odd}}{\text{odd}}$, then $\underset{\sim}{t}^k < t^k < \tilde{t}^k$.

③ If $\hat{t} < 0$ (i.e., $\underset{\sim}{t} \le t \le \tilde{t} < 0$) and $k>0$, $k=\frac{\text{even}}{\text{odd}}$, then $\tilde{t}^k < t^k < \underset{\sim}{t}^k$.

④ If $\hat{t}$ is else and $k>0$, then $\min\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\} \le \text{mid}\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\} \le \max\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}$.

⑤ If $\hat{t} \ge 0$ (i.e., $0 \le \underset{\sim}{t} \le t \le \tilde{t}$) and $k<0$, then $\tilde{t}^k < t^k < \underset{\sim}{t}^k$.

⑥ If $\hat{t} < 0$ (i.e., $\underset{\sim}{t} \le t \le \tilde{t} < 0$) and $k < 0$, $k = \frac{\text{odd}}{\text{odd}}$, then $\tilde{t}^k < t^k < \underset{\sim}{t}^k$.

⑦ If $\hat{t} < 0$ (i.e., $\underset{\sim}{t} \le t \le \tilde{t} < 0$) and $k < 0$, $k = \frac{\text{even}}{\text{odd}}$, then $\underset{\sim}{t}^k < t^k < \tilde{t}^k$.

⑧ If $\hat{t}$ is else, and $k < 0$, then $\min\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\} \le \text{mid}\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\} \le \max\{\underset{\sim}{t}^k, t^k, \tilde{t}^k\}$.

Then, the rationality of the membership degree and non membership degree of the calculation results is proved from two different cases.

① When $k > 0$, $0 \le (l_{\hat{t}}^{\max})^k \le 1$ can be obtained because $0 \le l_{\hat{t}}^{\max} \le 1$. Since $0 \le f_{\hat{t}}^{\min} \le 1$, $0 \le 1 - f_{\hat{t}}^{\min} \le 1$ and $0 \le 1 - (1 - f_{\hat{t}}^{\min})^k \le 1$ can be obtained.

Because $l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0, 1]$, $l_{\hat{t}}^{\max} \le 1 - f_{\hat{t}}^{\min}$ and $(l_{\hat{t}}^{\max})^k \le (1 - f_{\hat{t}}^{\min})^k$ can be obtained. Therefore $0 \le (l_{\hat{t}}^{\max})^k + 1 - (1 - f_{\hat{t}}^{\min})^k \le (1 - f_{\hat{t}}^{\min})^k + 1 - (1 - f_{\hat{t}}^{\min})^k = 1$.

② When $k < 0$, $0 < \frac{1}{(l_{\hat{t}}^{\max})^k} < 1$ can be obtained because $0 \le l_{\hat{t}}^{\max} \le 1$. Since $0 \le 1 - f_{\hat{t}}^{\min} \le 1$, $0 \le \frac{1}{(1 - f_{\hat{t}}^{\min})^k} \le 1$ can be obtained. Therefore $0 < 1 - \frac{1}{(1 - f_{\hat{t}}^{\min})^k} < 1$.

Because $l_{\hat{t}}^{\max} + f_{\hat{t}}^{\min} \in [0, 1]$, $l_{\hat{t}}^{\max} \le 1 - f_{\hat{t}}^{\min}$ and $\frac{1}{(l_{\hat{t}}^{\max})^k} \le \frac{1}{(1 - f_{\hat{t}}^{\min})^k}$ can be obtained. Therefore $0 \le \frac{1}{(l_{\hat{t}}^{\max})^k} + 1 - \frac{1}{(1 - f_{\hat{t}}^{\min})^k} \le \frac{1}{(1 - f_{\hat{t}}^{\min})^k} + 1 - \frac{1}{(1 - f_{\hat{t}}^{\min})^k} = 1$.

Therefore, the membership degree and non membership degree of the result calculated by Eq. (11) are reasonable.

To sum up, the result calculated by Eq. (11) is still a TIFN.

PROOF OF THEOREM 2

Theorem 2 is proved by the mathematical induction.

(i) When $n = 1$, $\text{TIFWAA}_W(\hat{t}_1) = w_1\hat{t}_1 =< (w_1\underset{\sim}{t}_1, w_1 t_1, w_1\tilde{t}_1); 1 - (1 - l_{\hat{t}_1}^{\max})^{w_1}, (f_{\hat{t}_1}^{\min})^{w_1} >$ can be obtained according to Eq. (6) due to $w_1 \in [0, 1]$; then, Eq. (14) holds. According to Theorem 1, $\text{TIFWAA}_W(\hat{t}_1)$ is a TIFN.

(ii) When $n = 2$, $w_2\hat{t}_2 =< (w_2\underset{\sim}{t}_2, w_2 t_2, w_2\tilde{t}_2); 1 - (1 - l_{\hat{t}_2}^{\max})^{w_2}, (f_{\hat{t}_2}^{\min})^{w_2} >$ is obtained by Eq. (6) due to $w_2 \in [0, 1]$. Furthermore, according to Eq. (5), we get

$$\begin{aligned}
&\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2) \\
&\quad =< (w_1\underset{\sim}{t}_1 + w_2\underset{\sim}{t}_2, w_1 t_1 + w_2 t_2, w_1\tilde{t}_1 + w_2\tilde{t}_2); \\
&\qquad 1 - (1 - l_{\hat{t}_1}^{\max})^{w_1} + 1 - (1 - l_{\hat{t}_2}^{\max})^{w_2} \\
&\qquad - [1 - (1 - l_{\hat{t}_1}^{\max})^{w_1}][1 - (1 - l_{\hat{t}_2}^{\max})^{w_2}], \\
&\qquad (f_{\hat{t}_1}^{\min})^{w_1}(f_{\hat{t}_2}^{\min})^{w_2} > \\
&\quad =< (\sum_{s=1}^{2} w_s\underset{\sim}{t}_s, \sum_{s=1}^{2} w_s t_s, \sum_{s=1}^{2} w_s\tilde{t}_s); \\
&\qquad 1 - \prod_{s=1}^{2}(1 - l_{\hat{t}_s}^{\max})^{w_s}, \prod_{s=1}^{2}(f_{\hat{t}_s}^{\min})^{w_s} >
\end{aligned}$$

Therefore, Eq. (14) can be established.

Since $l_{\hat{t}_s}^{\max}, f_{\hat{t}_s}^{\min}, l_{\hat{t}_s}^{\max} + f_{\hat{t}_s}^{\min} \in [0, 1]$ and $w_s \in [0, 1]$ $(s = 1, 2)$, we get $0 \le 1 - l_{\hat{t}_s}^{\max} \le 1$ $(s = 1, 2)$ and $0 \le \prod_{s=1}^{2}(f_{\hat{t}_s}^{\min})^{w_s} \le 1$; Hence $0 \le (1 - l_{\hat{t}_s}^{\max})^{w_s} \le 1$ $(s = 1, 2)$. Then, we have $0 \le 1 - \prod_{s=1}^{2}(1 - l_{\hat{t}_s}^{\max})^{w_s} \le 1$. Hence, $0 \le 1 - \prod_{s=1}^{2}(1 - l_{\hat{t}_s}^{\max})^{w_s} + \prod_{s=1}^{2}(f_{\hat{t}_s}^{\min})^{w_s} \le 1 - \prod_{s=1}^{2}(1 - l_{\hat{t}_s}^{\max})^{w_s} + \prod_{s=1}^{2}(1 - l_{\hat{t}_s}^{\max})^{w_s} = 1$. Therefore, $\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2)$ is a TIFN.

(iii) It is assumed that when $n = k$, Eq. (14) holds and $\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k)$ is a TIFN. Hence $\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k) =< (\sum_{s=1}^{k} w_s\underset{\sim}{t}_s, \sum_{s=1}^{k} w_s t_s, \sum_{s=1}^{k} w_s\tilde{t}_s);$
$1 - \prod_{s=1}^{k}(1 - l_{\hat{t}_s}^{\max})^{w_s}, \prod_{s=1}^{k}(f_{\hat{t}_s}^{\min})^{w_s} >$

Because $w_{k+1} \in [0, 1]$, $w_{k+1}\hat{t}_{k+1} =< (w_{k+1}\underset{\sim}{t}_{k+1}, w_{k+1}t_{k+1}, w_{k+1}\tilde{t}_{k+1}); 1 - (1 - l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}, (f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}} >$ is obtained by Eq. (6).

When $n = k + 1$, by Eq. (5), we get

$$\begin{aligned}
&\text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1}) \\
&\quad = \text{TIFWAA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k) + w_{k+1}\hat{t}_{k+1} \\
&\quad =< (\sum_{s=1}^{k} w_s\underset{\sim}{t}_s, \sum_{s=1}^{k} w_s t_s, \sum_{s=1}^{k} w_s\tilde{t}_s); \\
&\qquad 1 - \prod_{s=1}^{k}(1 - l_{\hat{t}_s}^{\max})^{w_s}, \prod_{s=1}^{k}(f_{\hat{t}_s}^{\min})^{w_s} > \\
&\quad + < (w_{k+1}\underset{\sim}{t}_{k+1}, w_{k+1}t_{k+1}, w_{k+1}\tilde{t}_{k+1}); \\
&\quad 1 - (1 - l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}, (f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}} > \\
&\quad =< (\sum_{s=1}^{k} w_s\underset{\sim}{t}_s + w_{k+1}\underset{\sim}{t}_{k+1}, \sum_{s=1}^{k} w_s t_s + w_{k+1}t_{k+1}, \\
&\qquad \sum_{s=1}^{k} w_s\tilde{t}_s + w_{k+1}\tilde{t}_{k+1}); \\
&\qquad 1 - \prod_{s=1}^{k}(1 - l_{\hat{t}_s}^{\max})^{w_s} + 1 - (1 - l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}} \\
&\qquad - [1 - \prod_{s=1}^{k}(1 - l_{\hat{t}_s}^{\max})^{w_s}][1 - (1 - l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}], \\
&\qquad \prod_{s=1}^{k}(f_{\hat{t}_s}^{\min})^{w_s}(f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}} > \\
&\quad =< (\sum_{s=1}^{k+1} w_s\underset{\sim}{t}_s, \sum_{s=1}^{k+1} w_s t_s, \sum_{s=1}^{k+1} w_s\tilde{t}_s); \\
&\qquad 1 - \prod_{s=1}^{k+1}(1 - l_{\hat{t}_s}^{\max})^{w_s}, \prod_{s=1}^{k+1}(f_{\hat{t}_s}^{\min})^{w_s} >
\end{aligned}$$

Therefore, Eq. (14) holds.

Furthermore, Since $0 \le \prod_{s=1}^{k}(1-l_{\hat{t}_s}^{\max})^{w_s} \le 1$, $0 \le \prod_{s=1}^{k}(f_{\hat{t}_s}^{\min})^{w_s} \le 1$, $0 \le (1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}} \le 1$, and $0 \le (f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}} \le 1$, we have $0 \le 1-\prod_{s=1}^{k+1}(1-l_{\hat{t}_s}^{\max})^{w_s} \le 1$ and $0 \le \prod_{s=1}^{k+1}(f_{\hat{t}_s}^{\min})^{w_s} \le 1$. Hence, $0 \le 1-\prod_{s=1}^{k+1}(1-l_{\hat{t}_s}^{\max})^{w_s} + \prod_{s=1}^{k+1}(f_{\hat{t}_s}^{\min})^{w_s} \le 1-[\prod_{s=1}^{k+1}(1-l_{\hat{t}_s}^{\max})^{w_s}]+[\prod_{s=1}^{k+1}(1-l_{\hat{t}_s}^{\max})^{w_s}] = 1$. Therefore, $\text{TIFWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a TIFN.

To sum up, Eq. (14) holds and the calculation result is a TIFN.

PROOF OF THEOREM 3

Theorem 3 is proved by the mathematical induction.

(i) When $n = 1$, $\text{TIFWAA}_W(\hat{t}_1) = w_1\hat{t}_1 =< (w_1\tilde{t}_1, w_1t_1, w_1\underset{\sim}{t}_1); 1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}, \frac{1}{(f_{\hat{t}_1}^{\min})^{w_1}} >$ can be obtained by Eq. (6) due to $w_1 \in [0,1]$; therefore, Eq. (16) holds.

Since $l_{\hat{t}_1}^{\max}, f_{\hat{t}_1}^{\min}, l_{\hat{t}_1}^{\max}+f_{\hat{t}_1}^{\min} \in [0,1]$ and $w_1 < 0$, we have $0 \le 1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}} \le 1$ and $0 \le \frac{1}{(f_{\hat{t}_1}^{\min})^{w_1}} \le 1$. Then, we have $0 \le 1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}+\frac{1}{(f_{\hat{t}_1}^{\min})^{w_1}} \le 1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}+\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}} = 1$. Therefore, $\text{TIFWAA}_W(\hat{t}_1)$ is a TIFN.

(ii) When $n = 2$, $w_2\hat{t}_2 =< (w_2\tilde{t}_2, w_2t_2, w_2\underset{\sim}{t}_2); 1-\frac{1}{(1-l_{\hat{t}_2}^{\max})^{w_2}}, \frac{1}{(f_{\hat{t}_2}^{\min})^{w_2}} >$ can be obtained by Eq. (6) due to $w_2 < 0$. Then, we also conclude that $w_2\hat{t}_2$ is a TIFN. Furthermore, according to Eq. (5), we get

$$\begin{aligned}
&\text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2)\\
&\quad = < (w_1\tilde{t}_1+w_2\tilde{t}_2, w_1t_1+w_2t_2, w_1\underset{\sim}{t}_1+w_2\underset{\sim}{t}_2);\\
&\qquad 1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}+1-\frac{1}{(1-l_{\hat{t}_2}^{\max})^{w_2}}\\
&\qquad -[1-\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}][1-\frac{1}{(1-l_{\hat{t}_2}^{\max})^{w_2}}],\\
&\qquad \frac{1}{(f_{\hat{t}_1}^{\min})^{w_1}}\cdot\frac{1}{(f_{\hat{t}_2}^{\min})^{w_2}} >\\
&\quad = < (\sum_{s=1}^{2}w_s\tilde{t}_s, \sum_{s=1}^{2}w_st_s, \sum_{s=1}^{2}w_s\underset{\sim}{t}_s);\\
&\qquad 1-\prod_{s=1}^{2}\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_s}}, \prod_{s=1}^{2}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = 2$, Eq. (16) holds.

Since $l_{\hat{t}_s}^{\max}, f_{\hat{t}_s}^{\min}, l_{\hat{t}_s}^{\max}+f_{\hat{t}_s}^{\min} \in [0,1]$ and $w_s \in [0,1]$ $(s=1,2)$, we get $0 \le 1-l_{\hat{t}_s}^{\max} \le 1$ $(s=1,2)$ and $0 \le \prod_{s=1}^{2}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} \le 1$; hence $0 \le \frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}} \le 1$ $(s=1,2)$. Then, we have $0 \le 1-\prod_{s=1}^{2}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}} \le 1$. Hence,

$$\begin{aligned}
0 &\le 1-\prod_{s=1}^{2}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}+\prod_{s=1}^{2}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}}\\
&\le 1-[\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}\frac{1}{(1-l_{\hat{t}_2}^{\max})^{w_2}}]\\
&\quad +[\frac{1}{(1-l_{\hat{t}_1}^{\max})^{w_1}}\frac{1}{(1-l_{\hat{t}_2}^{\max})^{w_2}}] = 1.
\end{aligned}$$

Therefore $\text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2)$ is a TIFN.

(iii) It is assumed that when $n = k$, Eq. (16) holds and $\text{TIFWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)$ is a TIFN. Hence

$$\begin{aligned}
&\text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)\\
&\quad = < (\sum_{s=1}^{k}w_s\tilde{t}_s, \sum_{s=1}^{k}w_st_s, \sum_{s=1}^{k}w_s\underset{\sim}{t}_s);\\
&\qquad 1-\prod_{s=1}^{k}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}, \prod_{s=1}^{k}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Because $w_{k+1} < 0$, $w_{k+1}\hat{t}_{k+1} =< (w_{k+1}\tilde{t}_{k+1}, w_{k+1}t_{k+1}, w_{k+1}\underset{\sim}{t}_{k+1}); 1-\frac{1}{(1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}, \frac{1}{(f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} >$ can be obtained by Eq. (6). Then, we also conclude that $w_{k+1}\hat{t}_{k+1}$ is a TIFN.

When $n = k+1$, by Eq. (5), we get

$$\begin{aligned}
&\text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})\\
&= \text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)+w_{k+1}\hat{t}_{k+1}\\
&= < (\sum_{s=1}^{k}w_s\tilde{t}_s, \sum_{s=1}^{k}w_st_s, \sum_{s=1}^{k}w_s\underset{\sim}{t}_s);\\
&\qquad 1-\prod_{s=1}^{k}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}, \prod_{s=1}^{k}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} >\\
&\quad + < (w_{k+1}\tilde{t}_{k+1}, w_{k+1}t_{k+1}, w_{k+1}\underset{\sim}{t}_{k+1});\\
&\qquad 1-\frac{1}{(1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}, \frac{1}{(f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} >\\
&= < (\sum_{s=1}^{k}w_s\tilde{t}_s+w_{k+1}\tilde{t}_{k+1}, \sum_{s=1}^{k}w_st_s+w_{k+1}t_{k+1},\\
&\qquad \sum_{s=1}^{k}w_s\underset{\sim}{t}_s+w_{k+1}\underset{\sim}{t}_{k+1});\\
&\qquad 1-\prod_{s=1}^{k}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}+1-\frac{1}{(1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}\\
&\qquad -[1-\prod_{s=1}^{k}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}][1-\frac{1}{(1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}],\\
&\qquad \prod_{s=1}^{k}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}}\cdot\frac{1}{(f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} >\\
&= < (\sum_{s=1}^{k+1}w_s\tilde{t}_s, \sum_{s=1}^{k+1}w_st_s, \sum_{s=1}^{k+1}w_s\underset{\sim}{t}_s); 1-\prod_{s=1}^{k+1}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}},
\end{aligned}$$

$$\prod_{s=1}^{k+1}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}}>$$

Therefore, Eq. (16) holds.

Furthermore, since $0 \leq \prod_{s=1}^{k}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}} \leq 1$, $0 \leq \prod_{s=1}^{k}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} \leq 1$, $0 \leq \frac{1}{(1-l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}} \leq 1$ and $0 \leq \frac{1}{(f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} \leq 1$, we have $0 \leq 1-\prod_{s=1}^{k+1}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}} \leq 1$ and $0 \leq \prod_{s=1}^{k+1}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} \leq 1$. Hence, $0 \leq 1-\prod_{s=1}^{k+1}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}+\prod_{s=1}^{k+1}\frac{1}{(f_{\hat{t}_s}^{\min})^{w_s}} \leq 1-[\prod_{s=1}^{k+1}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}]+[\prod_{s=1}^{k+1}\frac{1}{(1-l_{\hat{t}_s}^{\max})^{w_s}}]=1$. Therefore, $\text{TIFEWAA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a TIFN.

To sum up, Eq. (16) holds and the calculation result is a TIFN.

PROOF OF THEOREM 5

Theorem 5 is proved by the mathematical induction.

(i) When $n=1$, because $\hat{t}_1$ is a positive TIFN and $w_1 \in [0,1]$, $\text{TIFWGA}_W(\hat{t}_1)=\hat{t}_1^{w_1}=<(\underset{\sim}{t}_1^{w_1},t_1^{w_1},\tilde{t}_1^{w_1});(l_{\hat{t}_1}^{\max})^{w_1},1-(1-f_{\hat{t}_1}^{\min})^{w_1}>$ can be obtained by Eq. (11). Hence Eq. (20) holds. According to Theorem 1, $\text{TIFWGA}_W(\hat{t}_1)$ is a TIFN. Because of $\underset{\sim}{t}_1^{w_1} \geq 0$, $\text{TIFWGA}_W(\hat{t}_1)$ is a positive TIFN.

(ii) When $n=2$, because $\hat{t}_2$ is a positive TIFN, and $w_2 \in [0,1]$, $\hat{t}_2^{w_2}=<(\underset{\sim}{t}_2^{w_2},t_2^{w_2},\tilde{t}_2^{w_2});(l_{\hat{t}_2}^{\max})^{w_2},1-(1-f_{\hat{t}_2}^{\min})^{w_2}>$ can be obtained by Eq. (11). Then, we also conclude that $\hat{t}_2^{w_2}$ is a positive TIFN. Furthermore, according to Eq. (8), we get

$$\begin{aligned}
&\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2)\\
&=<(\underset{\sim}{t}_1^{w_1}\underset{\sim}{t}_2^{w_2},t_1^{w_1}t_2^{w_2},\tilde{t}_1^{w_1}\tilde{t}_2^{w_2});\\
&\quad(l_{\hat{t}_1}^{\max})^{w_1}(l_{\hat{t}_2}^{\max})^{w_2},1-(1-f_{\hat{t}_1}^{\min})^{w_1}\\
&\quad+1-(1-f_{\hat{t}_2}^{\min})^{w_2}\\
&\quad-[1-(1-f_{\hat{t}_1}^{\min})^{w_1}][1-(1-f_{\hat{t}_2}^{\min})^{w_2}]>\\
&=<(\prod_{s=1}^{2}\underset{\sim}{t}_s^{w_s},\prod_{s=1}^{2}t_s^{w_s},\prod_{s=1}^{2}\tilde{t}_s^{w_s});\\
&\quad\prod_{s=1}^{2}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{2}(1-f_{\hat{t}_s}^{\min})^{w_s}>
\end{aligned}$$

Therefore, when $n=2$, Eq. (20) holds. According to Theorem 1, $\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2)$ is a TIFN; Because of $\prod_{s=1}^{2}\underset{\sim}{t}_s^{w_s} \geq 0$, $\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2)$ is a positive TIFN.

(iii) It is assumed that when $n=k$, Eq. (20) holds and $\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)$ is a positive TIFN. Hence,

$$\begin{aligned}
&\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)\\
&=<(\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{k}(t_s)^{w_s},\prod_{s=1}^{k}(\tilde{t}_s)^{w_s});\\
&\quad\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}>
\end{aligned}$$

Since $\hat{t}_{k+1} \geq 0$ and $w_{k+1} \in [0,1]$, $(\hat{t}_{k+1})^{w_{k+1}}=<(\underset{\sim}{t}_{k+1}^{w_{k+1}},t_{k+1}^{w_{k+1}},\tilde{t}_{k+1}^{w_{k+1}});(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}},1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}>$ can be obtained by Eq. (11). Then, we also conclude that $(\hat{t}_{k+1})^{w_{k+1}}$ is a positive TIFN. Therefore, when $n=k+1$, according to Eq. (8), we get

$$\begin{aligned}
&\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})\\
&=<(\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{k}(t_s)^{w_s},\prod_{s=1}^{k}(\tilde{t}_s)^{w_s});\\
&\quad\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\
&\quad+<(\underset{\sim}{t}_{k+1}^{w_{k+1}},t_{k+1}^{w_{k+1}},\tilde{t}_{k+1}^{w_{k+1}});(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}},1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}>\\
&=<(\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s}\cdot\underset{\sim}{t}_{k+1}^{w_{k+1}},\prod_{s=1}^{k}(t_s)^{w_s}\cdot t_{k+1}^{w_{k+1}},\prod_{s=1}^{k}(\tilde{t}_s)^{w_s}\cdot\tilde{t}_{k+1}^{w_{k+1}});\\
&\quad\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}},\\
&\quad1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}+1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}\\
&\quad-[1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}][1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}]>\\
&=<(\prod_{s=1}^{k+1}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{k+1}(t_s)^{w_s},\prod_{s=1}^{k+1}(\tilde{t}_s)^{w_s});\\
&\quad\prod_{s=1}^{k+1}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{k+1}(1-f_{\hat{t}_s}^{\min})^{w_s}>
\end{aligned}$$

Therefore, when $n=k+1$, Eq.(20) holds. According to Theorem 1, $\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a TIFN. Also because of $\prod_{s=1}^{k+1}(\underset{\sim}{t}_s)^{w_s} \geq 0$, $\text{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a positive TIFN.

To sum up, Eq. (20) is established and the result is a positive TIFN.

PROOF OF THEOREM 6

Theorem 6 is proved by mathematical induction.

(i) When $n=1$, $\hat{t}_1$ is a positive TIFN, and $w_1 \in (-\infty,0)$, $\text{TIFEWGA}_W(\hat{t}_1)=<(\tilde{t}_1^{w_1},t_1^{w_1},\underset{\sim}{t}_1^{w_1});\frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}},1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}}>$ can be obtained by Eq. (11); Therefore, Eq. (22) holds. According to Theorem 1, $\text{TIFEWGA}_W(\hat{t}_1)$ is a TIFN. Because of $\tilde{t}_1^{w_1} \geq 0$, $\text{TIFWGA}_W(\hat{t}_1)$ is a positive TIFN.

(ii) When $n=2$, because $\hat{t}_2$ is a positive TIFN, and $w_2 \in (-\infty,0)$, $\hat{t}_2^{w_2}=<(\tilde{t}_2^{w_2},t_2^{w_2},\underset{\sim}{t}_2^{w_2});\frac{1}{(l_{\hat{t}_2}^{\max})^{w_2}},1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}>$ can be obtained according to Eq. (11). Then, we also conclude that $\hat{t}_2^{w_2}$ is a positive TIFN. Furthermore, according to Eq. (8), we get

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)\\&=<(\tilde{t}_1^{w_1}\tilde{t}_2^{w_2},t_1^{w_1}t_2^{w_2},\underset{\sim}{t}_1^{w_1}\underset{\sim}{t}_2^{w_2});\frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}(l_{\hat{t}_2}^{\max})^{w_2}},\\&\quad 1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}}+1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}\\&\quad-[1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}}][1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}]>\\&=<(\prod_{s=1}^{2}\tilde{t}_s^{w_s},\prod_{s=1}^{2}t_s^{w_s},\prod_{s=1}^{2}\underset{\sim}{t}_s^{w_s});\frac{1}{\prod_{s=1}^{2}(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\frac{1}{\prod_{s=1}^{2}(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

Therefore, when $n=2$, Eq. (22) holds. According to Theorem 1, $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$ is a TIFN. Because of $\prod_{s=1}^{2}\tilde{t}_s^{w_s}\geq 0$, $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$ is a positive TIFN.

(iii) It is assumed that when $n=k$, Eq. (22) holds and $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)$ is a positive TIFN, Hence

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_k)\\&=<(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s},\prod_{s=1}^{k}(t_s)^{w_s},\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s});\\&\quad\frac{1}{\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}},1-\frac{1}{\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

According to Eq. (11), $\hat{t}_{k+1}\geq 0$, $w_{k+1}<0$ and $(\hat{t}_{k+1})^{w_{k+1}}=<(\tilde{t}_{k+1}^{w_{k+1}},t_{k+1}^{w_{k+1}},\underset{\sim}{t}_{k+1}^{w_{k+1}});\frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}},1-\frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}}>$ can be obtained. Then, we also conclude that $(\hat{t}_{k+1})^{w_{k+1}}$ is a positive TIFN. Therefore, when $n=k+1$, we get

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})\\&=<(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s},\prod_{s=1}^{k}(t_s)^{w_s},\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s});\frac{1}{\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\frac{1}{\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}}>\\&\quad+<(\tilde{t}_{k+1}^{w_{k+1}},t_{k+1}^{w_{k+1}},\underset{\sim}{t}_{k+1}^{w_{k+1}});\frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}},\\&\quad 1-\frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}}>\\&=<(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s}\tilde{t}_{k+1}^{w_{k+1}},\prod_{s=1}^{k}(t_s)^{w_s}t_{k+1}^{w_{k+1}},\prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s}\underset{\sim}{t}_{k+1}^{w_{k+1}});\\&\quad\frac{1}{\prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}}\frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}},\\&\quad 1-\frac{1}{\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}}+1-\frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}}\\&\quad-[1-\frac{1}{\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}}][1-\frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}}]>\\&=<(\prod_{s=1}^{k+1}(\tilde{t}_s)^{w_s},\prod_{s=1}^{k+1}(t_s)^{w_s},\prod_{s=1}^{k+1}(\underset{\sim}{t}_s)^{w_s});\\&\quad\frac{1}{\prod_{s=1}^{k+1}(l_{\hat{t}_s}^{\max})^{w_s}},1-\frac{1}{\prod_{s=1}^{k+1}(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

Therefore, when $n=k+1$, Eq. (22) holds. According to Theorem 1, $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a TIFN. Because $\prod_{s=1}^{k+1}(\tilde{t}_s)^{w_s}\geq 0$, $\mathrm{TIFWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{k+1})$ is a positive TIFN.

To sum up, Eq. (22) holds and the result is a positive TIFN.

PROOF OF THEOREM 7

Let $W_1=(w_1,w_2,\ldots,w_g)$, $W_2=(w_{g+1},w_{g+2},\ldots,w_n)$, $T_1=\mathrm{TIFEWGA}_{W_1}(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_g)$ and $T_2=\mathrm{TIFEWGA}_{W_2}(\hat{t}_{g+1},\hat{t}_{g+2},\ldots,\hat{t}_n)$, then according to Theorem 5 and Theorem 6, we get

$$\begin{aligned}T_1&=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\T_2&=<(\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=g+1}^{n}(t_s)^{w_s},\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},1-\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

Furthermore, according to Eq. (8), we get

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)=T_1T_2\\&=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}+1-\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}\\&\quad-[1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}][1-\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}]>\end{aligned}$$

$$= <(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});$$
$$\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},$$
$$1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\cdot\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>$$

To sum up, Eq. (23) is established and the result is a TIFN.

PROOF OF THEOREM 8

Theorem 8 is proved by the mathematical induction.

First, prove Case (I).

(i) When $n = 1$, due to $\hat{t}_1 < 0$ and $w_1 = \frac{\text{odd}}{\text{odd}} > 0$, $\text{TIFEWGA}_W(\hat{t}_1) = \hat{t}_1^{w_1} =< (\underset{\sim}{t}_1^{w_1}, t_1^{w_1}, \tilde{t}_1^{w_1}); (l_{\hat{t}_1}^{\max})^{w_1}, 1-(1-f_{\hat{t}_1}^{\min})^{w_1} >$ can be obtained by Eq. (11); therefore, Eq. (24) holds. According to Theorem 1, $\text{TIFWAA}_W(\hat{t}_1)$ is a TIFN. Because of $\tilde{t}_1^{w_1} < 0$, $\text{TIFEWGA}_w(\hat{t}_1)$ is a negative TIFN.

(ii) When $n = 2$, due to $\hat{t}_2 < 0$ and $w_2 = \frac{\text{odd}}{\text{odd}} > 0$, $\hat{t}_2^{w_2} =< (\tilde{t}_2^{w_2}, t_2^{w_2}, \underset{\sim}{t}_2^{w_2}); (l_{\hat{t}_2}^{\max})^{w_2}, 1-(1-f_{\hat{t}_2}^{\min})^{w_2} >$ can be obtained according to Eq. (11); thereinto $\tilde{t}_2^{w_2} < 0$. Therefore, according to Eq. (8), Eq. (24) holds. According to Theorem 1, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$ is a TIFN; Because of $\prod_{s=1}^{2}\tilde{t}_s^{w_s} > 0$, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$ is a positive TIFN.

(iii) When $n = 3$, due to $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2) > 0$ and $\hat{t}_3^{w_3} < 0$, according to Eq. (8), we get

$$\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\hat{t}_3)$$
$$= \hat{t}_3^{w_3}\cdot\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$$
$$= <(\underset{\sim}{t}_3^{w_3}\prod_{s=1}^{2}\underset{\sim}{t}_s^{w_s}, t_3^{w_3}\prod_{s=1}^{2}t_s^{w_s}, \tilde{t}_3^{w_3}\prod_{s=1}^{2}\tilde{t}_s^{w_s});$$
$$(l_{\hat{t}_3}^{\max})^{w_3}\prod_{s=1}^{2}(l_{\hat{t}_s}^{\max})^{w_s},$$
$$1-(1-f_{\hat{t}_3}^{\min})^{w_3}+1-\prod_{s=1}^{2}(1-f_{\hat{t}_s}^{\min})^{w_s}$$
$$-[1-(1-f_{\hat{t}_3}^{\min})^{w_3}][1-\prod_{s=1}^{2}(1-f_{\hat{t}_s}^{\min})^{w_s}]$$
$$= <(\prod_{s=1}^{3}\underset{\sim}{t}_s^{w_s},\prod_{s=1}^{3}t_s^{w_s},\prod_{s=1}^{3}\tilde{t}_s^{w_s});$$
$$\prod_{s=1}^{3}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{3}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

Therefore, when $n = 3$, Eq. (24) holds. Hence, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\hat{t}_3)$ is a TIFN; Because of $\prod_{s=1}^{3}\tilde{t}_s^{w_s} < 0$, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2)$ is a negative TIFN.

(iv) It is assumed that when $n = 2k+1$, Eq. (24) holds and $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+1})$ is a negative TIFN. Hence

$$\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+1})$$
$$= <(\prod_{s=1}^{2k+1}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{2k+1}(t_s)^{w_s},\prod_{s=1}^{2k+1}(\tilde{t}_s)^{w_s});$$
$$\prod_{s=1}^{2k+1}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{2k+1}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

Since $\hat{t}_{2k+2} < 0$ and $w_{2k+2} > 0$, according to Eq. (11), we get $(\hat{t}_{2k+2})^{w_{2k+2}} =< (\underset{\sim}{t}_{2k+2}^{w_{2k+2}}, t_{2k+2}^{w_{2k+2}}, \tilde{t}_{2k+2}^{w_{2k+2}}); (l_{\hat{t}_{2k+2}}^{\max})^{w_{2k+2}}, 1-(1-f_{\hat{t}_{2k+2}}^{\min})^{w_{2k+2}} >$ and $(\hat{t}_{2k+2})^{w_{2k+2}} < 0$. Therefore, when $n = 2k+2$, according to Eq. (8), we get

$$\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+2})$$
$$= \text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+1})\cdot(\hat{t}_{2k+2})^{w_{2k+2}}$$
$$= <(\prod_{s=1}^{2k+1}(\tilde{t}_s)^{w_s}\tilde{t}_{2k+2}^{w_{2k+2}},\prod_{s=1}^{2k+1}(t_s)^{w_s}t_{2k+2}^{w_{2k+2}},\prod_{s=1}^{2k+1}(\underset{\sim}{t}_s)^{w_s}\underset{\sim}{t}_{2k+2}^{w_{2k+2}});$$
$$\prod_{s=1}^{2k+1}(l_{\hat{t}_s}^{\max})^{w_s}(l_{\hat{t}_{2k+2}}^{\max})^{w_{2k+2}},$$
$$1-\prod_{s=1}^{2k+1}(1-f_{\hat{t}_s}^{\min})^{w_s}(1-f_{\hat{t}_{2k+2}}^{\min})^{w_{2k+2}}>$$
$$= <(\prod_{s=1}^{2k+2}(\tilde{t}_s)^{w_s},\prod_{s=1}^{2k+2}(t_s)^{w_s},\prod_{s=1}^{2k+2}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{2k+2}(l_{\hat{t}_s}^{\max})^{w_s},$$
$$1-\prod_{s=1}^{2k+2}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

Therefore, when $n = 2k+2$, Eq. (24) holds. Hence, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+2})$ is a TIFN; Because of $\prod_{s=1}^{2k+2}(\tilde{t}_s)^{w_s} > 0$, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+2})$ is a positive TIFN.

(v) According to the above conclusion, when $n = 2k+2$, $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+2})$ is a positive TIFN. Since $\hat{t}_{2k+3} < 0$, $w_{2k+3} > 0$ and $(\hat{t}_{2k+3})^{w_{2k+3}} < 0$, $(\hat{t}_{2k+3})^{w_{2k+3}} =< (\underset{\sim}{t}_{2k+3}^{w_{2k+3}}, t_{2k+3}^{w_{2k+3}}, \tilde{t}_{2k+3}^{w_{2k+3}}); (l_{\hat{t}_{2k+3}}^{\max})^{w_{2k+3}}, 1-(1-f_{\hat{t}_{2k+3}}^{\min})^{w_{2k+3}} >$ can be obtained by Eq. (11). Therefore, when $n = 2k+3$, according to Eq. (8), we get

$$\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+3})$$
$$= (\hat{t}_{2k+3})^{w_{2k+3}}\cdot\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_{2k+2})$$
$$= <(\underset{\sim}{t}_{2k+3}^{w_{2k+3}}\prod_{s=1}^{2k+2}(\underset{\sim}{t}_s)^{w_s}, t_{2k+3}^{w_{2k+3}}\prod_{s=1}^{2k+2}(t_s)^{w_s}, \tilde{t}_{2k+3}^{w_{2k+3}}\prod_{s=1}^{2k+2}(\tilde{t}_s)^{w_s});$$
$$(l_{\hat{t}_{2k+3}}^{\max})^{w_{2k+3}}\prod_{s=1}^{2k+2}(l_{\hat{t}_s}^{\max})^{w_s},$$
$$1-(1-f_{\hat{t}_{2k+3}}^{\min})^{w_{2k+3}}\prod_{s=1}^{2k+2}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

$$= <(\prod_{s=1}^{2k+3}(\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{2k+3}(t_s)^{w_s}, \prod_{s=1}^{2k+3}(\tilde{t}_s)^{w_s}); \prod_{s=1}^{2k+3}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{2k+3}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

Therefore, when $n = 2k+3$, Eq. (24) holds. Hence, $TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+3})$ is a TIFN. Because of $\prod_{s=1}^{2k+3}(\tilde{t}_s)^{w_s} < 0$, $TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+3})$ is a negative TIFN.

To sum up, Eq. (24) holds for all n and the calculation result is a TIFN.

Second, prove Case (II).

(i) When $n = 1$, due to $\hat{t}_1 < 0$ and $w_1 = \frac{\text{even}}{\text{odd}} > 0$, $TIFEWGA_W(\hat{t}_1) = \hat{t}_1^{w_1} =< (\tilde{t}_1^{w_1}, t_1^{w_1}, \underset{\sim}{t}_1^{w_1}); (l_{\hat{t}_1}^{\max})^{w_1}, 1-(1-f_{\hat{t}_1}^{\min})^{w_1} >$ can be obtained by Eq. (11). Therefore, Eq. (25) holds. According to Theorem 1, $TIFWAA_W(\hat{t}_1)$ is a TIFN; Because of $\tilde{t}_1^{w_1} > 0$, $TIFEWGA_W(\hat{t}_1)$ is a positive TIFN.

(ii) When $n = 2$, due to $\hat{t}_2 < 0$ and $w_2 = \frac{\text{even}}{\text{odd}} > 0$, $\hat{t}_2^{w_2} =< (\tilde{t}_2^{w_2}, t_2^{w_2}, \underset{\sim}{t}_2^{w_2}); (l_{\hat{t}_2}^{\max})^{w_2}, 1-(1-f_{\hat{t}_2}^{\min})^{w_2} >$ can be obtained by Eq. (11); Because of $\tilde{t}_2^{w_2} > 0$, both $\hat{t}_1^{w_1}$ and $\hat{t}_2^{w_2}$ are positive TIFNs. Furthermore, according to Eq. (8), we get

$$\begin{aligned}
&TIFEWGA_W(\hat{t}_1, \hat{t}_2)\\
&\quad = <(\tilde{t}_1^{w_1}\tilde{t}_2^{w_2}, t_1^{w_1}t_2^{w_2}, \underset{\sim}{t}_1^{w_1}\underset{\sim}{t}_2^{w_2}); (l_{\hat{t}_1}^{\max})^{w_1}(l_{\hat{t}_2}^{\max})^{w_2},\\
&\qquad 1-(1-f_{\hat{t}_1}^{\min})^{w_1} + 1-(1-f_{\hat{t}_2}^{\min})^{w_2}\\
&\qquad -[1-(1-f_{\hat{t}_1}^{\min})^{w_1}][1-(1-f_{\hat{t}_2}^{\min})^{w_2}]>\\
&\quad = <(\prod_{s=1}^{2}\tilde{t}_s^{w_s}, \prod_{s=1}^{2}t_s^{w_s}, \prod_{s=1}^{2}\underset{\sim}{t}_s^{w_s});\\
&\qquad \prod_{s=1}^{2}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{2}(1-f_{\hat{t}_s}^{\min})^{w_s}>
\end{aligned}$$

Therefore, when $n = 2$, Eq. (25) holds. According to Theorem 1, $TIFEWGA_W(\hat{t}_1, \hat{t}_2)$ is a positive TIFN; Because of $\prod_{s=1}^{2}\tilde{t}_s^{w_s} > 0$, $TIFEWGA_W(\hat{t}_1, \hat{t}_2)$ is a positive TIFN.

(iii) It is assumed that when $n = k$, Eq. (25) holds and $TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k)$ is a positive TIFN. Hence

$$\begin{aligned}
&TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k)\\
&\quad = <(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s}, \prod_{s=1}^{k}(t_s)^{w_s}, \prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s});\\
&\qquad \prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}>.
\end{aligned}$$

Since $\hat{t}_{k+1} < 0$ and $w_{k+1} = \frac{\text{even}}{\text{odd}} > 0$, $(\hat{t}_{k+1})^{w_{k+1}} =< (\tilde{t}_{k+1}^{w_{k+1}}, t_{k+1}^{w_{k+1}}, \underset{\sim}{t}_{k+1}^{w_{k+1}}); (l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}, 1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}} >$ can be obtained by Eq. (11); Because of $\tilde{t}_{k+1}^{w_{k+1}} > 0$, $(\hat{t}_{k+1})^{w_{k+1}}$ is a positive TIFN. Therefore, when $n = k+1$, according to Eq. (8), we get

$$\begin{aligned}
&TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1})\\
&= <(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s}, \prod_{s=1}^{k}(t_s)^{w_s}, \prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s});\\
&\quad \prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\
&\quad + <(\tilde{t}_{k+1}^{w_{k+1}}, t_{k+1}^{w_{k+1}}, \underset{\sim}{t}_{k+1}^{w_{k+1}});\\
&\quad (l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}, 1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}>\\
&= <(\prod_{s=1}^{k}(\tilde{t}_s)^{w_s}\cdot\tilde{t}_{k+1}^{w_{k+1}}, \prod_{s=1}^{k}(t_s)^{w_s}\cdot t_{k+1}^{w_{k+1}}, \prod_{s=1}^{k}(\underset{\sim}{t}_s)^{w_s}\cdot\underset{\sim}{t}_{k+1}^{w_{k+1}});\\
&\quad \prod_{s=1}^{k}(l_{\hat{t}_s}^{\max})^{w_s}(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}},\\
&\quad 1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s} + 1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}\\
&\quad -[1-\prod_{s=1}^{k}(1-f_{\hat{t}_s}^{\min})^{w_s}][1-(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}]>\\
&= <(\prod_{s=1}^{k+1}(\tilde{t}_s)^{w_s}, \prod_{s=1}^{k+1}(t_s)^{w_s}, \prod_{s=1}^{k+1}(\underset{\sim}{t}_s)^{w_s}); \prod_{s=1}^{k+1}(l_{\hat{t}_s}^{\max})^{w_s},\\
&\quad 1-\prod_{s=1}^{k+1}(1-f_{\hat{t}_s}^{\min})^{w_s}>
\end{aligned}$$

Therefore, when $n = k+1$, Eq. (25) holds. According to Theorem 1, $TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1})$ is a TIFN; Because of $\prod_{s=1}^{k+1}(\tilde{t}_s)^{w_s} > 0$, $TIFEWGA_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1})$ is a positive TIFN.

To sum up, Eq. (25) holds for all n and the calculation result is a positive TIFN. □

Third, prove Case (III).

(i) When $n = 1$, due to$\hat{t}_1 < 0$ and $w_1 = \frac{\text{odd}}{\text{odd}} < 0$, $TIFEWGA_W(\hat{t}_1) = \hat{t}_1^{w_1} =< (\tilde{t}_1^{w_1}, t_1^{w_1}, \underset{\sim}{t}_1^{w_1}); \frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}}, 1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}} >$ can be obtained by Eq. (11); hence, Eq.(26) holds. According to Theorem 1, $TIFWAA_W(\hat{t}_1)$ is a TIFN; Because of $\underset{\sim}{t}_1^{w_1} < 0$, $TIFEWGA_W(\hat{t}_1)$ is a negative TIFN.

(ii) When $n = 2$, due to $\hat{t}_2 < 0$ and $w_2 = \frac{\text{odd}}{\text{odd}} < 0$, $\hat{t}_2^{w_2} =< (\tilde{t}_2^{w_2}, t_2^{w_2}, \underset{\sim}{t}_2^{w_2}); \frac{1}{(l_{\hat{t}_2}^{\max})^{w_2}}, 1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}} >$ can be obtained by Eq. (11). Because of $\underset{\sim}{t}_2^{w_2} < 0$, both $\hat{t}_1^{w_1}$ and $\hat{t}_2^{w_2}$ are negative TIFN. Furthermore, according to Eq. (8), we get

$$\begin{aligned}
&TIFEWGA_W(\hat{t}_1, \hat{t}_2)\\
&\quad = <(\underset{\sim}{t}_1^{w_1}\underset{\sim}{t}_2^{w_2}, t_1^{w_1}t_2^{w_2}, \tilde{t}_1^{w_1}\tilde{t}_2^{w_2}); \frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}(l_{\hat{t}_2}^{\max})^{w_2}},\\
&\qquad 1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}} + 1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}\\
&\qquad -[1-\frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}}][1-\frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}]>
\end{aligned}$$

$$= <(\prod_{s=1}^{2} \underset{\sim}{t}_s^{w_s}, \prod_{s=1}^{2} t_s^{w_s}, \prod_{s=1}^{2} \tilde{t}_s^{w_s}); \frac{1}{\prod_{s=1}^{2} (l_{\hat{t}_s}^{\max})^{w_s}}, 1 - \frac{1}{\prod_{s=1}^{2} (1-f_{\hat{t}_s}^{\min})^{w_s}} >$$

Therefore, when $n = 2$, Eq. (26) holds. According to Theorem 1, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2)$ is a TIFN; Because of $\prod_{s=1}^{2} \underset{\sim}{t}_s^{w_s} > 0$, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2)$ is a positive TIFN.

(iii) When $n = 3$, due to $\hat{t}_3 < 0$ and $w_3 = \frac{\text{odd}}{\text{odd}} < 0$, $\hat{t}_3^{w_3} =< (\tilde{t}_3^{w_3}, t_3^{w_3}, \underset{\sim}{t}_3^{w_3}); \frac{1}{(l_{\hat{t}_3}^{\max})^{w_3}}, 1 - \frac{1}{(1-f_{\hat{t}_3}^{\min})^{w_3}} >$ and $\hat{t}_3^{w_3} < 0$ can be obtained by Eq. (11). Because of $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2) > 0$, according to Eq. (8), we get

$$\begin{aligned}
&\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \hat{t}_3) \\
&= \hat{t}_3^{w_3} \cdot \mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2) \\
&= <(\tilde{t}_3^{w_3} \prod_{s=1}^{2} \tilde{t}_s^{w_s}, t_3^{w_3} \prod_{s=1}^{2} t_s^{w_s}, \underset{\sim}{t}_3^{w_3} \prod_{s=1}^{2} \underset{\sim}{t}_s^{w_s}); \\
&\quad \frac{1}{(l_{\hat{t}_3}^{\max})^{w_3}} \frac{1}{\prod_{s=1}^{2} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{(1-f_{\hat{t}_3}^{\min})^{w_3}} + 1 - \frac{1}{\prod_{s=1}^{2} (1-f_{\hat{t}_s}^{\min})^{w_s}} \\
&\quad - [1 - \frac{1}{(1-f_{\hat{t}_3}^{\min})^{w_3}}][1 - \frac{1}{\prod_{s=1}^{2} (1-f_{\hat{t}_s}^{\min})^{w_s}}] \\
&= <(\prod_{s=1}^{3} \tilde{t}_s^{w_s}, \prod_{s=1}^{3} t_s^{w_s}, \prod_{s=1}^{3} \underset{\sim}{t}_s^{w_s}); \frac{1}{\prod_{s=1}^{3} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{3} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = 3$, Eq. (26) holds. According to Theorem 1, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \hat{t}_3)$ is a TIFN. Because of $\prod_{s=1}^{3} \underset{\sim}{t}_s^{w_s} < 0$, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2)$ is a negative TIFN.

(iv) It is assumed that when $n = 2k+1$, Eq. (26) holds and $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+1})$ is a negative TIFN. Hence

$$\begin{aligned}
&\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+1}) \\
&= <(\prod_{s=1}^{2k+1} (\tilde{t}_s)^{w_s}, \prod_{s=1}^{2k+1} (t_s)^{w_s}, \prod_{s=1}^{2k+1} (\underset{\sim}{t}_s)^{w_s}); \\
&\quad \frac{1}{\prod_{s=1}^{2k+1} (l_{\hat{t}_s}^{\max})^{w_s}}, 1 - \frac{1}{\prod_{s=1}^{2k+1} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Since $\hat{t}_{2k+2} < 0$ and $w_{2k+2} = \frac{\text{odd}}{\text{odd}} < 0$, $(\hat{t}_{2k+2})^{w_{2k+2}} =< (\tilde{t}_{2k+2}^{w_{2k+2}}, t_{2k+2}^{w_{2k+2}}, \underset{\sim}{t}_{2k+2}^{w_{2k+2}}); \frac{1}{(l_{\hat{t}_{2k+2}}^{\max})^{w_{2k+2}}}, 1 - \frac{1}{(1-f_{\hat{t}_{2k+2}}^{\min})^{w_{2k+2}}} >$ and $(\hat{t}_{2k+2})^{w_{2k+2}} < 0$ can be obtained by Eq. (11). Therefore, when $n = 2k+2$, according to Eq. (8), we get

$$\begin{aligned}
&\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+2}) \\
&= \mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+1}) \cdot (\hat{t}_{2k+2})^{w_{2k+2}} \\
&= <(\prod_{s=1}^{2k+1} (\underset{\sim}{t}_s)^{w_s} \underset{\sim}{t}_{2k+2}^{w_{2k+2}}, \prod_{s=1}^{2k+1} (t_s)^{w_s} t_{2k+2}^{w_{2k+2}}, \prod_{s=1}^{2k+1} (\tilde{t}_s)^{w_s} \tilde{t}_{2k+2}^{w_{2k+2}}); \\
&\quad \frac{1}{\prod_{s=1}^{2k+1} (l_{\hat{t}_s}^{\max})^{w_s}} \frac{1}{(l_{\hat{t}_{2k+2}}^{\max})^{w_{2k+2}}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{2k+1} (1-f_{\hat{t}_s}^{\min})^{w_s}} \cdot \frac{1}{(1-f_{\hat{t}_{2k+2}}^{\min})^{w_{2k+2}}} > \\
&= <(\prod_{s=1}^{2k+2} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{2k+2} (t_s)^{w_s}, \prod_{s=1}^{2k+2} (\tilde{t}_s)^{w_s}); \frac{1}{\prod_{s=1}^{2k+2} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{2k+2} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = 2k+2$, Eq. (26) holds. According to Theorem 1, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+2})$ is a TIFN; $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+2})$ is a positive TIFN.

(v) According to the above conclusion, when $n = 2k+2$, $\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+2})$ is a positive TIFN. Since $\hat{t}_{2k+3} < 0$ and $w_{2k+3} = \frac{\text{odd}}{\text{odd}} < 0$, $(\hat{t}_{2k+3})^{w_{2k+3}} =< (\tilde{t}_{2k+3}^{w_{2k+3}}, t_{2k+3}^{w_{2k+3}}, \underset{\sim}{t}_{2k+3}^{w_{2k+3}}); (\frac{1}{(l_{\hat{t}_{2k+3}}^{\max})^{w_{2k+3}}}, 1 - \frac{1}{(1-f_{\hat{t}_{2k+3}}^{\min})^{w_{2k+3}}} >$ can be obtained according to Eq. (11); thereinto, $(\hat{t}_{2k+3})^{w_{2k+3}} < 0$. Therefore, when $n = 2k+3$, according to Eq. (8), we get

$$\begin{aligned}
&\mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+3}) \\
&= (\hat{t}_{2k+3})^{w_{2k+3}} \cdot \mathrm{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+2}) \\
&= <(\tilde{t}_{2k+3}^{w_{2k+3}} \prod_{s=1}^{2k+2} (\tilde{t}_s)^{w_s}, t_{2k+3}^{w_{2k+3}} \prod_{s=1}^{2k+2} (t_s)^{w_s}, \underset{\sim}{t}_{2k+3}^{w_{2k+3}} \prod_{s=1}^{2k+2} (\underset{\sim}{t}_s)^{w_s}); \\
&\quad \frac{1}{(l_{\hat{t}_{2k+3}}^{\max})^{w_{2k+3}}} \frac{1}{\prod_{s=1}^{2k+2} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{(1-f_{\hat{t}_{2k+3}}^{\min})^{w_{2k+3}}} \frac{1}{\prod_{s=1}^{2k+2} (1-f_{\hat{t}_s}^{\min})^{w_s}} > \\
&= <(\prod_{s=1}^{2k+3} (\tilde{t}_s)^{w_s}, \prod_{s=1}^{2k+3} (t_s)^{w_s}, \prod_{s=1}^{2k+3} (\underset{\sim}{t}_s)^{w_s}); \\
&\quad \frac{1}{\prod_{s=1}^{2k+3} (l_{\hat{t}_s}^{\max})^{w_s}}, 1 - \frac{1}{\prod_{s=1}^{2k+3} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = 2k+3$, Eq. (26) holds. According to Theorem 1, $n = 2k+3$ is a TIFN; Because of $\prod_{s=1}^{2k+3} (\underset{\sim}{t}_s)^{w_s} < 0$, $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{2k+3})$ is a negative TIFN.

To sum up, Eq. (26) holds for all n and the calculation result is a TIFN.

Fourth, prove Case (IV).

(i) When $n = 1$, Eq. (27) holds. Because $\hat{t}_1 < 0$ and $w_1 = \frac{\text{even}}{\text{odd}} < 0$, $\text{TIFEWGA}_W(\hat{t}_1) = \hat{t}_1^{w_1} =< (\underset{\sim}{t}_1^{w_1}, t_1^{w_1}, \tilde{t}_1^{w_1}); \frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}}, 1 - \frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}} >$ can be obtained by Eq. (11). According to Theorem 1, $\text{TIFEWGA}_W(\hat{t}_1)$ is a TIFN. Because of $\underset{\sim}{t}_1^{w_1} > 0$, $\text{TIFEWGA}_W(\hat{t}_1)$ is a positive TIFN

(ii) When $n = 2$, due to $\hat{t}_2 < 0$ and $w_2 = \frac{\text{even}}{\text{odd}} < 0$, $\hat{t}_2^{w_2} =< (\underset{\sim}{t}_2^{w_2}, t_2^{w_2}, \tilde{t}_2^{w_2}); \frac{1}{(l_{\hat{t}_2}^{\max})^{w_2}}, 1 - \frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}} >$ can be obtained by Eq. (11). Because of $\underset{\sim}{t}_2^{w_2} > 0$, both $\hat{t}_1^{w_1}$ and $\hat{t}_2^{w_2}$ are positive TIFNs. Furthermore, according to Eq. (8), we get

$$\begin{aligned}
&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2) \\
&= < (\underset{\sim}{t}_1^{w_1}\underset{\sim}{t}_2^{w_2}, t_1^{w_1}t_2^{w_2}, \tilde{t}_1^{w_1}\tilde{t}_2^{w_2}); \frac{1}{(l_{\hat{t}_1}^{\max})^{w_1}(l_{\hat{t}_2}^{\max})^{w_2}}, \\
&\quad 1 - \frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}} + 1 - \frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}} \\
&\quad - [1 - \frac{1}{(1-f_{\hat{t}_1}^{\min})^{w_1}}][1 - \frac{1}{(1-f_{\hat{t}_2}^{\min})^{w_2}}] > \\
&= < (\prod_{s=1}^{2} \underset{\sim}{t}_s^{w_s}, \prod_{s=1}^{2} t_s^{w_s}, \prod_{s=1}^{2} \tilde{t}_s^{w_s}); \frac{1}{\prod_{s=1}^{2} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{2} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = 2$, Eq. (27) holds. According to Theorem 1, $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2)$ is a TIFN. Because of $\prod_{s=1}^{2} \underset{\sim}{t}_s^{w_s} > 0$, $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2)$ is a positive TIFN.

(iii) It is assumed that when $n = k$, Eq. (27) holds and $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k)$ is a positive TIFN. Hence

$$\begin{aligned}
&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_k) \\
&= < (\prod_{s=1}^{k} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{k} (t_s)^{w_s}, \prod_{s=1}^{k} (\tilde{t}_s)^{w_s}); \frac{1}{\prod_{s=1}^{k} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{k} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Since $\hat{t}_{k+1} < 0$ and $w_{k+1} = \frac{\text{even}}{\text{odd}} < 0$, $(\hat{t}_{k+1})^{w_{k+1}} =< (\underset{\sim}{t}_{k+1}^{w_{k+1}}, t_{k+1}^{w_{k+1}}, \tilde{t}_{k+1}^{w_{k+1}}); \frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}, 1 - \frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} >$ can be obtained by Eq. (11). Therefore, when $n = k+1$, according to Eq. (8), we get

$$\begin{aligned}
&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1}) \\
&= < (\prod_{s=1}^{k} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{k} (t_s)^{w_s}, \prod_{s=1}^{k} (\tilde{t}_s)^{w_s}); \frac{1}{\prod_{s=1}^{k} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{k} (1-f_{\hat{t}_s}^{\min})^{w_s}} > \\
&\quad + < (\underset{\sim}{t}_{k+1}^{w_{k+1}}, t_{k+1}^{w_{k+1}}, \tilde{t}_{k+1}^{w_{k+1}}); \frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}, \\
&\quad 1 - \frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} > \\
&= < (\prod_{s=1}^{k} (\underset{\sim}{t}_s)^{w_s} \cdot \underset{\sim}{t}_{k+1}^{w_{k+1}}, \prod_{s=1}^{k} (t_s)^{w_s} \cdot t_{k+1}^{w_{k+1}}, \prod_{s=1}^{k} (\tilde{t}_s)^{w_s} \cdot \tilde{t}_{k+1}^{w_{k+1}}); \\
&\quad \frac{1}{\prod_{s=1}^{k} (l_{\hat{t}_s}^{\max})^{w_s}} \frac{1}{(l_{\hat{t}_{k+1}}^{\max})^{w_{k+1}}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{k} (1-f_{\hat{t}_s}^{\min})^{w_s}} + 1 - \frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}} \\
&\quad - [1 - \frac{1}{\prod_{s=1}^{k} (1-f_{\hat{t}_s}^{\min})^{w_s}}][1 - \frac{1}{(1-f_{\hat{t}_{k+1}}^{\min})^{w_{k+1}}}] > \\
&= < (\prod_{s=1}^{k+1} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{k+1} (t_s)^{w_s}, \prod_{s=1}^{k+1} (\tilde{t}_s)^{w_s}); \frac{1}{\prod_{s=1}^{k+1} (l_{\hat{t}_s}^{\max})^{w_s}}, \\
&\quad 1 - \frac{1}{\prod_{s=1}^{k+1} (1-f_{\hat{t}_s}^{\min})^{w_s}} >
\end{aligned}$$

Therefore, when $n = k+1$, Eq. (27) holds. According to Theorem 1, $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1})$ is a TIFN. Because of $\prod_{s=1}^{k+1} (\underset{\sim}{t}_s)^{w_s} > 0$, $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_{k+1})$ is a positive TIFN.

To sum up, Eq. (27) holds for all n and the calculation result is a positive TIFN.

Fifth, prove Case (V).

Let $T_1 = w_1\hat{t}_1 + w_2\hat{t}_2 + \cdots + w_b\hat{t}_b$, $T_2 = w_{b+1}\hat{t}_{b+1} + w_{b+2}\hat{t}_{b+2} + \cdots + w_g\hat{t}_g$, $T_3 = w_{g+1}\hat{t}_{g+1} + w_{g+2}\hat{t}_{g+2} + \cdots + w_c\hat{t}_c$ and $T_4 = w_{c+1}\hat{t}_{c+1} + w_{c+2}\hat{t}_{c+2} + \cdots + w_n\hat{t}_n$.

(i) When b and $c - g$ are odd numbers, according to Eqs. (24)-(27), we get

$$\begin{aligned}
T_1 &= < (\prod_{s=1}^{b} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{b} (t_s)^{w_s}, \prod_{s=1}^{b} (\tilde{t}_s)^{w_s}); \\
&\quad \prod_{s=1}^{b} (l_{\hat{t}_s}^{\max})^{w_s}, 1 - \prod_{s=1}^{b} (1-f_{\hat{t}_s}^{\min})^{w_s} >
\end{aligned}$$

$$\begin{aligned}T_2 = <(&\prod_{s=b+1}^{g} (\tilde{t}_s)^{w_s}, \prod_{s=b+1}^{g} (t_s)^{w_s}, \prod_{s=b+1}^{g} (\underset{\sim}{t}_s)^{w_s});\\ &\prod_{s=b+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=b+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} >\\ T_3 = <(&\prod_{s=g+1}^{c} (\tilde{t}_s)^{w_s}, \prod_{s=g+1}^{c} (t_s)^{w_s}, \prod_{s=g+1}^{c} (\underset{\sim}{t}_s)^{w_s});\\ &\prod_{s=g+1}^{c} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, 1-\prod_{s=g+1}^{c} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\\ T_4 = <(&\prod_{s=c+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=c+1}^{n} (t_s)^{w_s}, \prod_{s=c+1}^{n} (\tilde{t}_s)^{w_s});\\ &\prod_{s=c+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, 1-\prod_{s=c+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\end{aligned}$$

Furthermore, it can be seen that T_1 and T_3 are both negative TIFN, and T_2 and T_4 are both positive TIFNs. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1 \cdot T_2 = <(&\prod_{s=1}^{b} (\underset{\sim}{t}_s)^{w_s} \prod_{s=b+1}^{g} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{b} (t_s)^{w_s} \prod_{s=b+1}^{g} (t_s)^{w_s},\\ &\prod_{s=1}^{b} (\tilde{t}_s)^{w_s} \prod_{s=b+1}^{g} (\tilde{t}_s)^{w_s}); \prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s},\\ &1-\prod_{s=1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} >\\ = <(&\prod_{s=1}^{g} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{g} (t_s)^{w_s}, \prod_{s=1}^{g} (\tilde{t}_s)^{w_s}); \prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s},\\ &1-\prod_{s=1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} >\end{aligned}$$

and $T_1 \cdot T_2$ is a negative TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1 \cdot T_2 \cdot T_3 = <(&\prod_{s=1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{c} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{c} (t_s)^{w_s},\\ &\prod_{s=1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{c} (\tilde{t}_s)^{w_s});\\ &\prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{c} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &1-\prod_{s=1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{c} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\end{aligned}$$

and $T_1 \cdot T_2 \cdot T_3$ is a positive TIFN. Finally, according to Eq. (8), we get

$$\begin{aligned}T_1 \cdot T_2 \cdot T_3 \cdot T_4 = <(&\prod_{s=1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s},\\ &\prod_{s=1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s});\\ &\prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &1-\prod_{s=1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\end{aligned}$$

and $T_1 \cdot T_2 \cdot T_3 \cdot T_4$ is a positive TIFN. Namely,

$$\begin{aligned}&\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)\\ &= <(\prod_{s=1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s});\\ &\quad \prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &\quad 1-\prod_{s=1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\end{aligned}$$

and $\text{TIFEWGA}_W(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)$ is a positive TIFN.

(ii) When b is an odd number and $c-g$ is an even number, according to Eqs. (24)-(27), we get

$$\begin{aligned}T_1 = <(&\prod_{s=1}^{b} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{b} (t_s)^{w_s}, \prod_{s=1}^{b} (\tilde{t}_s)^{w_s}); \prod_{s=1}^{b} (l_{\hat{t}_s}^{\max})^{w_s},\\ &1-\prod_{s=1}^{b} (1-f_{\hat{t}_s}^{\min})^{w_s} >\\ T_2 = <(&\prod_{s=b+1}^{g} (\tilde{t}_s)^{w_s}, \prod_{s=b+1}^{g} (t_s)^{w_s}, \prod_{s=b+1}^{g} (\underset{\sim}{t}_s)^{w_s}); \prod_{s=b+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s},\\ &1-\prod_{s=b+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} >\\ T_3 = <(&\prod_{s=g+1}^{c} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=g+1}^{c} (t_s)^{w_s}, \prod_{s=g+1}^{c} (\tilde{t}_s)^{w_s});\\ &\prod_{s=g+1}^{c} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, 1-\prod_{s=g+1}^{c} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\\ T_4 = <(&\prod_{s=c+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=c+1}^{n} (t_s)^{w_s}, \prod_{s=c+1}^{n} (\tilde{t}_s)^{w_s});\\ &\prod_{s=c+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, 1-\prod_{s=c+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >\end{aligned}$$

It can be seen that T_1 is a negative TIFN, and T_2, T_3 and T_4 is a positive TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1 \cdot T_2 = <(&\prod_{s=1}^{b} (\underset{\sim}{t}_s)^{w_s} \prod_{s=b+1}^{g} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{b} (t_s)^{w_s} \prod_{s=b+1}^{g} (t_s)^{w_s},\\ &\prod_{s=1}^{b} (\tilde{t}_s)^{w_s} \prod_{s=b+1}^{g} (\tilde{t}_s)^{w_s}); \prod_{s=1}^{g} (l_{\hat{t}_s}^{\max})^{w_s},\end{aligned}$$

$$1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$
$$=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s});\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},\ 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>$$

and $T_1\cdot T_2$ is a negative TIFN. Therefore, according to Eq.(8), we get

$$\begin{aligned}T_1\cdot T_2\cdot T_3 &=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{c}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s}\prod_{s=g+1}^{c}(t_s)^{w_s},\\ &\quad\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{c}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $T_1\cdot T_2\cdot T_3$ is a negative TIFN. Finally, according to Eq. (8), we get

$$\begin{aligned}T_1\cdot T_2\cdot T_3\cdot T_4 &=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s}\prod_{s=g+1}^{n}(t_s)^{w_s},\\ &\quad\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $T_1\cdot T_2\cdot T_3$ is a negative TIFN. Namely,

$$\begin{aligned}&\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\ &=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s}\prod_{s=g+1}^{n}(t_s)^{w_s},\\ &\quad\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\ &\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $\text{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)$ is a negative TIFN.

(iii) When b is an even number and $c-g$ is an odd number, according to Eqs. (24)-(27), we get

$$\begin{aligned}T_1&=<(\prod_{s=1}^{b}(\tilde{t}_s)^{w_s},\prod_{s=1}^{b}(t_s)^{w_s},\prod_{s=1}^{b}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{b}(l_{\hat{t}_s}^{\max})^{w_s},\ 1-\prod_{s=1}^{b}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\ T_2&=<(\prod_{s=b+1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=b+1}^{g}(t_s)^{w_s},\prod_{s=b+1}^{g}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=b+1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},\ 1-\prod_{s=b+1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\ T_3&=<(\prod_{s=g+1}^{c}(\tilde{t}_s)^{w_s},\prod_{s=g+1}^{c}(t_s)^{w_s},\prod_{s=g+1}^{c}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=g+1}^{c}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\ 1-\prod_{s=g+1}^{c}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\\ T_4&=<(\prod_{s=c+1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=c+1}^{n}(t_s)^{w_s},\prod_{s=c+1}^{n}(\tilde{t}_s)^{w_s});\\ &\quad\prod_{s=c+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\ 1-\prod_{s=c+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

It can be seen that T_1, T_2 and T_4 are positive TIFNs and T_3 is a negative TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1\cdot T_2&=<(\prod_{s=1}^{b}(\tilde{t}_s)^{w_s}\prod_{s=b+1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=1}^{b}(t_s)^{w_s}\prod_{s=b+1}^{g}(t_s)^{w_s},\\ &\quad\prod_{s=1}^{b}(\underset{\sim}{t}_s)^{w_s}\prod_{s=b+1}^{g}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},\\ &\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\ &=<(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s},\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},\ 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\end{aligned}$$

and $T_1\cdot T_2$ is a positive TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1\cdot T_2\cdot T_3&=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{c}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s},\\ &\quad\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{c}(\underset{\sim}{t}_s)^{w_s});\\ &\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\end{aligned}$$

$$1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>$$

and $T_1\cdot T_2\cdot T_3$ is a negative TIFN. Finally, according to Eq. (8), we get

$$\begin{aligned}&T_1\cdot T_2\cdot T_3\cdot T_4\\&=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and, $T_1\cdot T_2\cdot T_3\cdot T_4$ is a negative TIFN. Namely,

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\&=<(\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and, $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)$ is a negative TIFN.

(iv) When b and $c-g$ are even numbers, according to Eqs. (24)-(27), we get

$$\begin{aligned}T_1&=<(\prod_{s=1}^{b}(\tilde{t}_s)^{w_s},\prod_{s=1}^{b}(t_s)^{w_s},\prod_{s=1}^{b}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{b}(l_{\hat{t}_s}^{\max})^{w_s},\\&\quad 1-\prod_{s=1}^{b}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\T_2&=<(\prod_{s=b+1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=b+1}^{g}(t_s)^{w_s},\prod_{s=b+1}^{g}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=b+1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=b+1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\T_3&=<(\prod_{s=g+1}^{c}(\underset{\sim}{t}_s)^{w_s},\prod_{s=g+1}^{c}(t_s)^{w_s},\prod_{s=g+1}^{c}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=g+1}^{c}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},1-\prod_{s=g+1}^{c}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\\T_4&=<(\prod_{s=c+1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=c+1}^{n}(t_s)^{w_s},\prod_{s=c+1}^{n}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=c+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},1-\prod_{s=c+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

It can be seen that T_1, T_2, T_3 and T_4 are positive TIFNs. Therefore, according to Eq. (8), we get

$$\begin{aligned}T_1\cdot T_2&=<(\prod_{s=1}^{b}(\tilde{t}_s)^{w_s}\prod_{s=b+1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=1}^{b}(t_s)^{w_s}\prod_{s=b+1}^{g}(t_s)^{w_s},\\&\quad\prod_{s=1}^{b}(\underset{\sim}{t}_s)^{w_s}\prod_{s=b+1}^{g}(\underset{\sim}{t}_s)^{w_s});\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\\&=<(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s},\prod_{s=1}^{g}(t_s)^{w_s},\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s},1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}>\end{aligned}$$

and $T_1\cdot T_2$ is a positive TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}&T_1\cdot T_2\cdot T_3\\&=<(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{c}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{c}(t_s)^{w_s},\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{c}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{c}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $T_1\cdot T_2\cdot T_3$ is a positive TIFN. Therefore, according to Eq. (8), we get

$$\begin{aligned}&T_1\cdot T_2\cdot T_3\cdot T_4\\&=<(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $T_1\cdot T_2\cdot T_3\cdot T_4$ is a positive TIFN. Namely,

$$\begin{aligned}&\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)\\&=<(\prod_{s=1}^{g}(\tilde{t}_s)^{w_s}\prod_{s=g+1}^{n}(\underset{\sim}{t}_s)^{w_s},\prod_{s=1}^{n}(t_s)^{w_s},\prod_{s=1}^{g}(\underset{\sim}{t}_s)^{w_s}\prod_{s=g+1}^{n}(\tilde{t}_s)^{w_s});\\&\quad\prod_{s=1}^{g}(l_{\hat{t}_s}^{\max})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}},\\&\quad 1-\prod_{s=1}^{g}(1-f_{\hat{t}_s}^{\min})^{w_s}\prod_{s=g+1}^{n}\frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>\end{aligned}$$

and $\mathrm{TIFEWGA}_W(\hat{t}_1,\hat{t}_2,\ldots,\hat{t}_n)$ is a positive TIFN.

To sum up, the calculation result is a TIFN, and Eqs. (24)-(28) hold for all n.

PROOF OF THEOREM 9

Let $T_1 = w_1\hat{t}_1+\cdots+w_r\hat{t}_r$, $T_2 = w_{r+1}\hat{t}_{r+1}+\cdots+w_a\hat{t}_a$, $T_3 = w_{a+1}\hat{t}_{a+1}+\cdots+w_n\hat{t}_n$, then according to Theorem 5, we get $T_1 =<(\prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=1}^{r} (t_s)^{w_s}, \prod_{s=1}^{r} (\tilde{t}_s)^{w_s}); \prod_{s=1}^{r} (l_{\hat{t}_s}^{\max})^{w_s}, 1-\prod_{s=1}^{r} (1-f_{\hat{t}_s}^{\min})^{w_s}>$ and T_1 is positive. According to Theorem 6, we get $T_2 =<(\prod_{s=r+1}^{a} (\tilde{t}_s)^{w_s}, \prod_{s=r+1}^{a} (t_s)^{w_s}, \prod_{s=r+1}^{a} (\underset{\sim}{t}_s)^{w_s}); \prod_{s=r+1}^{a} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, 1-\prod_{s=r+1}^{a} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}}>$ and T_2 is positive. According to Theorem 8, we get

$$T_3 = \begin{cases} <(\prod_{s=a+1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \prod_{s=a+1}^{n} (t_s)^{w_s}, \\ \prod_{s=a+1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s}); \\ \prod_{s=a+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, \\ 1-\prod_{s=a+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >, \\ \text{if } b-a \text{ and } c-g \text{ are both odd and even numbers} \\ <(\prod_{s=a+1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s}, \prod_{s=a+1}^{n} (t_s)^{w_s}, \\ \prod_{s=a+1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}); \\ \prod_{s=a+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, \\ 1-\prod_{s=a+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >, \\ \text{if the parity of } b-a \text{ and } c-g \text{ is opposite} \end{cases}$$

Furthermore, according to Eq. (8), we get

$$\begin{aligned} &T_1 \cdot T_2 \\ &= <(\prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s} \prod_{s=r+1}^{a} (\tilde{t}_s)^{w_s}, \prod_{s=1}^{a} (t_s)^{w_s}, \prod_{s=1}^{r} (\tilde{t}_s)^{w_s} \prod_{s=r+1}^{a} (\underset{\sim}{t}_s)^{w_s}); \\ &\quad \prod_{s=1}^{r} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, \\ &\quad 1-\prod_{s=1}^{r} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} > \end{aligned}$$

and $T_1 \cdot T_2$ is positive. Therefore, according to Eq. (8), we get

$$T_1 \cdot T_2 \cdot T_3 = \begin{cases} <(\prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s} \prod_{s=r+1}^{a} (\tilde{t}_s)^{w_s} \prod_{s=a+1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}, \\ \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{r} (\tilde{t}_s)^{w_s} \prod_{s=r+1}^{a} (\underset{\sim}{t}_s)^{w_s} \prod_{s=a+1}^{g} (\underset{\sim}{t}_s)^{w_s} \\ \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s}); \\ \prod_{s=1}^{r} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}} \prod_{s=a+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, \\ 1-\prod_{s=1}^{r} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} \\ \prod_{s=a+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >, \\ \text{if } b-a \text{ and } c-g \text{ are both odd and even numbers} \\ <(\prod_{s=1}^{r} (\tilde{t}_s)^{w_s} \prod_{s=r+1}^{a} (\underset{\sim}{t}_s)^{w_s} \prod_{s=a+1}^{g} (\underset{\sim}{t}_s)^{w_s} \prod_{s=g+1}^{n} (\tilde{t}_s)^{w_s}, \\ \prod_{s=1}^{n} (t_s)^{w_s}, \prod_{s=1}^{r} (\underset{\sim}{t}_s)^{w_s} \prod_{s=r+1}^{a} (\tilde{t}_s)^{w_s} \\ \prod_{s=a+1}^{g} (\tilde{t}_s)^{w_s} \prod_{s=g+1}^{n} (\underset{\sim}{t}_s)^{w_s}); \\ \prod_{s=1}^{r} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}} \prod_{s=a+1}^{g} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}}, \\ 1-\prod_{s=1}^{r} (l_{\hat{t}_s}^{\max})^{w_s} \prod_{s=r+1}^{a} \frac{1}{(l_{\hat{t}_s}^{\max})^{w_s}} \\ \prod_{s=a+1}^{g} (1-f_{\hat{t}_s}^{\min})^{w_s} \prod_{s=g+1}^{n} \frac{1}{(1-f_{\hat{t}_s}^{\min})^{w_s}} >, \\ \text{if the parity of } b-a \text{ and } c-g \text{ are opposite} \end{cases}$$

Therefore, Eq. (29) holds.

PROOF OF THEOREM 10

(i) Non-negativity: by Eqs. (39)-(45), we obtain $A_L^l(\hat{t}_1, \hat{t}_2) \geq 0$, $A_R^l(\hat{t}_1, \hat{t}_2) \geq 0$, $A_L^f(\hat{t}_1, \hat{t}_2) \geq 0$, $A_R^f(\hat{t}_1, \hat{t}_2) \geq 0$, $A_L^y(\hat{t}_1, \hat{t}_2) \geq 0$, $A_R^y(\hat{t}_1, \hat{t}_2) \geq 0$. Because $\lambda \in [1, +\infty)$ and $\omega_k \geq 0$, we have $\text{TIFWMD}_\lambda(\hat{t}_1, \hat{t}_2) \geq 0$.

(ii) Symmetry: by Eqs. (40)-(45), we obtain $A_L^l(\hat{t}_1, \hat{t}_2) = A_L^l(\hat{t}_2, \hat{t}_1)$, $A_R^l(\hat{t}_1, \hat{t}_2) = A_R^l(\hat{t}_2, \hat{t}_1)$, $A_L^f(\hat{t}_1, \hat{t}_2) = A_L^f(\hat{t}_2, \hat{t}_1)$, $A_R^f(\hat{t}_1, \hat{t}_2) = A_R^f(\hat{t}_2, \hat{t}_1)$, $A_L^y(\hat{t}_1, \hat{t}_2) = A_L^y(\hat{t}_2, \hat{t}_1)$, $A_L^y(\hat{t}_1, \hat{t}_2) = A_L^y(\hat{t}_2, \hat{t}_1)$. Therefore, $\text{TIFWMD}(\hat{t}_1, \hat{t}_2) = \text{TIFWMD}(\hat{t}_2, \hat{t}_1)$ by Eq. (39).

(iii) Trigonometric inequality: by Eqs. (35)-(39), we get

$$\begin{aligned} &\text{TIFWMD}(\hat{t}_2, \hat{t}_3) \\ &= \Big(\left(\omega_1^{1/\lambda} A_L^l(\hat{t}_2, \hat{t}_3)\right)^\lambda + \left(\omega_2^{1/\lambda} A_R^l(\hat{t}_2, \hat{t}_3)\right)^\lambda \\ &\quad + \left(\omega_3^{1/\lambda} A_L^f(\hat{t}_2, \hat{t}_3)\right)^\lambda + \left(\omega_4^{1/\lambda} A_R^f(\hat{t}_2, \hat{t}_3)\right)^\lambda \\ &\quad + \left(\omega_5^{1/\lambda} A_L^y(\hat{t}_2, \hat{t}_3)\right)^\lambda + \left(\omega_6^{1/\lambda} A_R^y(\hat{t}_2, \hat{t}_3)\right)^\lambda \Big)^{1/\lambda} \end{aligned}$$

$$\begin{aligned}&\mathrm{TIFWMD}(\hat{t}_1,\hat{t}_2)+\mathrm{TIFWMD}(\hat{t}_2,\hat{t}_3)\\ &\geq\left(\begin{array}{l}\left[\omega_1^{1/\lambda}A_L^l\left(\hat{t}_1,\hat{t}_2\right)+\omega_1^{1/\lambda}A_L^l\left(\hat{t}_2,\hat{t}_3\right)\right]^\lambda+\left(\omega_2^{1/\lambda}A_R^l\left(\hat{t}_1,\hat{t}_2\right)+\omega_2^{1/\lambda}A_R^l\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\left(\omega_3^{1/\lambda}A_L^f\left(\hat{t}_1,\hat{t}_2\right)+\omega_3^{1/\lambda}A_L^f\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda\\ +\left(\omega_4^{1/\lambda}A_R^f\left(\hat{t}_1,\hat{t}_2\right)+\omega_4^{1/\lambda}A_R^f\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\left(\omega_5^{1/\lambda}A_L^y\left(\hat{t}_1,\hat{t}_2\right)+\omega_5^{1/\lambda}A_L^y\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\left(\omega_6^{1/\lambda}A_R^y\left(\hat{t}_1,\hat{t}_2\right)+\omega_6^{1/\lambda}A_R^y\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda\end{array}\right)^{1/\lambda}\\ &=\left(\begin{array}{l}\omega_1\left(A_L^l\left(\hat{t}_1,\hat{t}_2\right)+A_L^l\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\omega_2\left(A_R^l\left(\hat{t}_1,\hat{t}_2\right)+A_R^l\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\omega_3\left(A_L^f\left(\hat{t}_1,\hat{t}_2\right)+A_L^f\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\\ \omega_4\left(A_R^f\left(\hat{t}_1,\hat{t}_2\right)+A_R^f\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\omega_5\left(A_L^y\left(\hat{t}_1,\hat{t}_2\right)+A_L^y\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda+\omega_6\left(A_R^y\left(\hat{t}_1,\hat{t}_2\right)+A_R^y\left(\hat{t}_2,\hat{t}_3\right)\right)^\lambda\end{array}\right)^{1/\lambda}\end{aligned}$$

Therefore, according to Minkowski inequality, we get, as shown in the equation at the top of the page.

Then, according to absolute value inequality, we get

$$\begin{aligned}&A_L^l\left(\hat{t}_1,\hat{t}_2\right)+A_L^l\left(\hat{t}_2,\hat{t}_3\right)\geq A_R^l\left(\hat{t}_1,\hat{t}_3\right),\\ &A_R^l\left(\hat{t}_1,\hat{t}_2\right)+A_R^l\left(\hat{t}_2,\hat{t}_3\right)\geq A_R^l\left(\hat{t}_1,\hat{t}_3\right),\\ &A_L^f\left(\hat{t}_1,\hat{t}_2\right)+A_L^f\left(\hat{t}_2,\hat{t}_3\right)\geq A_L^f\left(\hat{t}_1,\hat{t}_3\right),\\ &A_R^f\left(\hat{t}_1,\hat{t}_2\right)+A_R^f\left(\hat{t}_2,\hat{t}_3\right)\geq A_R^f\left(\hat{t}_1,\hat{t}_3\right),\\ &A_L^y\left(\hat{t}_1,\hat{t}_2\right)+A_L^y\left(\hat{t}_2,\hat{t}_3\right)\geq A_L^y\left(\hat{t}_1,\hat{t}_3\right),\\ &A_R^y\left(\hat{t}_1,\hat{t}_2\right)+A_R^y\left(\hat{t}_2,\hat{t}_3\right)\geq A_R^y\left(\hat{t}_1,\hat{t}_3\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathrm{TIFWMD}(\hat{t}_1,\hat{t}_2)+\mathrm{TIFWMD}(\hat{t}_2,\hat{t}_3)\\ &\geq\left(\omega_1\left(A_R^l\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda+\omega_2\left({}_R^l\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda\right.\\ &\quad+\omega_3\left(A_L^f\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda+\omega_4\left(A_R^f\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda\\ &\quad\left.+\omega_5\left(A_L^y\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda+\omega_6\left(A_R^y\left(\hat{t}_1,\hat{t}_3\right)\right)^\lambda\right)^{1/\lambda}\end{aligned}$$

Namely, $\mathrm{TIFWMD}(\hat{t}_1,\hat{t}_2)+\mathrm{TIFWMD}(\hat{t}_2,\hat{t}_3)\geq\mathrm{TIFWMD}(\hat{t}_1,\hat{t}_3)$.

PROOF OF THEOREM 11

(i) Non-negativity: According to Theorem 10, we have $\mathrm{TIFWMD}_\lambda(\hat{t}_j^1,\hat{t}_j^2)\geq 0, j=1,2,\ldots,n$. Since $\lambda\in[1,+\infty)$, $\mathrm{TIFSWMD}(\hat{T}_1,\hat{T}_2)\geq 0$ is obtained by Eq. (46).

(ii) Symmetry: According to Theorem 10, we have $\mathrm{TIFWMD}_\lambda(\hat{t}_j^1,\hat{t}_j^2)=\mathrm{TIFWMD}_\lambda(\hat{t}_j^2,\hat{t}_j^1), j=1,2,\ldots,n$. Therefore, $\mathrm{TIFSWMD}(\hat{T}_1,\hat{T}_2)=\mathrm{TIFSWMD}(\hat{T}_2,\hat{T}_1)$ is obtained by Eq. (46).

(iii) Trigonometric inequality: by Eq. (46), we get

$$\begin{aligned}&\mathrm{TIFSWMD}_\lambda(\hat{T}_1,\hat{T}_2)+\mathrm{TIFSWMD}_\lambda(\hat{T}_2,\hat{T}_3)\\ &=\sqrt[\lambda]{\frac{1}{n}\sum_{j=1}^n\left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^1,\hat{t}_j^2)\right)^\lambda}\\ &\quad+\sqrt[\lambda]{\frac{1}{n}\sum_{j=1}^n\left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^2,\hat{t}_j^3)\right)^\lambda}\\ &=\sqrt[\lambda]{\frac{1}{n}}\left[\sqrt[\lambda]{\sum_{j=1}^n\left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^1,\hat{t}_j^2)\right)^\lambda}\right.\\ &\quad\left.+\sqrt[\lambda]{\sum_{j=1}^n\left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^2,\hat{t}_j^3)\right)^\lambda}\right]\end{aligned}$$

Furthermore, according to Minkowski inequality, we have

$$\begin{aligned}&\mathrm{TIFSWMD}_\lambda(\hat{T}_1,\hat{T}_2)+\mathrm{TIFSWMD}_\lambda(\hat{T}_2,\hat{T}_3)\\ &\geq\sqrt[\lambda]{\frac{1}{n}}\sqrt[\lambda]{\sum_{j=1}^n\left(\mathrm{TIFWMD}_\lambda(\hat{t}_j^1,\hat{t}_j^2)+\mathrm{TIFWMD}_\lambda(\hat{t}_j^2,\hat{t}_j^3)\right)^\lambda}\end{aligned}$$

According to Theorem 10, $\mathrm{TIFWMD}(\hat{t}_j^1,\hat{t}_j^2)+\mathrm{TIFWMD}(\hat{t}_j^2,\hat{t}_j^3)\geq\mathrm{TIFWMD}(\hat{t}_j^1,\hat{t}_j^3), j=1,2,\ldots,n$. Hence,

$$\begin{aligned}&\mathrm{TIFSWMD}_\lambda(\hat{T}_1,\hat{T}_2)+\mathrm{TIFSWMD}_\lambda(\hat{T}_2,\hat{T}_3)\\ &\geq\sqrt[\lambda]{\frac{1}{n}}\sqrt[\lambda]{\sum_{j=1}^n\left(\mathrm{TIFWMD}(\hat{t}_j^1,\hat{t}_j^3)\right)^\lambda}\\ &=\sqrt[\lambda]{\frac{1}{n}\sum_{j=1}^n\left(\mathrm{TIFWMD}(\hat{t}_j^1,\hat{t}_j^3)\right)^\lambda}\end{aligned}$$

Therefore, $\mathrm{TIFSWMD}(\hat{T}_1,\hat{T}_2)+\mathrm{TIFSWMD}(\hat{T}_2,\hat{T}_3)\geq\mathrm{TIFSWMD}(\hat{T}_1,\hat{T}_3)$.

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