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RESEARCH ARTICLE

Spectrum and Ricci Curvature on the Weighted Strong Product Graphs

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ABSTRACT The strong product on graphs is also called the normal product or the AND product. It is the union of Cartesian product and tensor product, and also is a binary operation on graphs. This operation takes two graphs and produces a new graph. In this paper, we will study the strong product on weighted graphs. The key to study the relationship between the spectrum of two original weighted graphs and that of their strong product graph is to provide a reasonable weight function to the weighted strong product graph. We introduce a definition of the weight function to the strong product graph $G \boxtimes H$, where $G = (X, a)$ and $H = (Y, b)$ are two connected weighted graphs. And we derive an expression for the spectrum of $G \boxtimes H$ by using the spectrums of the weighted graph G and H. In this paper, we will also study the Ricci curvature of two adjacent points for the strong product. We prove that the Ricci curvature for strong product of two regular graphs with simple weight is bounded by the Ricci curvature of G and H.

INDEX TERMS Strong product, spectrum, Ricci curvature.

I. INTRODUCTION

A graph product is a binary operation on graphs. This operation takes two graphs G_1 and G_2 and produces a new graph H with the following properties: (1) The vertex set of H is the Cartesian product $V(G_1) \times V(G_2)$, where $V(G_1)$ and $V(G_2)$ are the vertex sets of G_1 and G_2, respectively. (2) Two vertices (u_1, v_1) and (u_2, v_2) of H are connected by an edge if and only if the vertices u_1, u_2, v_1, v_2 satisfy a condition that takes into account the edges of G_1 and G_2. The graph products differ in exactly which this condition is, such as Cartesian product [10], [30], strong product [5], [25], tensor product [24], [27], lexicographic product [6], [17].

The strong product palys a very important role in graph theory. It is well known that the king's graph, a graph whose vertices are squares of a chessboard and whose edges represent possible moves of a chess king, is a strong product of two path graphs. Many results are established on simple finite graphs. The strong product was introduced by Sabidussi in [26]. In that setting, the strong product is contrasted against a weak product, but the two are different only when applied to infinitely many factors. The strong product is also called the normal product or the AND product. It is the union of the Cartesian product and the tensor product. Abajo-Casablanca-Diánez-Vázquez [1] showed that the strong product of two maximally connected graphs with at least three vertices and girth at least 5 is maximally connected.

Strong product has also important applications in computer science. It can be used to combine multiple networks into a single, larger network. This can be useful in situations where it is necessary to communicate between different networks, such as in a distributed system or the Internet. By using the strong product, different networks can be merged together while maintaining their original properties, allowing for more efficient and effective communication.

Recently, many researchers' attentions were draw to various properties on weighted graphs, the readers can refer to [2], [3], [4], [11], [14], [15], [18], and [29]. Grigor'yan [13] studied Cartesian product on weighted graphs. They proved all the eigenvalues of the Laplace operator on the weighted

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Cartesian product graph are the convex combinations of eigenvalues of the Laplace operator of two original graphs.

Inspired by Grigor'yan' work in [13], we will consider the strong product on weighted graphs in this paper. The main difficulty to study the relationship between the spectrum of two original weighted graphs and that of their strong product graph lies in giving a reasonable weight function to the weighted strong product graph.

The Ricci curvature plays a very important role on geometric analysis on Riemannian manifolds. Many results are established on manifolds with non-negative Ricci curvature or on manifolds with Ricci curvature bounded below. The first definition of Ricci curvature on graphs was introduced by Chung-Yau in [8]. Lin-Yau [21] proved that the Ricci curvature for locally finite graph in the sense of Bakry and Emery is bounded below by -1. And they also showed that the Ricci curvature in the sence of Ollivier for simple random walk on graphs is bounded below. The Ricci flat graph in the sense of Chung and Yau was proved to be a graph with Ricci curvature bounded below by zero. Münch-Wojciechowski [22] showed that a lower bound on the Ollivier curvature is equivalent to a certain Lipschitz decay of solutions to the heat equation. Cushing-Liu-Peyerimhoff [9] proved that the curvature functions of the Cartesian product of two graphs G_1, G_2 are equal to an abstract product of curvature functions of G_1, G_2.

In this paper, we will also study the Ricci curvature of two adjacent points for the strong product. Since there are three kinds of edges for the strong product (Section II-A), we fall into three results to explain the bounds of the Ricci curvature.

The remaining part of this paper is organized as follows: In Section II, we give some notations and definitions on weighted graphs and state our main results. In Section II-A, we give an expression for the spectrum of $G \boxtimes H$ by using the spectrums of graph G and graph H. In Section II-B, by following the method in [20], we give the bound of Ricci curvature for the strong product of two regular locally finite graphs. In Section III, we give the proofs of our main results.

II. MAIN RESULTS

A graph is called simple if it has no loops and multiple edges. All graphs considered in this paper are connected, simple, undirected and weighted graphs. We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For any two vertices u and v of G, denote by $d_G(u, v)$ the distance between u and v in G. $G \boxtimes H$ denotes the strong product between graph G and H. Now, we recall basic definitions for weighted graphs, refer to [3], [7], and [13]. Let V be a finite discrete space serving as the set of vertices of a graph G, and E be the set of edges of the graph, $\mu : V \times V \ni (x, y) \mapsto \mu_{xy} \in [0, \infty)$ be an edge weight function satisfying: (1) $\mu_{xy} = \mu_{yx}$, $\forall x, y \in V$; (2) $\sum_{y\in V} \mu_{xy} < \infty$, $\forall x \in V$. These induce a combinatorial (undirect) graph structure $G = (V, E)$ with the set V of vertices and the set E of edges, such that for $x, y \in V$, $\{x, y\} \in E$ if and only if $\mu_{xy} > 0$, in symbols $x \sim y$. Alternatively, μ_{xy} can be considered as a positive function on the set E of edges, that is extended to be 0 on non-edge pairs (x, y). Therefore, we may denote a weighted graph by $G = (V, \mu)$. We call a graph G has simple weight if the weight function μ on G satisfying either $\mu_{xy} = 1$ for any $x \sim y$ or $\mu_{xy} = 0$ for any $x \not\sim y$ in G. Any weight μ_{xy} gives rise to a function on vertices as follows:

$$\mu(x) = \sum_{y\sim x} \mu_{xy}.$$

And $\mu(x)$ is called the weight of a vertex x. For example, if the weight μ is simple, then $\mu(x) = \deg(x)$.

A. SPECTRUM

Firstly, let us recollect the strong product of two unweighted graphs. Assume (X, E_1) and (Y, E_2) are two unweighted graphs. Their strong product is defined as follows:

$$(V, E) = (X, E_1) \boxtimes (Y, E_2)$$

where $V = X \times Y$ and the set E of edges is defined by

$$(x, y) \sim (x', y') \text{ if and only if } \begin{cases} x \sim x' \text{ and } y \sim y', \\ \text{or } x \sim x' \text{ and } y = y', \\ \text{or } x = x' \text{ and } y \sim y', \end{cases}$$

where $x, x' \in X$ and $y, y' \in Y$. It is easy to see that $|V| = |X||Y|$, $\deg(x, y) = \deg(x) + \deg(y) + \deg(x)\deg(y)$ for all $x \in X, y \in Y$ and $|E| = |X||E_2| + |Y||E_1| + 2|E_1||E_2|$. In the following, we construct a suitable weight function of product graph from two original graphs' weight functions.

Definition 1: Let $G = (X, a)$ be a locally finite connected weighted graph and $H = (Y, b)$ be a finite connected weighted graph. Fix three numbers $p_1, p_2, p_3 > 0$ and define the strong product graph

$$(V, \mu) = G \boxtimes H(p_1, p_2, p_3),$$

as follows: $V = X \times Y$ and the weight μ on V is defined by

$$\mu_{(x,y)(x',y')} = \begin{cases} p_1 a_{xx'} b_{yy'}, & x \sim x' \text{ in } G, y \sim y' \text{ in } H, \\ p_2 a_{xx'} b(y), & x \sim x' \text{ in } G, y = y' \text{ in } H, \\ p_3 a(x) b_{yy'}, & x = x' \text{ in } G, y \sim y' \text{ in } H, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Now we recall the definition of the Laplace operator and Markov operator on weighted graphs (refer to [13]). Let (V, μ) be a locally finite weighted graph without isolated points. For any function $f : V \to \mathbb{R}$, the function Δf is defined by

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_y \mu_{xy}\big(f(y) - f(x)\big). \tag{2}$$

The operator Δ acting on functions on V, is called the weighted Laplace operator of (V, μ). For any function $f : V \to \mathbb{R}$, the function Pf is defined by

$$Pf(x) = \sum_y P(x, y)f(y), \tag{3}$$

where the Markov kernel $P(x, y) = \mu_{xy}/\mu(x)$ is the random walk with transition probability of moving from a vertex x to each of its neighbours y. This operator P is called the *Markov operator*.

Theorem 2: Let $G = (X, a)$ and $H = (Y, b)$ be finite connected weighted graphs with m and n vertices respectively. Suppose that $\{\alpha_k\}_{k=1}^{m}$ and $\{\beta_l\}_{l=1}^{n}$ be the sequences of the eigenvalues of the Markov operators A on X and B on Y respectively, counted with multiplicities. Then all the eigenvalues of the Markov operator P on the strong product $G \boxtimes H(p_1, p_2, p_3)$ are given by

$$\left\{\frac{p_1\alpha_k\beta_l + p_2\alpha_k + p_3\beta_l}{p_1 + p_2 + p_3}\right\}$$

where $k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n$.

From (2) and (3), we know the Laplace operator Δ and the Markov operator P are related by a simple identity

$$\Delta = P - id,$$

where id is the identical operator in $\mathcal{F}$, which is the set of all real-valued functions on V. It is easy to see that the same relation holds for the eigenvalues of Δ and P. Hence, by Theorem 2, we have

Corollary 3: Let $G = (X, a)$ and $H = (Y, b)$ be finite connected weighted graphs with m and n vertices respectively. Suppose that $\{\alpha_k\}_{k=1}^{m}$ and $\{\beta_l\}_{l=1}^{n}$ are the sequences of the eigenvalues of the Laplace operators A on X and B on Y respectively, counted with multiplicities. Then all the eigenvalues of the Laplace operator Δ on the strong product $G \boxtimes H(p_1, p_2, p_3)$ are given by

$$\left\{\frac{p_1(\alpha_k\beta_l + \alpha_k + \beta_l) + p_2\alpha_k + p_3\beta_l}{p_1 + p_2 + p_3}\right\}$$

where $k = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, n$.

B. RICCI CURVATURE

We will use similar notations as in [20] and [23]. A *probability distribution* over the vertex-set $V(G)$ is a mapping $m : V(G) \to [0, 1]$ satisfying $\sum_{x \in V(G)} m(x) = 1$. Suppose two probability distributions m_1 and m_2 have finite support. A *coupling* between m_1 and m_2 is a mapping $A : V(G) \times V(G) \to [0, 1]$ with finite support so that

$$\sum_{y \in V(G)} A(x, y) = m_1(x) \text{ and } \sum_{x \in V(G)} A(x, y) = m_2(y).$$

The *transportation distance* between two probability distributions m_1 and m_2 is defined as follows:

$$W(m_1, m_2) = \inf_{A} \sum_{x,y \in V(G)} A(x, y)d(x, y),$$

where the infimum is taken over all coupling A between m_1 and m_2. A function f over graph G is c-Lipschitz if

$$|f(x) - f(y)| \leq cd(x, y),$$

for all $x, y \in V(G)$. By the duality theorem of a linear optimization problem, the transportation distance can also be written as follows:

$$W(m_1, m_2) = \sup_{f} \sum_{x \in V(G)} f(x)(m_1(x) - m_2(x)),$$

where the supremum is taken over all 1-Lipschitz function f. Noting that any c-Lipschitz function f over a metric subspace can be extended to a c-Lipschitz function over the whole metric space. The $W(m_1, m_2)$ only depends on distances among vertices in $\mathrm{supp}(m_1) \cup \mathrm{supp}(m_2)$. For any vertex $x \in V(G)$, let $N(x)$ denote the set of neighborhood of x, i.e., $N(x) = \{y \in V(G) : y \sim x \text{ in } G\}$, and $N[x] = N(x) \cup \{x\}$. For any $\alpha \in [0, 1]$ and any vertex x, the *probability measure* m_x^{α} is defined as

$$m_x^{\alpha}(v) = \begin{cases} \alpha, & \text{if } v = x, \\ \dfrac{1-\alpha}{\deg(x)}, & \text{if } v \in N(x), \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

For any $x, y \in V(G)$, we define α-Ricci-curvature k_α to be

$$k_\alpha(x, y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x, y)},$$

and the Ricci curvature at (x, y) in the graph is

$$k(x, y) = \lim_{\alpha \to 1} \frac{k_\alpha(x, y)}{1 - \alpha}. \tag{5}$$

Now we state our main results about the Ricci curvature of $G \boxtimes H$ as follows:

Theorem 4: Let G and H be r-regular and k-regular locally finite and connected graphs with simple weight respectively, where $G \neq K_{r+1}$. For $u_1 \sim u_2$ in G, $v \in V(H)$, the Ricci curvature of $G \boxtimes H$ is bounded, and if $N_G(u_1) \cap N_G(u_2) \subsetneqq N_G(u_1) \setminus \{u_2\}$,

$$\frac{(rk + r)k^G(u_1, u_2) - 2rk}{rk + k + r} \leq k^{G\boxtimes H}((u_1, v), (u_2, v)) \leq \frac{(kr + r)k^G(u_1, u_2) + 2k}{kr + k + r},$$

if $N_G(u_1) \cap N_G(u_2) = N_G(u_1) \setminus \{u_2\} = N_G(u_2) \setminus \{u_1\}$,

$$\frac{(rk + r)k^G(u_1, u_2)}{rk + k + r} - \frac{2kr^2 + kr - 3k}{r(rk + k + r)} \leq k^{G\boxtimes H}((u_1, v), (u_2, v)) \leq \frac{(kr + r)k^G(u_1, u_2) + 2k}{kr + k + r}.$$

Theorem 5: Let G and H be r-regular and k-regular locally finite and connected graphs with simple weight respectively, where $G \neq K_{r+1}$. For $u \in V(G)$, $v_1 \sim v_2$ in H, the Ricci curvature of $G \boxtimes H$ is bounded, and if $N_H(v_1) \cap N_H(v_2) \subsetneqq N_H(v_1) \setminus \{v_2\}$,

$$\frac{(rk + k)k^H(v_1, v_2) - 2rk}{rk + k + r} \leq k^{G\boxtimes H}((u, v_1), (u, v_2)) \leq \frac{(kr + k)k^H(v_1, v_2) + 2r}{kr + k + r},$$

if $N_H(v_1) \cap N_H(v_2) = N_H(v_1) \setminus \{v_2\} = N_H(v_2) \setminus \{v_1\}$,

$$\frac{(rk+k)k^H(v_1,v_2)}{rk+k+r} - \frac{2rk^2+kr-3r}{k(rk+k+r)} \le k^{G\boxtimes H}((u,v_1),(u,v_2)) \le \frac{(kr+k)k^H(v_1,v_2)+2r}{kr+k+r}.$$

Theorem 6: Let G and H be r-regular and k-regular locally finite and connected graphs with simple weight, respectively. For $u_1 \sim u_2$ in G, $v_1 \sim v_2$ in H, the Ricci curvature of $G \boxtimes H$ is bounded and

$$\frac{-4r^2k^2+10rk-2k-4r}{rk(rk+k+r)} \le k^{G\boxtimes H}((u_1,v_1),(u_2,v_2)) \le \frac{\min\{(kr+r)k^G(u_1,u_2)+2k,(kr+k)k^H(v_1,v_2)+2r\}}{kr+k+r}.$$

III. PROOFS

Theorem 7: Let $G=(X,a)$ be a locally finite connected weighted graph and $H=(Y,b)$ be a finite connected weighted graph. Suppose that A, B are the Markov kernels on X in G and Y in H, respectively. Then the Markov kernel P on the strong product $G \boxtimes H(p_1,p_2,p_3) = (V,\mu)$ is given by as shown in the equation at the bottom of the next page, where $x,x' \in X$, $y,y' \in Y$ and p_1,p_2 and p_3 are three given positive numbers.

Proof: From the definition, the weight on the vertices of V is

$$\begin{aligned}\mu(x,y) &= \sum_{(x',y')\sim(x,y)} \mu_{(x,y)(x',y')} \\ &= \sum_{\substack{x'\sim x\\ y'\sim y}} \mu_{(x,y)(x',y')} + \sum_{\substack{x'\sim x\\ y'=y}} \mu_{(x,y)(x',y')} + \sum_{\substack{x'=x\\ y'\sim y}} \mu_{(x,y)(x',y')} \\ &= p_1 \sum_{\substack{x'\sim x\\ y'\sim y}} a_{xx'}b_{yy'} + p_2 b(y)\sum_{x'\sim x} a_{xx'} + p_3 a(x) \sum_{y'\sim y} b_{yy'} \\ &= (p_1+p_2+p_3)a(x)b(y).\end{aligned}$$

In the case $x \sim x'$, $y \sim y'$, by (1), we have

$$\begin{aligned}P((x,y),(x',y')) &= \frac{\mu_{(x,y)(x',y')}}{\mu(x,y)} = \frac{p_1 a_{xx'}b_{yy'}}{(p_1+p_2+p_3)a(x)b(y)} \\ &= \frac{p_1}{p_1+p_2+p_3}A(x,x')B(y,y'),\end{aligned}$$

and other cases are treated similarly. □ ■

Lemma 8: If $G=(X,a)$ and $H=(Y,b)$ are r-regular, k-regular graphs with simple weights, respectively. Then their strong product

$$G \boxtimes H(1, \frac{1}{k}, \frac{1}{r})$$

is a $(r+k+rk)$-regular graph with a simple weight.

Proof: Since a, b are simple weights of G and H, with the regularity of graphs G and H, we have

$$a(x) = \deg(x) = r, \quad b(y) = \deg(y) = k.$$

Hence, by the definition of the weight function on strong product, we get

$$\mu_{(x,y)(x',y')} = \begin{cases} p_1, & x\sim x' \text{ in } G,\ y \sim y' \text{ in } H,\\ p_2 k, & x\sim x' \text{ in } G,\ y = y' \text{ in } H,\\ p_3 r, & x= x' \text{ in } G,\ y \sim y' \text{ in } H,\\ 0, & \text{otherwise.}\end{cases} \tag{6}$$

Therefore, when taking the parameters p_1, p_2 and p_3 as

$$p_1 = 1, \quad p_2 = \frac{1}{k},\ p_3 = \frac{1}{r},$$

we have $\mu_{(x,y)(x',y')} = 1$ for any $(x,y) \sim (x',y')$ and $\mu_{(x,y)(x',y')} = 0$ for any $(x,y) \nsim (x',y')$ in $G \boxtimes H$, that is the weight μ is also simple. □ ■

Corollary 9: Let (V,E) be a finite connected r-regular graph, and set $(V^n,E_n) = (V,E)^{\boxtimes n} = (V,E)\boxtimes(V,E)\boxtimes\cdots\boxtimes(V,E)$ be the strong product of n (V,E). Let μ be a simple weight on V, and $\{\alpha_k\}_{k=1}^{|V|}$ be a sequence of the eigenvalues of the Markov operator on (V,μ), counted with multiplicity. Let μ_n be a simple weight on V^n. Then the eigenvalue λ_{Υ_n} of the Markov operator on (V^n,μ_n) is given by

$$\lambda_{\Upsilon_n} = \frac{\sum_{j=1}^n r^j \sum_{I\subseteq[n],\ |I|=j} \prod_{i\in I}\alpha_{k_i}}{(r+1)^n-1} \tag{7}$$

where $\Upsilon_n = (k_1,k_2,\ldots,k_n) \in \{1,2,\ldots,|V|\}^n$, $[n] = \{1,2,\ldots,n\}$ and each eigenvalue is counted with multiplicity.

Proof: We use induction on n. If $n=1$, then there is nothing to prove. Let us make the inductive step from n to $n+1$. Denote D_n as the degree of (V^n,E_n), then D_n satisfies $D_{n+1} = rD_n + r + D_n$, $D_1 = r$. We can easy to see that

$$D_n = (r+1)^n - 1.$$

Note that $(V^{n+1},E_{n+1}) = (V^n,E_n)\boxtimes(V,E)$. It follows from Lemma 8 that

$$(V^{n+1},\mu_{n+1}) = (V^n,\mu_n)\boxtimes(V,\mu)(1,\frac{1}{r},\frac{1}{D_n}).$$

By the inductive hypothesis, the eigenvalues of the Markov operator on (V^n,μ_n) are given by formula (7). Hence, by Theorem 2, the eigenvalue $\lambda_{\Upsilon_{n+1}}$ on (V^{n+1},μ_{n+1}) is given by

$$\begin{aligned}\lambda_{\Upsilon_{n+1}} &= \frac{1}{1+\frac{1}{r}+\frac{1}{D_n}}\left\{\lambda_{\Upsilon_n}\alpha_{k_{n+1}} + \frac{1}{r}\lambda_{\Upsilon_n} + \frac{1}{D_n}\alpha_{k_{n+1}}\right\} \\ &= \frac{1}{1+1/r+1/((r+1)^n-1)} \\ &\quad\cdot\left\{\frac{\sum_{j=1}^n r^j \sum_{I\subseteq[n],\ |I|=j}\prod_{i\in I}\alpha_{k_i}}{(r+1)^n-1}\alpha_{k_{n+1}}\right.\end{aligned}$$

$$+\frac{1}{r}\frac{\sum_{j=1}^{n} r^j \sum_{I\subseteq[n],\ |I|=j}\prod_{i\in I}\alpha_{k_i}}{(r+1)^n-1}+\frac{1}{(r+1)^n-1}\alpha_{k_{n+1}}\Big\}$$
$$=\frac{1}{(r+1)^{n+1}-1}\Big\{\sum_{j=1}^{n} r^{j+1}\sum_{I\subseteq[n],\ |I|=j}\prod_{i\in I}\alpha_{k_i}\alpha_{k_{n+1}}$$
$$+\sum_{j=1}^{n} r^j\sum_{I\subseteq[n],\ |I|=j}\prod_{i\in I}\alpha_{k_i}+r\alpha_{k_{n+1}}\Big\}$$
$$=\frac{\sum_{j=1}^{n+1} r^j\sum_{I\subseteq[n+1],\ |I|=j}\prod_{i\in I}\alpha_{k_i}}{(r+1)^{n+1}-1},$$

which is just to be proved. □ ■

Now we shall prove our main results for the spectrum of the strong product.

Proof of Theorem 2: Let f be an eigenfunction of A with the eigenvalue α and g be an eigenfunction of B with the eigenvalue β. That is to say, for any $x \in X$ and $y \in Y$, there holds

$$Af(x)=\sum_{x'\in X}A(x,x')f(x')=\alpha f(x),$$
$$Bg(y)=\sum_{y'\in Y}B(y,y')g(y')=\beta g(y).$$

Now, let us show that the function $h(x,y)=f(x)g(y)$ is the eigenfunction of P with the eigenvalue $\frac{p_1\alpha\beta+p_2\alpha+p_3\beta}{p_1+p_2+p_3}$. For any $(x,y)\in X\times Y$, by Theorem 7, we have

$$\begin{aligned}
&Ph(x,y)\\
&=\sum_{\substack{x'\in X\\ y'\in Y}}P\left((x,y),(x',y')\right)h(x',y')\\
&=\frac{p_1}{p_1+p_2+p_3}\sum_{\substack{x'\sim x\\ y'\sim y}}A(x,x')B(y,y')f(x')g(y')\\
&\quad+\frac{p_2}{p_1+p_2+p_3}\sum_{\substack{x'\sim x\\ y'=y}}A(x,x')f(x')g(y')\\
&\quad+\frac{p_3}{p_1+p_2+p_3}\sum_{\substack{x'=x\\ y'\sim y}}B(y,y')f(x')g(y')\\
&=\frac{p_1}{p_1+p_2+p_3}\alpha f(x)\beta g(y)+\frac{p_2}{p_1+p_2+p_3}\alpha f(x)g(y)\\
&\quad+\frac{p_3}{p_1+p_2+p_3}f(x)\beta g(y)\\
&=\frac{p_1\alpha\beta+p_2\alpha+p_3\beta}{p_1+p_2+p_3}h(x,y),
\end{aligned}$$

which is to be proved.

Let $\{f_k\}$ be a basis in the space of functions on X such that $Af_k=\alpha_k f_k$, and $\{g_l\}$ be a basis in the space of functions on Y such that $Bg_l=\beta_l g_l$. Then $\{h_{kl}(x,y)=f_k(x)g_l(y)\}$ is a linearly independent sequence of functions on $X\times Y$. Since the number of such functions is $mn=|X\times Y|$, we see that h_{kl} is a basis in the space of functions on $X\times Y$. Since h_{kl} is the eigenfunction with the eigenvalue $\frac{p_1\alpha_k\beta_l+p_2\alpha_k+p_3\beta_l}{p_1+p_2+p_3}$, we conclude that the sequence $\frac{p_1\alpha_k\beta_l+p_2\alpha_k+p_3\beta_l}{p_1+p_2+p_3}$ exhausts all the eigenvalues of P. □

In the following, we shall prove our main results on Ricci curvature of the strong product of two regular graphs.

A. PROOF OF THEOREM 5

Claim 1: Let G and H be r-regular and k-regular locally finite connected graphs with simple weight respectively, where $G\neq K_{r+1}$. For $u_1\sim u_2$ in G, $v\in V(H)$, the Ricci curvature of $G\boxtimes H$ is bounded below, and if $N_G(u_1)\cap N_G(u_2)\subsetneqq N_G(u_1)\setminus\{u_2\}$,

$$k^{G\boxtimes H}((u_1,v),(u_2,v))\geq\frac{rk+r}{rk+k+r}k^G(u_1,u_2)-\frac{2rk}{rk+k+r},$$

if $N_G(u_1)\cap N_G(u_2)=N_G(u_1)\setminus\{u_2\}=N_G(u_2)\setminus\{u_1\}$,

$$k^{G\boxtimes H}((u_1,v),(u_2,v))\geq\frac{rk+r}{rk+k+r}k^G(u_1,u_2)-\frac{2kr^2+kr-3k}{r(rk+k+r)}.$$

Proof: Assume that A is a coupling function between $m_{u_1}^{\alpha}$ and $m_{u_2}^{\alpha}$ which defined as (4), and A achieves the infimum of $W(m_{u_1}^{\alpha},m_{u_2}^{\alpha})$, that is

$$W(m_{u_1}^{\alpha},m_{u_2}^{\alpha})=\sum_{x,y\in V(G)}A(x,y)d(x,y).$$

Since $u_1\sim u_2$ in G, $(u_1,v)\sim(u_2,v)$ in $G\boxtimes H$. We define a function $D_1: V(G\boxtimes H)\times V(G\boxtimes H)\to[0,1]$ as follows:

If $((x_1,y_1),(x_2,y_2))\in S_1$, then

$$D_1((x_1,y_1),(x_2,y_2))=\frac{rk+r}{rk+k+r}A(x_1,x_2)+\frac{\alpha k}{rk+k+r},$$

if $((x_1,y_1),(x_2,y_2))\in S_2$, then

$$D_1((x_1,y_1),(x_2,y_2))=\frac{rk+r}{rk+k+r}A(x_1,x_2)+\frac{(\alpha-1)k}{(rk+k+r)r},$$

$$P((x,y),(x',y'))=\begin{cases}\frac{p_1}{p_1+p_2+p_3}A(x,x')B(y,y'), & x\sim x' \text{ in } G,\ y\sim y' \text{ in } H,\\ \frac{p_2}{p_1+p_2+p_3}A(x,x'), & x\sim x' \text{ in } G,\ y=y' \text{ in } H,\\ \frac{p_3}{p_1+p_2+p_3}B(y,y'), & x=x' \text{ in } G,\ y\sim y' \text{ in } H,\\ 0, & \text{otherwise.}\end{cases}$$

if $((x_1, y_1), (x_2, y_2)) \in S_3$, then

$$D_1((x_1, y_1), (x_2, y_2)) = \frac{rk + r}{rk + k + r}A(x_1, x_2),$$

if $((x_1, y_1), (x_2, y_2)) \in S_4$, then

$$D_1((x_1, y_1), (x_2, y_2)) = \frac{1-\alpha}{k(rk + k + r)},$$

if $((x_1, y_1), (x_2, y_2)) \in S_5$, then

$$D_1((x_1, y_1), (x_2, y_2)) = \frac{1-\alpha}{k(rk + k + r)},$$

if $((x_1, y_1), (x_2, y_2)) \in S_6$, then

$$D_1((x_1, y_1), (x_2, y_2)) = \frac{(1-\alpha)(1-r)}{k(rk + k + r)},$$

otherwise, $D_1((x_1, y_1), (x_2, y_2)) = 0$.

Here $S_1 = \{((x_1, y_1), (x_2, y_2)) \in V(G \boxtimes H) \times V(G \boxtimes H) : x_1 = u_1, x_2 = u_2, y_1 = y_2 = v\}$, $S_2 = \{((x_1, y_1), (x_2, y_2)) \in V(G \boxtimes H) \times V(G \boxtimes H) : x_1 \sim u_1, x_2 \sim u_2, y_1 = y_2 = v\}$, $S_3 = \{((x_1, y_1), (x_2, y_2)) \in V(G\boxtimes H)\times V(G\boxtimes H) : (x_1, x_2) \neq (u_1, u_2), (x_1, x_2) \notin N_G(u_1) \times N_G(u_2), y_1 = y_2 = v\}$, $S_4 = \{((x_1, y_1), (x_2, y_2)) \in V(G \boxtimes H) \times V(G \boxtimes H) : x_1 = u_1, y_1 \sim v, x_2 \sim u_2, y_2 \sim v\}$, $S_5 = \{((x_1, y_1), (x_2, y_2)) \in V(G \boxtimes H) \times V(G \boxtimes H) : x_1 \sim u_1, y_1 \sim v, x_2 = u_2, y_2 \sim v\}$, $S_6 = \{((x_1, y_1), (x_2, y_2)) \in V(G \boxtimes H) \times V(G \boxtimes H) : x_1 = u_1, y_1 \sim v, x_2 = u_2, y_2 \sim v\}$.

Now we claim that D_1 is a coupling function between $m^{\alpha}_{(u_1,v)}$ and $m^{\alpha}_{(u_2,v)}$. Set a characteristic function as follows:

$$\mathbf{1}_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

and we denote $M_1 = \{(x, y) \in V(G \boxtimes H) : x \sim u_2,\ y = v\}$, $M_2 = \{(x, y) \in V(G \boxtimes H) : x \sim u_2,\ y \sim v\}$ and $M_3 = \{(x, y) \in V(G \boxtimes H) : x = u_2,\ y \sim v\}$.
Then

$$\begin{aligned}
&\sum_{(x_1,y_1)\in V(G\boxtimes H)} D_1((x_1, y_1), (x_2, y_2)) \\
&= \frac{rk + r}{rk + k + r} \sum_{x_1\in V(G)} A(x_1, x_2)\mathbf{1}_{\{v\}}(y_2) \\
&\quad + \frac{\alpha k}{rk + k + r}\mathbf{1}_{\{u_2\}}(x_2)\mathbf{1}_{\{v\}}(y_2) \\
&\quad + \sum_{x_1\in N_G(u_1)} \frac{(\alpha-1)k}{r(rk + k + r)}\mathbf{1}_{M_1}(x_2, y_2) \\
&\quad + \sum_{y_1\in N_H(v)} \frac{1-\alpha}{k(rk + k + r)}\mathbf{1}_{M_2}(x_2, y_2) \\
&\quad + \sum_{\substack{x_1\in N_G(u_1)\\ y_1\in N_H(v)}} \frac{1-\alpha}{k(rk + k + r)}\mathbf{1}_{M_3}(x_2, y_2) \\
&\quad + \sum_{y_1\in N_H(v)} \frac{(1-\alpha)(1-r)}{k(rk + k + r)}\mathbf{1}_{M_3}(x_2, y_2) \\
&= \frac{rk + r}{rk + k + r}m^{\alpha}_{u_2}(x_2)\mathbf{1}_{\{v\}}(y_2) \\
&\quad + \frac{\alpha k}{rk + k + r}\mathbf{1}_{\{u_2\}}(x_2)\mathbf{1}_{\{v\}}(y_2) + \frac{(\alpha-1)k}{rk + k + r}\mathbf{1}_{M_1}(x_2, y_2) \\
&\quad + \frac{1-\alpha}{rk + k + r}\mathbf{1}_{M_2}(x_2, y_2) + \frac{1-\alpha}{rk + k + r}\mathbf{1}_{M_3}(x_2, y_2) \\
&= m^{\alpha}_{(u_2,v)}(x_2, y_2).
\end{aligned}$$

Similarly, we have

$$\sum_{(x_2,y_2)\in V(G\boxtimes H)} D_1((x_1, y_1), (x_2, y_2)) = m^{\alpha}_{(u_1,v)}(x_1, y_1).$$

So D_1 is a coupling function between $m^{\alpha}_{(u_1,v)}$ and $m^{\alpha}_{(u_2,v)}$.

For any $x_1, x_2 \in V(G)$, $d((x_1, v), (x_2, v)) \leq d(x_1, x_2)$; for any $((x_1, y_1), (x_2, y_2)) \in S_4 \cup S_5$, $d((x_1, y_1), (x_2, y_2)) \leq 2$; for any $((x_1, y_1), (x_2, y_2)) \in S_6$, $d((x_1, y_1), (x_2, y_2)) \geq 1$. Then we obtain

$$\begin{aligned}
&W(m^{\alpha}_{(u_1,v)}, m^{\alpha}_{(u_2,v)}) \\
&\leq \sum_{\substack{(x_1,y_1)\in V(G\boxtimes H)\\ (x_2,y_2)\in V(G\boxtimes H)}} D_1((x_1, y_1), (x_2, y_2))d((x_1, y_1), (x_2, y_2)) \\
&= \frac{rk + r}{rk + k + r} \sum_{x_1,x_2\in V(G)} A(x_1, x_2)d((x_1, v), (x_2, v)) \\
&\quad + \frac{\alpha k}{rk + k + r} \\
&\quad + \frac{(\alpha-1)k}{r(rk + k + r)} \sum_{\substack{x_1\in N_G(u_1)\\ x_2\in N_G(u_2)}} d((x_1, v), (x_2, v)) \\
&\quad + \frac{1-\alpha}{k(rk + k + r)} \sum_{S_4} d((x_1, y_1), (x_2, y_2)) \\
&\quad + \frac{1-\alpha}{k(rk + k + r)} \sum_{S_5} d((x_1, y_1), (x_2, y_2)) \\
&\quad + \frac{(1-\alpha)(1-r)}{k(rk + k + r)} \sum_{S_6} d((x_1, y_1), (x_2, y_2)) \\
&\leq \frac{rk + r}{rk + k + r} \sum_{x_1,x_2\in V(G)} A(x_1, x_2)d(x_1, x_2) \\
&\quad + \frac{(\alpha-1)k}{r(rk + k + r)} \sum_{\substack{x_1\in N_G(u_1)\\ x_2\in N_G(u_2)}} d((x_1, v), (x_2, v)) + \frac{\alpha k}{rk + k + r} \\
&\quad + \frac{(1-\alpha)\big(2|S_4| + 2|S_5| + (1-r)|S_6|\big)}{k(rk + k + r)} \\
&= \frac{rk + r}{rk + k + r}W(m^{\alpha}_{u_1}, m^{\alpha}_{u_2}) \\
&\quad + \frac{(\alpha-1)k}{r(rk + k + r)} \sum_{\substack{x_1\in N_G(u_1)\\ x_2\in N_G(u_2)}} d((x_1, v), (x_2, v)) \\
&\quad + \frac{3kr(1-\alpha) + k}{rk + k + r}.
\end{aligned} \tag{9}$$

Denote $\Gamma = N_G(u_1) \cap N_G(u_2)$ and $t = |\Gamma|$. If $r > t + 1$, we claim for any $x \in \Gamma$, there exists at least one point $x' \in N_G(u_1) \cup N_G(u_2) \setminus \{x\}$ such that $x' \not\sim x$. Suppose it is not, then

$$\deg(x) = 2(r - t - 1) + t - 1 + 2 = 2r - t - 1 > r.$$

Which contradicts to the graph G is a r-regular graph. Then

$$d((x,v),(x',v)) = d((x',v),(x,v)) = 2.$$

Hence

$$\begin{aligned}\sum_{\substack{x_1\in N_G(u_1)\\ x_2\in N_G(u_2)}} d((x_1,v),(x_2,v)) &= \sum_{\substack{x_1\in N_G(u_1),\ x_2\in N_G(u_2)\\ x_1\neq x_2}} d((x_1,v),(x_2,v))\\ &\geq 2t + (r^2-2t)\cdot 1\\ &= r^2. \qquad (10)\end{aligned}$$

Taking (10) in (9), we have

$$W(m^\alpha_{(u_1,v)}, m^\alpha_{(u_2,v)}) \leq \frac{rk+r}{rk+k+r}W(m^\alpha_{u_1}, m^\alpha_{u_2}) + \frac{2kr(1-\alpha)+k}{rk+k+r}.$$

Thus, for any $\alpha\in[0,1]$, we get

$$\begin{aligned}&k^{G\boxtimes H}((u_1,v),(u_2,v))\\ &= \lim_{\alpha\to 1}\frac{k_\alpha^{G\boxtimes H}((u_1,v),(u_2,v))}{1-\alpha}\\ &= \lim_{\alpha\to 1}\frac{1-W(m^\alpha_{(u_1,v)}, m^\alpha_{(u_2,v)})}{1-\alpha}\\ &\geq \frac{rk+r}{rk+k+r}\lim_{\alpha\to 1}\frac{1-W(m^\alpha_{u_1}, m^\alpha_{u_2})}{1-\alpha} - \frac{2kr}{rk+k+r}\\ &= \frac{rk+r}{rk+k+r}k^G(u_1,u_2) - \frac{2kr}{rk+k+r}. \qquad (11)\end{aligned}$$

If $r=t+1$, then $\Gamma = N_G(u_1)\cap N_G(u_2) = N_G(u_1)\setminus\{u_2\} = N_G(u_2)\setminus\{u_1\}$. This means there exists two different points x, x' in Γ such that $x \not\sim x'$. Otherwise, the graph $G = K_{r+1}$ which contradicts $G\neq K_{r+1}$. Thus

$$\begin{aligned}\sum_{\substack{x_1\in N_G(u_1)\\ x_2\in N_G(u_2)}} d((x_1,v),(x_2,v)) &= \sum_{\substack{x_1\in N_G(u_1),\ x_2\in N_G(u_2)\\ x_1\neq x_2}} d((x_1,v),(x_2,v))\\ &\geq 2\cdot 2 + \big(r^2-2-(r-1)\big)\cdot 1\\ &= r^2-r+3. \qquad (12)\end{aligned}$$

Taking (12) in (9), we have

$$\begin{aligned}&W(m^\alpha_{(u_1,v)}, m^\alpha_{(u_2,v)})\\ &\leq \frac{rk+r}{rk+k+r}W(m^\alpha_{u_1}, m^\alpha_{u_2})\\ &\quad + \frac{(1-\alpha)(2kr^2+kr-3k)+rk}{r(rk+k+r)}.\end{aligned}$$

Thus, for any $\alpha\in[0,1]$, we get

$$\begin{aligned}&k^{G\boxtimes H}((u_1,v),(u_2,v))\\ &= \lim_{\alpha\to 1}\frac{1-W(m^\alpha_{(u_1,v)}, m^\alpha_{(u_2,v)})}{1-\alpha}\\ &\geq \frac{rk+r}{rk+k+r}\lim_{\alpha\to 1}\frac{1-W(m^\alpha_{u_1}, m^\alpha_{u_2})}{1-\alpha}\\ &\quad - \frac{2kr^2+kr-3k}{r(rk+k+r)}\\ &= \frac{rk+r}{rk+k+r}k^G(u_1,u_2) - \frac{2kr^2+kr-3k}{r(rk+k+r)}. \qquad (13)\end{aligned}$$

Combine (11) with (13), we get Claim 1. ☐

Claim 2: Let G and H be r-regular and k-regular locally finite connected graphs with simple weight, respectively. For $u_1\sim u_2$ in G, $v\in V(H)$, the Ricci curvature of $G\boxtimes H$ is bounded above, that is

$$k^{G\boxtimes H}((u_1,v),(u_2,v)) \leq \frac{kr+r}{kr+k+r}k^G(u_1,u_2) + \frac{2k}{kr+k+r}.$$

Proof: Let f be an 1-Lipschitz function which achieves the supremum in the duality theorem of $W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2})$, i.e.,

$$W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2}) = \sum_{x\in N_G[u_1]} f(x)m^{\alpha'}_{u_1}(x) - \sum_{y\in N_G[u_2]} f(y)m^{\alpha'}_{u_2}(y),$$

where α' satisfies

$$\frac{1-\alpha'}{1-\alpha} = \frac{kr+r}{kr+r+\alpha k}. \qquad (14)$$

We define a function $\phi : N[(u_1,v)]\cup N[(u_2,v)] \to \mathbb{R}$ as

$$\phi(x,y) = \begin{cases} f(x), & x\in (N_G(u_1)\cup N_G(u_2))\backslash\{u_1,u_2\},\\ & y\in N_H[v],\\ f(u_1), & x=u_1, y\in N_H[v],\\ f(u_2), & x=u_2, y\in N_H[v].\end{cases}$$

In fact, $\phi(x,y) = f(x)$ in $N[(u_1,v)]\cup N[(u_2,v)]$. Now, we show that ϕ is an 1-Lipschitz function over $N[(u_1,v)]\cup N[(u_2,v)]$. It suffices to prove that for any two points $(x_1,y_1),(x_2,y_2) \in N[(u_1,v)]\cup N[(u_2,v)]$, there holds $|\phi(x_1,y_1)-\phi(x_2,y_2)| \leq d((x_1,y_1),(x_2,y_2))$.

When $x_1,x_2 \in (N_G(u_1)\cup N_G(u_2))\backslash\{u_1,u_2\}$, $y_1,y_2 \in N_H[v]$, we have

$$|\phi(x_1,y_1)-\phi(x_2,y_2)| = |f(x_1)-f(x_2)| \leq d(x_1,x_2).$$

To prove

$$d(x_1,x_2) \leq d((x_1,y_1),(x_2,y_2)), \qquad (15)$$

we can discuss it according to the value of $d(x_1,x_2)$, i.e., 0, 1, 2, 3. If $d(x_1,x_2)=0$ or $d(x_1,x_2)=1$, then (15) holds clearly. If $d(x_1,x_2)=2$, then $x_1\neq x_2$ and $x_1\not\sim x_2$. This implies $d((x_1,y_1),(x_2,y_2)) \geq 2$. Hence (15) holds. If $d(x_1,x_2)=3$, then $x_1\neq x_2$ and $x_1\not\sim x_2$ and $d((x_1,y_1),(x_2,y_2))\geq 2$. If $d((x_1,y_1),(x_2,y_2))=2$, there exists a point $(x_3,y_3)\in N[(u_1,v)]\cup N[(u_2,v)]$ such that $(x_1,y_1)\sim(x_3,y_3)\sim(x_2,y_2)$, which means $d(x_1,x_2)\leq 2$. This contradicts to $d(x_1,x_2)=3$. So $d((x_1,y_1),(x_2,y_2)) \geq 3$ and (15) holds.

Hence, in the case $x_1,x_2 \in (N_G(u_1)\cup N_G(u_2))\backslash\{u_1,u_2\}$, $y_1,y_2 \in N_H[v]$, we have $|\phi(x_1,y_1)-\phi(x_2,y_2)| \leq$

$d((x_1,y_1),(x_2,y_2))$. And checking other cases, we also have $|\phi(x_1,y_1)-\phi(x_2,y_2)| \le d((x_1,y_1),(x_2,y_2))$. That is, ϕ is an 1-Lipschitz function over $N[(u_1,v)] \cup N[(u_2,v)]$, so that ϕ can be extended to an 1-Lipschitz function over $V(G \boxtimes H)$. Then

$$\begin{aligned}
&W(m^{\alpha}_{(u_1,v)}, m^{\alpha}_{(u_2,v)}) \\
&\ge \sum_{(x,y)\in N[(u_1,v)]} \phi(x,y) m^{\alpha}_{(u_1,v)}(x,y) \\
&\quad - \sum_{(x,y)\in N[(u_2,v)]} \phi(x,y) m^{\alpha}_{(u_2,v)}(x,y) \\
&= \sum_{\substack{x\in N_G(u_1)\\ y\in N_H[v]}} f(x) m^{\alpha}_{(u_1,v)}(x,y) - \sum_{\substack{x\in N_G(u_2)\\ y\in N_H[v]}} f(x) m^{\alpha}_{(u_2,v)}(x,y) \\
&\quad + \sum_{y\in N_H[v]} f(u_1) m^{\alpha}_{(u_1,v)}(u_1,y) - \sum_{y\in N_H[v]} f(u_2) m^{\alpha}_{(u_2,v)}(u_2,y) \\
&= \frac{kr+r+\alpha k}{kr+k+r} \sum_{x\in N_G[u_1]} f(x) m^{\alpha'}_{u_1}(x) \\
&\quad - \frac{kr+r+\alpha k}{kr+k+r} \sum_{x\in N_G[u_2]} f(x) m^{\alpha'}_{u_2}(x) \\
&\quad + \frac{(1-\alpha)k}{kr+k+r}(f(u_1)-f(u_2)) \\
&\ge \Big(\sum_{x\in N_G[u_1]} f(x) m^{\alpha'}_{u_1}(x) - \sum_{y\in N_G[u_2]} f(y) m^{\alpha'}_{u_2}(y) \Big) \\
&\quad \cdot \frac{kr+r+\alpha k}{kr+k+r} - \frac{(1-\alpha)k}{kr+k+r} \\
&= \frac{kr+r+\alpha k}{kr+k+r} W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2}) - \frac{(1-\alpha)k}{kr+k+r}.
\end{aligned}$$

Thus

$$\begin{aligned}
&k^{G\boxtimes H}_{\alpha}((u_1,v),(u_2,v)) \\
&= 1 - \frac{W(m^{\alpha}_{(u_1,v)}, m^{\alpha}_{(u_2,v)})}{d((u_1,v),(u_2,v))} \\
&\le 1 - \frac{kr+r+\alpha k}{kr+k+r} W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2}) + \frac{(1-\alpha)k}{kr+k+r} \\
&= \frac{kr+r+\alpha k}{kr+k+r}(1 - W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2})) + 2\frac{(1-\alpha)k}{kr+k+r} \\
&= \frac{kr+r+\alpha k}{kr+k+r} k^{G}_{\alpha'}(u_1,u_2) + \frac{2(1-\alpha)k}{kr+k+r}.
\end{aligned}$$

Using (14), we get

$$\begin{aligned}
k^{G\boxtimes H}((u_1,v),(u_2,v)) &= \lim_{\alpha\to 1} \frac{k^{G\boxtimes H}_{\alpha}((u_1,v),(u_2,v))}{1-\alpha} \\
&\le \lim_{\alpha\to 1} \frac{kr+r+\alpha k}{kr+k+r} \frac{k^{G}_{\alpha'}(u_1,u_2)}{1-\alpha} + \frac{2k}{kr+k+r} \\
&= \frac{kr+r}{kr+k+r} \lim_{\alpha'\to 1} \frac{k^{G}_{\alpha'}(u_1,u_2)}{1-\alpha'} + \frac{2k}{kr+k+r} \\
&= \frac{kr+r}{kr+k+r} k^{G}(u_1,u_2) + \frac{2k}{kr+k+r}. \qquad \square
\end{aligned}$$

Proof of Theorem 4: Combing Claim 1 with Claim 2, we get Theorem 4.

Proof of Theorem 5: By the symmetry, from Theorem 4, we have Theorem 5.

B. PROOF OF THEOREM 6

Claim 3: Let G and H be r-regular and k-regular locally finite connected graphs with simple weight, respectively. For $u_1 \sim u_2$ in G, $v_1 \sim v_2$ in H, the Ricci curvature of $G \boxtimes H$ is bounded below and

$$k^{G\boxtimes H}((u_1,v_1),(u_2,v_2)) \ge \frac{-4r^2k^2+10rk-2k-4r}{rk(rk+k+r)}.$$

Proof: Assume that B is a coupling function between $m^{\alpha}_{u_1}$ and $m^{\alpha}_{u_2}$ which defined as (4). Since $u_1 \sim u_2$ in G, $v_1 \sim v_2$ in H, $(u_1,v_1) \sim (u_2,v_2)$ in $G \boxtimes H$. We define a function $D_2 : V(G\boxtimes H) \times V(G\boxtimes H) \to [0,1]$ as follows:

If $((x_1,y_1),(x_2,y_2)) \in S'_1$, then

$$D_2((x_1,y_1),(x_2,y_2)) = \alpha,$$

If $((x_1,y_1),(x_2,y_2)) \in S'_2$, then

$$D_2((x_1,y_1),(x_2,y_2)) = \frac{1-\alpha}{(rk+k+r)r},$$

If $((x_1,y_1),(x_2,y_2)) \in S'_3 \cup S'_4$, then

$$D_2((x_1,y_1),(x_2,y_2)) = \frac{1-\alpha}{k(rk+k+r)},$$

If $((x_1,y_1),(x_2,y_2)) \in S'_5$, then

$$D_2((x_1,y_1),(x_2,y_2)) = \frac{(1-\alpha)(1-r)}{k(rk+k+r)},$$

otherwise, $D_2((x_1,y_1),(x_2,y_2)) = 0$. Where $S'_1 = \{((x_1,y_1),(x_2,y_2)) \in V(G\boxtimes H) \times V(G\boxtimes H) : x_1 = u_1, y_1 = v_1, x_2 = u_2, y_2 = v_2\}$, $S'_2 = \{((x_1,y_1),(x_2,y_2)) \in V(G\boxtimes H) \times V(G\boxtimes H) : x_1 \sim u_1, y_1 = v_1, x_2 \sim u_2, y_2 = v_2\}$, $S'_3 = \{((x_1,y_1),(x_2,y_2)) \in V(G\boxtimes H) \times V(G\boxtimes H) : x_1 = u_1, y_1 \sim v_1, x_2 \sim u_2, y_2 \sim v_2\}$, $S'_4 = \{((x_1,y_1),(x_2,y_2)) \in V(G\boxtimes H) \times V(G\boxtimes H) : x_1 \sim u_1, y_1 \sim v_1, x_2 = u_2, y_2 \sim v_2\}$, $S'_5 = \{((x_1,y_1),(x_2,y_2)) \in V(G\boxtimes H) \times V(G\boxtimes H) : x_1 = u_1, y_1 \sim v_1, x_2 = u_2, y_2 \sim v_2\}$.

Now we claim that D_2 is also a coupling function between $m^{\alpha}_{(u_1,v_1)}$ and $m^{\alpha}_{(u_2,v_2)}$. We denote $M'_1 = \{(x,y) \in V(G\boxtimes H) : x \sim u_2,\ y = v_2\}$, $M'_2 = \{(x,y) \in V(G\boxtimes H) : x \sim u_2,\ y_2 \sim v_2\}$ $M'_3 = \{(x,y) \in V(G\boxtimes H) : x = u_2,\ y \sim v_2\}$. Using the characteristic function $1_S(x)$ defined as (8), we have

$$\begin{aligned}
&\sum_{(x_1,y_1)\in V(G\boxtimes H)} D_2((x_1,y_1),(x_2,y_2)) \\
&= \alpha \mathbf{1}_{\{u_2\}}(x_2)\mathbf{1}_{\{v_2\}}(y_2) + \sum_{x_1\in N_G(u_1)} \frac{1-\alpha}{r(rk+k+r)} \mathbf{1}_{M'_1}(x_2,y_2) \\
&\quad + \sum_{y_1\in N_H(v_1)} \frac{1-\alpha}{k(rk+k+r)} \mathbf{1}_{M'_2}(x_2,y_2) \\
&\quad + \sum_{\substack{x_1\in N_G(u_1)\\ y_1\in N_H(v_1)}} \frac{1-\alpha}{k(rk+k+r)} \mathbf{1}_{M'_3}(x_2,y_2)
\end{aligned}$$

$$+\sum_{y_1\in N_H(v_1)}\frac{(1-\alpha)(1-r)}{k(rk+k+r)}\mathbf{1}_{M_3'}(x_2,y_2)$$
$$=\alpha\mathbf{1}_{\{u_2\}}(x_2)\mathbf{1}_{\{v_2\}}(y_2)+\frac{1-\alpha}{rk+k+r}\mathbf{1}_{M_1'}(x_2,y_2)$$
$$+\frac{1-\alpha}{rk+k+r}\mathbf{1}_{M_2'}(x_2,y_2)+\frac{1-\alpha}{rk+k+r}\mathbf{1}_{M_3'}(x_2,y_2)$$
$$=m^{\alpha}_{(u_2,v_2)}(x_2,y_2).$$

Similarly, we obtain

$$\sum_{(x_2,y_2)\in V(G\boxtimes H)}D_2((x_1,y_1),(x_2,y_2))=m^{\alpha}_{(u_1,v_1)}(x_1,y_1).$$

So D_2 is a coupling function between $m^{\alpha}_{(u_1,v_1)}$ and $m^{\alpha}_{(u_2,v_2)}$. Then

$$\begin{aligned}
&W(m^{\alpha}_{(u_1,v_1)},m^{\alpha}_{(u_2,v_2)})\\
&\le \sum_{\substack{(x_1,y_1)\in V(G\boxtimes H)\\(x_2,y_2)\in V(G\boxtimes H)}} D_2((x_1,y_1),(x_2,y_2))d((x_1,y_1),(x_2,y_2))\\
&=\alpha+\frac{1-\alpha}{r(rk+k+r)}\sum_{S_2'}d((x_1,y_1),(x_2,y_2))\\
&\quad+\frac{1-\alpha}{k(rk+k+r)}\sum_{S_3'}d((x_1,y_1),(x_2,y_2))\\
&\quad+\frac{1-\alpha}{k(rk+k+r)}\sum_{S_4'}d((x_1,y_1),(x_2,y_2))\\
&\quad+\frac{(1-\alpha)(1-r)}{k(rk+k+r)}\sum_{S_5'}d((x_1,y_1),(x_2,y_2)). \qquad (16)
\end{aligned}$$

We consider the distance sum in S_2' firstly. It is easy to see that, for any $(x_1,x_2)\in\{u_2\}\times N_G(u_2)\cup N_G(u_1)\times\{u_1\}$, there holds $d((x_1,v_1),(x_2,v_2))=1$; for $(x_1,x_2)\in(N_G(u_1)\times N_G(u_2))\backslash(\{u_2\}\times N_G(u_2)\cup N_G(u_1)\times\{u_1\})$, there holds $d((x_1,v_1),(x_2,v_2))\le 3$. Hence

$$\begin{aligned}
&\sum_{S_2'}d((x_1,y_1),(x_2,y_2))\\
&\le(2r-1)\cdot1+(r^2-2r+1)\cdot3\\
&=3r^2-4r+2. \qquad (17)
\end{aligned}$$

Now we consider the distance sum in S_3'. Noting that for any $y_2\in N_H(v_2)$, there holds $d((u_1,v_2),(u_1,y_2))=1$; for any $y_1\in N_H(v_1)$, there holds $d((u_1,y_1),(u_1,v_1))=1$; for any $x_2\in N_G(u_2)\setminus\{u_1\}$ and $y_2\in N_H(v_2)$, there holds $d((u_1,v_2),(x_2,y_2))\le2$; and for other cases in S_3', there holds $d((u_1,y_1),(x_2,y_2))\le3$. Hence

$$\begin{aligned}
&\sum_{S_3'}d((x_1,y_1),(x_2,y_2))\\
&\le(2k-1)\cdot1+k(r-1)\cdot2\\
&\quad+(rk^2-2k+1-k(r-1))\cdot3\\
&=3rk^2-rk-3k+2. \qquad (18)
\end{aligned}$$

Similarly,

$$\sum_{S_4'}d((x_1,y_1),(x_2,y_2))\le3rk^2-rk-3k+2. \qquad (19)$$

And for any $((x_1,y_1),(x_2,y_2))\in S_5'$, $d((x_1,y_1),(x_2,y_2))\ge1$, so

$$\sum_{S_5'}d((x_1,y_1),(x_2,y_2))\ge|S_5'|=k^2. \qquad (20)$$

Taking (17), (18),(19) and (20) in (16), we obtain

$$\begin{aligned}
&W(m^{\alpha}_{(u_1,v_1)},m^{\alpha}_{(u_2,v_2)})\\
&\le\alpha+\frac{(1-\alpha)(3r^2-4r+2)}{r(rk+k+r)}\\
&\quad+\frac{2(1-\alpha)(3rk^2-rk-3k+2)}{k(rk+k+r)}+\frac{(1-\alpha)(1-r)k^2}{k(rk+k+r)}\\
&=\alpha+\frac{(1-\alpha)(5r^2k^2+r^2k+rk^2-10rk+2k+4r)}{rk(rk+k+r)}.
\end{aligned} \qquad (21)$$

Thus, for any $\alpha\in[0,1]$, we get

$$\begin{aligned}
&k^{G\boxtimes H}((u_1,v_1),(u_2,v_2))\\
&=\lim_{\alpha\to1}\frac{1-W(m^{\alpha}_{(u_1,v_1)},m^{\alpha}_{(u_2,v_2)})}{1-\alpha}\\
&\ge1-\frac{5r^2k^2+r^2k+rk^2-10rk+2k+4r}{rk(rk+k+r)}\\
&=\frac{-4r^2k^2+10rk-2k-4r}{rk(rk+k+r)}. \qquad \square
\end{aligned}$$

Claim 4: Let G and H be r-regular and k-regular locally finite connected graphs with simple weight, respectively. For $u_1\sim u_2$ in G, $v_1\sim v_2$ in H, the Ricci curvature of $G\boxtimes H$ is bounded above and

$$\begin{aligned}
&k^{G\boxtimes H}((u_1,v_1),(u_2,v_2))\\
&\le\frac{\min\{(kr+r)k^G(u_1,u_2)+2k,(kr+k)k^H(v_1,v_2)+2r\}}{kr+k+r}.
\end{aligned}$$

Proof: Let g be a 1-Lipschitz function which achieves the supremum in the duality theorem of $W(m^{\alpha'}_{u_1},m^{\alpha'}_{u_2})$, i.e.,

$$W(m^{\alpha'}_{u_1},m^{\alpha'}_{u_2})=\sum_{x\in N_G[u_1]}g(x)m^{\alpha'}_{u_1}(x)-\sum_{y\in N_G[u_2]}g(y)m^{\alpha'}_{u_2}(y),$$

where α' satisfies

$$\frac{1-\alpha'}{1-\alpha}=\frac{kr+r}{kr+r+\alpha k}. \qquad (22)$$

We define a function $\varphi:N[(u_1,v_1)]\cup N[(u_2,v_2)]\to\mathbb{R}$ as $\varphi(x,y)=g(x)$. Through the similar analysis to the function ϕ in Claim 2, where partition $N[(u_1,v_1)]\cup N[(u_2,v_2)]$ into four mutually disjoint regions, i.e., $((N_G(u_1)\cup N_G(u_2))\backslash\{u_1,u_2\})\times((N_H(v_1)\cup N_H(v_2))\backslash\{v_1,v_2\})$, $(N_G(u_1)\times\{v_1\})\cup(N_G(u_2)\times\{v_2\})$, $\{u_1\}\times N_H[v_1]$, $\{u_2\}\times N_H[v_2]$, we can get that φ is an 1-Lipschitz function over $N[(u_1,v_1)]\cup N[(u_2,v_2)]$. So that φ can be extended to an 1-Lipschitz function over $V(G\boxtimes H)$.

Then, we have

$$\begin{aligned}
&W(m^{\alpha}_{(u_1,v_1)}, m^{\alpha}_{(u_2,v_2)})\\
&\geq \sum_{(x,y)\in N[(u_1,v_1)]} \varphi(x,y)m^{\alpha}_{(u_1,v_1)}(x,y)\\
&\quad - \sum_{(x,y)\in N[(u_2,v_2)]} \varphi(x,y)m^{\alpha}_{(u_2,v_2)}(x,y)\\
&= \sum_{\substack{x\in N_G(u_1)\\ y\in N_H(v_1)}} g(x)m^{\alpha}_{(u_1,v_1)}(x,y) - \sum_{\substack{x\in N_G(u_2)\\ y\in N_H(v_2)}} g(x)m^{\alpha}_{(u_2,v_2)}(x,y)\\
&\quad + \sum_{y\in N_H[v_1]} g(u_1)m^{\alpha}_{(u_1,v_1)}(u_1,y)\\
&\quad - \sum_{y\in N_H[v_2]} g(u_2)m^{\alpha}_{(u_2,v_2)}(u_2,y)\\
&\quad + \sum_{x\in N_G(u_1)} g(u_1)m^{\alpha}_{(u_1,v_1)}(x,v_1)\\
&\quad - \sum_{x\in N_G(u_2)} g(u_2)m^{\alpha}_{(u_2,v_2)}(x,v_2)\\
&= \Big(\sum_{x\in N_G[u_1]} g(x)m^{\alpha'}_{u_1}(x) - \sum_{y\in N_G[u_2]} g(y)m^{\alpha'}_{u_2}(y)\Big)\frac{kr+r+\alpha k}{kr+k+r}\\
&\quad + \frac{(1-\alpha)k}{kr+k+r}(g(u_1)-g(u_2))\\
&\geq \frac{kr+r+\alpha k}{kr+k+r}W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2}) - \frac{(1-\alpha)k}{kr+k+r}.
\end{aligned}$$

Thus

$$\begin{aligned}
k^{G\boxtimes H}_{\alpha}((u_1,v_1),(u_2,v_2)) &= 1 - \frac{W(m^{\alpha}_{(u_1,v_1)}, m^{\alpha}_{(u_2,v_2)})}{d((u_1,v_1),(u_2,v_2))}\\
&\leq 1 - \frac{kr+r+\alpha k}{kr+k+r}W(m^{\alpha'}_{u_1}, m^{\alpha'}_{u_2}) + \frac{(1-\alpha)k}{kr+k+r}\\
&= \frac{kr+r+\alpha k}{kr+k+r}k^{G}_{\alpha'}(u_1,u_2) + \frac{2(1-\alpha)k}{kr+k+r}.
\end{aligned}$$

Using (22), we get

$$\begin{aligned}
&k^{G\boxtimes H}((u_1,v_1),(u_2,v_2))\\
&\quad = \lim_{\alpha\to 1} \frac{k^{G\boxtimes H}_{\alpha}((u_1,v_1),(u_2,v_2))}{1-\alpha}\\
&\quad \leq \lim_{\alpha\to 1} \frac{kr+r+\alpha k}{kr+k+r}\frac{k^{G}_{\alpha'}(u_1,u_2)}{1-\alpha} + \frac{2k}{kr+k+r}\\
&\quad = \frac{kr+r}{kr+k+r}k^{G}(u_1,u_2) + \frac{2k}{kr+k+r}.
\end{aligned}$$

Similarly, we can also obtain

$$\begin{aligned}
&k^{G\boxtimes H}((u_1,v_1),(u_2,v_2))\\
&\qquad\qquad \leq \frac{kr+k}{kr+k+r}k^{H}(v_1,v_2) + \frac{2r}{kr+k+r}.
\end{aligned}$$

These give Claim 4. □

Proof of Theorem 6: Applying Claim 3 and Claim 4 gives Theorem 6.

IV. CONCLUSION

In this paper, we introduce a definition of the weight function to the strong product graph $G \boxtimes H$, and derive an expression for the spectrum of $G \boxtimes H$ by using the spectrums of G and H. It is proved that the Ricci curvature for strong product of two regular graphs with simple weight is bounded by the Ricci curvature of G and H. However, at present we cannot completely settle the problem for general weights. The main problem is that we need to find some new auxiliary methods to improve the bound for general weights.

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