IEEE Access
Multidisciplinary : Rapid Review : Open Access Journal

Received 3 May 2023, accepted 6 June 2023, date of publication 9 June 2023, date of current version 16 June 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3284549

RESEARCH ARTICLE

Optimal Real Estate Pricing and Offer Acceptance Strategy

EUGENE KHMELNITSKY[1] AND GONEN SINGER[2]
[1]Department of Industrial Engineering, Tel Aviv University, Tel Aviv 69978, Israel
[2]Faculty of Engineering, Bar-Ilan University, Ramat Gan 52900, Israel

Corresponding authors: Eugene Khmelnitsky (xmel@tauex.tau.ac.il) and Gonen Singer (gonen.singer@biu.ac.il)

ABSTRACT We consider the problem of choosing the best of a set of sequential offers proposed by the market in a house-selling process. During each decision epoch, the seller sets a listing price, observes the offers and decides whether to accept the maximum one or to reject all of them. We model a fixed holding cost, which is the constant marketing cost of searching for buyers, and a variable cost that is proportional to the number of offers received during each epoch. The objective is to maximize the expected revenue. Most previous studies assume a stationary known distribution from which the buyers' offers are generated and which reflects the market valuation of the house. In contrast, we assume that the number of incoming offers, and the distribution from which each individual offer is generated, are affected by the seller's listing price (i.e., price-based demand response). Thus, we propose a new approach for the selling policy, which consists of the listing price and the offer acceptance threshold in each period. We derive the seller's optimal selling policy and apply it to a scenario involving the sale of individual residential properties in Ames (Iowa), which yields results consistent with empirical observations.

INDEX TERMS Housing market policy, dynamic pricing, sequential decision making, optimal stopping.

I. INTRODUCTION

Private real-estate trading is conducted under uncertainty from the perspective of both the seller and the buyer in a competitive market. The buyers have access to all available real estate listings and past sale prices, but not the offers made for individual properties. The seller observes sequentially a series of randomly arriving offers and has to decide, in each time period, whether to accept the maximum offer proposed by the market or to reject the offers and continue with the search. The seller integrates a listing price into the decision-making process in order to influence the incoming offers and improve his performance. This paper develops a policy consisting of a listing price and a reservation price (i.e., the offer acceptance threshold) that together maximize the seller's expected revenue over a given time horizon. The functions representing the listing price and the reservation price may change over time until the asset is sold.

The associate editor coordinating the review of this manuscript and approving it for publication was Ton Duc Do.

We make use of the well-known optimal stopping problem as a mathematical formulation of this scenario. Problems of optimal stopping have a long history and have been widely studied in both probability theory and operations research [1]. Recent papers have used optimal stopping theory to investigate various applications, such as selling an asset [2], selling an ex-rental car in a rental business [3], stopping testing in software test management [4], stopping the interaction between a seller and a buyer who bargain to determine the transfer price [5], and selecting a cell in 5G networks [6].

The optimal stopping problem of the type we consider in this research was introduced by Bertsekas [7]. The model involves a control space that consists of a finite number of elements, one of which induces a termination (stopping) of the evolution of the system. In terms of a Markov decision process, at each decision epoch, the decision maker has two available actions in each state: to stop or to continue. If he decides to stop, he receives a reward (in our terms, he sells the asset), and if he decides to continue, the system evolves until the next decision epoch. The objective is to determine a policy that maximizes the expected total reward. This model has been extensively applied to the real estate market [8], [9]. In most research studies, the model assumes a stationary, known distribution from which the buyers' offers are generated and which cannot be influenced by the seller. In a competitive market, a seller integrates realistic knowledge into the decision-making process in order to improve performance [10], [11], when a listing price is one of the decision parameters of the seller's policy that can influence the offer distribution.

A number of articles have provided models for pricing an asset under different market conditions and have demonstrated the influence of the listing price on the incoming offers. Using novel survey data, Han and Strange [12] proved that the listing price has an impact in terms of both directing buyers and determining the closing selling price. Several studies have analyzed and compared the effect of different listing price strategies adopted by sellers. These include rounded-price, just-below-price and precise-price strategies [13], [14], low-price, fair-price and high-price strategies [15], as well as a list price–discount strategy [16], where the seller first publishes a list price, then during interactions with a buyer, offers discounts off of the list price [17]. Liu et al. [18] showed that the listing price affects the selling price and the negotiation process differently in cold market (in which housing supply is greater than housing demand) vs in hot market (in which housing demand exceeds housing supply). Arnold and Lippman [19] compared bargaining and posted-price mechanisms in a market with discounting and positive transaction costs in which the seller is imperfectly informed about both the buyer's valuation of the asset and the buyer's bargaining ability. These two mechanisms were chosen to reflect the fact that some markets witness posted prices while others witness bargaining, and it is important to predict when each of these mechanisms will be utilized. Yavas and Yang [20] addressed the case where a seller of a real estate property and his broker have two primary goals: to sell the property for as high a price as possible and to sell as quickly as possible. While these are separate objectives, they are closely related through the listing price of the seller. The listing price affects how long it takes to find a buyer (i.e., Time On the Market - TOM), and TOM influences the price that results from the bargaining between the seller and the buyer. Yavas and Yang's paper accomplished two tasks. Firstly, it provided a theoretical framework to (a) study the optimal listing price set by sellers in housing markets and (b) examine the impact of listing price on the time it takes to sell the property. Secondly, it provided an empirical analysis of the relationship between the listing price and the time it takes to sell the property. An interesting connection exists between our research on optimal property selling strategy and the problem of optimal trade execution [21], [22]. Both involve a trade-off between obtaining the highest price and reducing market risk. The success of both processes depends on the availability of buyers at the requested price and quick execution to minimize risk.

Another branch of the literature has addressed the problem of finding a time-dependent strategy for the house-selling process, using both theoretical and empirical methods. In a pioneering work, Salant [23] used dynamic programming to determine the optimal selling strategy for a risk-neutral seller in an environment with a finite time horizon. The seller chooses a listing price for each period in which the residential real estate is up for sale, and, when a bid arrives, decides whether to accept it or to wait and hope that a higher bid will arrive in the near future. Salant [23] showed that the optimal solution generally involves a strictly monotonically declining sequence of listing prices. The literature has also documented a set of stylized facts about the behavior of individual sellers [24], [25]. Merlo and Ortalo-Magné [25], for example, studied the residential real estate selling problem based on transaction history data in England between June 1995 and April 1998. They showed that sellers tend to adjust their listing prices downwards, even when market conditions do not change, and that sale prices for observationally equivalent residential real estates depend on TOM. Merlo et al. [26] presented a dynamic model, in a finite time horizon, of the behavior of the seller of a residential property. The model was based on their empirical findings which were obtained by analyzing a rich data set consisting of listing price changes and all offers made on the residential real estate between the initial listing and the final sale agreement. They developed a bargaining model with two-sided incomplete information where the listing price declines over the selling horizon due to the fact that the arrival rate of potential buyers exogenously declines over the selling horizon.

To the best of our knowledge, the present paper is the first to provide an optimal stopping policy for a house-selling problem, considering the dynamic listing pricing of the house, which influences the distribution of market offers in two ways: (1) the number of offers received and (2) the distribution from which each individual offer is generated. We apply the optimal selling policy to a numerical example and obtain results that are consistent with the empirical observations of previous studies, thus providing a theoretical underpinning for those observations. From a practical perspective, we demonstrate the applicability of the optimal selling policy based on a real case study of 1,460 individual residential properties in Ames, Iowa and gain insights that shed light on the decisions the seller has to make.

II. PROBLEM FORMULATION

The proposed model considers the following order of events. At the beginning of a given time period, the seller sets a price for the asset. At the end of the period, the seller decides whether to accept the maximum offer he received during the period (and sell the property) or decline all offers and solicit new offers in the next period. The objective is to determine a pricing policy and a policy for accepting/rejecting offers that together maximize the expected revenue of the seller.

The probability distribution of the maximum offer is affected by the number of offers accumulated during the time period and the distribution of each individual offer. It is common in the literature to assume that stochastic arrivals of potential buyers are independent, and that the strength of the market demand can be represented by a Poisson process, that is, assuming that the offers are drawn from a countably infinite population [2], [27], [28], [29], [30], [31], [32]. We assume also that the individual offers of potential buyers are generated from a known probability distribution that reflects the market valuation on the property [2], [28], [29]. The parameters that describe the arrival rate and the offer probability distribution depend on the listing price. Thus, the impact of the listing price is two-fold. On the one hand, a higher listing price results, statistically, in a higher individual offer. On the other hand, the higher the listing price of an asset, the lower the buyer arrival rate, which in turn causes the maximum offer to be statistically lower. This issue is discussed further in the following subsection.

A. PROBABILITY DISTRIBUTION OF THE MAXIMUM OFFER

Denote by p the listing price and by $\lambda(p)$ the price-sensitive rate of arrival of the offers. The Poisson probability of receiving n offers during a single time period, q_n, is

$$q_n = e^{-\lambda(p)} \frac{\lambda(p)^n}{n!}, \quad n = 0, 1, \ldots, \infty. \tag{1}$$

Denote by X_i the i-th offer, $i = 1, \ldots, n$ and by Z the maximum offer, $Z = max\,\{X_i\}_{i=1}^{n}$. Without loss of generality, we assume that $Z = 0$ if $n = 0$ (i.e., if no offer arrives). This may happen, for example, when the listing price is not appropriate because it is too far from the market expectation. The price-sensitive pdf and cdf of an individual offer are denoted by $f_x(p, \cdot)$ and $F_x(p, \cdot)$, respectively, and the pdf and cdf of Z are denoted by $f_z(p, \cdot)$ and $F_z(p, \cdot)$. The probability distribution of Z is

$$F_z(p, \xi) = \Pr(Z \le \xi) = \sum\nolimits_{n=0}^{\infty} q_n F_x(p, \xi)^n, \tag{2}$$

where we used the fact that the cdf of the maximum of a set of i.i.d. random variables is the n-th power of the individual cdf. By substituting (1), we have,

$$F_z(p, \xi) = e^{-\lambda(p)} \sum\nolimits_{n=0}^{\infty} \frac{\left(\lambda(p) F_x(p, \xi)\right)^n}{n!}.$$

Equivalently,

$$F_z(p, \xi) = e^{-\lambda(p)(1 - F_x(p, \xi))}. \tag{3}$$

Equation (3) transforms the distribution of an individual offer into the distribution of the maximum offer, given the rate of offer arrivals. Note that $F_z(p, \xi)$ is discontinuous at $\xi = 0$, with the discontinuity value of $e^{-\lambda(p)}$, which is the probability of having no offers.

We note that in a particular case where an individual offer is distributed exponentially, $F_x(p, \xi) = 1 - e^{-\xi/p}$, with the mean offer value $E[X] = p$, the maximum offer follows a Gumbel distribution,

$$F_z(p, \xi) = e^{-e^{-(\xi - p \ln \lambda(p))/p}},$$

which is generally used to model the distribution of the maximum of a fixed number of samples of various distributions. The Gumbel distribution in our case, however, is truncated at $\xi = 0$.

B. THE SELLER'S OBJECTIVE FUNCTION – STATIONARY CASE

This section considers a stationary case where the parameters of the seller's problem, $\lambda(p)$ and $F_x(p, \xi)$, do not change over an infinite time horizon. In such a case, the optimal seller's policy is shown to be of a threshold type (see Proposition 1 below). That is, given p, there exists a unique threshold level, r, such that the offer z is accepted if $z > r$, and rejected if $z \le r$ (by lower case z we denote a realization of random variable Z). This threshold is actually the reservation price of the seller. The reservation price property was discussed and proved in [33] for a seller's problem where the parameters of offer's probability density are updated over time in a Bayesian way.

Regarding the policy of the seller for showing the asset, it is assumed that the seller receives an offer only after showing the asset to the buyer. Therefore, the cost incurred for holding the asset in each decision epoch is represented by a linear function of the number of buyers, $w_0 + w_1 n$, where w_0 denotes the constant marketing cost of maintaining the search for buyers and w_1 denotes the communication and showing costs associated with each prospective buyer. If the seller chooses to adopt a more flexible showing policy, such as grouping buyers together or setting pre-showing conditions, the cost function for holding the asset can become more complex and may increase in a non-linear manner with respect to n.

Proposition 1: Given p, the optimal seller's policy is of a threshold type.

Proof: In Appendix A.

In this section, we formulate the seller's objective, denoted by $J(p, r)$, which aims to maximize the seller's expected revenue,

$$\max J(p, r) \quad \text{subject to } p \ge 0, \quad r \ge 0. \tag{4}$$

We denote the solution of problem (4) by p^* and r^*, and note that r^* is not necessarily smaller than p^*. In the case where the seller underestimates the asset and suggests a lower p^*, the bidders compete with each other, which allows the seller to increase the threshold. Having adopted a threshold policy, the seller communicates the same listing price in all decision epochs, until the terminal epoch, where the maximum offer exceeds the threshold.

The revenue is the maximum offer, Z, received at the terminal epoch. In terms of a Markov decision process, we have a system with two states – non-terminal and terminal, and the equation for the total expected revenue

$$J(p, r) = -w_0 - w_1 E[n] + F_z(p, r) J(p, r) + (1 - F_z(p, r)) \cdot E[Z | Z > r]. \tag{5}$$

The first two terms on the right-hand side of (5) represent the expected holding cost, while the last two terms are associated with a transition to the non-terminal and terminal states, respectively. The solution of (5) is

$$J(p,r) = E[Z|Z > r] - \frac{w_0 + w_1\lambda(p)}{1 - F_z(p,r)}. \tag{6}$$

We assume that the expected maximum offer $E[Z]$ is large enough to cover the expected holding cost incurred during a single period, i.e.,

$$E[Z] > w_0 + w_1\lambda(p). \tag{7}$$

This assumption defines a minimum requirement for this business to be profitable. In the next proposition, we use (7) to prove the properties of the expected revenue function $J(p,r)$.

Proposition 2: Given p,

(i) the function $J(p,r)$ is quasi-concave w.r.t. r;

(ii) a solution of the equation

$$J(p,r) = r, \tag{8}$$

with respect to r, if it exists, maximizes $J(p,r)$.

Proof: In Appendix A.

In Section III we illustrate Proposition 2 with a numerical example where an individual offer of a potential buyer is distributed normally. From Proposition 2 it follows that $J(p, r^*(p))$ for each p can be found using a convex search algorithm. In the numerical examples below, we use the standard golden search method. The optimal listing price is then determined as $p^* = \underset{p\geq 0}{\text{argmax}}\, J(p, r^*(p))$.

Propositions 3 and 4 below derive some structural properties of the optimal solution, which are then illustrated in Example 1.

Proposition 3: If (p^*, r^*) is an optimal policy, then the following two conditions hold,

$$r^* = E[Z] - w_0 - w_1\lambda(p^*) + \int_0^{r^*} F_z(p^*, z)\, dz, \tag{9}$$

$$w_1 \left.\frac{\partial\lambda(p)}{\partial p}\right]_{p=p^*} + \int_{r^*}^{\infty} \left.\frac{\partial F_z(p,z)}{\partial p}\right]_{p=p^*} dz = 0. \tag{10}$$

Proof: In Appendix A.

Proposition 4: The optimal revenue decreases with the parameters w_0 and w_1.

Proof: In Appendix A.

Example 1: Let an individual offer be distributed exponentially, $F_x(p,\xi) = 1 - e^{-\xi/p}$, and the arrival rate be also exponential, $\lambda(p) = \lambda_0 e^{-\chi p}$, where λ_0 and χ are positive parameters. Then, as noted in Section II-A, the cdf of the maximum offer is,

$$F_z(p,\xi) = e^{-\lambda_0 e^{-\chi p - \xi/p}}, \quad \xi > 0 \text{ and } F_z(p,0) = 0.$$

By substituting that in (9) and (10), we conclude that r^* and p^* satisfy the following algebraic equations,

$$r^* = -w_0 - w_1\lambda_0 e^{-\chi p^*} + p^*\left(\gamma - \chi p^* + Ln(\lambda_0) - Ei\left(-\lambda_0 e^{-\chi p^* - \frac{r^*}{p^*}}\right)\right), \tag{11}$$

and

$$\left(\chi p^{*2} - r^*\right)\left(1 - F_z(p^*, r^*)\right) = w_0 + w_1\lambda_0 e^{-\chi p^*}(1 + \chi p^*), \tag{12}$$

where γ is Euler's constant and $Ei(\cdot)$ is the exponential integral function. By neglecting the communication cost, $w_1 = 0$, and by assuming that the marketing cost per time period is small enough, $w_0 \ll p^*$, equations (11) and (12) can be further simplified (see technical details in Appendix B) and result in

$$r^* = \chi p^{*2} - p^* \tag{13}$$

where p^* is the largest root of the equation

$$p^* e^{1-2\chi p^*} = w_0/\lambda_0. \tag{14}$$

From (13) and (14) one can observe that it is optimal to let $p^* = r^*$ with these two values equal to $2/\chi$, if

$$\frac{\lambda_0}{\chi} = \frac{e^3}{2} w_0.$$

The listing price should over-report the threshold, i.e. $p^* > r^*$, if

$$e w_0 < \frac{\lambda_0}{\chi} < \frac{e^3}{2} w_0.$$

The listing price should under-report the threshold, i.e. $p^* < r^*$, if

$$\frac{\lambda_0}{\chi} > \frac{e^3}{2} w_0.$$

C. THE SELLER'S OBJECTIVE FUNCTION – FINITE TIME HORIZON CASE

In this section we assume that the seller must sell the property within a given finite time horizon. The objective is to determine a time-dependent pricing policy, as well as a policy for accepting/rejecting offers, which together maximize the expected revenue. Let p_k be the price set by the seller in epoch k, and let r_k be the threshold level at k, $k = 1, \ldots, N$, where N is the given time horizon.

At the terminal period, $k = N$, the threshold is trivial, $r_N^* = 0$, since the seller must sell the asset provided the process has not been stopped earlier. The seller's expected revenue in this case is

$$J_N(p) = E[Z] - w_0 - w_1\lambda(p). \tag{15}$$

The optimal price at $k = N$ maximizes $J_N(p)$,

$$p_N^* = \arg\max_p J_N(p).$$

We denote by J_N^* the maximum expected revenue at $k = N$, $J_N^* = J_N(p_N^*)$.

Having determined the objective and the two decision variables at the terminal period, we can recursively calculate

r_k^*, p_k^* and J_k^* for $k = N-1, N-2, \ldots, 1$ as follows. First, we define the expected revenue at k given p,

$$J_k(p) = -w_0 - w_1\lambda(p) + E\left[\max\left\{J_{k+1}^*, Z\right\}\right]. \quad (16)$$

The first term in the max operator in (16) corresponds to the decision to "reject" the maximum offer and "continue" the process, while the second term corresponds to "sell" and "stop". The second step is to obtain the optimal price by maximizing $J_k(p)$,

$$p_k^* = \arg\max_p J_k(p).$$

Third, the threshold follows from (16),

$$r_k^* = J_{k+1}^*.$$

The next section demonstrates the applicability of the stationary and time-dependent solution methods developed above to two practical cases.

III. COMPUTATIONAL STUDIES

A. NUMERICAL EXAMPLE

Let the seller estimate the market value of the asset and the variability of the market value as μ_0 and σ_0, respectively. The seller also estimates that an individual offer of a potential buyer is distributed normally regardless of the listing price. However, the mean $\mu(p)$ and standard deviation $\sigma(p)$ of the probability distribution of an individual offer are affected by the listing price, p. To be specific, the seller assumes the following dependence:

$$\mu(p) = \mu_0 - \sigma_0 + 2\sigma_0 \frac{e^{\frac{p-\mu_0}{\sigma_0}}}{1+e^{\frac{p-\mu_0}{\sigma_0}}}, \quad \sigma(p) = \sigma_0 \frac{\mu(p)}{\mu_0}, \quad (17)$$

which satisfies the following properties:

(i) $\mu(p)$ increases with p, i.e., a higher listing price results in stochastically higher individual offers.

(ii) $\sigma(p)$ increases with p, i.e., a higher listing price usually reflects a house with more unique characteristics, which results in a wider distribution of offer prices, as presented in several previous studies based on experimental evidence [34], [35].

(iii) Regardless of p, the expected value of an individual offer remains within a "reasonable" interval, defined as

$$\mu_0 - \sigma_0 < \mu(p) < \mu_0 + \sigma_0 \quad \forall p. \quad (18)$$

Even if the seller underestimates the asset and lists an extremely low price, the competition between multiple buyers ensures that $\mu(p)$ does not fall below $\mu_0 - \sigma_0$. On the other hand, if the seller overestimates the asset and asks an extremely high price, the potential buyers remain rational in the sense that $\mu(p)$ does not increase above $\mu_0 + \sigma_0$.

(iv) $\mu(p = \mu_0) = \mu_0$. That is, the appropriate listing price, $p = \mu_0$, makes the market respond appropriately.

(v) The relative standard deviation of the individual offer distribution is constant, $\frac{\sigma(p)}{\mu(p)} = \frac{\sigma_0}{\mu_0} \forall p$.

The seller estimates that when listing the appropriate price, the rate of offers is λ_0, and that thereafter, as the price increases, the rate of offers decreases exponentially. That is,

$$\lambda(p) = \lambda_0 e^{\chi(\mu_0 - p)},$$

where χ is the sensitivity parameter of $\lambda(p)$.

Fig. 1 presents the pdf of an individual offer and of the maximum offer for the following demand-side parameters:

$$\mu_0 = 100, \quad \sigma_0 = 25, \ \lambda_0 = 10, \ \chi = 0.03$$

and for $p = 80$, 100 and 120 (low, appropriate, and high listing prices). Fig. 2 illustrates Proposition 2 and shows that the revenue function is quasi-concave with the threshold, and that the maximum revenue equals the threshold. Figs. 3 and 4 present the impact of the demand-side factors (λ_0 and σ_0) and supply-side factors (w_0 and w_1) on the optimal seller's policy (p^*, r^*). Fig. 5 presents the division of the (λ_0, σ_0) parameter space into the regions where the threshold is lower ($r^* < p^*$) or higher ($r^* > p^*$) than the listing price.

Fig. 3 presents the optimal seller's policy (p^*, r^*) with the cost parameters $w_0 = 2$, $w_1 = 0.3$ when the demand-side parameters are kept constant (at the values shown above), except for the parameter that is the target of the sensitivity analysis. The graph on the left-hand side shows the optimal seller's policy as a function of the variability of the market value, σ_0. As mentioned in property number (ii) above, a wider distribution of the offer prices represents disagreement regarding the market value of the asset, which usually reflects a unique asset whereby the perception of its market value varies among buyers. From Fig. 3, we observe that both p^* and r^* increase in σ_0. However, r^* increases faster and overtakes p^* at $\sigma_0 \approx 19$. This shows that a higher market variability of an asset allows the seller to increase the reservation price (i.e., threshold), since the higher the variability of the market value, the higher the probability of generous offers. These findings support the empirical observations and claims of several previous research studies [35], [36], [37], which concluded that atypical houses that face a larger variance of offer prices tend to set a higher reservation price, and are sold at a higher price, although they require a relatively longer time period to be sold. The graph on the right hand-side shows the optimal selling policy as function of the offer arrival rate, λ_0. A higher buyer arrival rate indicates stronger market demand, allowing the seller to increase both p^* and r^*.

Fig. 4 shows the optimal selling policy (p^*, r^*) when holding all demand-side and sell-side parameters constant, except for either the constant marketing cost w_0 (left-hand graph) or the variable cost per prospective buyer w_1 (right-hand graph). It can be seen that a higher constant marketing cost may cause the seller to reduce the time on the market and sell the house faster by setting lower reservation and listing prices. Similarly, a higher variable cost will cause the seller to sell the house faster by reducing the reservation price. However,

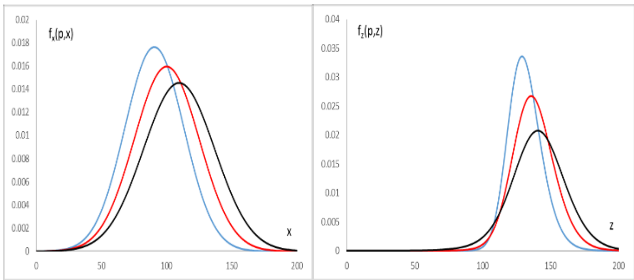

FIGURE 1. The probability distribution of an individual offer (left side) and maximum offer (right side) for listing price $p = 80$ (blue line), 100 (red line) and 120 (black line).

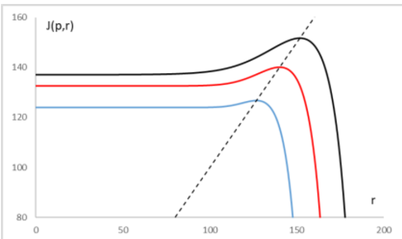

FIGURE 2. The expected revenue $J(p, r)$ as a function of threshold r for listing price $p = 80$ (blue line), 100 (red line) and 120 (black line). The dashed line is for $J(p, r) = r$.

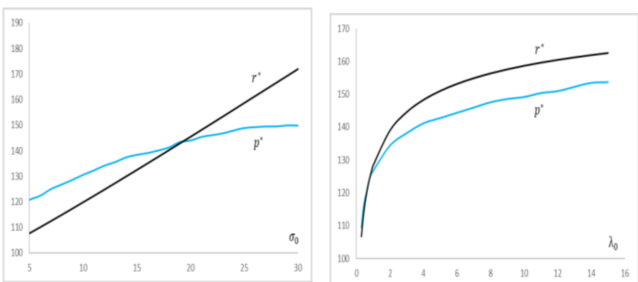

FIGURE 3. Sensitivity of the optimal seller's policy to the demand-side parameters: variability of the market value (left) and offer arrival rate (right).

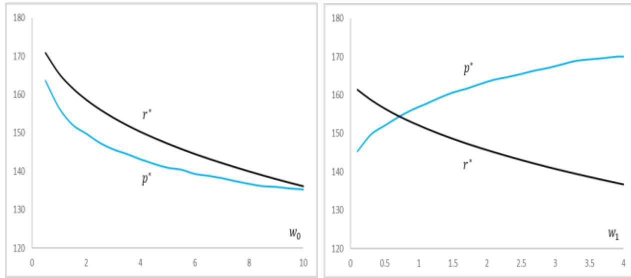

FIGURE 4. Sensitivity of the optimal seller's policy to the holding cost parameters: constant marketing cost (left) and variable cost per prospective buyer (right).

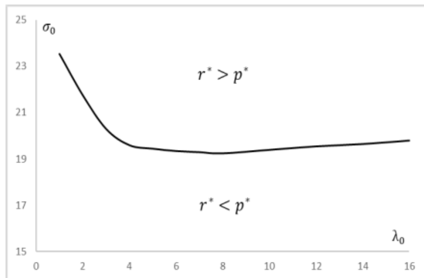

FIGURE 5. Diagram in the (λ_0, σ_0) parameter space specifying the regions where, respectively, the threshold is lower and higher than the listing price.

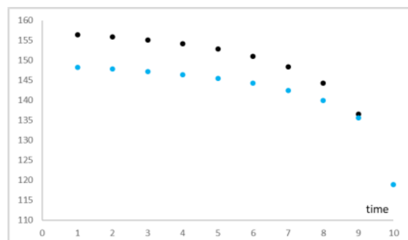

FIGURE 6. Optimal dynamic policy r_k^* (black dots) and p_k^* (blue dots) over a finite time horizon of $N = 10$ time units.

the seller sets a higher listing price, to reduce the number of prospective buyers in each epoch, and as a result, there is a decrease in the total showing and communication cost in each epoch.

Fig. 6 presents the optimal dynamic policy in a non-stationary case with the time horizon $N = 10$ and the parameters listed above. As expected, the (r_k^*, p_k^*) policy tends to the stationary one as the remaining time until the end of the process increases.

B. REAL-LIFE CASE STUDY

The main goals of the real-life case study are twofold: to demonstrate the steps with which the optimal sales strategy can be determined and implemented in a practical setting, and to deepen our comprehension of the results produced by this approach.

The case study is based on a dataset describing the sale of individual residential properties in Ames, Iowa from 2006 to 2010. The database is taken from Kaggle, which is a public source for high quality datasets from different disciplines. It contains 1460 observations and 80 explanatory variables that have been shown to affect property prices. The 80 explanatory variables include 20 continuous variables related to various area dimensions (such as basement area and lot size), 14 discrete variables typically quantifying the number of items within the house (such as the number of bathrooms or bedrooms), 23 ordinals (such as the quality level of the kitchen or garage) and 23 nominals (such as the garage location and the general zoning classification of the property).

Since we don't have data that includes the list of offers for individual property, we propose a linear regression model that predicts the market value of the properties and evaluate the distribution of offers.

As a preprocessing step, we convert all the categorical variables to discrete numbers, and evaluate the regression model by using a ten-fold cross validation test. Thus, the data are divided into ten subsets, where nine-tenths are used for training the prediction model and the remaining one-tenth is used, in turn, as the validation set. The prediction accuracy is calculated 10 times, each time one of the sub-samples is used as the test sample. This technique is common in classification and prediction models [38], [39], [40]. The regression model is given by

$$p_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_n x_{in} + \varepsilon_i,$$

where p_i is the price of the i-th observation (i.e., the property price, which is the dependent variable), x_{ij} is the i-th observation on the j-th independent variable, β_j is the regression parameter of the j-th independent variable, β_0 is the y-intercept, and ε_i is the error term for the i-th observation. The regression model finds that 23 of the 80 explanatory variables have an association with price that is significant with a p-value of less than 0.05.

In order to apply the pricing policy developed in Section II for residential property i in the one-tenth of the database used for validation, we need to determine the following parameters: (i) the three factors related to the demand, i.e., the market value of the asset represented by the mean and standard deviation of the offer prices (μ_0, σ_0), and the arrival rate λ_0 assuming a listing price $p = \mu_0$, which indicates the strength of the market demand; and (ii) the two factors representing the holding cost incurred by the seller during each decision epoch, namely the constant marketing cost, w_0 and the showing cost per prospective buyer, w_1.

Since, in linear regression, the errors between the observed price and the predicted price (i.e., the residuals of the regression) are assumed to be normally distributed, we can evaluate the distribution of offer prices for a specific residential property. In particular, we observe that the standard deviation of the errors is $\hat{\sigma}_0 = 31,998\$$. We follow Han and Strange [12], who claimed that for identical homes, increasing the listing price by 1% reduces the arrival rate of potential offers by ~0.6% and who assumed the following linear relation, $\lambda(p) = \lambda_0 \left(1.6 - \frac{0.6 \cdot p}{\mu_0}\right)$. We set the arrival rate $\lambda_0 = 0.27 \frac{offers}{day}$, by assuming that the seller receives an average of 30 offers before the house is sold, and that the average mean time on the market is 111 days, as was reported by Levitt and Syverson [41]. Since most sellers in the housing market still use a traditional high-street estate agent [42], who typically charges a total commission of 3% of the sale price [41] in return for bearing the marketing costs (e.g., advertising, conducting open houses), we assume an average constant marketing cost of $w_0 = \frac{0.03 \cdot \mu_0}{111} \frac{\$}{day}$. Finally, a showing cost of $w_1 = 100\$$ per prospective buyer is assumed, estimated by the loss of alternative income of the seller during the time when they are required to prepare the house for a viewing. We consider an individual property from the validation set, which, according to the explanatory values, yields a predicted sale price of $\hat{\mu}_0 = 321,555\$$. The distribution of each individual offer, assuming a listing price equal to the market value of the asset, is estimated as $f_x(\cdot) \sim N(\hat{\mu}_0, \hat{\sigma}_0)$. Using the stationary case described in Section II-B, we calculate the optimal threshold for this property for listing prices in the range $300,000 \leq p \leq 600,000$ (see Fig. 7) and find the optimal selling policy, which corresponds to the listing price $p^* \approx 460,000$ and the reservation price $r^*(p^*) \approx 415,000$. The revenue of the optimal policy is $J(p^*, r^*) = r^*(p^*) \approx 415,000$.

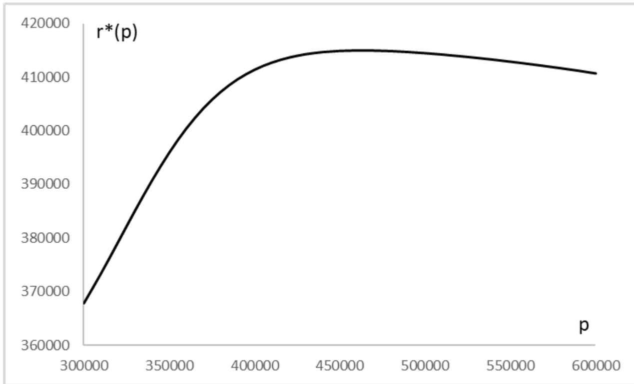

FIGURE 7. **The stationary policy.**

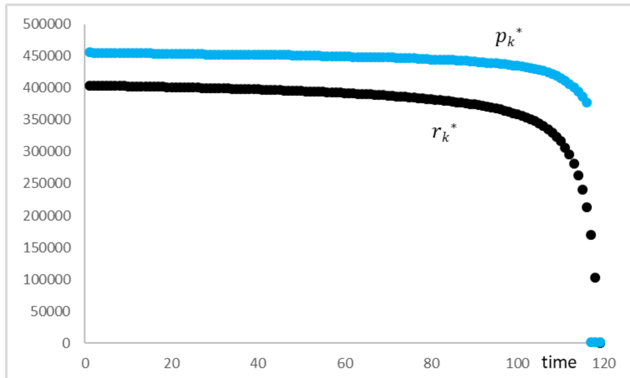

FIGURE 8. **The dynamic policy, r_k^* (black dots) and p_k^* (blue dots), over a finite time horizon of $N = 120$ days.**

This case clearly illustrates the trade-off discussed in Section II. On the one hand, a higher listing price results in higher individual offers distributed normally with $\mu(p^*) = 352,718$ and $\sigma(p^*) = 35,099$ (see (17)). On the other hand, the buyer arrival rate decreases to $\lambda(p^*) = 0.2$. The combination of these two factors enables the seller to establish the threshold r^* within the interval $[\mu(p^*) + \sigma(p^*), \mu(p^*) + 2\sigma(p^*)]$. This implies that r^* is set at a level that is both low enough to accept a significant offer and high enough to optimize the objective.

Fig. 8 shows the convergence of the selling policy in the non-stationary case for $N = 120$ days. It can be observed that the (r_k^*, p_k^*) policy tends to the stationary one as the remaining time until the end of the process increases.

IV. CONCLUSION

In a house-selling problem, a crucial task for many sellers is to develop listing and reservation pricing policies that together maximize the expected revenue. In this paper, we generalize the house-selling model by allowing the seller to set a price for the asset dynamically, at each period, which influences the distribution of market offers. By making use of optimal stopping theory, we find the seller's optimal selling solution and provide explicit expressions for determining the listing and reservation pricing policy over an infinite planning horizon. We also present theoretical properties of this solution. In addition, we show how the optimal solution can be adjusted in the case where the seller must sell the property within a given, finite time horizon. In such a case, the listing and reservation pricing policies are dynamically updated. Thus, the results are useful both to house sellers who are not pressed for time in the sale of their house, and to sellers who need to sell their house within a finite time horizon, perhaps because they have already purchased another house and may be repaying a bridging loan.

We implement the solution method proposed in this paper for the case of a house with a normally distributed market value, and arrive at some specific conclusions that support the findings of previous empirical studies. For the infinite planning horizon, we draw the following conclusions: (i) When the distribution of offer prices is wider, which usually reflects a unique asset for which the market value is perceived differently by different prospective buyers, the reservation and listing prices are higher. For a very high dispersion of offer prices, the dispersion of reservation prices exceeds the dispersion of listing prices. (ii) In the case of stronger market demand and a higher buyer arrival rate, the seller increases both the listing and the reservation price, with the reservation price always higher than the listing price. (iii) Higher holding costs cause the seller to set a lower reservation price in order to reduce the number of epochs before the sale of the house. (iv) The seller sets a lower listing price when the constant costs are higher, but sets a higher listing price when the variable costs are higher (so as to reduce the number of prospective buyers arriving during each epoch).

Finally, we carry out a case study in which we estimate the market price distribution, as well as the listing and reservation pricing policies, based on real sales data of individual residential properties in Ames, Iowa from 2006 to 2010. Accordingly, we demonstrate the applicability of the optimal selling solution to a rich and real dataset.

Future research could further investigate the methodology suggested in this paper by applying the solution method to other types of market value distribution. Other directions could include extending the problem to the case where it is assumed that the distribution of offers is unknown to the seller. In such a case, the seller would have to estimate the (non-stationary) distribution shape and parameters by a learning process over time.

APPENDIX A

Proof of Proposition 1: The cost-to-go criterion of the problem satisfies the equation,

$$V(z) = max\{z, E[V(z)] - w_0 - w_1 E[n]\}, \tag{A1}$$

where z is the maximum offer at a time period. The first term in the *max* function in (A1) corresponds to the "accept" decision. The second term in the *max* function corresponds to the "reject" decision, where seller's expected revenue at the next time period equals $E[V(z)]$ subtracting the holding cost. Let the second term in the *max* function be denoted by α,

$$\alpha = E[V(z)] - w_0 - w_1 E[n].$$

Then, by applying the expectation operator to (A1), we obtain,

$$\alpha + w_0 + w_1 E[n] = \int_{\alpha}^{\infty} f_z(p, z)\, z dz + \alpha F_z(p, \alpha).$$

Consequently,

$$\alpha + w_0 + w_1 E[n] = E[Z] - \int_{0}^{\alpha} f_z(p, z)\, z dz + \alpha F_z(p, \alpha),$$

and after integrating by parts,

$$\alpha + w_0 + w_1 E[n] = E[Z] + \int_{0}^{\alpha} F_z(p, z)\, dz. \tag{A2}$$

By considering the left- and right-hand sides of (A2) as a function of α, we observe

- the left-hand side grows with the rate of 1;
- the right-hand side grows with the smaller rate of $F_z(p, \alpha)$;
- at $\alpha = 0$, the left-hand side is smaller than the right-hand side. This follows since we assume that the expected maximum offer is large enough to cover the expected holding cost incurred during a single period (see (7));
- when α tends to infinity, the expression

$$\alpha - \int_{0}^{\alpha} F_z(p, z)\, dz$$

tends to $E[Z]$, and as a result, the left-hand side is larger than the right-hand side.

Therefore, there exists a unique α which satisfies (A2). This proves that α is the threshold, and the "accept" ("reject") decision is made at any period when $z > \alpha$ ($z < \alpha$).

Proof of Proposition 2: By differentiating (6) w.r.t. r, we obtain

$$\frac{\partial J(p, r)}{\partial r} = h(p, r)(J(p, r) - r), \tag{A3}$$

where $h(p, r)$ is the hazard rate of the distribution function $F_z(p, r)$,

$$h(p, r) = \frac{f_z(p, r)}{1 - F_z(p, r)}.$$

At $r = 0$, $J(p, r = 0) = E[Z] - w_0 - w_1 \lambda(p)$, which is positive according to the assumption given in (7). Therefore, from (A3), we conclude that the derivative of $J(p, r)$ at $r = 0$

is positive. The case where $J(p, r)$ remains greater than r along an infinite interval is not of interest, since in practice, the reservation price of an asset is finite. Therefore, we focus on the case where

$$J\left(p, r\right) = r$$

at some r, which we denote by $r^*(p)$. Since the second derivative of $J(p, r)$ at $r^*(p)$ is strictly negative, we conclude that $\partial J\left(p, r\right) / \partial r$ must be negative for all $r > r^*(p)$. This proves the quasi-concavity of the function $J\left(p, r\right)$ w.r.t. r for all p, and the uniqueness of $r^*(p)$, where $J\left(p, r\right)$ attains its maximum value. This proves also that

$$J\left(p, r^*(p)\right) = r^*(p). \tag{A4}$$

Proof of Proposition 3: By substituting (A4) in (6) we obtain the expression for $r^*(p)$ for all p,

$$r^*\left(p\right) = \frac{\int_{r^*(p)}^{\infty} z f_z\left(p, z\right) dz - w_0 - w_1 \lambda\left(p\right)}{1 - F_z\left(p, r^*\left(p\right)\right)}.$$

Consequently,

$$r^*\left(p\right) = E\left[Z\right] - w_0 - w_1 \lambda\left(p\right) + \int_0^{r^*(p)} F_z\left(p, z\right) dz. \tag{A5}$$

By differentiating (A5) w.r.t p, and equating the derivative to zero, we obtain

$$w_1 \frac{\partial \lambda\left(p\right)}{\partial p} + \int_{r^*(p)}^{\infty} \frac{\partial F_z\left(p, z\right)}{\partial p} dz = 0. \tag{A6}$$

The two statements of the proposition are now obtained by substituting p with p^* in (A5) and (A6).

Proof of Proposition 4: By differentiating (12) w.r.t. w_0 we obtain

$$\frac{\partial r^*}{\partial w_0} = -\frac{1}{1 - F_z\left(r^*\right)}.$$

Similarly, by differentiating (12) w.r.t. w_1,

$$\frac{\partial r^*}{\partial w_1} = -\frac{\lambda}{1 - F_z\left(r^*\right)}.$$

The two partial derivatives are negative that proves the proposition.

APPENDIX B

By assuming $w_1 = 0$, equations (11) and (12) simplify to

$$r^* = -w_0 + p^*\left(\gamma - \chi p^* + Ln\left(\lambda_0\right) - Ei\left(-\lambda_0 e^{-\chi p^* - \frac{r^*}{p^*}}\right)\right) \tag{B1}$$

$$r^* = \chi p^{*2} - \frac{w_0}{1 - F_z\left(p^*, r^*\right)}, \tag{B2}$$

We note (see Wikipedia, Exponential integral) that $Ei\left(-y\right) = \gamma + Ln\left(y\right) + \sum_{k=1}^{\infty} \frac{(-y)^k}{k \cdot k!}$ for $y > 0$. That is, for $y \ll 1$ the terms of the orders higher than $k = 1$ can be neglected, and, as a result, (B1) is rewritten as

$$r^* = -w_0 + r^* + p^* \lambda_0 e^{-\chi p^* - \frac{r^*}{p^*}}.$$

Equivalently,

$$r^* = -\chi p^{*2} - p^* Ln\left(\frac{w_0}{p^* \lambda_0}\right). \tag{B3}$$

Combining (B2) and (B3) yields

$$2\chi p^{*2} + e^{\frac{w_0}{p^*}}\left(w_0 - 2\chi p^{*2}\right) = p^*\left(e^{\frac{w_0}{p^*}} - 1\right) Ln\left(\frac{w_0}{p^* \lambda_0}\right).$$

Assuming that

$$\frac{w_0}{p^*} \ll 1,$$

we conclude,

$$2\chi p^{*2} + \left(1 + \frac{w_0}{p^*}\right)\left(w_0 - 2\chi p^{*2}\right) = w_0 Ln\left(\frac{w_0}{p^* \lambda_0}\right).$$

Consequently,

$$1 - 2\chi p^* = Ln\left(\frac{w_0}{p^* \lambda_0}\right). \tag{B4}$$

By combining (B3) and (B4), we obtain that

$$r^* = \chi p^{*2} - p^*. \tag{B5}$$

Equations (B4) and (B5) are equivalent to (13) and (14).

REFERENCES

[1] A. N. Shiryaev, *Optimal Stopping Rules*, vol. 8. New York, NY, USA: Springer, 2007.
[2] P. Cheng, Z. Lin, and Y. Liu, "Competing selling strategies in the housing market," *J. Real Estate Finance Econ.*, vol. 63, no. 3, pp. 394–413, Oct. 2021.
[3] B. Fontem, "An optimal stopping policy for car rental businesses with purchasing customers," *Ann. Oper. Res.*, vol. 317, no. 1, pp. 1–30, 2016.
[4] B. Zachariah, "Optimal stopping time in software testing based on failure size approach," *Ann. Oper. Res.*, vol. 235, no. 1, pp. 771–784, Dec. 2015.
[5] L. Guo, "Strategic communication before price haggling: A tale of two orientations," *Marketing Sci.*, vol. 41, no. 5, pp. 922–940, Sep. 2022.
[6] N. Wei, G. Yue, and Z. Zhang, "An optimal stopping approach to cell selection in 5G networks," *IEEE Access*, vol. 7, pp. 116094–116106, 2019.
[7] D. P. Bertsekas, *Dynamic Programming and Stochastic Control*. New York, NY, USA: Academic, 1976.
[8] P. Cheng, Z. Lin, Y. Liu, and M. J. Seiler, "The benefit of search in housing markets," *J. Real Estate Res.*, vol. 37, no. 4, pp. 597–622, Oct. 2015.
[9] X. He, Z. Lin, Y. Liu, and M. J. Seiler, "Search benefit in housing markets: An inverted U-shaped price and TOM relation," *Real Estate Econ.*, vol. 48, no. 3, pp. 772–807, Sep. 2020.
[10] S. He and Z. Ma, "Service quality and price competition in crowd-sourced delivery markets," *Int. Trans. Oper. Res.*, Aug. 2022, doi: 10.1111/itor.13192.
[11] Y. Zeng and W. Hou, "An analysis of price strategies with price matching in the presence of showrooming," *Int. Trans. Oper. Res.*, Jul. 2022, doi: 10.1111/itor.13187 .
[12] L. Han and W. C. Strange, "What is the role of the asking price for a house?" *J. Urban Econ.*, vol. 93, pp. 115–130, May 2016.
[13] E. Beracha and M. J. Seiler, "The effect of listing price strategy on transaction selling prices," *J. Real Estate Finance Econ.*, vol. 49, no. 2, pp. 237–255, Aug. 2014.
[14] E. Cardella and M. J. Seiler, "The effect of listing price strategy on real estate negotiations: An experimental study," *J. Econ. Psychol.*, vol. 52, pp. 71–90, Feb. 2016.
[15] G. W. Bucchianeri and J. A. Minson, "A homeowner's dilemma: Anchoring in residential real estate transactions," *J. Econ. Behav. Org.*, vol. 89, pp. 76–92, May 2013.
[16] A. A. Shaikh, M. A.-A. Khan, G. C. Panda, and I. Konstantaras, "Price discount facility in an EOQ model for deteriorating items with stock-dependent demand and partial backlogging," *Int. Trans. Oper. Res.*, vol. 26, no. 4, pp. 1365–1395, Jul. 2019.

[17] Z. E. Ning, "List price and discount in a stochastic selling process," *Marketing Sci.*, vol. 40, no. 2, pp. 366–387, Mar. 2021.

[18] C. H. Liu, A. Nowak, and S. S. Rosenthal, "Housing price bubbles, new supply, and within-city dynamics," *J. Urban Econ.*, vol. 96, pp. 55–76, Nov. 2016.

[19] M. A. Arnold and S. A. Lippman, "Posted prices versus bargaining in markets with asymmetric information," *Econ. Inquiry*, vol. 36, no. 3, pp. 450–457, Jul. 1998.

[20] A. Yavas and S. Yang, "The strategic role of listing price in marketing real estate: Theory and evidence," *Real Estate Econ.*, vol. 23, no. 3, pp. 347–368, Sep. 1995.

[21] O. Guéant, *The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making*, vol. 33. Boca Raton, FL, USA: CRC Press, 2016.

[22] S. Gökay, A. F. Roch, and H. M. Soner, *Liquidity Models in Continuous and Discrete Time*. Berlin, Germany: Springer, 2011.

[23] S. Salant, "For sale by owner: When to use a broker and how to price the house," *J. Real Estate Finance Econ.*, vol. 4, no. 2, pp. 157–173, Jun. 1991.

[24] A. Díaz and B. Jerez, "House prices, sales, and time on the market: A search-theoretic framework," *Int. Econ. Rev.*, vol. 54, no. 3, pp. 837–872, Aug. 2013.

[25] A. Merlo and F. Ortalo-Magné, "Bargaining over residential real estate: Evidence from England," *J. Urban Econ.*, vol. 56, no. 2, pp. 192–216, Sep. 2004.

[26] A. Merlo, F. Ortalo-Magné, and J. Rust, "The home selling problem: Theory and evidence," *Int. Econ. Rev.*, vol. 56, no. 2, pp. 457–484, May 2015.

[27] M. Glower, D. Haurin, and P. Hendershott, "Selling price and selling time: The impact of investor motivation," *Real Estate Econ.*, vol. 26, no. 4, pp. 719–740, 1998.

[28] M. A. Arnold, "Search, bargaining and optimal asking prices," *Real Estate Econ.*, vol. 27, no. 3, pp. 453–481, Sep. 1999.

[29] Z. Lin and Y. Liu, "Real estate returns and risk with heterogeneous investors," *Real Estate Econ.*, vol. 36, no. 4, pp. 753–776, Dec. 2008.

[30] S. M. Shavarani, M. Golabi, and G. Izbirak, "A capacitated biobjective location problem with uniformly distributed demands in the UAV-supported delivery operation," *Int. Trans. Oper. Res.*, vol. 28, no. 6, pp. 3220–3243, Nov. 2021.

[31] P. Cao, W. Zhou, and J. Wu, "Optimal pricing strategy for a service provider in the presence of repetitive usage," *Int. Trans. Oper. Res.*, vol. 29, no. 4, pp. 2586–2612, Jul. 2022.

[32] Z. Li, Q. Li, H. Sun, and L. Sun, "Diffused effort, asset heterogeneity, and real estate brokerage," *Real Estate Econ.*, vol. 50, no. 3, pp. 707–742, Sep. 2022.

[33] A. Seierstad, "Reservation prices in optimal stopping," *Oper. Res.*, vol. 40, no. 2, pp. 409–415, Apr. 1992.

[34] C. J. Mayer, "Taxes, income distribution, and the real estate cycle: Why all houses do not appreciate at the same rate," *New England Econ. Rev.*, pp. 39–50, May 1993.

[35] D. Haurin, "The duration of marketing time of residential housing," *AREUEA J.*, vol. 16, no. 4, pp. 396–410, 1988.

[36] S. C. Bourassa, D. R. Haurin, J. L. Haurin, M. Hoesli, and J. Sun, "House price changes and idiosyncratic risk: The impact of property characteristics," *Real Estate Econ.*, vol. 37, no. 2, pp. 259–278, 2009.

[37] Y. Zhou and D. Haurin, "On the determinants of house value volatility," *J. Real Estate Res.*, vol. 32, no. 4, pp. 377–395, 2010.

[38] G. Singer and M. Golan, "Applying data mining algorithms to encourage mental health disclosure in the workplace," *Int. J. Bus. Inf. Syst.*, vol. 36, no. 4, pp. 553–571, 2021.

[39] G. Singer, M. Golan, R. Shiff, and D. Kleper, "Evaluating the effectiveness of accommodations given to students with learning impairments: Ordinal and interpretable machine-learning-based methodology," *IEEE Trans. Learn. Technol.*, vol. 15, no. 6, pp. 736–746, Dec. 2022.

[40] M. Marudi, I. Ben-Gal, and G. Singer, "A decision tree-based method for ordinal classification problems," *IISE Trans.*, pp. 1–15, Jul. 2022. [Online]. Available: https://www.tandfonline.com/doi/full/10.1080/24725854.2022.2081745

[41] S. D. Levitt and C. Syverson, "Market distortions when agents are better informed: The value of information in real estate transactions," *Rev. Econ. Statist.*, vol. 90, no. 4, pp. 599–611, Nov. 2008.

[42] S. Agarwal, J. He, T. F. Sing, and C. Song, "Do real estate agents have information advantages in housing markets?" *J. Financial Econ.*, vol. 134, no. 3, pp. 715–735, Dec. 2019.

EUGENE KHMELNITSKY received the Ph.D. degree in industrial engineering from Tel Aviv University. He studied applied mathematics with the Institute of Physics and Technology, Moscow. He is currently an Associate Professor of industrial engineering with Tel Aviv University. His research interests include analysis of dynamical systems, optimal control theory and applications, stochastic modeling and methods, and production planning and scheduling.

GONEN SINGER received the B.Sc., M.Sc., and Ph.D. degrees in industrial engineering and management from Tel Aviv University, Israel, in 2000, 2001, and 2006, respectively. He is currently an Assistant Professor of industrial engineering and data science with the Faculty of Engineering, Bar-Ilan University, where he co-founded the Process Mining Laboratory which promotes innovative adaptions of machine learning and optimization methods to support the analysis of real-world processes. He is also the Co-Founder of CB4 ("See Before") a startup that provides granular predictive analytics solutions to retail organizations and was acquired by GAP, in 2021.