IEEE Access
Multidisciplinary | Rapid Review | Open Access Journal

Received 9 June 2023, accepted 16 June 2023, date of publication 21 June 2023, date of current version 10 July 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3287926

RESEARCH ARTICLE

Modeling Voltage Real Data Set by a New Version of Lindley Distribution

NAWEL KHODJA[1], AHMED M. GEMEAY[2], HALIM ZEGHDOUDI[1], KADIR KARAKAYA[3], ARWA M. ALSHANGITI[4], M. E. BAKR[4], OLUWAFEMI SAMSON BALOGUN[5], ABDISALAM HASSAN MUSE[6], AND ESLAM HUSSAM[7]

[1]LaPS Laboratory, Badji Mokhtar-Annaba University, Annaba 23000, Algeria
[2]Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt
[3]Department of Statistics, Faculty of Sciences, Selçuk University, 42130 Konya, Turkey
[4]Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[5]School of Computing, University of Eastern Finland, 70211 Kuopio, Finland
[6]Faculty of Science and Humanities, School of Postgraduate Studies and Research (SPGSR), Amoud University, Borama 25263, Somalia
[7]Department of Mathematics, Faculty of Science, Helwan University, Cairo 11795, Egypt

Corresponding author: Abdisalam Hassan Muse (abdisalam.hassan@amoud.edu.so)

This work was supported by the King Saud University, Riyadh, Saudi Arabia, through Researchers Supporting under Project RSPD2023R538.

ABSTRACT This paper presents a novel probability distribution, namely the new XLindley distribution, derived from a unique combination of exponential and gamma distributions through a special mixture formulation. The study extensively investigates the mathematical properties of the proposed distribution, including but not limited to the moment generation function, moments of different orders, mode identification, and the quantile function. Furthermore, the research employs a Monte Carlo simulation to assess and compare the performance of various estimators in estimating the unknown parameter of the new XLindley distribution. These estimators are carefully evaluated and analyzed in terms of their efficiency and accuracy, providing valuable insights into the practical application of the new distribution in statistical modeling and data analysis contexts. The voltage and failure time data in the field of engineering are used to model the proposed distribution. The new model is compared with many current distributions such as Xlindley, gamma, Weibull, exponential, Lindley, Shanker, Akash, Zeghdoudi, Chris-Jerry, and Xgamma. Among all models, it is concluded that the new one-parameter distribution performed the best in modeling based on criteria such as the Akaike information criterion, Bayesian information criterion, and others. The real data results show that the proposed distribution exhibits greater flexibility and improved goodness of fit compared to alternative distributions. The new XLindley distribution could be useful in modeling real-life data and may warrant further exploration in future research. Overall, this study contributes to the field of probability distributions and provides new insights for statistical modeling.

INDEX TERMS Exponential distribution, XLindley distribution, quantile function, estimation, simulation, voltage data.

I. INTRODUCTION

New distributions may be formed by merging a finite number of probability distributions with a mixed proportion, known as "finite mixture distributions". These distributions are used regularly to simulate a wide range of random occurrences, in particular, to account for the heterogeneity of the undetected data. In the field of statistical data modeling, the mixed distribution model is widely recognized.

Many researchers are very interested in studying mixes of distributions because datasets can be considered mixed populations. Work on mixtures has been done in the literature. Some of the more important references that cover various forms of distribution mixtures include: authors in [4], [11],

The associate editor coordinating the review of this manuscript and approving it for publication was Jahangir Hossain.

[13], [15], and [17] considered the mixture of the exponential and gamma or Lindley and gamma distributions.

The Lindley distribution, named after British statistician Dennis Lindley [11], is a probability distribution that is widely used in statistical modeling and data analysis. It is a one-parameter distribution that has been found to provide a good fit for various real-world datasets, especially those involving non-negative, right-skewed data. The Lindley distribution has been used in a wide range of applications, including in the fields of finance, environmental studies, and medical research, among others. The Lindley distribution is characterized by its probability density function, which exhibits a rapid drop-off at zero and a long tail to the right. The distribution is particularly useful in modeling count data with excess zeros, as well as in survival analysis and reliability modeling. In recent years, there has been an increasing interest in the Lindley distribution and its applications, leading to the development of new estimation methods and the exploration of its properties in greater detail. This new paper will contribute to this growing body of research by providing a comprehensive overview of the Lindley distribution, its properties, and its applications. One of the key advantages of the Lindley distribution is its ability to capture the effects of covariates or explanatory variables. In particular, the distribution can be used in regression modeling, where it can account for the influence of one or more predictor variables on the distribution of the response variable. This makes it a valuable tool in various applications, from epidemiology to finance. Also, for some generalizations of the Lindley distribution, one can look at [1], [5], and [16].

The one-parameter Lindley distribution can be seen as a mixture of exp(θ) and gamma(2,θ) distributions. Later, Ghitany et al. [7] worked extensively on the statistical properties of the Lindley distribution and established that it performs better than the well-known one-parameter exponential distribution in many ways. The Lindley distribution has only one scale parameter and can model data with monotonic increasing failure rates. Due to this, the analysis of different types of lifetime data may require a more flexible distribution than the Lindley distribution provides. Several one-parameter models have been proposed in the statistical literature to modify lifetime data, including the Lindley [11], exponential, Zeghdoudi [13], Ishita [27], Shanker [19], Rama [25], Pranav [9], Rani [26], Akash [18], Aradhana [21], Sujatha [24], Amarendra [20], Devya, [22] Shambhu [23], Chris-Jerry [15], XLindley [4], and Xgamma [17].

The purpose of this article is, firstly, to propose and study a new distribution with one parameter that combines the benefits of Lindley and exponential distributions. It may be used in various areas, including biology, engineering, astronomy, actuarial science, and medicine. On the other hand, the new distribution has an increased risk rate and a decreasing average residual life function. This new distribution may attract research attention.

The following is the format of this research paper: In Section II, the formulation of the proposed distribution is presented. Some distributional properties of the new model are discussed in Section III. In Section IV, the estimation procedure of the model parameter is performed by many estimators. The performances of these estimators are evaluated via a Monte Carlo simulation in Section V. In Section VI, several actuarial properties of the proposed model are investigated. The usability of the new distribution is illustrated via a real data application in Section VII. Section VIII ends the paper with a conclusion.

II. FORMULATION OF THE NEW DISTRIBUTION

Recently, Beghriche et al. [2] introduced a new statistical family named new one-parameter polynomial exponential distribution (NPED) having the probability density function (pdf)

$$f(x,\theta)=\frac{\exp(-\theta x)\sum_{k=0}^{n}x^k\,a_{k,\theta}}{\sum_{k=0}^{n}\frac{k!}{\theta^{k+1}}\,a_{k,\theta}};\quad x>0,\ \theta>0. \quad (1)$$

Our proposed model which is called new XLindley distribution (NXLD) is obtained as a special case of (1), when $n=1$, $a_{0,\theta}=1$, and $a_{1,\theta}=\theta$ as follows

$$f_{NXL}(x;\theta)=\frac{\theta}{2}(1+\theta x)\exp(-\theta x),\quad x,\ \theta>0, \quad (2)$$

and it can be obtained by mixture of $f_1(x)\sim Exp(\theta)$ and $f_2(x)\sim gamma(2,\theta)$ with $p_1=p_2=\frac{1}{2}$.

III. STATISTICAL PROPERTIES

A. MODE

The pdf (2) behavior at zero and infinity are, respectively, discussed by the following two limits as follows

$$\lim_{x\to 0}f_{NXL}(x)=\frac{\theta}{2}\ \text{and}\ \lim_{x\to\infty}f(x)=0.$$

The first derivative of $f_{NXL}(x)$ is

$$\frac{df_{NXL}(x)}{dx}=-\frac{1}{2}x\theta^3e^{-x\theta}<0, \quad (3)$$

and its second derivative is

$$\frac{d^2f_{NXL}(x)}{dx^2}=\frac{1}{2}\theta^3(x\theta-1)e^{-x\theta},$$

with $\frac{d^2f_{NXL}(x)}{dx^2}<0,\ x<\frac{1}{\theta}$, and $\frac{d^2f_{NXL}(x)}{dx^2}>0,\ x>\frac{1}{\theta}$

where $\left(\frac{1}{\theta},\theta e^{-1}\right)$ is inflection point and the mode of NXLD is $\frac{\theta}{2}$. Note that the mode of the exponential distribution is always at 0 like the mode of the NXLD.

B. SURVIVAL AND HAZARD RATE FUNCTION

The cumulative distribution function (c.d.f.) of the NXLD is defined as follows

$$F_{NXL}(x;\theta)=1-\left(\frac{1}{2}\theta x+1\right)e^{-x\theta}, \quad (4)$$

then, the survival function $S_{NXL}(x)$ and hazard rate function (hrf) $h_{NXL}(x)$ for the NXLD are, respectively, defined as follows

$$S_{NXL}(x) = 1 - F_{NXL}(x) = S_{NXL}(x) = \left(\frac{1}{2}\theta x + 1\right) e^{-x\theta}, \tag{5}$$

$$h_{NXL}(x) = \frac{f_{NXL}(x)}{1 - F_{NXL}(x)} = \frac{\theta + \theta^2 x}{\theta x + 2}. \tag{6}$$

Proposition 1: The hrf $h_{NXL}(x)$ (6) is an increasing function.

Proof: By Glaser [8] and the pdf (2), we have

$$\rho(x) = \frac{\theta^3 x}{\theta\,(1 + \theta x)}, \tag{7}$$

and its first derivative is

$$\rho^{'}(x) = \frac{\theta^2}{(x\theta + 1)^2}, \tag{8}$$

then, the $h_{NXL}(x)$ is an increasing function.

The pdf and hrf plots of the NXLD distribution are given in Figure 1. In Figure 1, it observed that the pdf is decreasing and the hrf is increasing shaped.

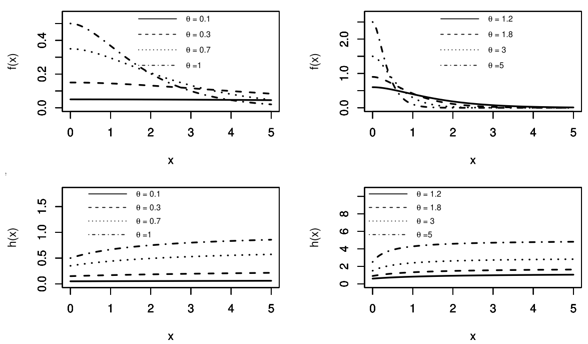


Figure 1. The pdf and hrf plots for some choices of parameter θ.

C. MOMENTS

The *rth* moment of the NXLD is defined as follows

$$\mu_r^{'} = \mathbb{E}(X^r) = \int_0^\infty x^r f_{NXL}(x)dx = \int_0^\infty x^r \frac{\theta}{2}\,(1+\theta x)\exp(-\theta x)\,dx$$
$$= \frac{1}{2\theta^r}\left(\Gamma(r+1) + \Gamma(r+2)\right). \tag{9}$$

Proposition 2: Let $X \sim$ NXLD, the mean, variance, coefficients of variation, skewness, and kurtosis for X are, respectively defined as follows

$$\mathbb{E}(X) = \frac{3}{2\theta},\; Var(X) = \frac{7}{4\theta^2},$$

$$Skewness = \sqrt{\beta_1} = \frac{\mathbb{E}(X^3)}{(Var(X))^{\frac{3}{2}}} = \frac{\frac{15}{\theta^3}}{\left(\frac{7}{4\theta^2}\right)^{\frac{3}{2}}}$$
$$= \frac{120}{49}\sqrt{7} = 6.4793,$$

$$Kurtosis = \beta_2 = \frac{\mathbb{E}(X^4)}{(Var(X))^2} = \frac{\frac{72}{\theta^4}}{\left(\frac{7}{4\theta^2}\right)^2} = \frac{1152}{49} = 23.5102,$$

$$C.V = \gamma = \frac{\sqrt{Var(X)}}{\mathbb{E}(X)} = \frac{\sqrt{\frac{7}{4\theta^2}}}{\frac{3}{2\theta}} = \frac{\sqrt{7}}{3}.$$

The new distribution is leptokurtic and right-skewed according to the skewness and kurtosis.

Theorem 1: Let $X \sim NXLD(\theta)$. Then the median$(X) < E(X)$.

Proof: Let $m = median(X)$ and $\mu = E(X) = \frac{3}{2\theta}$. Since the c.d.f. is given by (4), it follows that $F(m) = \frac{1}{2}$ and $F(\mu) = 1 - \frac{7}{4}e^{-\frac{3}{2}}$. Note that $\frac{1}{2} < 1 - \frac{7}{4}e^{-\frac{3}{2}}$. Finally, since $F(x)$ is an increasing function in $x > 0$ for all $\theta > 0$, we have $m < \mu$.

D. ENTROPY

It is generally agreed that entropy and information can be used to calculate the amount of uncertainty in a probability distribution. But many correlations have been created from the characteristics of entropy. The entropy of a random variable X measures the uncertainty's variation. The entropy of Rényi is defined as follows

$$I_R(s) = \frac{1}{1-s}\log\left\{\int_0^\infty f^s(x)\,dx\right\},$$

where $s(integer) > 0$ and $s \neq 1$. For the NXLD, we have

$$I_R(s) = \frac{1}{1-s}\log\left(\int_0^\infty \left(\frac{\theta}{2}(1+\theta x)\exp(-\theta x)\right)^s dx\right)$$
$$= \frac{1}{1-s}\log\left(\int_0^\infty \frac{\theta^s}{2^s}(1+\theta x)^s e^{-\theta s x}dx\right),$$

where

$$\int_0^\infty \frac{\theta^s}{2^s}(1+\theta x)^s e^{-\theta s x}dx$$
$$= \frac{\theta^s}{2^s}\sum_{i=0}^{n}\frac{n!}{(n-i)!i!}\int_0^\infty (\theta x)^{(n-i)} e^{-\theta s x}dx$$
$$= \frac{\theta^s}{2^s}\sum_{i=0}^{n}\frac{n!}{(n-i)!i!}\frac{\theta^{n-i}}{s\theta}\Gamma(n+1-i)(s\theta)^{i-n}.$$

Now, the Rényi entropy for the NXLD is determined as follows

$$I_R(s) = \frac{1}{1-s}\log\left(\frac{\theta^s}{2^s}\sum_{i=0}^{n}\frac{n!}{(n-i)!i!}\frac{\theta^{n-i}\Gamma(n-i+1)}{(s\theta)^{n-i+1}}\right).$$

E. STRESS-STRENGTH RELIABILITY

Stress-strength reliability is a concept used to describe the lifespan of a component that experiences random strength, represented by the variable X, and random stress, represented by the variable Y. If the stress exceeds the component's strength, it will fail immediately. Otherwise, the component will function correctly until the stress exceeds its strength. In statistical terms, the stress-strength parameter, denoted as $R = P[Y < X]$, measures component reliability. This concept has broad applications in many fields, particularly in engineering, where it is used to study the deterioration of rocket motors, static fatigue of ceramic components, aging of concrete pressure vessels, and other related issues.

One way to calculate the stress-strength reliability R for a component with independent strength and stress random variables X and Y, both follow NXLD with parameters θ_1 and θ_2, respectively, is as follows.

$$R = P[Y < X] = \int_0^{\infty} P[Y < X | X = x] f_{NXL}(x;\theta)dx$$
$$= \int_0^{\infty} f_{NXL}(x;\theta_1)F_{NXL}(x;\theta_2)$$
$$= \int_0^{\infty} \left(\frac{\theta_1}{2}(1+\theta_1 x)\exp(-\theta_1 x)\right)$$
$$\times \left(1 - \left(\frac{1}{2}\theta_2 x + 1\right)e^{-x\theta_2}\right)dx$$
$$= \frac{\theta_2\left(3\theta_1^2 + 9\theta_1\theta_2 + 4\theta_2^2\right)}{4(\theta_1+\theta_2)^3}.$$

F. STOCHASTIC ORDERING

Definition 1: Consider two random variables X and Y. Then X is said smaller than Y in the following

a) *Stochastic order ($X <_S Y$), if $F_X(t) < F_Y(t)$,$\forall t$.*
b) *Convex order ($X \leq_{cx} Y$), if for all convex functions ϕ and .provided expectation exist, $E[\phi(X)] \leq E[\phi(Y)]$*
c) *Hazard rate order ($X \leq_{hr} Y$), if $h_X(t) \geq h_Y(t)$, $\forall t$.*
d) *Likelihood ratio order ($X <_{lr} Y$), if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t.*

Remark 1: Likelihood ratio order ⇒Hazard rate order ⇒ Stochastic order If $E[X] = E[Y]$; then Convex order ⇔ Stochastic order.

Theorem 2: Let $X_i \sim NXLD(\theta_i)$; $i = 1; 2$ be two random variables. If $\theta_1 \geq \theta_2$, then: $X_1 <_{lr} X_2$; $X_1 <_{hr} X_2$; $X_1 <_S X_2$ and $X_1 \leq_{cx} X_2$.

Proof: We have

$$\frac{f_X(t)}{f_Y(t)} = \frac{\theta_1(1+\theta_1 t)}{\theta_2(1+\theta_2 t)}e^{-(\theta_1-\theta_2)t}.$$

For simplification, we use $\ln\frac{f_X(t)}{f_Y(t)}$ Now, we can find:

$$\frac{d}{dt}\ln\left(\frac{f_X(t)}{f_Y(t)}\right) = -\frac{(\theta_1-\theta_2)(\theta_1+\theta_2+t\theta_1\theta_2)t}{(t\theta_1+1)(t\theta_2+1)}.$$

To this end, if $\theta_1 \geq \theta_2$, we have $\frac{d}{dt}\ln\left(\frac{f_X(t)}{f_Y(t)}\right) \leq 0$. This means that $X_1 <_{lr} X_2$: Also, according to Remark 1 the theorem is proved.

G. LORENZ CURVE

Let X be a random variable pdf $f(x)$ and the CDF $F(x)$, the Lorenz curve L is given by

$$L(F(x)) = \frac{\int_{-\infty}^{x} tf(t)\,dt}{E(X)},$$

where $E(X)$ denotes the average. The Lorenz curve $L(F)$ may then be plotted as a function parametric in x: $L(x)$ vs. $F(x)$. In other contexts, the quantity computed here is known as size-biased distribution; it also has an important role in renewal theory.

We have it for NXLD

$$\int_0^x tf(t)\,dt = \frac{3}{2\theta} - \frac{1}{2\theta}e^{-x\theta}\left(\theta^2x^2 + 3\theta x + 3\right).$$

We obtain the Lorenz curve for the NXLD as follows

$$L(p) = 1 - \frac{(1-p)\left(\frac{1}{3}\theta x^2 + \theta x + 1\right)}{\left(\frac{1}{2}x\theta + 1\right)}.$$

where $x = F^{-1}(p)$ with $F(\cdot)$ given by (4).

H. EXTREME ORDER STATISTICS OF NXLD

Let $X_1, \ldots, X_n$ a sample of n random variables that follow the NXLD and if $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$ represents the sample mean then by the central limit theorem $\frac{\sqrt{n}(\overline{X}-E(X))}{\sqrt{Var(X)}}$ approximates the standard normal distribution when $n \to \infty$.

Theorem 3: We will study the asymptotic law of extreme values $S_n = max(X_1, \ldots, X_n)$ and $I_n = min(X_1, \ldots, X_n)$, For the distribution function defined in (4), we find that

$$\lim_{t\to\infty}\frac{1 - F_{NXL}(t+x)}{1 - F_{NXL}(t)} = exp(-\theta x).$$

and

$$\lim_{t\to 0}\frac{F_{NXL}(tx)}{F_{NXL}(t)} = x.$$

According to Theorem 1.6.2 in Leadbetter et al. [10] that there must be norming constants $\alpha_n > 0, \beta_n, \gamma_n > 0$ and δ_n such that

$$P\{\alpha_n(S_n - \beta_n) \leq x\} \to exp\left(-\exp(-\theta x)\right).$$

and

$$P\{\gamma_n(I_n - \delta_n) \leq x\} \to 1 - \exp(-x).$$

as $n \to \infty$. Using Corollary 1.6.3 in Leadbetter et al. [10], we can see that $\alpha_n = 1$ and $\beta_n = F^{-1}(1 - 1/n)$ with $F(\cdot)$ given by (4).

I. FUZZY RELIABILITY

Let T be a continuous random variable representing a system's failure time (component). The fuzzy dependability can then be calculated using the fuzzy probability in the formula

$$R_F(t) = P(T > t) = \int_t^{\infty} \nu(x) f_{NXL}(x)dx, 0 \leq t \leq x < \infty,$$

where $\nu(x)$ is a membership function that describes the degree to which each element of a given universe belongs to a fuzzy set (for more details see Chen et al. [3]. Now, assume that $\nu(x)$ is

$$\nu(x)=\begin{cases}0, & x\leq t_1\\ \dfrac{x-t_1}{t_2-t_1}, & 0\leq t_1<x<t\\ 1, & x\geq t_2\end{cases}$$

For $\nu(x)$, by the computational analysis of the function of fuzzy numbers, the lifetime $x(\lambda)$ can be obtained to correspond to a certain value of $\lambda - Cut$, $\lambda\in[0,1]$, can be obtained as $\nu(x)=\lambda\rightarrow\frac{x-t_1}{t_2-t_1}=\lambda$, then

$$\begin{cases}x(\lambda)\leq t_1, & \lambda=0\\ x(\lambda)=t_1+\gamma(t_2-t_1), & 0<\lambda<1\\ x(\lambda)\geq t_2, & \lambda=1\end{cases}$$

As a result, the fuzzy reliability values may be determined for all λ values. The fuzzy reliability definition determines the fuzzy dependability of the *NXLD*. The fuzzy reliability of the *NXLD* can be defined as,

$$R_F(t)=\left(1+\left(\frac{1}{2}\theta t_1+1\right)\right)e^{-\theta t_1}-\left(1+\left(\frac{1}{2}\theta\alpha+1\right)\right)e^{-\theta\lambda}.$$

Then $R_F(t)_{\lambda=0}=0$.

IV. ESTIMATION OF NXLD PARAMETER

This section will focus on estimating the proposed model parameter using various methods. We will obtain the estimator by maximizing or minimizing an objective function, as we will demonstrate.

Let $X_i \sim NXLD(\theta)$, $i=1\ldots n$ be n random variables, the likelihood function is defined as follows

$$L(\theta)=\left(\frac{\theta}{2}\right)^n\prod_{i=1}^{n}(1+\theta x_i))e^{-\theta\sum_{i=1}^{n}x_i}.$$

The log-likelihood function is given as follows

$$\ln\ l\,(x_i;\theta)=n\log\frac{\theta}{2}+\sum_{i=1}^{n}\log(1+\theta x_i)-\theta\sum_{i=1}^{n}x_i. \quad (10)$$

The derivatives of $ln\ l\ (x_i;\theta)$ with respect to θ is determined as follows

$$=\frac{n}{\theta}+\sum_{i=1}^{n}\left(\frac{x_i}{1+\theta x_i}\right)-\sum_{i=1}^{n}x_i.$$

To obtain the maximum likelihood estimation (MLE, E_1) of θ, $\hat{\theta}_{MLE}$, we can maximize equation (10) directly with respect to θ, or we can solve the non-linear equation $\frac{\partial\ln l(x_i;\theta)}{\partial\theta}=0$. Note that $\hat{\theta}_{MLE}$ cannot be solved analytically, so we can use numerical iteration techniques for determining it such as the Newton-Raphson algorithm.

Suppose we have an ordered random sample of size n given by $X_1, X_2, \ldots, X_n$, where each X_i is distributed according to an NXLD with parameter θ.

Then estimation of the NXLD parameter $\hat{\theta}$ is carried out using the Anderson-Darling estimation (E2) approach, which requires the minimization of the following equation.

$$A=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_i)+\log S(x_i)]$$

The estimation of the NXLD parameter $\hat{\theta}$ is carried out using the Cramér-von Mises estimation (E3) approach, which requires the minimization of the following equation.

$$C=-\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_i)-\frac{2i-1}{2n}\right]^2.$$

The estimation of the NXLD parameter $\hat{\theta}$ is carried out using the maximum product of the spacings estimation (E4) approach, which requires maximizing the following equation.

$$T=\frac{1}{n+1}\sum_{i=1}^{n+1}\log M_i,\ M_i=F\left(x_{(i)}\right)-F\left(x_{(i-1)}\right).$$

The NXLD parameter $\hat{\theta}$ is estimated using the least-squares estimation (E5) approach, which requires the minimization of the following equation.

$$S=\sum_{i=1}^{n}\left[F(x_i)-\frac{i}{n+1}\right]^2.$$

The NXLD parameter $\hat{\theta}$ is estimated using the right-tail Anderson-Darling estimation (E6) approach, which requires the minimization of the following equation.

$$RL=\frac{n}{2}-2\sum_{i=1}^{n}F\left(x_i\right)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log S\left(x_{n+1-i}\right).$$

The NXLD parameter $\hat{\theta}$ is estimated using the weighted least-squares estimation (E7) approach, which requires the minimization of the following equation.

$$W=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_i)-\frac{i}{n+1}\right]^2.$$

Suppose we have an ordered random sample of size n given by $X_1, X_2, \ldots, X_n$, where each X_i is distributed according to the NXLD with parameter θ, the estimation of the NXLD parameter $\hat{\theta}$ is carried out using the left tailed Anderson–Darling estimation (E8) approach, which requires the minimization of the following equation.

$$LT=-\frac{3}{2}n+2\sum_{i=1}^{n}F(x_{i:n})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F(x_{i:n}).$$

The NXLD parameter $\hat{\theta}$ is estimated using the minimum spacing absolute distance estimation (E9) approach, which requires the minimization of the following equation.

$$\omega_1\left(x_i\right)=\sum_{i=1}^{n+1}|M_i-\frac{1}{n+1}|.$$

Table 1. Simulation values of BIAS, MSE, and MRE for $\theta = 0.1$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.01938^{\{2\}}$	$0.02048^{\{4\}}$	$0.02219^{\{7\}}$	$0.01878^{\{1\}}$	$0.02256^{\{8\}}$	$0.01994^{\{3\}}$	$0.02167^{\{6\}}$	$0.02646^{\{10\}}$	$0.02456^{\{9\}}$	$0.02084^{\{5\}}$
	MSE	$0.00069^{\{3\}}$	$0.00075^{\{5\}}$	$0.00093^{\{7.5\}}$	$0.00057^{\{1\}}$	$0.00093^{\{7.5\}}$	$0.00069^{\{3\}}$	$0.00082^{\{6\}}$	$0.00233^{\{10\}}$	$0.00123^{\{9\}}$	$0.00069^{\{3\}}$
	MRE	$0.19377^{\{2\}}$	$0.20481^{\{4\}}$	$0.22193^{\{7\}}$	$0.18778^{\{1\}}$	$0.22559^{\{8\}}$	$0.19941^{\{3\}}$	$0.21674^{\{6\}}$	$0.26464^{\{10\}}$	$0.24555^{\{9\}}$	$0.20845^{\{5\}}$
	$\sum Ranks$	$7^{\{2\}}$	$13^{\{4.5\}}$	$21.5^{\{7\}}$	$3^{\{1\}}$	$23.5^{\{8\}}$	$9^{\{3\}}$	$18^{\{6\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$13^{\{4.5\}}$
30	BIAS	$0.01373^{\{3\}}$	$0.01421^{\{5\}}$	$0.01588^{\{8\}}$	$0.01313^{\{1\}}$	$0.01484^{\{7\}}$	$0.01326^{\{2\}}$	$0.01393^{\{4\}}$	$0.01746^{\{10\}}$	$0.01654^{\{9\}}$	$0.0146^{\{6\}}$
	MSE	$0.00032^{\{3\}}$	$0.00035^{\{6\}}$	$0.00044^{\{8\}}$	$0.00028^{\{1\}}$	$0.00037^{\{7\}}$	$0.00029^{\{2\}}$	$0.00034^{\{5\}}$	$0.00055^{\{10\}}$	$0.00045^{\{9\}}$	$0.00033^{\{4\}}$
	MRE	$0.13734^{\{3\}}$	$0.14209^{\{5\}}$	$0.15882^{\{8\}}$	$0.13132^{\{1\}}$	$0.14838^{\{7\}}$	$0.1326^{\{2\}}$	$0.13934^{\{4\}}$	$0.17459^{\{10\}}$	$0.16539^{\{9\}}$	$0.14601^{\{6\}}$
	$\sum Ranks$	$9^{\{3\}}$	$16^{\{5.5\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$6^{\{2\}}$	$13^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$16^{\{5.5\}}$
100	BIAS	$0.00701^{\{1\}}$	$0.0076^{\{4\}}$	$0.00821^{\{8\}}$	$0.00725^{\{3\}}$	$0.00783^{\{6\}}$	$0.00724^{\{2\}}$	$0.00764^{\{5\}}$	$0.00911^{\{10\}}$	$0.00902^{\{9\}}$	$0.00818^{\{7\}}$
	MSE	$8e-05^{\{2\}}$	$9e-05^{\{4.5\}}$	$0.00011^{\{8\}}$	$8e-05^{\{2\}}$	$1e-04^{\{6.5\}}$	$8e-05^{\{2\}}$	$9e-05^{\{4.5\}}$	$0.00013^{\{9.5\}}$	$0.00013^{\{9.5\}}$	$1e-04^{\{6.5\}}$
	MRE	$0.07009^{\{1\}}$	$0.07604^{\{4\}}$	$0.0821^{\{8\}}$	$0.07254^{\{3\}}$	$0.07833^{\{6\}}$	$0.07242^{\{2\}}$	$0.07641^{\{5\}}$	$0.09114^{\{10\}}$	$0.09022^{\{9\}}$	$0.08176^{\{7\}}$
	$\sum Ranks$	$9^{\{1\}}$	$17.5^{\{6\}}$	$17^{\{5\}}$	$13^{\{3\}}$	$16.5^{\{4\}}$	$11^{\{2\}}$	$19.5^{\{8\}}$	$22.5^{\{10\}}$	$20.5^{\{9\}}$	$18.5^{\{7\}}$
150	BIAS	$0.00581^{\{1\}}$	$0.00633^{\{4\}}$	$0.00661^{\{7\}}$	$0.00593^{\{2\}}$	$0.00655^{\{6\}}$	$0.00617^{\{3\}}$	$0.0064^{\{5\}}$	$0.00747^{\{10\}}$	$0.00735^{\{9\}}$	$0.00678^{\{8\}}$
	MSE	$5e-05^{\{1.5\}}$	$6e-05^{\{3.5\}}$	$7e-05^{\{6.5\}}$	$5e-05^{\{1.5\}}$	$7e-05^{\{6.5\}}$	$6e-05^{\{3.5\}}$	$7e-05^{\{6.5\}}$	$9e-05^{\{9.5\}}$	$9e-05^{\{9.5\}}$	$7e-05^{\{6.5\}}$
	MRE	$0.05806^{\{1\}}$	$0.0633^{\{4\}}$	$0.06613^{\{7\}}$	$0.05925^{\{2\}}$	$0.06549^{\{6\}}$	$0.06166^{\{3\}}$	$0.06397^{\{5\}}$	$0.07471^{\{10\}}$	$0.07352^{\{9\}}$	$0.06777^{\{8\}}$
	$\sum Ranks$	$3.5^{\{1\}}$	$11.5^{\{4\}}$	$20.5^{\{7\}}$	$5.5^{\{2\}}$	$18.5^{\{6\}}$	$9.5^{\{3\}}$	$16.5^{\{5\}}$	$29.5^{\{10\}}$	$27.5^{\{9\}}$	$22.5^{\{8\}}$
200	BIAS	$0.00484^{\{2\}}$	$0.0054^{\{4.5\}}$	$0.00562^{\{7\}}$	$0.00483^{\{1\}}$	$0.00552^{\{6\}}$	$0.00515^{\{3\}}$	$0.0054^{\{4.5\}}$	$0.00625^{\{9\}}$	$0.00673^{\{10\}}$	$0.00581^{\{8\}}$
	MSE	$4e-05^{\{2\}}$	$5e-05^{\{6\}}$	$5e-05^{\{6\}}$	$4e-05^{\{2\}}$	$5e-05^{\{6\}}$	$4e-05^{\{2\}}$	$5e-05^{\{6\}}$	$6e-05^{\{9\}}$	$7e-05^{\{10\}}$	$5e-05^{\{6\}}$
	MRE	$0.04836^{\{2\}}$	$0.05399^{\{4\}}$	$0.05622^{\{7\}}$	$0.04827^{\{1\}}$	$0.05522^{\{6\}}$	$0.0515^{\{3\}}$	$0.05401^{\{5\}}$	$0.06248^{\{9\}}$	$0.06727^{\{10\}}$	$0.05813^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$14.5^{\{4\}}$	$20^{\{7\}}$	$4^{\{1\}}$	$18^{\{6\}}$	$8^{\{3\}}$	$15.5^{\{5\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$22^{\{8\}}$
300	BIAS	$0.00418^{\{2\}}$	$0.00466^{\{6\}}$	$0.0046^{\{5\}}$	$0.00396^{\{1\}}$	$0.00474^{\{8\}}$	$0.0042^{\{3\}}$	$0.0045^{\{4\}}$	$0.00508^{\{9\}}$	$0.00526^{\{10\}}$	$0.00467^{\{7\}}$
	MSE	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$3e-05^{\{4.5\}}$	$4e-05^{\{9.5\}}$	$4e-05^{\{9.5\}}$	$3e-05^{\{4.5\}}$
	MRE	$0.04182^{\{2\}}$	$0.04655^{\{6\}}$	$0.04598^{\{5\}}$	$0.0396^{\{1\}}$	$0.0474^{\{8\}}$	$0.04201^{\{3\}}$	$0.045^{\{4\}}$	$0.05078^{\{9\}}$	$0.05262^{\{10\}}$	$0.04666^{\{7\}}$
	$\sum Ranks$	$8.5^{\{2\}}$	$16.5^{\{6\}}$	$14.5^{\{5\}}$	$6.5^{\{1\}}$	$20.5^{\{8\}}$	$10.5^{\{3\}}$	$12.5^{\{4\}}$	$27.5^{\{9\}}$	$29.5^{\{10\}}$	$18.5^{\{7\}}$

Table 2. Simulation values of BIAS, MSE and MRE for $\theta = 0.5$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.10121^{\{4\}}$	$0.10097^{\{3\}}$	$0.11391^{\{7\}}$	$0.09387^{\{1\}}$	$0.11747^{\{9\}}$	$0.09915^{\{2\}}$	$0.10687^{\{6\}}$	$0.13971^{\{10\}}$	$0.11515^{\{8\}}$	$0.10536^{\{5\}}$
	MSE	$0.02029^{\{5\}}$	$0.01794^{\{3\}}$	$0.02442^{\{7\}}$	$0.01415^{\{1\}}$	$0.02929^{\{9\}}$	$0.0171^{\{2\}}$	$0.02064^{\{6\}}$	$0.04157^{\{10\}}$	$0.02518^{\{8\}}$	$0.01895^{\{4\}}$
	MRE	$0.20242^{\{4\}}$	$0.20195^{\{3\}}$	$0.22783^{\{7\}}$	$0.18774^{\{1\}}$	$0.23495^{\{9\}}$	$0.19831^{\{2\}}$	$0.21374^{\{6\}}$	$0.27942^{\{10\}}$	$0.2303^{\{8\}}$	$0.21071^{\{5\}}$
	$\sum Ranks$	$13^{\{4\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$27^{\{9\}}$	$6^{\{2\}}$	$18^{\{6\}}$	$30^{\{10\}}$	$24^{\{8\}}$	$14^{\{5\}}$
30	BIAS	$0.0667^{\{2\}}$	$0.07008^{\{4\}}$	$0.0722^{\{5\}}$	$0.06244^{\{1\}}$	$0.07366^{\{7\}}$	$0.06701^{\{3\}}$	$0.07504^{\{8\}}$	$0.08784^{\{10\}}$	$0.08545^{\{9\}}$	$0.07329^{\{6\}}$
	MSE	$0.00749^{\{2\}}$	$0.00824^{\{4\}}$	$0.00886^{\{6\}}$	$0.00617^{\{1\}}$	$0.00933^{\{7\}}$	$0.00754^{\{3\}}$	$0.00943^{\{8\}}$	$0.01424^{\{9\}}$	$0.01474^{\{10\}}$	$0.00865^{\{5\}}$
	MRE	$0.13339^{\{2\}}$	$0.14015^{\{4\}}$	$0.1444^{\{5\}}$	$0.12488^{\{1\}}$	$0.14731^{\{7\}}$	$0.13402^{\{3\}}$	$0.15007^{\{8\}}$	$0.17567^{\{10\}}$	$0.1709^{\{9\}}$	$0.14658^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$16^{\{5\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$9^{\{3\}}$	$24^{\{8\}}$	$29^{\{10\}}$	$28^{\{9\}}$	$17^{\{6\}}$
100	BIAS	$0.03402^{\{1\}}$	$0.03695^{\{4\}}$	$0.04077^{\{8\}}$	$0.03525^{\{2\}}$	$0.03883^{\{6\}}$	$0.03527^{\{3\}}$	$0.03701^{\{5\}}$	$0.04536^{\{9\}}$	$0.04663^{\{10\}}$	$0.03894^{\{7\}}$
	MSE	$0.00192^{\{1\}}$	$0.00226^{\{5\}}$	$0.00264^{\{8\}}$	$0.00194^{\{2\}}$	$0.00246^{\{7\}}$	$0.00207^{\{3\}}$	$0.00223^{\{4\}}$	$0.0034^{\{9\}}$	$0.00356^{\{10\}}$	$0.0024^{\{6\}}$
	MRE	$0.06804^{\{1\}}$	$0.0739^{\{4\}}$	$0.08154^{\{8\}}$	$0.0705^{\{2\}}$	$0.07766^{\{6\}}$	$0.07055^{\{3\}}$	$0.07403^{\{5\}}$	$0.09072^{\{9\}}$	$0.09326^{\{10\}}$	$0.07789^{\{7\}}$
	$\sum Ranks$	$3^{\{1\}}$	$13^{\{4\}}$	$24^{\{8\}}$	$6^{\{2\}}$	$19^{\{6\}}$	$9^{\{3\}}$	$14^{\{5\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$20^{\{7\}}$
150	BIAS	$0.02994^{\{2\}}$	$0.03143^{\{5\}}$	$0.03226^{\{6\}}$	$0.02861^{\{1\}}$	$0.03382^{\{7\}}$	$0.03051^{\{3\}}$	$0.03103^{\{4\}}$	$0.0369^{\{10\}}$	$0.03602^{\{9\}}$	$0.0344^{\{8\}}$
	MSE	$0.00144^{\{2\}}$	$0.00158^{\{5\}}$	$0.00168^{\{6\}}$	$0.00127^{\{1\}}$	$0.00175^{\{7\}}$	$0.0015^{\{3\}}$	$0.00155^{\{4\}}$	$0.00212^{\{10\}}$	$0.00207^{\{9\}}$	$0.00187^{\{8\}}$
	MRE	$0.05989^{\{2\}}$	$0.06287^{\{5\}}$	$0.06453^{\{6\}}$	$0.05722^{\{1\}}$	$0.06764^{\{7\}}$	$0.06102^{\{3\}}$	$0.06207^{\{4\}}$	$0.0738^{\{10\}}$	$0.07203^{\{9\}}$	$0.06881^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$24^{\{8\}}$
200	BIAS	$0.02466^{\{1\}}$	$0.02631^{\{3\}}$	$0.02952^{\{8\}}$	$0.02649^{\{4\}}$	$0.02887^{\{7\}}$	$0.02565^{\{2\}}$	$0.02738^{\{5\}}$	$0.03149^{\{9\}}$	$0.03235^{\{10\}}$	$0.02837^{\{6\}}$
	MSE	$0.00097^{\{1\}}$	$0.00113^{\{4\}}$	$0.00138^{\{8\}}$	$0.0011^{\{3\}}$	$0.00133^{\{7\}}$	$0.00107^{\{2\}}$	$0.00115^{\{5\}}$	$0.0016^{\{9\}}$	$0.00165^{\{10\}}$	$0.00129^{\{6\}}$
	MRE	$0.04932^{\{1\}}$	$0.05262^{\{3\}}$	$0.05903^{\{8\}}$	$0.05298^{\{4\}}$	$0.05774^{\{7\}}$	$0.05131^{\{2\}}$	$0.05477^{\{5\}}$	$0.06299^{\{9\}}$	$0.0647^{\{10\}}$	$0.05674^{\{6\}}$
	$\sum Ranks$	$3^{\{1\}}$	$10^{\{3\}}$	$24^{\{8\}}$	$11^{\{4\}}$	$21^{\{7\}}$	$6^{\{2\}}$	$15^{\{5\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$18^{\{6\}}$
300	BIAS	$0.01914^{\{1\}}$	$0.02265^{\{5\}}$	$0.02337^{\{6\}}$	$0.02127^{\{3\}}$	$0.0236^{\{7\}}$	$0.02081^{\{2\}}$	$0.0225^{\{4\}}$	$0.02668^{\{10\}}$	$0.0257^{\{9\}}$	$0.02446^{\{8\}}$
	MSE	$0.00058^{\{1\}}$	$0.00083^{\{5\}}$	$0.00088^{\{6\}}$	$0.00071^{\{2.5\}}$	$0.00091^{\{7\}}$	$0.00071^{\{2.5\}}$	$0.00079^{\{4\}}$	$0.00113^{\{10\}}$	$0.00104^{\{9\}}$	$0.00092^{\{8\}}$
	MRE	$0.03829^{\{1\}}$	$0.04531^{\{5\}}$	$0.04673^{\{6\}}$	$0.04254^{\{3\}}$	$0.0472^{\{7\}}$	$0.04163^{\{2\}}$	$0.045^{\{4\}}$	$0.05336^{\{10\}}$	$0.0514^{\{9\}}$	$0.04892^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$8.5^{\{3\}}$	$21^{\{7\}}$	$6.5^{\{2\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$24^{\{8\}}$

The estimation of the NXLD parameter $\hat{\theta}$ is carried out using the minimum spacing absolute-log distance estimation (E10) approach, which requires the minimization of the following equation.

$$\omega_2(x_i) = \sum_{i=1}^{n+1} |\log M_i - \log \frac{1}{n+1}|.$$

V. NUMERICAL SIMULATION

This section will utilize all the estimation methods previously discussed. We will investigate the performance of these various techniques when applied to estimate the parameter of NXLD. Furthermore, we will compare the numerical values obtained from each approach to evaluate and contrast their effectiveness by determining an average of bias (BIAS) $|Bias(\widehat{\boldsymbol{\theta}})| = \frac{1}{M}\sum_{i=1}^{M} |\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2$, and mean relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M} |\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|/\boldsymbol{\theta}$.

To determine the optimal method for estimating model parameters, one potential approach is relying on simulation results. In our case, we generated 1000 random samples from the NXLD distribution using the R programming language. The sample sizes varied between 15, 30, 100, 150, 200, and 300. Tables 1-6 present the outcomes obtained from our simulation, with each value representing the ranking of a particular estimation method relative to the others. Meanwhile, Table 7 shows our estimator's partial and overall rankings. Based on the results of our simulations using random samples from the NXLD, we conclude that the MPSE approach is the most effective for estimating the NXLD parameter.

Table 3. Simulation values of BIAS, MSE and MRE for $\theta = 1.0$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.19213^{\{2\}}$	$0.19928^{\{4\}}$	$0.21785^{\{7\}}$	$0.18068^{\{1\}}$	$0.22533^{\{9\}}$	$0.19645^{\{3\}}$	$0.20634^{\{6\}}$	$0.26011^{\{10\}}$	$0.21863^{\{8\}}$	$0.20455^{\{5\}}$
	MSE	$0.06526^{\{3\}}$	$0.06621^{\{4\}}$	$0.09131^{\{8\}}$	$0.05171^{\{1\}}$	$0.08881^{\{7\}}$	$0.07214^{\{5\}}$	$0.07926^{\{6\}}$	$0.1407^{\{10\}}$	$0.09525^{\{9\}}$	$0.06481^{\{2\}}$
	MRE	$0.19213^{\{2\}}$	$0.19928^{\{4\}}$	$0.21785^{\{7\}}$	$0.18068^{\{1\}}$	$0.22533^{\{9\}}$	$0.19645^{\{3\}}$	$0.20634^{\{6\}}$	$0.26011^{\{10\}}$	$0.21863^{\{8\}}$	$0.20455^{\{5\}}$
	$\sum Ranks$	$7^{\{2\}}$	$12^{\{4.5\}}$	$22^{\{7\}}$	$3^{\{1\}}$	$25^{\{8.5\}}$	$11^{\{3\}}$	$18^{\{6\}}$	$30^{\{10\}}$	$25^{\{8.5\}}$	$12^{\{4.5\}}$
30	BIAS	$0.12632^{\{1\}}$	$0.14259^{\{5\}}$	$0.15065^{\{8\}}$	$0.1288^{\{2\}}$	$0.14613^{\{6\}}$	$0.13473^{\{3\}}$	$0.14163^{\{4\}}$	$0.17816^{\{10\}}$	$0.16722^{\{9\}}$	$0.15011^{\{7\}}$
	MSE	$0.02698^{\{2\}}$	$0.03339^{\{4\}}$	$0.04088^{\{8\}}$	$0.02626^{\{1\}}$	$0.03584^{\{6\}}$	$0.02961^{\{3\}}$	$0.03353^{\{5\}}$	$0.0575^{\{10\}}$	$0.0497^{\{9\}}$	$0.03741^{\{7\}}$
	MRE	$0.12632^{\{1\}}$	$0.14259^{\{5\}}$	$0.15065^{\{8\}}$	$0.1288^{\{2\}}$	$0.14613^{\{6\}}$	$0.13473^{\{3\}}$	$0.14163^{\{4\}}$	$0.17816^{\{10\}}$	$0.16722^{\{9\}}$	$0.15011^{\{7\}}$
	$\sum Ranks$	$4^{\{1\}}$	$14^{\{5\}}$	$24^{\{8\}}$	$5^{\{2\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$13^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$21^{\{7\}}$
100	BIAS	$0.06851^{\{1\}}$	$0.07468^{\{4\}}$	$0.07939^{\{6\}}$	$0.07195^{\{2\}}$	$0.08257^{\{7\}}$	$0.07349^{\{3\}}$	$0.07877^{\{5\}}$	$0.09307^{\{10\}}$	$0.08868^{\{9\}}$	$0.08395^{\{8\}}$
	MSE	$0.00756^{\{1\}}$	$0.00912^{\{4\}}$	$0.01006^{\{6\}}$	$0.00792^{\{2\}}$	$0.01097^{\{8\}}$	$0.0086^{\{3\}}$	$0.0098^{\{5\}}$	$0.01411^{\{10\}}$	$0.01228^{\{9\}}$	$0.01074^{\{7\}}$
	MRE	$0.06851^{\{1\}}$	$0.07468^{\{4\}}$	$0.07939^{\{6\}}$	$0.07195^{\{2\}}$	$0.08257^{\{7\}}$	$0.07349^{\{3\}}$	$0.07877^{\{5\}}$	$0.09307^{\{10\}}$	$0.08868^{\{9\}}$	$0.08395^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$22^{\{7\}}$	$9^{\{3\}}$	$15^{\{5\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$23^{\{8\}}$
150	BIAS	$0.05796^{\{1\}}$	$0.0631^{\{4\}}$	$0.06568^{\{6\}}$	$0.05906^{\{3\}}$	$0.06577^{\{7\}}$	$0.05817^{\{2\}}$	$0.06347^{\{5\}}$	$0.07278^{\{9\}}$	$0.07514^{\{10\}}$	$0.0668^{\{8\}}$
	MSE	$0.00533^{\{3\}}$	$0.00646^{\{5\}}$	$0.0071^{\{8\}}$	$0.00529^{\{1.5\}}$	$0.00687^{\{6\}}$	$0.00529^{\{1.5\}}$	$0.00633^{\{4\}}$	$0.00825^{\{9\}}$	$0.00907^{\{10\}}$	$0.00702^{\{7\}}$
	MRE	$0.05796^{\{1\}}$	$0.0631^{\{4\}}$	$0.06568^{\{6\}}$	$0.05906^{\{3\}}$	$0.06577^{\{7\}}$	$0.05817^{\{2\}}$	$0.06347^{\{5\}}$	$0.07278^{\{9\}}$	$0.07514^{\{10\}}$	$0.0668^{\{8\}}$
	$\sum Ranks$	$5^{\{1\}}$	$13^{\{4\}}$	$20^{\{6.5\}}$	$7.5^{\{3\}}$	$20^{\{6.5\}}$	$5.5^{\{2\}}$	$14^{\{5\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$23^{\{8\}}$
200	BIAS	$0.05011^{\{1\}}$	$0.05477^{\{5\}}$	$0.05807^{\{7\}}$	$0.05028^{\{2\}}$	$0.05639^{\{6\}}$	$0.0527^{\{3\}}$	$0.05462^{\{4\}}$	$0.06038^{\{9\}}$	$0.06303^{\{10\}}$	$0.05958^{\{8\}}$
	MSE	$0.00389^{\{1\}}$	$0.00489^{\{5\}}$	$0.00538^{\{7\}}$	$0.00407^{\{2\}}$	$0.00511^{\{6\}}$	$0.00426^{\{3\}}$	$0.00477^{\{4\}}$	$0.00602^{\{9\}}$	$0.00633^{\{10\}}$	$0.00545^{\{8\}}$
	MRE	$0.05011^{\{1\}}$	$0.05477^{\{5\}}$	$0.05807^{\{7\}}$	$0.05028^{\{2\}}$	$0.05639^{\{6\}}$	$0.0527^{\{3\}}$	$0.05462^{\{4\}}$	$0.06038^{\{9\}}$	$0.06303^{\{10\}}$	$0.05958^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$21^{\{7\}}$	$6^{\{2\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$24^{\{8\}}$
300	BIAS	$0.04159^{\{3\}}$	$0.04383^{\{4\}}$	$0.04633^{\{7\}}$	$0.03966^{\{1\}}$	$0.04632^{\{6\}}$	$0.04092^{\{2\}}$	$0.04454^{\{5\}}$	$0.05197^{\{10\}}$	$0.05076^{\{9\}}$	$0.04752^{\{8\}}$
	MSE	$0.00272^{\{3\}}$	$0.003^{\{4\}}$	$0.00338^{\{6\}}$	$0.00242^{\{1\}}$	$0.00352^{\{7\}}$	$0.00259^{\{2\}}$	$0.00305^{\{5\}}$	$0.00428^{\{10\}}$	$0.004^{\{9\}}$	$0.00355^{\{8\}}$
	MRE	$0.04159^{\{3\}}$	$0.04383^{\{4\}}$	$0.04633^{\{7\}}$	$0.03966^{\{1\}}$	$0.04632^{\{6\}}$	$0.04092^{\{2\}}$	$0.04454^{\{5\}}$	$0.05197^{\{10\}}$	$0.05076^{\{9\}}$	$0.04752^{\{8\}}$
	$\sum Ranks$	$9^{\{3\}}$	$12^{\{4\}}$	$20^{\{7\}}$	$3^{\{1\}}$	$19^{\{6\}}$	$6^{\{2\}}$	$15^{\{5\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$24^{\{8\}}$

Table 4. Simulation values of BIAS, MSE and MRE for $\theta = 2.0$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.38486^{\{3\}}$	$0.41306^{\{4\}}$	$0.45271^{\{9\}}$	$0.3848^{\{2\}}$	$0.44012^{\{8\}}$	$0.38257^{\{1\}}$	$0.43248^{\{7\}}$	$0.53837^{\{10\}}$	$0.42516^{\{5\}}$	$0.42999^{\{6\}}$
	MSE	$0.2678^{\{3\}}$	$0.28693^{\{4\}}$	$0.40406^{\{9\}}$	$0.24069^{\{1\}}$	$0.3668^{\{7\}}$	$0.2672^{\{2\}}$	$0.39292^{\{8\}}$	$0.60651^{\{10\}}$	$0.33543^{\{6\}}$	$0.2974^{\{5\}}$
	MRE	$0.19243^{\{3\}}$	$0.20653^{\{4\}}$	$0.22636^{\{9\}}$	$0.1924^{\{2\}}$	$0.22006^{\{8\}}$	$0.19129^{\{1\}}$	$0.21624^{\{7\}}$	$0.26918^{\{10\}}$	$0.21258^{\{5\}}$	$0.215^{\{6\}}$
	$\sum Ranks$	$9^{\{3\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$5^{\{2\}}$	$23^{\{8\}}$	$4^{\{1\}}$	$22^{\{7\}}$	$30^{\{10\}}$	$16^{\{5\}}$	$17^{\{6\}}$
30	BIAS	$0.27471^{\{3\}}$	$0.28927^{\{6\}}$	$0.28329^{\{4\}}$	$0.2656^{\{1\}}$	$0.30146^{\{8\}}$	$0.27051^{\{2\}}$	$0.28945^{\{7\}}$	$0.33688^{\{10\}}$	$0.31662^{\{9\}}$	$0.28646^{\{5\}}$
	MSE	$0.12844^{\{3\}}$	$0.14013^{\{7\}}$	$0.13922^{\{6\}}$	$0.11302^{\{1\}}$	$0.1475^{\{8\}}$	$0.12089^{\{2\}}$	$0.13721^{\{5\}}$	$0.19635^{\{10\}}$	$0.1703^{\{9\}}$	$0.13131^{\{4\}}$
	MRE	$0.13735^{\{3\}}$	$0.14464^{\{6\}}$	$0.14164^{\{4\}}$	$0.1328^{\{1\}}$	$0.15073^{\{8\}}$	$0.13526^{\{2\}}$	$0.14473^{\{7\}}$	$0.16844^{\{10\}}$	$0.15831^{\{9\}}$	$0.14323^{\{5\}}$
	$\sum Ranks$	$9^{\{3\}}$	$19^{\{6.5\}}$	$14^{\{4.5\}}$	$3^{\{1\}}$	$24^{\{8\}}$	$6^{\{2\}}$	$19^{\{6.5\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$14^{\{4.5\}}$
100	BIAS	$0.14382^{\{2\}}$	$0.1571^{\{4\}}$	$0.16412^{\{8\}}$	$0.13828^{\{1\}}$	$0.1591^{\{6\}}$	$0.14621^{\{3\}}$	$0.15741^{\{5\}}$	$0.17572^{\{9\}}$	$0.17647^{\{10\}}$	$0.16329^{\{7\}}$
	MSE	$0.0325^{\{2\}}$	$0.03839^{\{4\}}$	$0.0424^{\{8\}}$	$0.03084^{\{1\}}$	$0.04161^{\{7\}}$	$0.03297^{\{3\}}$	$0.04057^{\{5\}}$	$0.05226^{\{10\}}$	$0.05014^{\{9\}}$	$0.04086^{\{6\}}$
	MRE	$0.07191^{\{2\}}$	$0.07855^{\{4\}}$	$0.08206^{\{8\}}$	$0.06914^{\{1\}}$	$0.07955^{\{6\}}$	$0.0731^{\{3\}}$	$0.0787^{\{5\}}$	$0.08786^{\{9\}}$	$0.08823^{\{10\}}$	$0.08165^{\{7\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$19^{\{6\}}$	$9^{\{3\}}$	$15^{\{5\}}$	$28^{\{9\}}$	$29^{\{10\}}$	$20^{\{7\}}$
150	BIAS	$0.11448^{\{2\}}$	$0.12539^{\{4\}}$	$0.13264^{\{7\}}$	$0.11379^{\{1\}}$	$0.13272^{\{8\}}$	$0.11972^{\{3\}}$	$0.12553^{\{5\}}$	$0.14309^{\{9\}}$	$0.14745^{\{10\}}$	$0.13173^{\{6\}}$
	MSE	$0.02055^{\{2\}}$	$0.02459^{\{4\}}$	$0.02723^{\{6\}}$	$0.01992^{\{1\}}$	$0.02731^{\{7\}}$	$0.02273^{\{3\}}$	$0.02528^{\{5\}}$	$0.03302^{\{9\}}$	$0.03376^{\{10\}}$	$0.02783^{\{8\}}$
	MRE	$0.05724^{\{2\}}$	$0.0627^{\{4\}}$	$0.06632^{\{7\}}$	$0.0569^{\{1\}}$	$0.06636^{\{8\}}$	$0.05986^{\{3\}}$	$0.06277^{\{5\}}$	$0.07154^{\{9\}}$	$0.07372^{\{10\}}$	$0.06586^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$20^{\{6.5\}}$	$3^{\{1\}}$	$23^{\{8\}}$	$9^{\{3\}}$	$15^{\{5\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$20^{\{6.5\}}$
200	BIAS	$0.1^{\{1\}}$	$0.11061^{\{5\}}$	$0.11563^{\{7\}}$	$0.10269^{\{2\}}$	$0.11394^{\{6\}}$	$0.10388^{\{3\}}$	$0.10849^{\{4\}}$	$0.13058^{\{10\}}$	$0.12359^{\{9\}}$	$0.11657^{\{8\}}$
	MSE	$0.01598^{\{1\}}$	$0.01949^{\{5\}}$	$0.02125^{\{8\}}$	$0.01707^{\{2\}}$	$0.02034^{\{6\}}$	$0.01784^{\{3\}}$	$0.01829^{\{4\}}$	$0.02752^{\{10\}}$	$0.02396^{\{9\}}$	$0.02104^{\{7\}}$
	MRE	$0.05^{\{1\}}$	$0.0553^{\{5\}}$	$0.05782^{\{7\}}$	$0.05135^{\{2\}}$	$0.05697^{\{6\}}$	$0.05194^{\{3\}}$	$0.05424^{\{4\}}$	$0.06529^{\{10\}}$	$0.0618^{\{9\}}$	$0.05829^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$22^{\{7\}}$	$6^{\{2\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$23^{\{8\}}$
300	BIAS	$0.07791^{\{2\}}$	$0.09121^{\{5\}}$	$0.09542^{\{7\}}$	$0.07557^{\{1\}}$	$0.09543^{\{8\}}$	$0.08157^{\{3\}}$	$0.08895^{\{4\}}$	$0.10304^{\{10\}}$	$0.10082^{\{9\}}$	$0.09537^{\{6\}}$
	MSE	$0.00991^{\{2\}}$	$0.01302^{\{5\}}$	$0.01461^{\{8\}}$	$0.00922^{\{1\}}$	$0.01421^{\{6\}}$	$0.01072^{\{3\}}$	$0.01236^{\{4\}}$	$0.01692^{\{10\}}$	$0.01613^{\{9\}}$	$0.0145^{\{7\}}$
	MRE	$0.03896^{\{2\}}$	$0.0456^{\{5\}}$	$0.04771^{\{7.5\}}$	$0.03778^{\{1\}}$	$0.04771^{\{7.5\}}$	$0.04079^{\{3\}}$	$0.04448^{\{4\}}$	$0.05152^{\{10\}}$	$0.05041^{\{9\}}$	$0.04768^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$15^{\{5\}}$	$22.5^{\{8\}}$	$3^{\{1\}}$	$21.5^{\{7\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$19^{\{6\}}$

Table 5. Simulation values of BIAS, MSE and MRE for $\theta = 2.5$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.49252^{\{3\}}$	$0.51747^{\{4\}}$	$0.53682^{\{7\}}$	$0.45198^{\{1\}}$	$0.55288^{\{9\}}$	$0.47819^{\{2\}}$	$0.53671^{\{6\}}$	$0.6571^{\{10\}}$	$0.53848^{\{8\}}$	$0.52641^{\{5\}}$
	MSE	$0.45927^{\{3\}}$	$0.46279^{\{4\}}$	$0.51903^{\{6\}}$	$0.31955^{\{1\}}$	$0.55864^{\{8\}}$	$0.40998^{\{2\}}$	$0.58946^{\{9\}}$	$0.86018^{\{10\}}$	$0.5289^{\{7\}}$	$0.47642^{\{5\}}$
	MRE	$0.19701^{\{3\}}$	$0.20699^{\{4\}}$	$0.21473^{\{7\}}$	$0.18079^{\{1\}}$	$0.22115^{\{9\}}$	$0.19128^{\{2\}}$	$0.21468^{\{6\}}$	$0.26284^{\{10\}}$	$0.21539^{\{8\}}$	$0.21056^{\{5\}}$
	$\sum Ranks$	$9^{\{3\}}$	$12^{\{4\}}$	$20^{\{6\}}$	$3^{\{1\}}$	$26^{\{9\}}$	$6^{\{2\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$23^{\{8\}}$	$15^{\{5\}}$
30	BIAS	$0.32305^{\{1\}}$	$0.37251^{\{6\}}$	$0.39685^{\{9\}}$	$0.3347^{\{2\}}$	$0.38179^{\{7\}}$	$0.34322^{\{3\}}$	$0.35974^{\{4\}}$	$0.42666^{\{10\}}$	$0.39021^{\{8\}}$	$0.36665^{\{5\}}$
	MSE	$0.17771^{\{2\}}$	$0.23562^{\{6\}}$	$0.28157^{\{9\}}$	$0.17588^{\{1\}}$	$0.24317^{\{7\}}$	$0.19781^{\{3\}}$	$0.21293^{\{5\}}$	$0.3228^{\{10\}}$	$0.27011^{\{8\}}$	$0.2077^{\{4\}}$
	MRE	$0.12922^{\{1\}}$	$0.14901^{\{6\}}$	$0.15874^{\{9\}}$	$0.13388^{\{2\}}$	$0.15271^{\{7\}}$	$0.13729^{\{3\}}$	$0.1439^{\{4\}}$	$0.17066^{\{10\}}$	$0.15608^{\{8\}}$	$0.14666^{\{5\}}$
	$\sum Ranks$	$4^{\{1\}}$	$18^{\{6\}}$	$27^{\{9\}}$	$5^{\{2\}}$	$21^{\{7\}}$	$9^{\{3\}}$	$13^{\{4\}}$	$30^{\{10\}}$	$24^{\{8\}}$	$14^{\{5\}}$
100	BIAS	$0.17301^{\{2\}}$	$0.1883^{\{5\}}$	$0.20787^{\{7\}}$	$0.17202^{\{1\}}$	$0.20832^{\{8\}}$	$0.17739^{\{3\}}$	$0.18686^{\{4\}}$	$0.22111^{\{9\}}$	$0.22999^{\{10\}}$	$0.20341^{\{6\}}$
	MSE	$0.04774^{\{2\}}$	$0.05752^{\{5\}}$	$0.06898^{\{8\}}$	$0.04619^{\{1\}}$	$0.06832^{\{7\}}$	$0.04976^{\{3\}}$	$0.05648^{\{4\}}$	$0.07997^{\{9\}}$	$0.08724^{\{10\}}$	$0.06346^{\{6\}}$
	MRE	$0.0692^{\{2\}}$	$0.07532^{\{5\}}$	$0.08315^{\{7\}}$	$0.06881^{\{1\}}$	$0.08333^{\{8\}}$	$0.07096^{\{3\}}$	$0.07474^{\{4\}}$	$0.08845^{\{9\}}$	$0.092^{\{10\}}$	$0.08136^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$15^{\{5\}}$	$22^{\{7\}}$	$3^{\{1\}}$	$23^{\{8\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$18^{\{6\}}$
150	BIAS	$0.14808^{\{2\}}$	$0.15693^{\{5\}}$	$0.16948^{\{8\}}$	$0.1489^{\{3\}}$	$0.16555^{\{6\}}$	$0.14521^{\{1\}}$	$0.15335^{\{4\}}$	$0.18847^{\{10\}}$	$0.18526^{\{9\}}$	$0.16792^{\{7\}}$
	MSE	$0.03519^{\{3\}}$	$0.03986^{\{5\}}$	$0.04496^{\{8\}}$	$0.03376^{\{1\}}$	$0.04451^{\{7\}}$	$0.03425^{\{2\}}$	$0.03785^{\{4\}}$	$0.05684^{\{10\}}$	$0.05348^{\{9\}}$	$0.04423^{\{6\}}$
	MRE	$0.05923^{\{2\}}$	$0.06277^{\{5\}}$	$0.06779^{\{8\}}$	$0.05956^{\{3\}}$	$0.06622^{\{6\}}$	$0.05808^{\{1\}}$	$0.06134^{\{4\}}$	$0.07539^{\{10\}}$	$0.0741^{\{9\}}$	$0.06717^{\{7\}}$
	$\sum Ranks$	$7^{\{2.5\}}$	$15^{\{5\}}$	$24^{\{8\}}$	$7^{\{2.5\}}$	$19^{\{6\}}$	$4^{\{1\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$20^{\{7\}}$
200	BIAS	$0.12336^{\{2\}}$	$0.13071^{\{3\}}$	$0.14303^{\{7\}}$	$0.1231^{\{1\}}$	$0.14066^{\{6\}}$	$0.13505^{\{5\}}$	$0.13092^{\{4\}}$	$0.16106^{\{9\}}$	$0.16284^{\{10\}}$	$0.14677^{\{8\}}$
	MSE	$0.02398^{\{2\}}$	$0.02719^{\{3\}}$	$0.03183^{\{6.5\}}$	$0.02337^{\{1\}}$	$0.03183^{\{6.5\}}$	$0.02884^{\{5\}}$	$0.02834^{\{4\}}$	$0.04099^{\{9\}}$	$0.04191^{\{10\}}$	$0.03239^{\{8\}}$
	MRE	$0.04934^{\{2\}}$	$0.05228^{\{3\}}$	$0.05721^{\{7\}}$	$0.04924^{\{1\}}$	$0.05626^{\{6\}}$	$0.05402^{\{5\}}$	$0.05237^{\{4\}}$	$0.06442^{\{9\}}$	$0.06514^{\{10\}}$	$0.05871^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$9^{\{3\}}$	$20.5^{\{7\}}$	$3^{\{1\}}$	$18.5^{\{6\}}$	$15^{\{5\}}$	$12^{\{4\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$24^{\{8\}}$
300	BIAS	$0.0992^{\{1\}}$	$0.1096^{\{5\}}$	$0.11219^{\{6\}}$	$0.10123^{\{2\}}$	$0.1152^{\{7\}}$	$0.10279^{\{3\}}$	$0.10899^{\{4\}}$	$0.13075^{\{10\}}$	$0.12539^{\{9\}}$	$0.11602^{\{8\}}$
	MSE	$0.01554^{\{1\}}$	$0.01913^{\{5\}}$	$0.01994^{\{6\}}$	$0.01612^{\{2\}}$	$0.02082^{\{7\}}$	$0.01688^{\{3\}}$	$0.01911^{\{4\}}$	$0.02795^{\{10\}}$	$0.02554^{\{9\}}$	$0.02132^{\{8\}}$
	MRE	$0.03968^{\{1\}}$	$0.04384^{\{5\}}$	$0.04488^{\{6\}}$	$0.04049^{\{2\}}$	$0.04608^{\{7\}}$	$0.04111^{\{3\}}$	$0.0436^{\{4\}}$	$0.0523^{\{10\}}$	$0.05016^{\{9\}}$	$0.04641^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$21^{\{7\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$24^{\{8\}}$

Table 6. Simulation values of BIAS, MSE and MRE for $\theta = 3.5$.

n	Est.	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
15	BIAS	$0.68569^{\{3\}}$	$0.71023^{\{4\}}$	$0.7643^{\{8\}}$	$0.65^{\{1\}}$	$0.75908^{\{6\}}$	$0.66943^{\{2\}}$	$0.7748^{\{9\}}$	$0.9538^{\{10\}}$	$0.76303^{\{7\}}$	$0.71788^{\{5\}}$
	MSE	$0.85124^{\{4\}}$	$0.88161^{\{5\}}$	$1.18711^{\{9\}}$	$0.66575^{\{1\}}$	$0.99427^{\{6\}}$	$0.7587^{\{2\}}$	$1.112^{\{7\}}$	$2.20835^{\{10\}}$	$1.11849^{\{8\}}$	$0.82867^{\{3\}}$
	MRE	$0.19591^{\{3\}}$	$0.20292^{\{4\}}$	$0.21837^{\{8\}}$	$0.18571^{\{1\}}$	$0.21688^{\{6\}}$	$0.19127^{\{2\}}$	$0.22137^{\{9\}}$	$0.27251^{\{10\}}$	$0.21801^{\{7\}}$	$0.20511^{\{5\}}$
	$\sum Ranks$	$10^{\{3\}}$	$13^{\{4.5\}}$	$25^{\{8.5\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$25^{\{8.5\}}$	$30^{\{10\}}$	$22^{\{7\}}$	$13^{\{4.5\}}$
30	BIAS	$0.47307^{\{3\}}$	$0.49505^{\{5\}}$	$0.53515^{\{8\}}$	$0.47122^{\{2\}}$	$0.54052^{\{9\}}$	$0.43335^{\{1\}}$	$0.49342^{\{4\}}$	$0.6052^{\{10\}}$	$0.52885^{\{7\}}$	$0.51877^{\{6\}}$
	MSE	$0.37528^{\{3\}}$	$0.44254^{\{6\}}$	$0.49651^{\{9\}}$	$0.34819^{\{2\}}$	$0.47914^{\{7\}}$	$0.30249^{\{1\}}$	$0.41859^{\{4\}}$	$0.63849^{\{10\}}$	$0.49236^{\{8\}}$	$0.43142^{\{5\}}$
	MRE	$0.13516^{\{3\}}$	$0.14144^{\{5\}}$	$0.1529^{\{8\}}$	$0.13463^{\{2\}}$	$0.15443^{\{9\}}$	$0.12382^{\{1\}}$	$0.14098^{\{4\}}$	$0.17291^{\{10\}}$	$0.1511^{\{7\}}$	$0.14822^{\{6\}}$
	$\sum Ranks$	$9^{\{3\}}$	$16^{\{5\}}$	$25^{\{8.5\}}$	$6^{\{2\}}$	$25^{\{8.5\}}$	$3^{\{1\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$22^{\{7\}}$	$17^{\{6\}}$
100	BIAS	$0.25613^{\{2\}}$	$0.25971^{\{3\}}$	$0.29209^{\{8\}}$	$0.24898^{\{1\}}$	$0.27805^{\{6\}}$	$0.26338^{\{4\}}$	$0.26765^{\{5\}}$	$0.32394^{\{10\}}$	$0.31267^{\{9\}}$	$0.28608^{\{7\}}$
	MSE	$0.10348^{\{2\}}$	$0.10492^{\{3\}}$	$0.13818^{\{8\}}$	$0.09836^{\{1\}}$	$0.12344^{\{6\}}$	$0.10885^{\{4\}}$	$0.11243^{\{5\}}$	$0.16918^{\{10\}}$	$0.15554^{\{9\}}$	$0.12526^{\{7\}}$
	MRE	$0.07318^{\{2\}}$	$0.0742^{\{3\}}$	$0.08345^{\{8\}}$	$0.07114^{\{1\}}$	$0.07944^{\{6\}}$	$0.07525^{\{4\}}$	$0.07647^{\{5\}}$	$0.09255^{\{10\}}$	$0.08933^{\{9\}}$	$0.08174^{\{7\}}$
	$\sum Ranks$	$6^{\{2\}}$	$9^{\{3\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$12^{\{4\}}$	$15^{\{5\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$21^{\{7\}}$
150	BIAS	$0.19275^{\{1\}}$	$0.2248^{\{5\}}$	$0.22753^{\{6\}}$	$0.1991^{\{2\}}$	$0.2315^{\{7\}}$	$0.19969^{\{3\}}$	$0.21904^{\{4\}}$	$0.26341^{\{10\}}$	$0.24797^{\{9\}}$	$0.24589^{\{8\}}$
	MSE	$0.06042^{\{1\}}$	$0.08063^{\{5\}}$	$0.08113^{\{6\}}$	$0.06292^{\{2\}}$	$0.08557^{\{7\}}$	$0.06471^{\{3\}}$	$0.07721^{\{4\}}$	$0.11109^{\{10\}}$	$0.10002^{\{9\}}$	$0.09315^{\{8\}}$
	MRE	$0.05507^{\{1\}}$	$0.06423^{\{5\}}$	$0.06501^{\{6\}}$	$0.05689^{\{2\}}$	$0.06614^{\{7\}}$	$0.05706^{\{3\}}$	$0.06258^{\{4\}}$	$0.07526^{\{10\}}$	$0.07085^{\{9\}}$	$0.07025^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$21^{\{7\}}$	$9^{\{3\}}$	$12^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$24^{\{8\}}$
200	BIAS	$0.16944^{\{1\}}$	$0.18874^{\{5\}}$	$0.20154^{\{6\}}$	$0.17736^{\{2\}}$	$0.21146^{\{8\}}$	$0.18211^{\{3\}}$	$0.18872^{\{4\}}$	$0.22455^{\{10\}}$	$0.21619^{\{9\}}$	$0.20259^{\{7\}}$
	MSE	$0.04587^{\{1\}}$	$0.0555^{\{4\}}$	$0.06538^{\{7\}}$	$0.05093^{\{2\}}$	$0.06835^{\{8\}}$	$0.05342^{\{3\}}$	$0.05691^{\{5\}}$	$0.08141^{\{10\}}$	$0.07444^{\{9\}}$	$0.06417^{\{6\}}$
	MRE	$0.04841^{\{1\}}$	$0.05393^{\{5\}}$	$0.05758^{\{6\}}$	$0.05068^{\{2\}}$	$0.06042^{\{8\}}$	$0.05203^{\{3\}}$	$0.05392^{\{4\}}$	$0.06416^{\{10\}}$	$0.06177^{\{9\}}$	$0.05788^{\{7\}}$
	$\sum Ranks$	$3^{\{1\}}$	$14^{\{5\}}$	$19^{\{6\}}$	$6^{\{2\}}$	$24^{\{8\}}$	$9^{\{3\}}$	$13^{\{4\}}$	$30^{\{10\}}$	$27^{\{9\}}$	$20^{\{7\}}$
300	BIAS	$0.14094^{\{2\}}$	$0.15762^{\{5\}}$	$0.15709^{\{4\}}$	$0.14042^{\{1\}}$	$0.15828^{\{6\}}$	$0.14403^{\{3\}}$	$0.16179^{\{7\}}$	$0.17413^{\{9\}}$	$0.19088^{\{10\}}$	$0.16854^{\{8\}}$
	MSE	$0.03096^{\{1\}}$	$0.03901^{\{4\}}$	$0.03928^{\{5\}}$	$0.03166^{\{2\}}$	$0.03979^{\{6\}}$	$0.03316^{\{3\}}$	$0.04084^{\{7\}}$	$0.04784^{\{9\}}$	$0.0558^{\{10\}}$	$0.04415^{\{8\}}$
	MRE	$0.04027^{\{2\}}$	$0.04503^{\{5\}}$	$0.04488^{\{4\}}$	$0.04012^{\{1\}}$	$0.04522^{\{6\}}$	$0.04115^{\{3\}}$	$0.04623^{\{7\}}$	$0.04975^{\{9\}}$	$0.05454^{\{10\}}$	$0.04815^{\{8\}}$
	$\sum Ranks$	$5^{\{2\}}$	$14^{\{5\}}$	$13^{\{4\}}$	$4^{\{1\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$30^{\{10\}}$	$24^{\{8\}}$

VI. ACTUARIAL MEASURES

In this section, some actuarial properties of the NXLD are discussed and derived mathematically.

Table 7. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of the model parameter.

Parameter	n	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
$\theta = 0.1$	15	2.0	4.5	7.0	1.0	8.0	3.0	6.0	10.0	9.0	4.5
	30	3.0	5.5	8.0	1.0	7.0	2.0	4.0	10.0	9.0	5.5
	100	1.0	6.0	5.0	3.0	4.0	2.0	8.0	10.0	9.0	7.0
	150	1.0	4.0	7.0	2.0	6.0	3.0	5.0	10.0	9.0	8.0
	200	2.0	4.0	7.0	1.0	6.0	3.0	5.0	9.0	10.0	8.0
	300	2.0	6.0	5.0	1.0	8.0	3.0	4.0	9.0	10.0	7.0
$\theta = 0.5$	15	4.0	3.0	7.0	1.0	9.0	2.0	6.0	10.0	8.0	5.0
	30	2.0	4.0	5.0	1.0	7.0	3.0	8.0	10.0	9.0	6.0
	100	1.0	4.0	8.0	2.0	6.0	3.0	5.0	9.0	10.0	7.0
	150	2.0	5.0	6.0	1.0	7.0	3.0	4.0	10.0	9.0	8.0
	200	1.0	3.0	8.0	4.0	7.0	2.0	5.0	9.0	10.0	6.0
	300	1.0	5.0	6.0	3.0	7.0	2.0	4.0	10.0	9.0	8.0
$\theta = 1.0$	15	2.0	4.5	7.0	1.0	8.5	3.0	6.0	10.0	8.5	4.5
	30	1.0	5.0	8.0	2.0	6.0	3.0	4.0	10.0	9.0	7.0
	100	1.0	4.0	6.0	2.0	7.0	3.0	5.0	10.0	9.0	8.0
	150	1.0	4.0	6.5	3.0	6.5	2.0	5.0	9.0	10.0	8.0
	200	1.0	5.0	7.0	2.0	6.0	3.0	4.0	9.0	10.0	8.0
	300	3.0	4.0	7.0	1.0	6.0	2.0	5.0	10.0	9.0	8.0
$\theta = 2.0$	15	3.0	4.0	9.0	2.0	8.0	1.0	7.0	10.0	5.0	6.0
	30	3.0	6.5	4.5	1.0	8.0	2.0	6.5	10.0	9.0	4.5
	100	2.0	4.0	8.0	1.0	6.0	3.0	5.0	9.0	10.0	7.0
	150	2.0	4.0	6.5	1.0	8.0	3.0	5.0	9.0	10.0	6.5
	200	1.0	5.0	7.0	2.0	6.0	3.0	4.0	10.0	9.0	8.0
	300	2.0	5.0	8.0	1.0	7.0	3.0	4.0	10.0	9.0	6.0
$\theta = 2.5$	15	3.0	4.0	6.0	1.0	9.0	2.0	7.0	10.0	8.0	5.0
	30	1.0	6.0	9.0	2.0	7.0	3.0	4.0	10.0	8.0	5.0
	100	2.0	5.0	7.0	1.0	8.0	3.0	4.0	9.0	10.0	6.0
	150	2.5	5.0	8.0	2.5	6.0	1.0	4.0	10.0	9.0	7.0
	200	2.0	3.0	7.0	1.0	6.0	5.0	4.0	9.0	10.0	8.0
	300	1.0	5.0	6.0	2.0	7.0	3.0	4.0	10.0	9.0	8.0
$\theta = 3.5$	15	3.0	4.5	8.5	1.0	6.0	2.0	8.5	10.0	7.0	4.5
	30	3.0	5.0	8.5	2.0	8.5	1.0	4.0	10.0	7.0	6.0
	100	2.0	3.0	8.0	1.0	6.0	4.0	5.0	10.0	9.0	7.0
	150	1.0	5.0	6.0	2.0	7.0	3.0	4.0	10.0	9.0	8.0
	200	1.0	5.0	6.0	2.0	8.0	3.0	4.0	10.0	9.0	7.0
	300	2.0	5.0	4.0	1.0	6.0	3.0	7.0	9.0	10.0	8.0
∑Ranks		67.5	164.5	247.5	58.5	249.5	95.0	184.0	349.0	323.5	241.0
Overall Rank		2	4	7	1	8	3	5	10	9	6

A. THE QUANTILE FUNCTION OR VALUE AT RISK OF THE NXLD

From c.d.f. in Equation (4), the quantile function of the NXLD is defined as follows

$$Q_X(u) = VaR = x_u = -\frac{2}{\theta} - \frac{1}{\theta} W_{-1}\left[\frac{2(u-1)}{e^2}\right], \ u \in [0, 1],$$

where W_{-1} is the negative branch Lambert function

Definition 2: Risk managers use value at risk (VaR) to measure and control the level of risk exposure. The mathematical definition is

$$VaR = \inf(x \in \mathbb{R}, \ P(X > x \leq 1-p)),$$

where $p \in (0, 1)$ is the level. The formula tells us what the maximum loss we can expect tomorrow, with normal market conditions, or what amount of loss we should not exceed with a given level of probability, thus VaR is also known as a quantile risk measure and is defined as $VaR = F^{-1}(p)$ for a continuous distribution is.

B. MEAN EXCESS FUNCTION

For a claim amount random variable X, the mean excess or residual life function is the expected payment per claim on a policy with a fixed amount deductible of x, where claims with amounts less than or equal to x is completely ignored. It is defined for the NXLD as follows

$$e(x) = E(X - x/X > x) = \frac{1}{1-F(x)} \int_x^{\infty} (1 - F(u))du$$
$$= \frac{\theta x + 3}{\theta\,(\theta x + 2)}.$$

C. LIMITED EXPECTED VALUE FUNCTION

The limited expected value function L of a claim size variable X, or of the corresponding c.d.f $F(x)$, is defined as follows

$$L(u) = E\{\min(X, u)\} = \int_0^u x dF(x) + u\{1 - F(u)\}, \ u > 0.$$

The value of the function L at point x is equal to the expectation of the c.d.f $F(x)$ truncated at this point. Given a policy limit or deductible from a reinsurance perspective, say u, a limited loss random variable is defined as follows

$$X \wedge u = \min(X, u) = \begin{cases} X, & X \leq u \\ u, & X > u \end{cases}$$

The limited expected value function is defined as the expectation of the limited which is calculated as follows

$$E(X \wedge u) = \int_0^u xf(x)dx + u(1 - F(u))$$
$$= m_1(u) + u(1 - F(u)),$$

where

$$m_1(u) = \int_0^u xf(x)dx = \frac{3}{2\theta} + \frac{1}{2}\left(\theta u^2 - (1 + 2\theta)u - \frac{3}{\theta}\right) e^{-\theta u}.$$

Then, we have

$$E(X \wedge u) = \frac{3}{2\theta} + \left(\theta u^2 + (\frac{1}{2} - \theta)u - \frac{3}{2\theta}\right) e^{-\theta u}.$$

D. TAIL VALUE AT RISK

The tail value at risk (TVaR) also known as the tail conditional expectation is a risk measure associated with the general value at risk. TVaR measures the expectation of the losses beyond VaR. The TVaR is defined for the NXLD as follows

$$\begin{aligned} TVaR &= E(X|X > VaR) \\ &= \frac{1}{1-p} \int_{VaR}^{\infty} xf(x)dx \\ &= \frac{1}{1-p} \int_{VaR}^{\infty} x \frac{\theta}{2}(1 + \theta x)e^{-\theta x} dx \\ &= \frac{\theta e^{-\theta VaR}}{2(1-p)} \left(VaR^2 + \frac{3}{\theta} VaR + \frac{3}{\theta^2}\right). \end{aligned}$$

Although it virtually always represents a loss, *VaR* is conventionally reported as a positive number.

E. TAIL VARIANCE

Tail variance (TV) measures losses' conditional variance, given that they exceed VaR at a given probability P. TV is defined for the NXLD as follows

$$\begin{aligned} TV &= E(X^2|X > VaR) - (TVaR)^2 \\ &= \frac{1}{1-p} \int_{VaR}^{\infty} x^2 f(x)dx - (TVaR)^2 \\ &= \frac{\theta e^{-\theta VaR}}{2(1-p)}((VaR)^3 + \frac{4}{\theta}(VaR)^2 + \frac{8}{\theta^2}(VaR) + \frac{8}{\theta^3}) \\ &\quad - \frac{9e^{-2\theta VaR}}{4(1-p)^2}(\frac{\theta}{3}(VaR)^2 + (VaR) + \frac{1}{\theta})^2. \end{aligned}$$

VII. REAL DATA ANALYSIS

In this section, the real-life applicability of the new one-parameter model is demonstrated by two real datasets. The first real dataset represents the failure and running times of a sample of devices from a larger system field-tracking research. The data studied by [12] and for detailed information about the data, see [6]. The first data are given by: 275, 13, 147, 23, 181, 30, 65, 10, 300, 173, 106, 300, 300, 212, 300, 300, 300, 2, 261, 293, 88, 247, 28, 143, 300, 23, 300, 80, 245, 266. The second dataset indicates the failure times of eight components at three different temperatures 100, 120, 140 and taken from [14]. The second data are 14.712, 32.644, 61.979, 65.521, 105.50, 114.60, 120.40, 138.50, 8.610, 11.741, 54.535, 55.047, 58.928, 63.391, 105.18, 113.02, 2.998, 5.016, 15.628, 23.040, 27.851, 37.843, 38.050, 48.226.

We evaluate the NXLD from Xlindley (XL), gamma (G), Weibull (W), exponential (E), Lindley(L), Shanker (S), Akash (A), Zeghdoudi (Z), Chris-Jerry(CJ), and Xgamma (XG) distributions for this data. Information of pdf about competitor models is provided as follows:

$$f_{XL}(x) = \frac{\theta^2}{(1+\theta)^2}(\theta + x + 2)\exp(-\theta x), \quad x, \theta > 0$$

$$f_G(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right), \quad x, \alpha, \theta > 0$$

$$f_W(x) = \frac{\alpha}{\theta}\left(\frac{x}{\theta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\theta}\right)^\alpha\right), \quad x, \alpha, \theta > 0$$

$$f_E(x) = \theta \exp(-\theta x), \quad x, \theta > 0$$

$$f_L(x) = \frac{\theta^2}{(1+\theta)^2}(1 + x)\exp(-\theta x), \quad x, \theta > 0$$

$$f_S(x) = \frac{\theta^2}{\theta^2 + 1}(\theta + x)\exp(-\theta x), \quad x, \theta > 0$$

$$f_A(x) = \frac{\theta^3}{\theta^2 + 2}\left(1 + x^2\right)\exp(-\theta x), \quad x, \theta > 0$$

$$f_Z(x) = \frac{x\theta^3}{2 + \theta}(1 + x)\exp(-\theta x), \quad x, \theta > 0$$

$$f_{CJ}(x) = \frac{\theta^2}{2 + \theta}\left(1 + \theta x^2\right)\exp(-\theta x), \quad x, \theta > 0$$

$$f_{XG}(x) = \frac{\theta^2}{1 + \theta}\left(1 + \frac{\theta x^2}{2}\right)\exp(-\theta x), \quad x, \theta > 0$$

The MLE method is used to carry out the parameter estimation procedure for all models based on voltage data. We compute the MLEs of the parameter of models with their standard errors (SE), as well as $\widehat{\ell}$, Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent AIC (CAIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov statistic (KS), Cramér von Mises statistic (CVM), Anderson-Darling statistics (AD) and p-value of these statistics (KS p-value, CVM p-value, and AD p-value).

Modeling results for two real data are presented in Tables 8-11. It is noteworthy from Tables 8-11 that NXLD gives better results than the other ten models when all criteria are considered. Based on this result, the proposed model gave better results than the two-parameter Weibull and gamma distributions. As NXLD performs well over a wide range of one-parameter distributions as well as Weibull and gamma distributions, NXLD will become a very competitive model.

Table 8. The goodness of fit results in voltage data for models NXLD, XL, G, W, and E.

	NXLD	XL	G	W	E
$\widehat{\ell}$	-184.1358	-187.0717	-185.0207	-184.3138	-185.2901
AIC	370.2716	376.1434	374.0413	372.6277	372.5803
BIC	371.6728	377.5446	376.8437	375.4301	373.9815
CAIC	370.4144	376.2862	374.4858	373.0721	372.7231
HQIC	370.7198	376.5916	374.9379	373.5242	373.0285
KS (KS p-value)	0.2151(0.1246)	0.2286(0.0869)	0.2173(0.1178)	0.2194(0.1112)	0.2161(0.1214)
AD (AD p-value)	1.8720(0.1085)	3.0734(0.0254)	2.0004(0.0922)	2.1098(0.0804)	2.0022(0.0920)
CVM (CVM p-value)	0.3260(0.1140)	0.3797(0.0812)	0.3464(0.1001)	0.3315(0.1101)	0.3678(0.0875)
θ	0.0086	0.0112	0.0067	188.0545	0.0056
α			1.1896	1.2650	
SE of θ	0.0013	0.0014	0.2669	0.2044	0.0010
SE of α			0.0018	28.2174	

Table 9. The goodness of fit results in voltage data for models L, S, A, Z, CJ, and XG.

	L	S	A	Z	CJ	XG
$\widehat{\ell}$	-187.4624	-187.9747	-195.5284	-195.1676	-192.0710	-190.7116
AIC	376.9248	377.9493	393.0567	392.3352	386.1419	383.4232
BIC	378.3260	379.3505	394.4579	393.7364	387.5431	384.8244
CAIC	377.0677	378.0922	393.1996	392.4781	386.2848	383.5661
HQIC	377.3731	378.3976	393.5050	392.7835	386.5902	383.8715
KS (KS p-value)	0.2294(0.0851)	0.2302(0.0831)	0.2499(0.0471)	0.2495(0.0477)	0.2374(0.0679)	0.2329(0.0772)
AD (AD p-value)	3.1803(0.0225)	3.3080(0.0194)	5.9712(0.0010)	5.8873(0.0011)	5.2249(0.0023)	4.7973(0.0037)
CVM (CVM p-value)	0.3842(0.0790)	0.3891(0.0766)	0.5029(0.0384)	0.5003(0.0390)	0.4642(0.0485)	0.4471(0.0537)
θ	0.0112	0.0113	0.0169	0.0169	0.0165	0.0163
α						
SE of θ	0.0014	0.0014	0.0018	0.0018	0.0018	0.0018
SE of α						

Table 10. The goodness of fit results in failure times data for models NXLD, XL, G, W, and E.

	NXLD	XL	G	W	E
$\widehat{\ell}$	-119.4669	-119.8106	-119.3782	-119.1195	-120.2298
AIC	240.9338	241.6212	242.7565	242.2390	242.4595
BIC	242.1118	242.7992	245.1126	244.5951	243.6376
CAIC	241.1156	241.8030	243.3279	242.8105	242.6413
HQIC	241.2463	241.9337	243.3816	242.8641	242.7720
KS (KS p-value)	0.1146(0.8760)	0.1389 (0.6921)	0.1212 (0.8312)	0.1272 (0.7871)	0.1282 (0.7792)
AD (KS p-value)	0.3442 (0.9007)	0.5883 (0.6573)	0.3460 (0.8991)	0.3410 (0.9036)	0.5557 (0.6891)
CVM (KS p-value)	0.0524 (0.8660)	0.0673 (0.7729)	0.0490 (0.8868)	0.0448 (0.9109)	0.0928 (0.6254)
θ	0.0275	0.0351	1.4314	1.3003	0.0181
α			0.0260	59.5171	
SE of θ	0.0049	0.0051	0.3740	0.2167	0.0037
SE of α			0.0081	9.8175	

Table 11. The goodness of fit results in failure times data for models L, S, A, Z, CJ, and XG.

	L	S	A	Z	CJ	XG
$\widehat{\ell}$	-120.0061	-120.2673	-124.2598	-123.9266	-122.7025	-121.8237
AIC	242.0122	242.5345	250.5197	249.8532	247.4051	245.6474
BIC	243.1902	243.7126	251.6977	251.0312	248.5831	246.8254
CAIC	242.1940	242.7164	250.7015	250.0350	247.5869	245.8292
HQIC	242.3247	242.8471	250.8322	250.1657	247.7176	245.9599
KS (KS p-value)	0.1414 (0.6720)	0.1439 (0.6506)	0.1944 (0.2858)	0.1922 (0.2984)	0.1862 (0.3339)	0.1769 (0.3942)
AD (KS p-value)	0.6813 (0.5730)	0.8043 (0.4766)	2.3537 (0.0597)	2.2555 (0.0673)	1.8341 (0.1140)	1.5044 (0.1756)
CVM (KS p-value)	0.0743 (0.7302)	0.0824 (0.6824)	0.1846 (0.3007)	0.1794 (0.3125)	0.1693 (0.3375)	0.1527 (0.3834)
θ	0.0357	0.0363	0.0544	0.0539	0.0526	0.0516
α						
SE of θ	0.0051	0.0052	0.0064	0.0064	0.0063	0.0063
SE of α						

VIII. CONCLUSION

This study introduced a novel one-parameter distribution called the New XLindley Distribution (NXLD). One-parameter distributions hold significant value as they serve as fundamental models for developing future distributions. We have investigated various mathematical properties of the NXLD, including mode, moments, entropy, and stress-strength reliability. Moreover, we have explored the NXLD's applicability in actuarial science by examining its actuarial properties. Furthermore, we have addressed the estimation of the unknown parameter in the NXLD using multiple estimation techniques. To assess the performance of these estimators, we conducted a comprehensive Monte Carlo simulation and compared them based on criteria such as bias, mean squared error (MSE), and mean relative error (MRE). The examined mathematical properties contribute to a deeper understanding of the distribution's behavior, while the actuarial analyses highlight its utility in actuarial contexts. The findings of our study provide valuable insights into the characteristics and practical applications of the NXLD. After analyzing the voltage and failure time data, it is seen that the new one-parameter distribution demonstrates higher flexibility compared to several one-parameter distributions such as exponential, Lindley, Shanker, Akash, Xlindley, and others, as well as two-parameter distributions like gamma and Weibull. If a data analyst aims to utilize distribution with one parameter to model real data effectively, the new model serves as a good alternative to achieve this objective.

ACKNOWLEDGMENT

This research project was supported by the Researchers Supporting Project Number (RSPD2023R538), King Saud University, Riyadh, Saudi Arabia.

References

[1] G. Altun, M. Alizadeh, E. Altun, and O. Gamze, "Odd burr Lindley distribution with properties and applications," *Hacettepe J. Math. Statist.*, vol. 46, no. 2, pp. 255–276, 2017.

[2] A. Beghriche, H. Zeghdoudi, V. Raman, and S. Chouia, "New polynomial exponential distribution: properties and applications," *Statist. Transition New Ser.*, vol. 23, no. 3, pp. 95–112, 2022.

[3] G. Chen, T. T. Pham, and N. M. Boustany, "Introduction to fuzzy sets, fuzzy logic, and fuzzy control systems," *Appl. Mech. Rev.*, vol. 54, no. 6, pp. B102–B103, 2001.

[4] A. Beghriche, Y. A. Tashkandy, M. E. Bakr, Z. Halim, A. M. Gemeay, Md. M. Hossain, and A. H. Muse, "The inverse XLindley distribution: Properties and application," *IEEE Access*, vol. 11, pp. 47272–47281, 2023.

[5] G. M. Cordeiro, M. Alizadeh, G. Ozel, B. Hosseini, E. M. M. Ortega, and E. Altun, "The generalized odd log-logistic family of distributions: Properties, regression models and applications," *J. Stat. Comput. Simul.*, vol. 87, no. 5, pp. 908–932, Mar. 2017.

[6] G. M. Cordeiro, E. M. M. Ortega, and S. Nadarajah, "The Kumaraswamy Weibull distribution with application to failure data," *J. Franklin Inst.*, vol. 347, no. 8, pp. 1399–1429, Oct. 2010.

[7] M. E. Ghitany, B. Atieh, and S. Nadarajah, "Lindley distribution and its application," *Math. Comput. Simul.*, vol. 78, no. 4, pp. 493–506, 2008.

[8] R. E. Glaser, "Bathtub and related failure rate characterizations," *J. Amer. Stat. Assoc.*, vol. 75, no. 371, pp. 667–672, Sep. 1980.

[9] K. K. Shukla, "Pranav distribution with properties and its applications," *Biometrics Biostatistics Int. J.*, vol. 7, no. 3, pp. 244–254, 2018.

[10] M. R. Leadbetter, G. Lindgren, and H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes*. Berlin, Germany: Springer, 2012.

[11] D. V. Lindley, "Fiducial distributions and Bayes' theorem," *J. Roy. Stat. Soc., B Methodol.*, vol. 20, no. 1, pp. 102–107, Jan. 1958.

[12] W. Q. Meeker, L. A. Escobar, and F. G. Pascual, *Statistical Methods for Reliability Data*. Hoboken, NJ, USA: Wiley, 2022.

[13] H. Messaadia and H. Zeghdoudi, "Zeghdoudi distribution and its applications," *Int. J. Comput. Sci. Math.*, vol. 9, no. 1, pp. 58–65, 2018.

[14] D. P. Murthy, M. Xie, and R. Jiang, *Weibull Models*. Hoboken, NJ, USA: Wiley, 2004.

[15] C. K. Onyekwere and O. J. Obulezi, "Chris-Jerry distribution and its applications," *Asian J. Probab. Statist.*, vol. 20, no. 1, pp. 16–30, 2022.

[16] G. Ozel, M. Alizadeh, S. Cakmakyapan, G. Hamedani, E. M. Ortega, and V. G. Cancho, "The odd log-logistic Lindley Poisson model for lifetime data," *Commun. Statistics-Simulation Comput.*, vol. 46, no. 8, pp. 6513–6537, 2017.

[17] S. Sen, S. S. Maiti, and N. Chandra, "The xgamma distribution: Statistical properties and application," *J. Modern Appl. Stat. Methods*, vol. 15, no. 1, pp. 1–17, 2016.

[18] R. Shanker, "Akash distribution and its applications," *Int. J. Probab. Statist.*, vol. 4, no. 3, pp. 65–75, 2015.

[19] R. Shanker, "Shanker distribution and its applications," *Int. J. Statist. Appl. Math.*, vol. 5, no. 6, pp. 338–348, 2015.

[20] R. Shanker, "Amarendra distribution and its applications," *Amer. J. Math. Statist.*, vol. 6, no. 1, pp. 44–56, 2016.

[21] R. Shanker, "Aradhana distribution and its applications," *Int. J. Statist. Probab.*, vol. 6, no. 1, pp. 23–34, 2016.

[22] R. Shanker, "Devya distribution and its applications," *Int. J. Statist. Appl.*, vol. 6, no. 4, pp. 189–202, 2016.

[23] R. Shanker, "Shambhu distribution and its applications," *Int. J. Statist. Appl.*, vol. 5, no. 2, pp. 48–63, 2016.

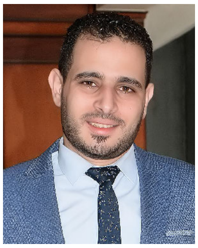
[24] R. Shanker, "Sujatha distribution and its applications," *Statist. Transition. New Ser.*, vol. 17, no. 3, pp. 391–410, 2016.

[25] R. Shanker, "Rama distribution and its application," *Int. J. Statist. Appl.*, vol. 7, no. 1, pp. 26–35, 2017.

[26] R. Shanker, "Rani distribution and its application," *Biometrics Biostatistics Int. J.*, vol. 6, no. 1, pp. 1–10, May 2017.

[27] R. Shanker, "Ishita distribution and its applications," *Biometrics Biostatistics Int. J.*, vol. 5, no. 2, pp. 1–9, Feb. 2017.

NAWEL KHODJA received the Ph.D. degree in mathematics specializing in financial mathematics from Badji Mokhtar-Annaba University, Annaba, Algeria. She is currently a Faculty Member of the Department of Mathematics, Badji Mokhtar-Annaba University. Her research interests include financial mathematics, applied statistics, and mathematics sciences.



AHMED M. GEMEAY received the M.Sc. degree in statistics from Tanta University, Tanta, Egypt, in 2020. He is currently an Assistant Lecturer in statistics and mathematics with the Department of Mathematics, Faculty of Science, Tanta University. His current research interests include the generalized classes of distributions and their special models.

HALIM ZEGHDOUDI received the Ph.D. degree in mathematics and the H.D.R. degree specializing in probability and statistics from Badji Mokhtar-Annaba University, Annaba, Algeria. He is currently a Faculty Member of the Department of Mathematics, Badji Mokhtar-Annaba University. He also did his postdoctoral research with the Waterford Institute of Technology, Waterford, Ireland. He has published more than 100 research papers in international/national journals and conferences. His research interests include actuarial science, particles systems, dynamics systems, and applied statistics. He is also a member of two editorial boards, such as *Frontiers in Applied Mathematics and Statistics* and *Asian Journal of Probability and Statistics*.

KADIR KARAKAYA received the Ph.D. degree from the Statistics Department, Faculty of Science, Selçuk University, Konya, Turkey, in 2019. His research interests include the theory of reliability, distribution, statistical inference, and quality control.

ARWA M. ALSHANGITI is an Associated Professor with the Department of Statistics and OR, King Saud University, Riyadh. Her research interests include reliability, distributions, and biostatistics.

M. E. BAKR received the Ph.D. degree in mathematical statistics from the Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt, in 2018. His research interests include the theory of reliability, the classes of life distributions, censored data, and statistical inference.

OLUWAFEMI SAMSON BALOGUN is currently a Postdoctoral Researcher with the School of Computing, University of Eastern Finland, Kuopio, Finland. His research interests include data mining, machine learning, learning analytics, biostatistics, and probability distribution models.

ABDISALAM HASSAN MUSE was born in Borama, Somalia, in 1988. He received the bachelor's degree in mathematics and physics from the Faculty of Science, Amoud University, Borama, in 2012, the M.Sc. degree in mathematics and statistics from the School of Postgraduate Studies and Research, Amoud University, in 2017, and the Ph.D. degree in statistics from the Institute of Basic Science, Technology, and Innovation, Pan African University, hosted with the Jomo Kenyatta University of Agriculture and Technology, Nairobi, Kenya, in 2022. He is currently the Head of the Faculty of Science and Humanities, School of Postgraduate Studies and Research, Amoud University. His research interests include applied statistics, survival analysis, Bayesian statistics, distribution theory, disease mapping, spatial statistics, and regression modeling. He is a maintainer of three R packages and has over 30 refereed journal articles published or accepted and serves as a reviewer for several journals. He received the Post Graduate African Union Scholarship Award during his Ph.D. (statistics) Program.

ESLAM HUSSAM received the degree in statistics from Helwan University in 2018. His current research interests include distribution theory and survival analysis.