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RESEARCH ARTICLE

Distributed State and Input Estimation Over Sensor Networks Under Deception Attacks

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ABSTRACT This paper is concerned with the joint state and input estimation problem for linear discrete time-varying systems over peer-to-peer sensor networks, in which deception attacks are taken into account. By resorting to singular value decomposition, a local estimator structure is proposed to jointly estimate the system states and unknown inputs. Then, a distributed state estimator is constructed by fusing local estimates and covariance matrices. In addition, a sufficient condition is provided to ensure the uniformly bounded estimation error (in mean square) in each node. Finally, a numerical example is provided to show the effectiveness of the proposed estimation algorithm.

INDEX TERMS Distributed estimation, sensor networks, unknown inputs, deception attacks.

I. INTRODUCTION

For several decades, the problems of distributed state estimation/filtering have received great research attention due primarily to the widespread application of various sensors. In general, the distributed state estimation (DSE) over a sensor network refers to estimate the states of a dynamical system by utilizing measurements acquired by all sensors. Up to now, many elaborated DSE strategies have been reported, including distributed Kalman filtering [1], [2], distributed H_∞ filtering [3], [4], distributed set-membership estimation [5], [6], distributed moving-horizon estimation [7], [8] and so on.

In the context of DSE, significant developments have been made by designing distributed estimation algorithms under minimal requirements, such as consensus-on-measurement estimator [9], [10], [11], consensus-on-information (CI) estimator [12], [13] and information weighted consensus estimator [14]. The interested reader is referred to the above-cited papers and references therein for the related developments. In particular, the CI estimator performs a consensus on the information matrices and the information vectors such that the estimation algorithm is convergent even if only one consensus step per iteration is performed [12].

In many practical systems, the statistical properties of some external disturbances are generally unknown and cannot be approximated to Gaussian distributions. In this case, such disturbances, which can be regarded as exogenous inputs, are likely to result in unreliable estimates when Kalman filter and its extended versions are enforced. As for the state estimation with unknown inputs, an optimal minimum-variance unbiased (MVU) filter has been originally developed in [15], only taking into account the unknown inputs appearing in the state equation. In [16], a simultaneous state-input estimation scheme has been proposed to design the MVU estimator for linear systems with direct feedthrough. In [18], the singular value decomposition technique has been exploited to construct the MVU estimator with a milder requirement on the direct feedthrough matrix than that in [16]. Based on the results in [16] and [18], an optimal three-step recursive filter has been proposed for time-varying systems with direct feedthrough [19].

On the other hand, the security problems are inevitable for a sensor network duet to its complex application environment [20], [21]. To withstand and mitigate the impacts of certain types of attacks on system performance, an efficient

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remedy is to formulate distributed secure state estimation schemes. In this case, the distributed secure state estimators are designed by modeling attack signals as disturbances with known bounds [22], [23] or well-defined statistical properties [24]. Nevertheless, it is intractable to obtain a priori knowledge on the attack signal, in the presence of intelligent attackers which inject false data into measurement channels without following specific statistics [25]. In this sense, a computational efficient recursive state estimator has been developed in [26] to deal with the distributed secure estimation problem with unknown deception attacks, using innovation analysis and the projection technique in Krein space. In [27], a state-input secure estimator has been designed for the unknown attacks appearing simultaneously in controller-actuator channel and sensor-controller channel. To the best of authors' knowledge, there have been very few results in the literature on the distributed state and input estimation under deception attacks, which constitutes one of the motivations of our current investigation.

Motivated by the above discussions, in this paper, we consider the joint state and input estimation subject to deception attacks. Following [16] and [19], a joint estimator structure is developed to simultaneously estimate the system states and unknown inputs by using singular value decomposition. Then, the DSE algorithm is designed by combining local estimates and covariance matrices from neighbors in a convex manner. The main contributions of this paper are summarized as follows:

1) Based on singular value decomposition, a novel distributed state estimator is developed against unknown inputs and deception attacks.
2) A sufficient condition is provided by showing that the estimation error in each node is uniformly bounded in mean square.

Notations: $\mathbb{R}$ denotes the set of real numbers. Given a matrix $P \in \mathbb{R}^{m\times n}$, P^T, $P^\dagger$, and $\mathrm{rank}(P)$ represent its transpose, Moore–Penrose pseudoinverse and rank, respectively. Given a square matrix $P \in \mathbb{R}^{n\times n}$, P^{-1} and $\mathrm{tr}(P)$ are its inverse and trace, respectively. Let $\|\cdot\|$ denote the induced matrix 2-norm or the Euclidean vector norm. Given a vector $x \in \mathbb{R}^n$ and a positive definite matrix $\Omega \in \mathbb{R}^{n\times n}$, $\|x\|_\Omega = \sqrt{x^T\Omega x}$ is the Ω-norm of x. $[x_1; \ldots; x_n]$ stands for the vertical concatenation of vectors $x_1, \ldots, x_n$, and for a random variable a, $\mathbb{E}\{a\}$ stands for its expectation. Finally, I and 0 represent the identity matrix and the zero matrix with appropriate dimensions.

II. PROBLEM FORMULATION

The sensor network considered in this paper is described by an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \Pi)$, where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set, $\mathcal{E} \subset \mathcal{N}\times\mathcal{N}$ is an edge set and $\Pi = [\pi_{ij}]_{N\times N}$ is a weight matrix. The elements π_{ij} are nonnegative and satisfy that $\pi_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $\pi_{ij} = 0$ otherwise. The edge $(i, j) \in \mathcal{E}$ from node j to node i means that node i can receive the information from node j. For each node $i \in \mathcal{N}$, denote its in-neighbor set as $\mathcal{N}_i \triangleq \{j : (i, j) \in \mathcal{E}\}$. Let $\mathcal{D}(\mathcal{G})$ define the diameter of graph $\mathcal{G}$ (i.e., the maximum distance between any two vertices in $\mathcal{N}$). In the sequel, it is always assumed that the network does not contain self-loop (i.e., $i \notin \mathcal{N}_i$) and the graph $\mathcal{G}$ is connected throughout the paper.

Consider a class of linear discrete time-varying systems described by

$$\begin{cases} x_{k+1} = A_k x_k + G_k d_k + \omega_k, \\ y_k^i = C_k^i x_k + \nu_k^i, \quad i \in \mathcal{N}, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the system state to be estimated, $d_k \in \mathbb{R}^{n_d}$ is the unknown input term, $y_k^i \in \mathbb{R}^{n_{y_i}}$ is the measurement output of node i, $\omega_k \in \mathbb{R}^{n_x}$ and $\nu_k^i \in \mathbb{R}^{n_{y_i}}$ are uncorrelated zero-mean Gaussian noises with covariances $Q_k > 0$ and $R_k^i > 0$, respectively, A_k, G_k and C_k^i are real-valued time-varying matrices with appropriate dimensions. Then, let us take into account the situation in which an adversary is capable to corrupt the measurements transmitted from the sensor to the estimator. In this sense, the attacked measurement $y_k^{a,i} \in \mathbb{R}^{n_{y_i}}$ can be described by

$$y_k^{a,i} = y_k^i + H_k^i a_k, \quad i \in \mathcal{N}, \tag{2}$$

where $a_k \in \mathbb{R}^{n_a}$ is the unknown deception attack signal and H_k^i is a real-valued time-varying matrix with appropriate dimension. In this work, we assume that both d_k and a_k are uniformly bounded for any $k \geq 0$ and no information is available for them.

Remark 1: In this paper, the attacker is assumed to be capable of injecting deception attacks into all nodes in the whole sensor network. For the sake of simplicity, the injected deception attack in each node $i \in \mathcal{N}$ is dependent on the unknown signal a_k in (2), and such an attack model can be found in [26]. In addition, the results in this paper can be easily extended to the case in which measurements of different sensors are corrupted by different attack signals, by replacing a_k with a_k^i.

By integrating the unknown input term d_k and the deception attack signal a_k into a new vector u_k (i.e., $u_k := [d_k; a_k] \in \mathbb{R}^{n_d+n_a}$), the attacked system is described by

$$\begin{cases} x_{k+1} = A_k x_k + \bar{G}_k u_k + \omega_k, \\ y_k^{a,i} = C_k^i x_k + \bar{H}_k^i u_k + \nu_k^i, \quad i \in \mathcal{N}, \end{cases} \tag{3}$$

where $\bar{G}_k = [G_k \ 0] \in \mathbb{R}^{n_x\times n_u}$, $\bar{H}_k^i = [0 \ H_k^i] \in \mathbb{R}^{n_{y_i}\times n_u}$, and $n_u = n_d + n_a$. Without loss of generality, it is assumed that the condition $n_{y_i} \geq n_u$ holds for any $i \in \mathcal{N}$, which means that the number of channels for $y_k^{a,i}$ is equal to or larger than that of channels for u_k.

III. DISTRIBUTED STATE AND INPUT ESTIMATION UNDER DECEPTION ATTACKS

A. MEASUREMENT EQUATION TRANSFORMATION

Inspired by [18], [19], we transform the measurement equation in (3) into a new form, which can be divided into two components: one with an unknown input term and the other without unknown input. To this end, define $p_{i,k} := \mathrm{rank}(\bar{H}_k^i)$ and apply singular value decomposition to $\bar{H}_k^i$

$$\bar{H}_k^i = U_k^i \begin{bmatrix} \Sigma_k^i & 0 \\ 0 & 0 \end{bmatrix} V_k^{i,T} = \begin{bmatrix} U_{1,k}^i & U_{2,k}^i \end{bmatrix} \begin{bmatrix} \Sigma_k^i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{1,k}^{i,T} \\ V_{2,k}^{i,T} \end{bmatrix},$$

where $U_{1,k}^i \in \mathbb{R}^{n_{y_i} \times p_{i,k}}$, $U_{2,k}^i \in \mathbb{R}^{n_{y_i} \times (n_{y_i} - p_{i,k})}$, $V_{1,k}^i = \mathbb{R}^{n_u \times p_{i,k}}$, $V_{2,k}^i = \mathbb{R}^{n_u \times (n_u - p_{i,k})}$, both U_k^i and V_k^i are unitary matrices, $\Sigma_k^i \in \mathbb{R}^{p_{i,k} \times p_{i,k}}$ is a diagonal matrix of full rank. Further, let us define two orthogonal components of u_k as

$$u_{1,k}^i = V_{1,k}^{i,T} u_k, \; u_{2,k}^i = V_{2,k}^{i,T} u_k.$$

Following the fact that $V_k^i V_k^{i,T} = I$, we obtain $u_k = V_{1,k}^i u_{1,k}^i + V_{2,k}^i u_{2,k}^i$. Then, system (3) is rewritten as

$$\begin{cases} x_{k+1} = A_k x_k + \bar{G}_{1,k}^i u_{1,k}^i + \bar{G}_{2,k}^i u_{2,k}^i + \omega_k, \\ y_k^{a,i} = C_k^i x_k + \bar{H}_{1,k}^i u_{1,k}^i + \nu_k^i, \quad i \in \mathcal{N}, \end{cases} \tag{4}$$

where $\bar{G}_{1,k}^i = \bar{G}_k V_{1,k}^i$, $\bar{G}_{2,k}^i = \bar{G}_k V_{2,k}^i$ and $\bar{H}_{1,k} = \bar{H}_k V_{1,k}^i = U_{1,k}^i \Sigma_k^i$. Using a nonsingular matrix $T_k^i = [T_{1,k}^i; T_{2,k}^i]$, we decouple the measurement $y_k^{a,i}$ into two components $z_{1,k}^i \in \mathbb{R}^{p_{i,k}}$ and $z_{2,k}^i \in \mathbb{R}^{n_{y_i} - p_{i,k}}$, i.e.,

$$\begin{bmatrix} z_{1,k}^i \\ z_{2,k}^i \end{bmatrix} = \begin{bmatrix} T_{1,k}^i \\ T_{2,k}^i \end{bmatrix} y_k^{a,i} = \begin{bmatrix} T_{1,k}^i y_k^{a,i} \\ T_{2,k}^i y_k^{a,i} \end{bmatrix}$$

with

$$T_k^i = \begin{bmatrix} I_{p_{i,k}} & -U_{1,k}^{i,T} R_k^i U_{2,k}^i (U_{2,k}^{i,T} R_k^i U_{2,k}^i)^{-1} \\ 0 & I_{(n_{y_i} - p_{i,k})} \end{bmatrix} \begin{bmatrix} U_{1,k}^{i,T} \\ U_{2,k}^{i,T} \end{bmatrix}.$$

Then, the transformed $z_{1,k}^i$ and $z_{2,k}^i$ are

$$z_{1,k}^i = C_{1,k}^i x_k + \Sigma_k^i u_{1,k}^i + \nu_{1,k}^i, \tag{5}$$
$$z_{2,k}^i = C_{2,k}^i x_k + \nu_{2,k}^i, \tag{6}$$

where $C_{1,k}^i = T_{1,k}^i C_k^i$, $C_{2,k}^i = T_{2,k}^i C_k^i = U_{2,k}^{i,T} C_k^i$, $\nu_{1,k}^i = T_{1,k}^i \nu_k^i$ and $\nu_{2,k}^i = T_{2,k}^i \nu_k^i = U_{2,k}^{i,T} \nu_k^i$.

B. JOINT STATE AND INPUT ESTIMATION UNDER DECEPTION ATTACKS

In this subsection, a recursive estimator structure is presented to deal with the joint state and input estimation in node $i \in \mathcal{N}$

$$\begin{cases} \hat{u}_{1,k}^i &= M_{1,k}^i (z_{1,k}^i - C_{1,k}^i \hat{x}_k^i) \\ \hat{u}_{2,k-1}^i &= M_{2,k}^i (z_{2,k}^i - C_{2,k}^i \hat{x}_{k|k-1}^i) \\ \hat{x}_{k|k-1}^i &= A_{k-1} \hat{x}_{k-1}^i + \bar{G}_{1,k-1}^i \hat{u}_{1,k-1}^i \\ \hat{x}_k^{*i} &= \hat{x}_{k|k-1}^i + \bar{G}_{2,k-1}^i \hat{u}_{2,k-1}^i \\ \hat{x}_k^{i,i} &= \hat{x}_k^{*i} + K_k^i (y_k^i - C_k^i \hat{x}_k^{*i}), \end{cases} \tag{7}$$

where $\hat{u}_{1,k}^i$ and $\hat{u}_{2,k-1}^i$ are the estimates of $u_{1,k}^i$ and $u_{2,k-1}^i$, respectively, $\hat{x}_k^{*i}$ and $\hat{x}_k^{i,i}$ are the propagated and updated estimates of x_k, respectively, $\hat{x}_k^i$ is the fused estimate of x_k (see (21) in the fusion step below), $M_{1,k}^i \in \mathbb{R}^{p_{i,k} \times p_{i,k}}$, $M_{2,k}^i \in \mathbb{R}^{(n_u - p_{i,k}) \times (n_{y_i} - p_{i,k})}$ and $K_k^i \in \mathbb{R}^{n_x \times n_{y_i}}$ are estimator gain matrices to be designed.

In addition, to facilitate the subsequent estimator design, we define $\tilde{x}_k^{*i} := x_k - \hat{x}_k^{*i}$, $\tilde{x}_k^{i,i} := x_k - \hat{x}_k^{i,i}$, $\tilde{x}_k^i := x_k - \hat{x}_k^i$, $P_k^{*i} := \mathbb{E}\{\tilde{x}_k^{*i} \tilde{x}_k^{*i,T}\}$, $P_k^{i,i} := \mathbb{E}\{\tilde{x}_k^{i,i} \tilde{x}_k^{i,i,T}\}$, $P_k^i := \mathbb{E}\{\tilde{x}_k^i \tilde{x}_k^{i,T}\}$, $\tilde{u}_{1,k}^i := u_{1,k}^i - \hat{u}_{1,k}^i$ and $\tilde{u}_{2,k}^i := u_{2,k}^i - \hat{u}_{2,k}^i$. In the sequel, we obtain that from (7)

$$\hat{u}_{1,k}^i = M_{1,k}^i \big(C_{1,k}^i \tilde{x}_k^i + \Sigma_k^i u_{1,k}^i + \nu_{1,k}^i \big), \tag{8}$$
$$\hat{u}_{2,k-1}^i = M_{2,k}^i \big(C_{2,k}^i A_{k-1} \tilde{x}_{k-1}^i + C_{2,k}^i \bar{G}_{1,k-1}^i \tilde{u}_{1,k-1}^i + C_{2,k}^i \bar{G}_{2,k-1}^i u_{2,k-1}^i + C_{2,k}^i \omega_{k-1} + \nu_{2,k}^i \big). \tag{9}$$

To ensure the unbiasedness of $\hat{u}_{1,k}^i$ and $\hat{u}_{2,k-1}^i$, it is required that

$$M_{1,k}^i \Sigma_k^i = I, \; M_{2,k}^i C_{2,k}^i \bar{G}_{2,k-1}^i = I. \tag{10}$$

Next, defining $\tilde{z}_{1,k}^i := z_{1,k}^i - C_{1,k}^i \hat{x}_k^i$ and $\tilde{z}_{2,k}^i := z_{2,k}^i - C_{2,k}^i \hat{x}_{k|k-1}^i$, we have

$$\tilde{z}_{1,k}^i = \Sigma_k^i u_{1,k}^i + e_{1,k}^i,$$
$$\tilde{z}_{2,k}^i = C_{2,k}^i \bar{G}_{2,k-1}^i u_{2,k-1}^i + e_{2,k}^i,$$

where $e_{1,k}^i = C_{1,k}^i \tilde{x}_k^i + \nu_{1,k}^i$ and $e_{2,k}^i = C_{2,k}^i (A_{k-1} \tilde{x}_{k-1}^i + \bar{G}_{1,k-1}^i \tilde{u}_{1,k-1}^i + \omega_{k-1}) + \nu_{2,k}^i$. Based on the unbiasedness of state estimates, we have $\mathbb{E}\{e_{1,k}^i\} = 0$ and $\mathbb{E}\{e_{2,k}^i\} = 0$, and the corresponding covariances are

$$\Xi_{1,k}^i := \mathbb{E}\{e_{1,k}^i e_{1,k}^{i,T}\} = C_{1,k}^i P_k^i C_{1,k}^{i,T} + R_{1,k}^i,$$
$$\Xi_{2,k}^i := \mathbb{E}\{e_{2,k}^i e_{2,k}^{i,T}\} = C_{2,k}^i P_{k|k-1}^i C_{2,k}^{i,T} + R_{2,k}^i,$$

where $R_{1,k}^i := \mathbb{E}\{\nu_{1,k}^i \nu_{1,k}^{i,T}\} = T_{1,k}^i R_k^i T_{1,k}^{i,T}$, $R_{2,k}^i := \mathbb{E}\{\nu_{2,k}^i \nu_{2,k}^{i,T}\} = T_{2,k}^i R_k^i T_{2,k}^{i,T} = U_{2,k}^{i,T} R_k^i U_{2,k}^i$,

$$P_{k|k-1}^i = \hat{A}_{k-1}^i P_{k-1}^i \hat{A}_{k-1}^{i,T} + \hat{Q}_{k-1}^i,$$
$$\hat{A}_{k-1}^i = A_{k-1} - \bar{G}_{1,k-1}^i M_{1,k-1}^i C_{1,k-1}^i,$$
$$\hat{Q}_{k-1}^i = Q_{k-1} + \bar{G}_{1,k-1}^i M_{1,k-1}^i R_{1,k-1}^i M_{1,k-1}^{i,T} \bar{G}_{1,k-1}^{i,T}.$$

Furthermore, by applying the well known generalized least squares estimate [17], the gain matrices $M_{1,k}^i$ and $M_{2,k}^i$ in (7) are determined by

$$M_{1,k}^i = \big[\Sigma_k^i (\Xi_{1,k}^i)^{-1} \Sigma_k^i \big]^{-1} \Sigma_k^i (\Xi_{1,k}^i)^{-1} = (\Sigma_k^i)^{-1}, \tag{11}$$
$$M_{2,k}^i = \mathcal{G}_{2,k-1}^i \bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\Xi_{2,k}^i)^{-1}, \tag{12}$$

where $\mathcal{G}_{2,k-1}^i = [\bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\Xi_{2,k}^i)^{-1} C_{2,k}^i \bar{G}_{2,k-1}^i]^{-1}$. In (12), it is intractable to calculate the explicit expression of P_{k-1}^i because the cross correlations between different sensors are unknown in this paper. Hence, the gain matrix $M_{2,k}^i$ is substituted with $\bar{M}_{2,k}^i$ given by

$$\bar{M}_{2,k}^i = \bar{\mathcal{G}}_{2,k-1}^i \bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\bar{\Xi}_{2,k}^i)^{-1}, \tag{13}$$

where

$$\bar{\mathcal{G}}_{2,k-1}^i = \big[\bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\bar{\Xi}_{2,k}^i)^{-1} C_{2,k}^i \bar{G}_{2,k-1}^i \big]^{-1},$$
$$\bar{\Xi}_{2,k}^i = U_{2,k}^{i,T} \bar{\Xi}_k^i U_{2,k}^i, \; \bar{\Xi}_k^i = C_k^i \bar{P}_{k|k-1}^i C_k^{i,T} + R_k^i,$$
$$\bar{P}_{k|k-1}^i = \hat{A}_{k-1}^i \bar{P}_{k-1}^i \hat{A}_{k-1}^{i,T} + \hat{Q}_{k-1}^i.$$

Then, the recursive estimator structure in (7) is rearranged as

$$\begin{cases} \hat{u}^i_{1,k} &= M^i_{1,k}(z^i_{1,k} - C^i_{1,k}\hat{x}^i_k) \\ \hat{u}^i_{2,k-1} &= \bar{M}^i_{2,k}(z^i_{2,k} - C^i_{2,k}\hat{x}^i_{k|k-1}) \\ \hat{x}^i_{k|k-1} &= A_{k-1}\hat{x}^i_{k-1} + \bar{G}^i_{1,k-1}\hat{u}^i_{1,k-1} \\ \hat{x}^{*i}_k &= \hat{x}^i_{k|k-1} + \bar{G}^i_{2,k-1}\hat{u}^i_{2,k-1} \\ \hat{x}^{i,i}_k &= \hat{x}^{*i}_k + K^i_k(y^i_k - C^i_k\hat{x}^{*i}_k), \end{cases} \tag{14}$$

and the condition in (10) is converted into $M^i_{1,k}\Sigma^i_k = I$ and $\bar{M}^i_{2,k}C^i_{2,k}\bar{G}^i_{2,k-1} = I$. From (14), we have $\tilde{u}^i_{1,k} = -M^i_{1,k}e^i_{1,k}$ and $\tilde{u}^i_{2,k-1} = -\bar{M}^i_{2,k}e^i_{2,k}$.

To sum up, the propagated and updated estimation errors are given by

$$\begin{aligned} \tilde{x}^{*i}_k &= A_{k-1}\tilde{x}^i_{k-1} + \bar{G}^i_{1,k-1}\tilde{u}^i_{1,k-1} + \bar{G}^i_{2,k-1}\tilde{u}^i_{2,k-1} + \omega_{k-1} \\ &= (I - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}C^i_{2,k})(\hat{A}^i_{k-1}\tilde{x}^i_{k-1} + \hat{\omega}^i_{k-1}) \\ &\quad - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}\nu^i_{2,k}, \end{aligned} \tag{15}$$

$$\tilde{x}^{i,i}_k = (I - K^i_kC^i_k)\tilde{x}^{*i}_k - K^i_kU^i_{1,k}\Sigma^i_ku^i_{1,k} - K^i_k\nu^i_k, \tag{16}$$

where $\hat{\omega}^i_{k-1} = \omega_{k-1} - \bar{G}^i_{1,k-1}M^i_{1,k-1}\nu^i_{1,k-1}$. According to (11), we have

$$\begin{aligned} \bar{G}^i_{1,k}M^i_{1,k}T^i_{1,k} &= [G_k\ 0]([0\ H^i_k])^\dagger \\ &\quad \times [I - R^i_kU^i_{2,k}(U^{i,T}_{2,k}R^i_kU^i_{2,k})^{-1}U^{i,T}_{2,k}] = 0, \end{aligned}$$

which implies that $\hat{\omega}^i_k = \omega_k, \hat{A}^i_k = A_k$ and $\hat{Q}^i_k = Q_k$. Without loss of generality, we assume $K^i_kU^i_{1,k} = 0$ such that the local estimate $\hat{x}^{i,i}_k$ is unbiased. Then, the propagated and updated error covariances are derived by

$$\begin{aligned} P^{*i}_k &= (I - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}C^i_{2,k})P^i_{k|k-1}(I - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k} \\ &\quad \times C^i_{2,k})^T + \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}R^i_{2,k}\bar{M}^{i,T}_{2,k}\bar{G}^{i,T}_{2,k-1}, \end{aligned} \tag{17}$$

$$\begin{aligned} P^{i,i}_k &= (I - K^i_kC^i_k)P^{*i}_k(I - K^i_kC^i_k)^T + K^i_kR^i_kK^{i,T}_k \\ &\quad + (I - K^i_kC^i_k)\bar{G}^i_{2,k-1}\bar{M}^i_{2,k}U^{i,T}_{2,k}R^i_kK^{i,T}_k \\ &\quad + K^i_kR^i_kU^i_{2,k}\bar{M}^{i,T}_{2,k}\bar{G}^{i,T}_{2,k-1}(I - K^i_kC^i_k)^T. \end{aligned} \tag{18}$$

C. DISTRIBUTED STATE ESTIMATION

First, let us define $\bar{P}^{i,i}_k$ as an upper bound on $P^{i,i}_k$ for any $i \in \mathcal{N}$. Then, the fused estimate and covariance are computed in a distributed manner:

$$\bar{P}^i_k = \left[\pi_{i,i}(\bar{P}^{i,i}_k)^{-1} + \sum_{j\in\mathcal{N}_i}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\right]^{-1}, \tag{19}$$

$$W^{i,j}_k = \pi_{i,j}\bar{P}^i_k(\bar{P}^{j,j}_k)^{-1},\ j \in \mathcal{N}_i \cup \{i\}, \tag{20}$$

$$\hat{x}^i_k = W^{i,i}_k\hat{x}^{i,i}_k + \sum_{j\in\mathcal{N}_i}W^{i,j}_k\hat{x}^{j,j}_k, \tag{21}$$

where $W^{i,j}_k, j \in \mathcal{N}_i \cup \{i\}$ is the fusion weight matrix.

To present the convergence property of the proposed DSE algorithm, the following consistency definition is introduced [12].

Definition 1: For a random vector x, let $\hat{x}$ be an unbiased estimate of x and P be an estimate of the corresponding error covariance. Then, the pair $(\hat{x}, P)$ is consistent if

$$\mathbb{E}\{(x - \hat{x})(x - \hat{x})^T\} \leq P. \tag{22}$$

Now the consistency of the pair $(\hat{x}^i_k, \bar{P}^i_k)$ is shown by checking whether the condition $P^i_k \leq \bar{P}^i_k$ holds.

Theorem 1: Considering the estimator structure in (14) and the fusion scheme in (19)-(21), the pair $(\hat{x}^i_k, \bar{P}^i_k)$ is consistent, provided by

$$\bar{P}^i_{k|k-1} = A_{k-1}\bar{P}^i_{k-1}A^T_{k-1} + Q_{k-1}, \tag{23}$$

$$\begin{aligned} \bar{P}^{*i}_k &= (I - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}C^i_{2,k})\bar{P}^i_{k|k-1}(I - \bar{G}^i_{2,k-1}\bar{M}^i_{2,k} \\ &\quad \times C^i_{2,k})^T + \bar{G}^i_{2,k-1}\bar{M}^i_{2,k}R^i_{2,k}\bar{M}^{i,T}_{2,k}\bar{G}^{i,T}_{2,k-1}, \end{aligned} \tag{24}$$

$$\begin{aligned} \bar{P}^{i,i}_k &= (I - K^i_kC^i_k)\bar{P}^{*i}_k(I - K^i_kC^i_k)^T + K^i_kR^i_kK^{i,T}_k \\ &\quad + (I - K^i_kC^i_k)\bar{G}^i_{2,k-1}\bar{M}^i_{2,k}U^{i,T}_{2,k}R^i_kK^{i,T}_k \\ &\quad + K^i_kR^i_kU^i_{2,k}\bar{M}^{i,T}_{2,k}\bar{G}^{i,T}_{2,k-1}(I - K^i_kC^i_k)^T. \end{aligned} \tag{25}$$

Proof of Theorem 1: From (19)-(21), the fused estimation error is given by

$$\begin{aligned} \tilde{x}^i_k &= x^i_k - \bar{P}^i_k\left[\pi_{i,i}(\bar{P}^{i,i}_k)^{-1}\hat{x}^{i,i}_k + \sum_{j\in\mathcal{N}_i}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\hat{x}^{j,j}_k\right] \\ &= \bar{P}^i_k\left[\pi_{i,i}(\bar{P}^{i,i}_k)^{-1}\tilde{x}^{i,i}_k + \sum_{j\in\mathcal{N}_i}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\tilde{x}^{j,j}_k\right]. \end{aligned}$$

Further, the fused error covariance is obtained that

$$\begin{aligned} P^i_k &= \bar{P}^i_k\mathbb{E}\Bigg\{\left[\pi_{i,i}(\bar{P}^{i,i}_k)^{-1}\tilde{x}^{i,i}_k + \sum_{j\in\mathcal{N}_i}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\tilde{x}^{j,j}_k\right] \\ &\quad \times \left[\pi_{i,i}(\bar{P}^{i,i}_k)^{-1}\tilde{x}^{i,i}_k + \sum_{j\in\mathcal{N}_i}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\tilde{x}^{j,j}_k\right]^T\Bigg\}\bar{P}^i_k \\ &\leq \frac{1}{2}\bar{P}^i_k\Bigg\{\sum_{j\in\mathcal{N}}\sum_{\ell\in\mathcal{N}}\pi_{i,j}\pi_{i,\ell}\big[(\bar{P}^{j,j}_k)^{-1}\mathbb{E}\{\tilde{x}^{j,j}_k\tilde{x}^{j,j,T}_k\}(\bar{P}^{j,j}_k)^{-1} \\ &\quad + (\bar{P}^{\ell,\ell}_k)^{-1}\mathbb{E}\{\tilde{x}^{\ell,\ell}_k\tilde{x}^{\ell,\ell,T}_k\}(\bar{P}^{\ell,\ell}_k)^{-1}\big]\Bigg\}\bar{P}^i_k \\ &= \bar{P}^i_k\left[\sum_{j\in\mathcal{N}}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\mathbb{E}\{\tilde{x}^{j,j}_k\tilde{x}^{j,j,T}_k\}(\bar{P}^{j,j}_k)^{-1}\right]\bar{P}^i_k. \end{aligned}$$

Since $\mathbb{E}\{\tilde{x}^{j,j}_k\tilde{x}^{j,j,T}_k\} \leq \bar{P}^{j,j}_k$ for any $j \in \mathcal{N}$, we have

$$P^i_k \leq \bar{P}^i_k\left[\sum_{j\in\mathcal{N}}\pi_{i,j}(\bar{P}^{j,j}_k)^{-1}\right]\bar{P}^i_k = \bar{P}^i_k.$$

This completes the proof. ■

D. ESTIMATOR GAIN DESIGN

Now we are in a position to provide the filter gain K^i_k by minimizing the trace of $\bar{P}^{i,i}_k$. We rearrange (25) as

$$\bar{P}^{i,i}_k = \bar{P}^i_{k|k-1} + K^i_k\tilde{\Xi}^i_kK^{i,T}_k - K^i_kS^{i,T}_k - S^i_kK^{i,T}_k, \tag{26}$$

where

$$\tilde{\Xi}^i_k = (I - C^i_k\bar{G}^i_{2,k-1}\bar{M}^i_{2,k}U^{i,T}_{2,k})\Xi^i_k(I - C^i_k\bar{G}^i_{2,k-1}\bar{M}^i_{2,k}U^{i,T}_{2,k}),$$

$$S_k^i = \bar{P}_k^{*i} C_k^{i,T} - \bar{G}_{2,k-1}^i \bar{M}_{2,k}^i U_{2,k}^{i,T} R_k^i,$$

and $\bar{\Xi}_k^i$ is defined in (13). It can be seen that $\bar{\Xi}_k^i$ is singular since $I - C_k^i G_{k-1} \bar{M}_k^i U_{2,k}^{i,T}$ is rank deficient, and the optimal filter gain K_k^i is in general not unique. In this case, the gain K_k^i is rewritten as a new form [17]

$$K_k^i = \tilde{K}_k^i \Gamma_k^i,$$

where $\Gamma_k^i \in \mathbb{R}^{\hbar_{i,k} \times n_{y_i}}$ is an arbitrary matrix ensuring that $\Gamma_k^i \tilde{\Xi}_k^i \Gamma_k^{i,T}$ is nonsingular and $\hbar_{i,k} = \mathrm{rank}(I - C_k^i G_{2,k-1} \bar{M}_k^i U_{2,k}^{i,T})$.

Lemma 1: Under the condition $K_k^i U_{1,k}^i = 0$, the optimal filter gain K_k^i is derived by

$$K_k^i = S_k^i \Theta_k^i (I - \Psi_k^i \Theta_k^i) = S_k^i (I - \Theta_k^i \Psi_k^i) \Theta_k^i, \tag{27}$$

such that the trace of $\bar{P}_k^{i,i}$ is minimized for any $i \in \mathcal{N}$, where

$$\Theta_k^i = \Gamma_k^{i,T} (\Gamma_k^i \tilde{\Xi}_k^i \Gamma_k^{i,T})^{-1} \Gamma_k^i,$$
$$\Psi_k^i = U_{1,k}^i (U_{1,k}^{i,T} \Theta_k^i U_{1,k}^i)^{-1} U_{1,k}^{i,T}.$$

Proof of Lemma 1: To minimize the trace of $\bar{P}_k^{i,i}$ subject to the constraint $K_k^i U_{1,k}^i = 0$, a Lagrange multiplier $\Lambda_k^i \in \mathbb{R}^{n_x \times p_{i,k}}$ is exploited. In view of (26) and $K_k^i = \tilde{K}_k^i \Gamma_k^i$, the Lagrangian is constructed as

$$\mathrm{tr}\{\bar{P}_{k|k-1}^i + \tilde{K}_k^i \Gamma_k^i \tilde{\Xi}_k^i \Gamma_k^{i,T} \tilde{K}_k^{i,T} - \tilde{K}_k^i \Gamma_k^i S_k^{i,T} - S_k^i \Gamma_k^{i,T} \tilde{K}_k^{i,T} - 2\tilde{K}_k^i \Gamma_k^i U_{1,k}^i \Lambda_k^{i,T}\}.$$

Then, take the partial derivative of it with respect to $\tilde{K}_k^i$ and let the derivative be zero

$$\tilde{K}_k^i \Gamma_k^i \tilde{\Xi}_k^i \Gamma_k^{i,T} - S_k^i \Gamma_k^{i,T} - \Lambda_k^i U_{1,k}^{i,T} \Gamma_k^{i,T} = 0.$$

Further, the following linear equation is obtained that

$$\begin{bmatrix} \Gamma_k^i \tilde{\Xi}_k^i \Gamma_k^{i,T} & -\Gamma_k^i U_{1,k}^i \\ U_{1,k}^{i,T} \Gamma_k^{i,T} & 0 \end{bmatrix} \begin{bmatrix} \tilde{K}_k^{i,T} \\ \Lambda_k^{i,T} \end{bmatrix} = \begin{bmatrix} \Gamma_k^i S_k^{i,T} \\ 0 \end{bmatrix}.$$

Multiplying left- and right-hand sides of above equation by the inverse of the coefficient matrix, the gain K_k^i is derived. ■

Lemma 2: Given $\Gamma_k^i = [0\ I_{\hbar_{i,k}}] \bar{U}_k^{i,T} (\bar{\Xi}_k^i)^{-\frac{1}{2}}$, the gain K_k^i in (27) reduces to

$$K_k^i = S_k^i \tilde{\Xi}_k^i [I - U_{1,k}^i (U_{1,k}^{i,T} \tilde{\Xi}_k^i U_{1,k}^i)^{-1} U_{1,k}^{i,T} \tilde{\Xi}_k^i], \tag{28}$$

and the corresponding bound on error covariance is

$$\bar{P}_k^{i,i} = \bar{P}_k^{*i} - K_k^i S_k^{i,T}, \tag{29}$$

where $\bar{U}_k^i$ is an orthogonal matrix obtained by using the singular value decomposition of $(\bar{\Xi}_k^i)^{-\frac{1}{2}} C_k^i \bar{G}_{2,k-1}^i = \bar{U}_k^i \bar{\Sigma}_k^i \bar{V}_k^{i,T}$ and $\tilde{\Xi}_k^i$ is defined in (26).

Proof of Lemma 2: Based on the given Γ_k^i, we have

$$\begin{aligned}
&\Gamma_k^i \breve{\Xi}_k^i \Gamma_k^{i,T} \\
&= [0\ I_{\hbar_{i,k}}] \bar{U}_k^{i,T} (\bar{\Xi}_k^i)^{-\frac{1}{2}} (I_{n_{y_i}} - C_k^i G_{2,k-1} \bar{M}_k^i U_{2,k}^{i,T}) \bar{\Xi}_k^i \\
&\quad \times (I_{n_{y_i}} - C_k^i G_{2,k-1} \bar{M}_k^i U_{2,k}^{i,T})^T (\bar{\Xi}_k^i)^{-\frac{1}{2}} \bar{U}_k^i [0\ I_{\hbar_{i,k}}]^T \\
&= [0\ I_{\hbar_{i,k}}] \bar{U}_k^{i,T} \Big(I_{n_{y_i}} - (\bar{\Xi}_k^i)^{-\frac{1}{2}} C_k^i \bar{G}_{2,k-1}^i \\
&\quad \times \Big[(\bar{\Xi}_k^i)^{-\frac{1}{2}} C_k^i \bar{G}_{2,k-1}^i \Big]^{\dagger} \Big) \bar{U}_k^i [0\ I_{\hbar_{i,k}}]^T \\
&= [0\ I_{\hbar_{i,k}}] \bar{U}_k^{i,T} \left(I_{n_{y_i}} - \bar{U}_k^i \begin{bmatrix} I_{(n_{y_i} - \hbar_{i,k})} & 0 \\ 0 & 0 \end{bmatrix} \bar{U}_k^{i,T} \right) \\
&\quad \times \bar{U}_k^i [0\ I_{\hbar_{i,k}}]^T \\
&= I_{\hbar_{i,k}}.
\end{aligned}$$

Then, substituting the above equation into Θ_k^i and rearranging, the following holds

$$\begin{aligned}
\Theta_k^i &= (\bar{\Xi}_k^i)^{-\frac{1}{2}} \bar{U}_k^i \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \bar{U}_k^{i,T} (\bar{\Xi}_k^i)^{-\frac{1}{2}} \\
&= (\bar{\Xi}_k^i)^{-\frac{1}{2}} \left(I - \bar{U}_k^i \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \bar{U}_k^{i,T} \right) (\bar{\Xi}_k^i)^{-\frac{1}{2}} \\
&= (\bar{\Xi}_k^i)^{-1} (I - C_k^i G_{2,k-1} \bar{M}_k^i U_{2,k}^{i,T}).
\end{aligned} \tag{30}$$

Following $\Theta_k^i \tilde{\Xi}_k^i \Theta_k^i = \Theta_k^i$ and $\Psi_k^i \Theta_k^i \Psi_k^i = \Psi_k^i$, we have

$$\begin{aligned}
\bar{P}_k^{i,i} &= \bar{P}_k^{*i} - K_k^i S_k^{i,T} - S_k^i (I - \Psi_k^i \Theta_k^i)^T \Theta_k^{i,T} S_k^{i,T} \\
&\quad + S_k^i \Theta_k^i (I - \Psi_k^i \Theta_k^i) \breve{\Xi}_k^i (I - \Psi_k^i \Theta_k^i)^T \Theta_k^{i,T} S_k^{i,T} \\
&= \bar{P}_k^{*i} - K_k^i S_k^{i,T} + S_k^i [-\Theta_k^i + \Theta_k^i \Psi_k^i \Theta_k^i \\
&\quad + \Theta_k^i \breve{\Xi}_k^i \Theta_k^i - \Theta_k^i \Psi_k^i \Theta_k^i \breve{\Xi}_k^i \Theta_k^i - \Theta_k^i \breve{\Xi}_k^i \Theta_k^i \Psi_k^i \Theta_k^i \\
&\quad + \Theta_k^i \Psi_k^i \Theta_k^i \breve{\Xi}_k^i \Theta_k^i \Psi_k^i \Theta_k^i] S_k^{i,T} \\
&= \bar{P}_k^{*i} - K_k^i S_k^{i,T}.
\end{aligned}$$

This completes the proof. ■

The DSE algorithm with unknown inputs and deception attacks is given in Algorithm 1.

IV. STABILITY ANALYSIS

Now, let us take into account the stability of the proposed distributed estimator. Define the collectively fused estimation error as $\tilde{x}_k = [\tilde{x}_k^1; \ldots; \tilde{x}_k^N]$ and construct the quadratic function as $\mathcal{V}(\tilde{x}_k) = \sum_{i \in \mathcal{N}} \|\tilde{x}_k^i\|^2_{(\bar{P}_k^i)^{-1}}$.

Before proceeding further, we need to introduce the following preliminary assumptions.

Assumption 1: The system matrix A_k is invertible.

Assumption 2: The pair $(A_k, C_{2,k})$ is observable, where $C_{2,k} \triangleq [C_{2,k}^1; \ldots; C_{2,k}^N]$.

Assumption 3: The noise sequences ω_k and v_k^i are bounded in mean square for any $k \geq 0$ and $i \in \mathcal{N}$.

Lemma 3: Under Assumptions 1-3, let $\bar{P}_k^i$ be calculated via Algorithm 1 with the initial condition $\bar{P}_0^i > 0$. Then, there exist positive real numbers $\underline{p}$ and $\bar{p}$ such that $0 < \underline{p} I \leq \bar{P}_k^i \leq \bar{p} I$ for any $i \in \mathcal{N}$.

Proof of Lemma 3: In view of Theorem 1, it is obtained that $\bar{P}_k^i \geq P_k^i > 0$, and thus a uniform lower bound $\underline{p} I$ on $\bar{P}_k^i$ can be readily obtained.

In the sequel, the existence of an upper bound on $\bar{P}_k^i$ remains to be proved. From (19), we can write

$$(\bar{P}_k^i)^{-1} = \pi_{i,i} (\bar{P}_k^{i,i})^{-1} + \sum_{j \in \mathcal{N}_i} \pi_{i,j} (\bar{P}_k^{j,j})^{-1}.$$

Algorithm 1 Distributed Simultaneous State-Input Estimation

Initialize: For any $i \in \mathcal{N}$, $\hat{x}_0^i = \hat{x}_0$, $\bar{P}_0^i = P_0$ and $\hat{u}_{1,0}^i = (\Sigma_0^i)^{-1}(z_{1,0}^i - C_{1,0}^i \hat{x}_0^i)$;
for k=1 to K **do**
  ▷ Estimation of $u_{2,k-1}^i$
  $\bar{P}_{k|k-1}^i = A_{k-1}\bar{P}_{k-1}^i A_{k-1}^T + Q_{k-1}$;
  $\bar{\Xi}_{2,k}^i = C_{2,k}^i \bar{P}_{k|k-1}^i C_{2,k}^{i,T} + R_{2,k}^i$;
  $\bar{\mathcal{G}}_{2,k-1}^i = [\bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\bar{\Xi}_{2,k}^i)^{-1} C_{2,k}^i \bar{G}_{2,k-1}^i]^{-1}$;
  $\bar{M}_{2,k}^i = \bar{\mathcal{G}}_{2,k-1}^i \bar{G}_{2,k-1}^{i,T} C_{2,k}^{i,T} (\bar{\Xi}_{2,k}^i)^{-1}$;
  $\hat{x}_{k|k-1}^i = A_{k-1}\hat{x}_{k-1}^i + \bar{G}_{1,k-1}\hat{u}_{1,k-1}^i$;
  $\hat{u}_{2,k-1}^i = \bar{M}_{2,k}^i (z_{2,k}^i - C_{2,k}^i \hat{x}_{k|k-1}^i)$;
  ▷ Time and measurement update
  $\hat{x}_k^{*i} = \hat{x}_{k|k-1}^i + \bar{G}_{2,k-1}\hat{u}_{2,k-1}^i$;
  $\bar{P}_k^{*i} = (I - \bar{G}_{2,k-1}^i \bar{M}_{2,k}^i C_{2,k}^i)\bar{P}_{k|k-1}^i (I - \bar{G}_{2,k-1}^i \times \bar{M}_{2,k}^i C_{2,k}^i)^T + \bar{G}_{2,k-1}^i \bar{M}_{2,k}^i R_{2,k}^i \bar{M}_{2,k}^{i,T} \bar{G}_{2,k-1}^{i,T}$;
  $S_k^i = \bar{P}_k^{*i} C_k^{i,T} - \bar{G}_{2,k-1}^i \bar{M}_{2,k}^i U_{2,k}^{i,T} R_k^i$;
  $\bar{\Xi}_k^i = C_k^i \bar{P}_{k|k-1}^i C_k^{i,T} + R_k^i$;
  $\tilde{\Xi}_k^i = (\bar{\Xi}_k^i)^{-1}(I - C_k^i \bar{G}_{2,k-1}^i \bar{M}_{2,k}^i U_{2,k}^{i,T})$;
  $K_k^i = S_k^i \tilde{\Xi}_k^i [I - U_{1,k}^i (U_{1,k}^{i,T} \tilde{\Xi}_k^i U_{1,k}^i)^{-1} U_{1,k}^{i,T} \tilde{\Xi}_k^i]$;
  $\hat{x}_k^{i,i} = \hat{x}_k^{*i} + K_k^i (y_k^{a,i} - C_k^i \hat{x}_k^{*i})$;
  $\bar{P}_k^{i,i} = \bar{P}_k^{*i} - K_k^i S_k^{i,T}$;
  ▷ Distributed fusion estimation
  $\bar{P}_k^i = \left[\pi_{i,i}(\bar{P}_k^{i,i})^{-1} + \sum_{j\in\mathcal{N}_i} \pi_{i,j}(\bar{P}_k^{j,j})^{-1}\right]^{-1}$;
  $W_k^{i,j} = \pi_{i,j}\bar{P}_k^i (\bar{P}_k^{j,j})^{-1}$, $j \in \mathcal{N}_i \cup \{i\}$;
  $\hat{x}_k^i = W_k^{i,i}\hat{x}_k^{i,i} + \sum_{j\in\mathcal{N}_i} W_k^{i,j}\hat{x}_k^{j,j}$;
  ▷ Estimation of $u_{1,k}^i$
  $M_{1,k}^i = (\Sigma_k^i)^{-1}$;
  $\hat{u}_{1,k}^i = M_{1,k}^i (z_{1,k}^i - C_{1,k}^i \hat{x}_k^i)$;
end for

Following the facts that $\bar{P}_k^{i,i} \le \bar{P}_k^{*i}$ and $\bar{P}_k^i$ is bounded by a uniform lower bound, there exist positive real numbers $\tau^i, \mu^i > 0$ such that

$$\begin{aligned}\bar{P}_k^{*i} &\le \bar{P}_{k|k-1}^i - \tau^i \bar{G}_{2,k-1}^i \bar{\mathcal{G}}_{2,k-1}^i (\bar{\mathcal{G}}_{2,k-1}^i)^{-1} \bar{\mathcal{G}}_{2,k-1}^i \bar{G}_{2,k-1}^{i,T} \\ &\le \bar{P}_{k|k-1}^i - \bar{P}_{k|k-1}^i C_{2,k}^{i,T} (C_{2,k}^i \bar{P}_{k|k-1}^i C_{2,k}^{i,T} + \mu^i R_{2,k}^i)^{-1} \\ &\quad \times C_{2,k}^i \bar{P}_{k|k-1}^i. \end{aligned} \tag{31}$$

Taking the inverse on (31) and applying the matrix inversion lemma yields

$$(\bar{P}_k^{*i})^{-1} \ge (\bar{P}_{k|k-1}^i)^{-1} + (\mu^i)^{-1} C_{2,k}^{i,T} (R_{2,k}^i)^{-1} C_{2,k}^i.$$

From (23), the following holds

$$\bar{P}_{k|k-1}^i = A_{k-1}\big(\bar{P}_{k-1}^i + A_{k-1}^{-1} Q_{k-1} A_{k-1}^{-T}\big) A_{k-1}^T.$$

Since Q_k is bounded, there exists a positive real number η^i such that $A_{k-1}^{-1} Q_{k-1} A_{k-1}^{-T} \le \eta^i \bar{P}_{k-1}^i$. Then, we have $\bar{P}_{k|k-1}^i \le (1+\eta^i) A_{k-1}\bar{P}_{k-1}^i A_{k-1}^T$. As a consequence, applying the above inequalities yields

$$\begin{aligned}(\bar{P}_k^i)^{-1} &> \varepsilon \sum_{j\in\mathcal{N}} \pi_{i,j} A_{k-1}^{-T} (\bar{P}_{k-1}^j)^{-1} A_{k-1}^{-1} \\ &\quad + \varepsilon \sum_{j\in\mathcal{N}} \pi_{i,j} C_{2,k}^{j,T} (R_{2,k}^j)^{-1} C_{2,k}^j, \end{aligned} \tag{32}$$

where $\varepsilon = \min\left\{\frac{1}{1+\eta^j}, \frac{1}{\mu^j}\right\}$ for any $j \in \mathcal{N}$ and $k \ge 0$. For sufficiently large k, by recursively applying inequality (32) L times, we have

$$\begin{aligned}(\bar{P}_k^i)^{-1} &\ge \varepsilon^L \sum_{j\in\mathcal{N}} \Pi_{[i,j]}^L A_{k-1,k-L}^{-T} (\bar{P}_{k-1}^j)^{-1} A_{k-1,k-L}^{-1} \\ &\quad + \sum_{\ell=k-L+1}^{k} \varepsilon^{k-\ell} A_{k-1,\ell}^{-T} \\ &\quad \times \left(\sum_{j\in\mathcal{N}} \Pi_{[i,j]}^{k-\ell+1} C_{2,\ell}^{j,T} (R_{2,\ell}^j)^{-1} C_{2,\ell}^j\right) A_{k-1,\ell}^{-1}, \end{aligned}$$

where $A_{k-1,s} = A_{k-1}A_{k-2}\cdots A_s$ if $s \le k-1$ or $A_{k-1,s} = I$ otherwise, and $\Pi_{[i,j]}^m$ denotes the (i,j)th element of the matrix Π^m. When $L > \mathcal{D}(\mathcal{G}) + n_x$, there exists a positive real number φ such that

$$(\bar{P}_k^i)^{-1} \ge \varphi \sum_{\ell=k-L+1}^{k} A_{k-1,\ell}^{-T} \left(\sum_{j\in\mathcal{N}} C_{2,\ell}^{j,T} (R_{2,\ell}^j)^{-1} C_{2,\ell}^j\right) A_{k-1,\ell}^{-1}.$$

It is seen that under Assumption 2, the right-hand side of the above inequality is positive definite. As a result, there exists a uniform upper bound $\bar{p}I$ such that $\bar{P}_k^i \le \bar{p}I$ for sufficiently large k. ■

We now derive an upper bound on $\|\tilde{x}_{k+1}^i\|_{(\bar{P}_{k+1}^i)^{-1}}^2$ for any $i \in \mathcal{N}$. To begin with, recalling (19) and (21), we can write

$$\begin{aligned}\|\tilde{x}_{k+1}^i\|_{(\bar{P}_{k+1}^i)^{-1}}^2 &= \left\| \sum_{j\in\mathcal{N}} \pi_{i,j} (\bar{P}_{k+1}^{j,j})^{-1} \tilde{x}_{k+1}^{j,j} \right\|_{\bar{P}_{k+1}^i}^2 \\ &\le \sum_{j\in\mathcal{N}} \pi_{i,j} \|\tilde{x}_{k+1}^{j,j}\|_{(\bar{P}_{k+1}^{j,j})^{-1}}^2 \\ &= \sum_{j\in\mathcal{N}} \pi_{i,j} \|\bar{A}_k^j \tilde{x}_k^j + \kappa_k^j\|_{(\bar{P}_{k+1}^{j,j})^{-1}}^2, \end{aligned}$$

where the inequality follows the Lemma 2 in [12], and $\bar{A}_k^j$ as well as κ_k^j are

$$\begin{aligned}\bar{A}_k^j &= (I - K_{k+1}^j C_{k+1}^j)(I - \bar{G}_{2,k}^j \bar{M}_{2,k+1}^j C_{2,k+1}^j) A_k, \\ \kappa_k^j &= (I - K_{k+1}^j C_{k+1}^j)[(I - \bar{G}_{2,k}^j \bar{M}_{2,k+1}^j C_{2,k+1}^j)\omega_k \\ &\quad - \bar{G}_{2,k}^j \bar{M}_{2,k+1}^j v_{2,k+1}^j] - K_{k+1}^j v_{k+1}^j. \end{aligned}$$

Following the facts that $\bar{P}_{k+1}^{j,j} = \bar{A}_k^j \bar{P}_k^j \bar{A}_k^{j,T} + \mathbb{E}\{\kappa_k^j \kappa_{k+1}^{j,T}\} \ge \eta \bar{A}_k^j \bar{P}_k^j \bar{A}_k^{j,T}$ with $\eta > 1$ and $(\bar{A}_k^j)^{\dagger}\bar{A}_k^j$ is idempotent, we have

$$\|\tilde{x}_{k+1}^i\|_{(\bar{P}_{k+1}^i)^{-1}}^2 \le \frac{1}{\eta} \sum_{j\in\mathcal{N}} \pi_{i,j} \|\tilde{x}_k^j + \tilde{\kappa}_k^j\|_{(\bar{P}_k^j)^{-1}}^2, \tag{33}$$

where $\tilde{\kappa}_k^j$ is the least-squares solution of $\bar{A}_k^j \tilde{\kappa}_k^j = \kappa_k^j$, and such a solution can be obtained by solving rank deficient linear

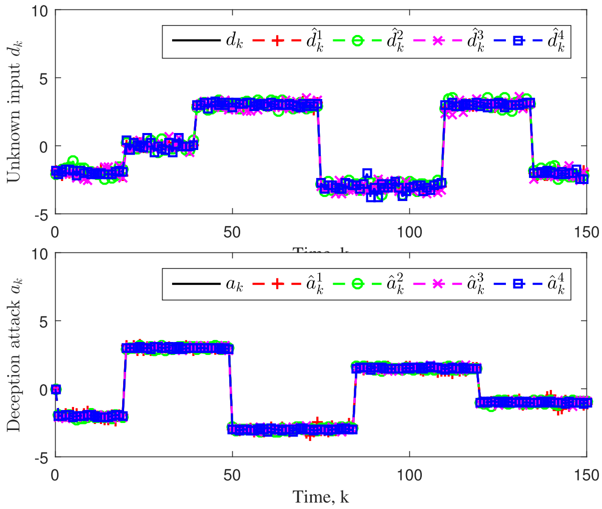

FIGURE 1. **The values of unknown input d_k and deception attack a_k.**

least-squares problems [28], [29]. Under Assumption 3, it is an easy matter to obtain that the quantities $\tilde{\kappa}_k^j, j \in \mathcal{N}$ are bounded in mean square since ω_k and v_k^j are bounded in mean square. Hence, there exists a positive real number ρ_κ such that $\mathbb{E}\{\|\tilde{\kappa}_k^j\|^2\} \leq \rho_\kappa^2$ for any $k \geq 0$ and $j \in \mathcal{N}$.

Theorem 2: Let Assumptions 1-3 hold, then the fused estimation error $\tilde{x}_k^i$ in each node $i \in \mathcal{N}$ is uniformly bounded in mean square, in that

$$\limsup_{k\to\infty} \mathbb{E}\{\|\tilde{x}_k^i\|^2\} \leq \frac{N\bar{p}\rho_\kappa^2}{(\sqrt{\eta}-1)^2\underline{p}} \tag{34}$$

with $\eta > 1$.

Proof of Theorem 2: From (33), the function $\mathcal{V}(\tilde{x}_{k+1})$ is bounded by

$$\begin{aligned}\mathcal{V}(\tilde{x}_{k+1}) &\leq \frac{1}{\eta}\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}} \pi_{i,j}\big\|\tilde{x}_k^j + \tilde{\kappa}_k^j\big\|^2_{(\bar{P}_k^j)^{-1}} \\ &= \frac{1}{\eta}\sum_{i\in\mathcal{N}} \big\|\tilde{x}_k^i + \tilde{\kappa}_k^i\big\|^2_{(\bar{P}_k^i)^{-1}}.\end{aligned}$$

In the sequel, by defining $\tilde{\kappa}_k = [\tilde{\kappa}_k^1; \ldots; \tilde{\kappa}_k^N]$, we have $\mathcal{V}(\tilde{x}_{k+1}) \leq \frac{1}{\eta}\mathcal{V}(\tilde{x}_k + \tilde{\kappa}_k)$. Applying the linearity of expectation and the triangular inequality yields

$$\sqrt{\mathbb{E}\{\mathcal{V}(\tilde{x}_{k+1})\}} \leq \frac{1}{\sqrt{\eta}}\sqrt{\mathbb{E}\{\mathcal{V}(\tilde{x}_k)\}} + \frac{1}{\sqrt{\eta}}\sqrt{\mathbb{E}\{\mathcal{V}(\tilde{\kappa}_k)\}}.$$

Since $0 < \frac{1}{\sqrt{\eta}} < 1$, the above inequality implies that

$$\limsup_{k\to\infty} \sqrt{\mathbb{E}\{\mathcal{V}(\tilde{x}_k)\}} \leq \frac{\sqrt{N}\rho_\kappa}{(\sqrt{\eta}-1)\sqrt{\underline{p}}}.$$

In view of Lemma 3, the following holds

$$\mathbb{E}\{\mathcal{V}(\tilde{x}_k)\} \geq \frac{1}{\bar{p}}\mathbb{E}\Big\{\sum_{i\in\mathcal{N}}\|\tilde{x}_k^i\|^2\Big\},$$

which implies that $\mathbb{E}\{\|\tilde{x}_k^i\|^2\} \leq \bar{p}\mathbb{E}\{\mathcal{V}(\tilde{x}_k)\}$. ■

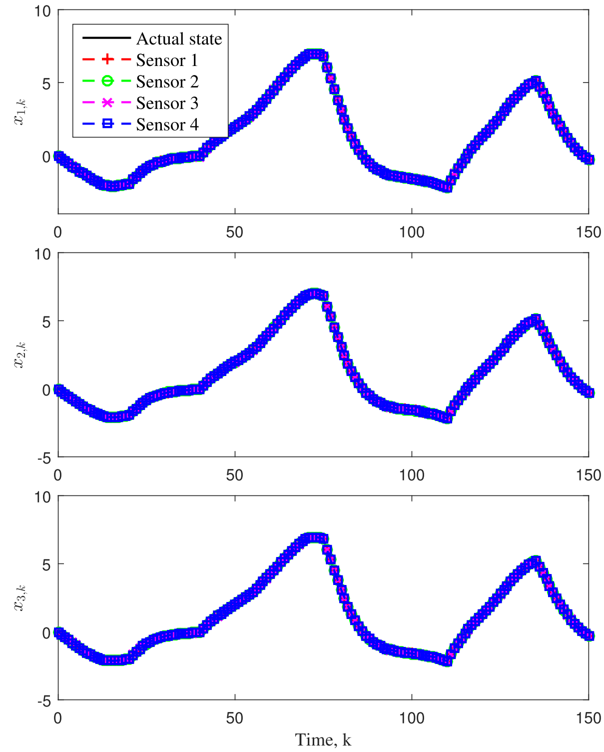

FIGURE 2. **Actual and estimated values of system states.**

V. NUMERICAL EXAMPLE

In this section, a numerical example is exploited to verify the proposed distributed estimation algorithm subject to unknown inputs and deception attacks. Let us consider a linear time-varying system described as follows [30]:

$$x_{k+1} = \begin{bmatrix} a_{11,k} & a_{12,k} & a_{13,k} \\ a_{21,k} & a_{22,k} & a_{23,k} \\ a_{31,k} & a_{32,k} & a_{33,k} \end{bmatrix} x_k + \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix} d_k + \omega_k,$$

where

$$\begin{aligned} a_{11,k} &= \exp[-h + \sin(kh) - \sin(kh-h)], \\ a_{12,k} &= 0, \quad a_{13,k} = 0, \\ a_{21,k} &= 2\sinh(0.5h)\exp[-1.5h + \sin(kh) - \sin(kh-h)], \\ a_{22,k} &= \exp[-2h + \sin(kh) - \sin(kh-h)], \\ a_{23,k} &= 0, \quad h = 0.1, \\ a_{31,k} &= 0, \quad a_{32,k} = 0, \\ a_{33,k} &= \exp[-h + \sin(kh) - \sin(kh-h)], \\ Q_k &= 10^{-4} \times [1\ 0\ 0; 0\ 1\ 0; 0\ 0\ 1]. \end{aligned}$$

A sensor network with 4 nodes is used to measure the system state vector, and the corresponding measurement equation of each sensor is

$$y_k^{a,i} = C_k^i x_k + H_k^i a_k + v_k^i, \quad i = 1, 2, 3, 4,$$

where

$$C_k^1 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & \sin(kh) & 0 \end{bmatrix}, \quad C_k^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$C_k^3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_k^4 = \begin{bmatrix} \cos(kh) & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$H_k^1 = [0.1; 0.1], \quad H_k^2 = [0.1; 0.2], \; H_k^3 = [0.2; 0.1],$$
$$H_k^4 = [0.2; 0.2], \quad R_k^1 = R_k^2 = R_k^3 = R_k^4 = 10^{-4} \times [1\;0; 0\;1].$$

The weight is $\Pi = [\frac{1}{3}\;\frac{1}{3}\;0\;\frac{1}{3}; \frac{1}{3}\;\frac{1}{3}\;\frac{1}{3}\;0; 0\;\frac{1}{3}\;\frac{1}{3}\;\frac{1}{3}; \frac{1}{3}\;0\;\frac{1}{3}\;\frac{1}{3}]$. The unknown input signal d_k and deception attacks a_k have the forms plotted in Figure 1 (the black solid ones). The initial state and covariance are set to $\hat{x}_0 = [0; 0; 0]$ and $P_0 = \text{diag}\{0.01, 0.01, 0.01\}$, respectively.

The estimated values of unknown inputs and deception attacks are plotted in Figure 1, and actual and estimated states of sensor nodes $i = 1, 2, 3, 4$ are shown in Figure 2. It can be seen that the estimated values can track the actual unknown inputs, deception attacks and states well, which indicates that the proposed distributed estimation algorithm possesses satisfactory performance in state estimation.

In particular, it can be observed that nodes 2 and 3 own insufficient capabilities to ensure the local observability of system states, while the collective observability is guaranteed by the whole sensor network. In the case, the proposed distributed estimation algorithm remains to be effective to estimate the system states.

VI. CONCLUSION

In this work, we have addressed the distributed state estimation problem for linear discrete time-varying systems in the presence of unknown inputs and deception attacks. By utilizing singular value decomposition, a joint estimator has been designed to simultaneously estimate the system states and unknown inputs. A distributed estimation algorithm has been constructed by combing the local information in a convex manner. Moreover, the stability analysis of the distributed estimator has been carried out by showing that the fused estimation error in each node is uniformly bounded.

In the present distributed estimation framework, the number of channels for measurements in each node is equal to or larger than that of channels for the sum of unknown inputs and deception attacks. Nevertheless, it is intractable to ensure this condition when the number of channels for measurements is too small. As such, the consideration of relaxing this condition is of important theoretical and practical significance, which forms an interesting direction for our future work.

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