IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 4 May 2023, accepted 16 May 2023, date of publication 19 May 2023, date of current version 26 May 2023.
Digital Object Identifier 10.1109/ACCESS.2023.3277986

RESEARCH ARTICLE

Exponential Synchronization of Cohen-Grossberg Neural Networks With Delays

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ABSTRACT This article analyzes exponential synchronization for a class of Cohen-Grossberg neural networks with time-varying delays. Firstly, according to the concept of synchronization, a controlled response system is constructed, and the error system is obtained. Secondly, by establishing suitable Lyapunov functions and using inequality techniques, sufficient conditions for exponential synchronization of the error system under different controllers are provided, and the exponential convergence rate of the system is given. Finally, two examples are used to verify the effectiveness of the theoretical results.

INDEX TERMS Cohen-grossberg neural network, time-varying delays, exponential synchronization, impulsive control.

I. INTRODUCTION

Cohen-Grossberg neural network(CGNN) is a kind of neural network first proposed by Cohen and Grossberg in [1]. In recent decades, CGNN has attracted much attention due to its potential and widespread applications in the fields of pattern recognition, model prediction, optimization problems, signal processing. The realization of neural networks depends on their dynamic behavior, such as stability, convergence, oscillation and periodicity, and many research results have also appeared [2], [3], [4], [5], [6], [7], [8], [9], [10]. Time delay may result from the onset of non-vanishing oscillations and may affect the dynamic behavior. In addition, due to the limited speed of amplifier switching and signal transmission, time delays are often unavoidable. And the application of time delay in the system are shown in [11], [12], [13], and [14]. Therefore, it is meaningful to study time-delays systems.

It is known that impulsive control, as a discontinuous control strategy, is activated only at some isolated moments, and impulse control has a huge impact on the stability of the system. Especially, some systems are stable without considering impulse control. Once impulsive control is introduced, the original system may become unstable, the structure of the system has changed. It is worth noting that some practical systems are inevitably affected by time delay and impulse disturbance, which may lead to undesirable phenomena such as oscillation and instability in the system. Therefore, it is necessary to consider the influence of these two factors on the system's stability. In recent years, some results have been obtained on the asymptotic behavioral properties of impulsive neural networks [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. The exponential convergence of impulsive inertial neural networks is explored in [17], the exponential stability of impulsive complex-valued neural networks is discussed in [18] and [19], the robust passivity and stability of uncertain complex-valued impulsive neural network is investigated in [20], and the exponential synchronization of neural networks is proposed in [21].

Generally speaking, synchronization is an important dynamic characteristic of the networks. The research on synchronization can further reveal the dynamic characteristics of networks in the real world and help to understand various real-world phenomena. It can also be applied to the fields of network control, information processing, and complex computing. This kind of research can be widely applied in multiple sciences and has received attention from researchers in many fields. The main method to realize network synchronization is to design the appropriate controller, such as impulsive control, feedback control and pinning control.

The associate editor coordinating the review of this manuscript and approving it for publication was Mauro Gaggero.

Meanwhile, some scholars have studied different types of synchronization, such as asymptotic synchronization [25], [26], [27], [28], finite-time synchronization [29], [30], and fixed-time synchronization [5], [31].

Based on the above analysis, there are few studies on CGNN synchronization under the impulse control strategy. Based on the existing research results, it is meaningful to further explore the exponential synchronization of CGNN with impulsive effects. Therefore, this paper mainly designs two different types of control (including impulsive control) to explore the exponential synchronization of CGNN with time-varying delays. It contains the following three contributions:
(i) The synchronization of CGNN are investigated in [27] and [30], where the derivative of time-varying delay is less than one. Theorem 1 studies the exponential synchronization of CGNN with impulsive effects, and removes the restriction on time-varying delay derivatives;
(ii) The authors discusses the case of $p = 2$ in [28], this paper focuses on the case where p is a positive number, and gives the exponential convergence rate of the system;
(iii) When the model is simplified to a Hopfield neural network, the results with impulse effect studied in this paper also improve and extend some existing results.

This article is structured as follows. Section II introduces the model description, definition and lemma. The exponential synchronization of the system is studied in Section III. Two examples are obtained in Section IV. Finally, Section V gives the conclusion.

For simplicity of description, the following notations are given: $\mathcal{Z}$ is the set of positive integers. $\dot{w}(t)$ represents the derivative of $w(t)$ with respect to the time t. $\mathcal{N} = \{1, 2, \cdots, n\}$.

II. PRELIMINARIES

Consider the CGNN with delays:

$$\dot{w}_j(t) = \alpha_j(w_j(t))\Big(- h_j(w_j(t)) + \sum_{r=1}^{n} a_{jr} f_r(w_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(w_r(t-\rho(t))) + u_j(t)\Big), \qquad (1)$$

where $j \in \mathcal{N}$, $w_j(t)$ denote the system state, $\alpha_j(\cdot)$, $h_j(\cdot)$, $f_j(\cdot)$ represent amplification function, behave function and activation function, respectively. a_{jr}, d_{jr} are the connection weights of neurons and $u_j(t)$ is the external input, $\rho(t)$ is the time-varying delay and satisfies $0 \leq \rho(t) \leq \rho$. The initial condition is $w_j(s) = \varphi_j(s)$, $s \in [-\rho, 0]$ and $\varphi_j(s)$ is a continuous bounded function.

Remark 1: The model (1) includes Hopfield neural network, cellular neural network, and BAM neural network as special cases. The system (1) for $\alpha_j(w_j(t)) = 0$ are investigated in [23] and [24], the derivative of the delay is less than 1 at (1) is needed in [7]. Compared with [7], [23], and [24], the model in this paper is more general.

To study the synchronization of the system, let the system (1) be the drive system, and the response system is

$$\dot{z}_j(t) = \alpha_j(z_j(t))\Big(- h_j(z_j(t)) + \sum_{r=1}^{n} a_{jr} f_r(z_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(z_r(t-\rho(t))) + u_j(t)\Big) + I_j(t), \qquad (2)$$

where $z_j(t)$ is the state of the neuron, the remaining symbols are the same as system (1), $I_j(t)$ is the control input. $z_j(s) = \phi_j(s) (s \in [-\rho, 0])$ is the initial condition of (2) and $\phi_j(s)$ is a continuous bounded function.

Let $e_j(t) = z_j(t) - w_j(t)$, the following error system is obtained by subtracting (1) from (2)

$$\begin{aligned}\dot{e}_j(t) = &-\big(\alpha_j(z_j(t))h_j(z_j(t)) - \alpha_j(w_j(t))h_j(w_j(t))\big) \\ &+ \alpha_j(z_j(t))\Big(\sum_{r=1}^{n} a_{jr} f_r(e_r(t)) \\ &+ \sum_{r=1}^{n} d_{jr} f_r(e_r(t-\rho(t)))\Big) \\ &+ \big(\alpha_j(z_j(t)) - \alpha_j(w_j(t))\big)\Big(\sum_{r=1}^{n} a_{jr} f_r(w_r(t)) \\ &+ \sum_{r=1}^{n} d_{jr} f_r(w_r(t-\rho(t))) + u_j(t)\Big) + I_j(t), \end{aligned} \qquad (3)$$

where $f_r(e_r(t)) = f_r(z_r(t)) - f_r(w_r(t))$, $f_r(e_r(t - \rho(t))) = f_r(z_r(t - \rho(t))) - f_r(w_r(t - \rho(t)))$.

The assumptions of this paper are as follows:

Assumption A1: If $\alpha_j(\cdot)(\forall j \in \mathcal{N})$ is a differentiable function, and there exist positive constant $\underline{\alpha}_j$, $\overline{\alpha}_j$, $\widetilde{\alpha}_j$ such that

$$0 < \underline{\alpha}_j < \alpha_j(\cdot) < \overline{\alpha}_j, \ |\dot{\alpha}_j(\cdot)| < \widetilde{\alpha}_j.$$

Assumption A2: If $g_j(\cdot) = \alpha_j(\cdot)h_j(\cdot)$ is a differentiable function and its derivative is bounded, there exist $\widetilde{g}_j > 0$ and $\underset{\sim}{g}_j > 0$ satisfying the inequality

$$0 < \underset{\sim}{g}_j < \dot{g}_j(\cdot) < \widetilde{g}_j.$$

Assumption A3: Assume that $f_j(\cdot)$ is globally Lipschitz continuous, that is, for any real number μ, ν, there exist nonnegative constant l_j and $\overline{f}_j$ such that

$$|f_j(\mu) - f_j(\nu)| < l_j|\mu - \nu|, \ |f_j(\mu)| < \overline{f}_j.$$

Assumption A4: If $u_j(t)$ is a bounded function, and there exists $\overline{u}_j > 0$ such that $|u_j(t)| \leq \overline{u}_j$.

Definition 1 ([27]): If there are $M > 0$ and $\varepsilon > 0$ satisfy

$$\sum_{j=1}^{n} |z_j(s) - w_j(s)|^p \leq M|\phi_z - \varphi_w|^p e^{-\varepsilon t}, \ t \geq t_0,$$

where $|\phi_z - \varphi_w|^p = \sup_{t_0 - \rho \leq s \leq t_0} \sum_{j=1}^{n} |\phi_j(s) - \varphi_j(s)|^p$. Then system (1) and system (2) can reach exponential synchronization, *Lemma 1 ([32]):* Consider the following differential equations

$$\begin{cases} D^+g(t) \leq -ag(t) + b \sup_{t-\rho\leq s\leq t} g(s), \ \ t \neq t_k, \\ g(t_k) = \widehat{a}_k g(t_k^-) + \widehat{b}_k \sup_{t_k-\rho\leq s\leq t_k} g(s), \ k \in \mathcal{Z}, \end{cases}$$

where $g(t) \geq 0$ is a continuous function, when $t = t_k$, if $0 \leq b < a$ and exists $\delta > 1$, for $\forall t \in [t_k - t_{k+1})$, such that $t_{k+1} - t_k \geq \delta\rho$, then

$$g(t) \leq \eta_1 \cdots \eta_{k+1} e^{k\gamma\rho} \sup_{t_0-\rho\leq s\leq t_0} g(s) e^{-\gamma(t-t_0)},$$

where $\eta_i = \max\{1, \ \widehat{a}_i + \widehat{b}_i e^{\gamma\rho}\}(i = 1, 2, \cdots, k+1)$, γ is the only positive real root of $\gamma = a - be^{\gamma\rho}$. In particular, if $\xi = \sup_{k=1,2\cdots}\{1, \ \widehat{a}_i + \widehat{b}_i e^{\gamma\rho}\}$, then

$$g(t) \leq \xi \sup_{t_0-\rho\leq s\leq t_0} g(s) e^{-\left(\gamma - \frac{\ln(\xi e^{\gamma\rho})}{\delta\rho}\right)(t-t_0)}, \ \ \forall t \geq t_0.$$

III. EXPONENTIAL SYNCHRONIZATION OF THE SYSTEM

The design control input is $I_j(t) = -\kappa_{jk} e_j(t)\delta(t - t_k^-)$, where $\delta(t)$ is the Dirac function, $\{t_k, k \in \mathcal{Z}\}$ is the impulse sequence, and t_k is the fixed impulse moment satisfy $0 < t_{k-1} < t_k$ and $\lim_{k\to\infty} t_k = \infty$. This means that the state of the system (3) jumps at t_k, and κ_{jk} represents the impulsive control gain. At this point, it is necessary to design a suitable control gain and impulsive sequence so that (1) and (2) are exponentially synchronized. Based on the control input $I_j(t)$, the system (3) can be written in the form of

$$\begin{cases} \dot{e}_j(t) = -\big(\alpha_j(z_j(t))h_j(z_j(t)) - \alpha_j(w_j(t))h_j(w_j(t))\big) \\ +\alpha_j(z_j(t))\Big(\sum_{r=1}^{n} a_{jr} f_r(e_r(t)) \\ +\sum_{r=1}^{n} d_{jr} f_r(e_r(t-\rho(t)))\Big) \\ +\big(\alpha_j(z_j(t)) - \alpha_j(w_j(t))\big)\Big(\sum_{r=1}^{n} a_{jr} f_r(w_r(t)) \\ +\sum_{r=1}^{n} d_{jr} f_r(w_r(t-\rho(t))) + u_j(t)\Big), \\ e_j(t_k) = -(\kappa_{jk} - 1)e_j(t_k^-), \ t = t_k. \end{cases} \tag{4}$$

Theorem 1: Under the A1−A4, system (1) and system (2) are exponentially synchronized via impulsive effects, in other words, system (4) is exponentially synchronized, and the exponential convergence rate is $\gamma - \frac{\ln(\xi e^{\gamma\rho})}{\delta\rho}$, if there exist positive number $a_j, \ b_j(\forall j \in \mathcal{N})$, $\delta > 1$ and p is a positive integer, the following conditions hold
(i) $a_j = p\underset{\sim}{g}_j - p\widetilde{\alpha}_j \bar{u}_j - \overline{\alpha}_j \sum_{r=1}^{n} |a_{rj}| l_j - \sum_{r=1}^{n}(|a_{jr}| + |d_{jr}|)(p\widetilde{\alpha}_j \overline{f}_r + (p-1)\overline{\alpha}_r l_r)$, $b_j = \sum_{r=1}^{n} \overline{\alpha}_r |d_{rj}| l_j$, where $a = \min_{1\leq j\leq n}\{a_j\} > b = \max_{1\leq j\leq n}\{b_j\}$;
(ii) There exist scalars $\gamma > \frac{\ln(\xi e^{\gamma\rho})}{\delta\rho}$, such that $t_k - t_{k-1} \geq \delta\rho$, where $\eta_k = \max_{1\leq j\leq n} | - (\kappa_{jk} - 1)|^p$ and $\xi = \max\{1, \max\{\eta_k\}\}$, γ is the only positive real root of $\gamma = a - be^{\gamma\rho}$.

Constructing the following Lyapunov function:

$$V(t) = \sum_{j=1}^{n} |e_j(t)|^p. \tag{5}$$

Proof: On account of Lagrange mean value theorem $g(z + \triangle z) - g(z) = \dot{g}(z + \theta\triangle z) \cdot \triangle z (0 < \theta < 1)$, one obtains

$$\begin{aligned} &\alpha_j(z_j(t)) - \alpha_j(w_j(t)) \\ &\quad= \dot{\alpha}_j\big(w_j(t) + \theta_1(z_j(t) - w_j(t))\big)e_j(t), \\ &g_j(z_j(t)) - g_j(w_j(t)) \\ &\quad= \dot{g}_j\big(w_j(t) + \theta_2(z_j(t) - w_j(t))\big)e_j(t) \end{aligned} \tag{6}$$

where $0 < \theta_1, \theta_2 < 1$.

When $t \neq t_k$, calculating the derivative of (5) along (4), and combining (6), one gets

$$\begin{aligned} D^+V(t) \leq \sum_{j=1}^{n}\Big[& p\widetilde{\alpha}_j\Big(\sum_{r=1}^{n}(|a_{jr}| + |d_{jr}|)\overline{f}_r + \bar{u}_j\Big)|e_j(t)|^p \\ & + p\overline{\alpha}_j|e_j(t)|^{p-1}\Big(\sum_{r=1}^{n}|a_{jr}| l_r |e_r(t)| \\ & + \sum_{r=1}^{n}|d_{jr}| l_r |e_r(t-\rho(t))|\Big) - p\underset{\sim}{g}_j|e_j(t)|^p\Big]. \end{aligned} \tag{7}$$

Using $p\mu\nu^{p-1} \leq \mu^p + (p-1)\nu^p(\mu, \nu > 0)$, yields

$$\begin{aligned} p|e_r(t)||e_j(t)|^{p-1} &\leq |e_r(t)|^p + (p-1)|e_j(t)|^p, \\ p|e_r(t-\rho(t))||e_j(t)|^{p-1} &\leq |e_r(t-\rho(t))|^p + (p-1)|e_j(t)|^p. \end{aligned} \tag{8}$$

Combining (7) and (8), one gets

$$\begin{aligned} D^+V(t) &\leq \sum_{j=1}^{n}\Big(\sum_{r=1}^{n}(|a_{jr}| + |d_{jr}|)(p\widetilde{\alpha}_j\overline{f}_r + (p-1)\overline{\alpha}_r l_r) \\ &\qquad - p\underset{\sim}{g}_j + p\widetilde{\alpha}_j\bar{u}_j + \overline{\alpha}_j\sum_{r=1}^{n}|a_{rj}| l_j\Big)|e_j(t)|^p \\ &\qquad + \sum_{j=1}^{n}\sum_{r=1}^{n}\overline{\alpha}_r|d_{rj}| l_j |e_r(t-\rho(t))|^p \\ &\leq -a\sum_{j=1}^{n}|e_j(t)|^p + b\sum_{j=1}^{n}|e_j(t-\rho(t))|^p \\ &\leq -aV(t) + b\sup_{t-\rho\leq s\leq t} V(s). \end{aligned} \tag{9}$$

When $t = t_k$, one has

$$V(t_k) = \sum_{j=1}^{n}| - (\kappa_{jk} - 1)|^p|e_j(t_k^-)|^p \leq \eta_k V(t_k^-). \tag{10}$$

Combining (9)-(10) and Lemma 1, one obtains

$$V(t) \leq \xi \sup_{t_0-\rho \leq s \leq t_0} V(s) e^{-(\gamma - \frac{\ln(\xi e^{\gamma\rho})}{\delta\rho})(t-t_0)}, \ \forall t > t_0. \quad (11)$$

According to Definition 1 and (11), it follows that system (1) and system (2) are exponentially synchronized under impulsive control.

Remark 2: Aouiti and Dridi investigate the global exponential stability on impulsive CGNN with constant time delay in [15]. Zhang et al. study the exponential synchronization of CGNN with impulse control in [27], and Peng et al. investigate finite-time synchronization for CGNN with mixed time delays in [30]. However, the derivative of the time delay in [27] and [30] are required to be no greater than one. It should be noted that the restricted condition is removed in Theorem 1 of this paper.

Design feedback control $I_j(t) = -\kappa_j e_j(t)$, where $\kappa_j > 0$ is the control gain. That is, it is only necessary to design a suitable control gain κ_j so that (1) and (2) are exponentially synchronized.

Theorem 2: Under A1−A4 and $\dot{\rho}(t) \leq \widetilde{\rho} < 1$, given the constants $\kappa_j > 0$, $\varepsilon > 0$, $q_{jr}(j, r \in \mathcal{N})$ and p is a positive integer, the feedback control is $I_j(t) = -\kappa_j e_j(t)$, the following conditions are satisfied

(i) $\varepsilon - p\underset{\sim}{g}_j - p\kappa_j + p\widetilde{\alpha}_j\bar{u}_j + \sum_{r=1}^{n} |a_{rj}| l_j \overline{\alpha}_r + \sum_{r=1}^{n} |q_{jr}| + \sum_{r=1}^{n}(|a_{jr}| + |d_{jr}|)(p\widetilde{\alpha}_j \overline{f}_r + (p-1) l_r \overline{\alpha}_j) \leq 0$;

(ii) $\sum_{r=1}^{n}(|d_{rj}| l_j \overline{\alpha}_r - (1-\widetilde{\rho})|q_{jr}| e^{-\varepsilon\rho}) \leq 0$.

Then system (1) and system (2) can realize exponential synchronization.

Proof: Establishing the following Lyapunov functional

$$V(t) = \sum_{j=1}^{n} \left(e^{\varepsilon t} |e_j(t)|^p + \sum_{r=1}^{n} |q_{jr}| \int_{t-\rho(t)}^{t} e^{\varepsilon s} |e_j(s)|^p ds \right). \quad (12)$$

Calculating the derivative of (12) along (3), one gets

$$\begin{aligned} D^+V(t) \leq \sum_{j=1}^{n} \Bigg[\Bigg(& \varepsilon - p\underset{\sim}{g}_j - p\kappa_j + p\widetilde{\alpha}_j \bar{u}_j + \sum_{r=1}^{n} |q_{jr}| \\ & + p\widetilde{\alpha}_j \sum_{r=1}^{n} (|a_{jr}| + |d_{jr}|) \overline{f}_r \Bigg) e^{\varepsilon t} |e_j(t)|^p \\ & + p e^{\varepsilon t} |e_j(t)|^{p-1} \overline{\alpha}_j \Bigg(\sum_{r=1}^{n} |a_{jr}| l_r |e_r(t)| \\ & + \sum_{r=1}^{n} |d_{jr}| l_r |e_r(t-\rho(t))| \Bigg) \\ & - (1-\dot{\rho}(t)) \sum_{r=1}^{n} |q_{jr}| e^{\varepsilon(t-\rho(t))} |e_j(t-\rho(t))|^p \Bigg]. \end{aligned} \quad (13)$$

Combining (8) and (13), one has

$$\begin{aligned} D^+V(t) \leq \sum_{j=1}^{n} \Bigg[\Bigg(& \varepsilon - p\underset{\sim}{g}_j - p\kappa_j + p\widetilde{\alpha}_j \bar{u}_j + \sum_{r=1}^{n} |a_{rj}| l_j \overline{\alpha}_r \\ & + \sum_{r=1}^{n} |q_{jr}| + \sum_{r=1}^{n} (|a_{jr}| + |d_{jr}|)(p\widetilde{\alpha}_j \overline{f}_r \\ & + (p-1) l_r \overline{\alpha}_j) \Bigg) |e_j(t)|^p \\ & + \sum_{r=1}^{n} |d_{jr}| l_r \overline{\alpha}_j |e_r(t-\rho(t))|^p \\ & - (1-\widetilde{\rho}) \sum_{r=1}^{n} |q_{jr}| e^{-\varepsilon\rho} |e_j(t-\rho(t))|^p \Bigg] e^{\varepsilon t} \\ \leq\ & 0. \end{aligned} \quad (14)$$

For $\forall t \geq 0$, it can be obtained that

$$\begin{aligned} V(0) &= \sum_{j=1}^{n} \left(|e_j(0)|^p + \sum_{r=1}^{n} |q_{jr}| \int_{-\rho(0)}^{0} e^{\varepsilon s} |e_j(s)|^p ds \right) \\ &\leq m |\phi_z - \varphi_w|^p, \end{aligned}$$

where $|\phi_z - \varphi_w|^p = \sup_{-\rho \leq t \leq 0} \sum_{j=1}^{n} |\phi_j(t) - \varphi_j(t)|^p$, $m = 1 + \rho \max_{1 \leq j \leq n} \left\{ \sum_{r=1}^{n} |q_{jr}| \right\}$. Then $V(t)$ is a decreasing function and

$$\sum_{j=1}^{n} e^{\varepsilon t} |e_j(t)|^p \leq V(t) \leq V(0) \leq m |\phi_z - \varphi_w|^p.$$

Therefore,

$$\sum_{j=1}^{n} |e_j(t)|^p \leq m e^{-\varepsilon t} |\phi_z - \varphi_w|^p. \quad (15)$$

It can be concluded that system (1) and system (2) can realize exponential synchronization according to Definition 1.

Remark 3: Ke and Li [28] consider the exponential synchronization of the error system with constant time delay, and the authors mainly discusses the case of $p = 2$ in Theorem 1. However, it is noteworthy that p is a positive integer in this paper, the results of which are more general.

Remark 4: Theorem 1 and Theorem 2 investigate different control inputs. Theorem 1 focuses on exponential synchronization of the system under impulse control, while Theorem 2 discusses exponential synchronization of the system under linear feedback control. Therefore, Theorem 1 and Theorem 2 cannot contain each other.

Remark 5: Through the analysis of this paper, d_{jr} is a constant, the complexity of $f_r(w_r(t-\rho(t)))$ depends on the number of neurons and the selection of the activation function $f_r(\cdot)$. As far as we know, the activation function in the neural network is generally selected as sigmoid(.), tanh(.) etc.. Therefore, the algebraic conditions obtained in Theorem 1 and Theorem 2 are simple and easy to implement.

If $\alpha_j(\cdot) = \alpha_j$ and $h_j(\cdot) = h_j$, α_j and h_j are normal numbers, the system (1) becomes a ordinary cellular neural network with delays

$$\dot{w}_j(t) = -\alpha_j h_j w_j(t) + \alpha_j\Big(\sum_{r=1}^{n} a_{jr} f_r(w_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(w_r(t-\rho(t))) + u_j(t)\Big). \quad (16)$$

System (16) is the drive system, and the following is the response system

$$\dot{z}_j(t) = -\alpha_j h_j z_j(t) + \alpha_j\Big(\sum_{r=1}^{n} a_{jr} f_r(z_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(z_r(t-\rho(t))) + u_j(t)\Big) + I_j(t). \quad (17)$$

Let $e_j(t) = z_j(t) - w_j(t)$, and the error system is obtained by subtracting (16) from (17)

$$\dot{e}_j(t) = -\alpha_j h_j e_j(t) + \alpha_j\Big(\sum_{r=1}^{n} a_{jr} f_r(e_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(e_r(t-\rho(t)))\Big) + I_j(t). \quad (18)$$

Corollary 1: The design feedback control is $I_j(t) = -\kappa_j e_j(t)$, where k_j is positive constant. Under the A1−A4 and $\dot{\rho}(t) \le \widetilde{\rho} < 1$, for given constant $\varepsilon > 0$ and $q_{jr} (j, r \in \mathcal{N})$, such that

(i) $\varepsilon - 2\underset{\sim}{g}_j h_j - 2\kappa_j + \sum_{r=1}^{n}(|a_{jr}| + |d_{jr}|)\alpha_j l_r + \sum_{r=1}^{n} |q_{jr}| + \sum_{r=1}^{n} |a_{rj}| l_j \alpha_r \le 0$;

(ii) $\sum_{r=1}^{n}(|d_{rj}| l_j \alpha_r - (1-\widetilde{\rho})|q_{jr}| e^{-\varepsilon\rho}) \le 0$.

Then system (16) and system (17) are exponentially synchronized. And the Lyapunov functional is established

$$V(t) = \sum_{j=1}^{n}\Big(e^{\varepsilon t}|e_j(t)|^2 + \sum_{r=1}^{n}|q_{jr}|\int_{t-\rho(t)}^{t} e^{\varepsilon s}|e_j(s)|^2 ds\Big) \quad (19)$$

Remark 6: When $p = 2$ in Theorem 2, Corollary 1 is the general case of Theorem 2, then it can be found that the case of theorem 2 is more general.

IV. EXAMPLES

Consider the following n-dimensional CGNN

$$\dot{w}_j(t) = \alpha_j(w_j(t))\Big(-h_j(w_j(t)) + \sum_{r=1}^{n} a_{jr} f_r(w_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(w_r(t-\rho(t))) + u_j(t)\Big). \quad (20)$$

The response system is described as

$$\dot{z}_j(t) = \alpha_j(z_j(t))\Big(-h_j(z_j(t)) + \sum_{r=1}^{n} a_{jr} f_r(z_r(t)) + \sum_{r=1}^{n} d_{jr} f_r(z_r(t-\rho(t))) + u_j(t)\Big) + I_j(t). \quad (21)$$

Example 1: Choosing the 3-dimensional CGNN.
$a_{11} = 0.2$, $a_{12} = -0.14$, $a_{13} = -0.14$, $a_{21} = 0.13$, $a_{22} = -0.25$, $a_{23} = 0.1$, $a_{31} = -0.11$, $a_{32} = -0.1$, $a_{33} = 0.19$, $d_{11} = 0.15$, $d_{12} = -0.2$, $d_{13} = 0.09$, $d_{21} = -0.16$, $d_{22} = 0.17$, $d_{23} = 0.2$, $d_{31} = 0.1$, $d_{32} = -0.14$, $d_{33} = -0.2$, $u_1(t) = 0.2\sin(t)$, $u_2(t) = 0.2\cos(t)$, $u_3(t) = 0.3\cos(t)$, $h_j(\cdot) = 1.5(\cdot)$, $\alpha_j(\cdot) = 1 + \frac{1}{2(1+(\cdot)^2)}$, $f_j(\cdot) = \frac{1}{2}\sin(\frac{\cdot}{2})$, $I_j(t) = -\kappa_{jk}(z_j(t) - w_j(t))\delta(t - t_k^-)$, $\kappa_{jk} = 1.6$, $\rho(t) = 1.5\sin^2(t)$, $j = 1, 2, 3$. So, one can get

$$1 \le \alpha_j(\cdot) \le 1.5, \quad |\dot{\alpha}_j(\cdot)| \le 0.5,$$
$$g_j(\cdot) = \frac{3}{2}(\cdot)\Big(1 + \frac{1}{2(1+(\cdot)^2)}\Big),$$
$$\frac{3}{4} \le \dot{g}_j(\cdot) = \frac{3}{2}\Big(1 + \frac{1-(\cdot)^2}{2(1+(\cdot)^2)^2}\Big) \le \frac{9}{4},$$
$$|f_j(\mu) - f_j(\nu)| = \frac{1}{2}\Big|(\sin(\frac{\xi}{2}))'(\mu-\nu)\Big| \le \frac{1}{4}|\mu-\nu|,$$

where $\xi \in [\min\{\mu,\nu\}, \max\{\mu,\nu\}]$. Then, $l_j = 0.25$, $\overline{f}_j = 0.5$, $\underset{\sim}{g}_j = 0.75$, $\widetilde{g}_j = 2.25$, $\underline{\alpha}_j = 1$, $\overline{\alpha}_j = 1,5$, $\widetilde{\alpha}_j = 0.5$, $u_1 = 0.2$, $u_2 = 0.2$, $u_3 = 0.3$, $\rho = 1.5$, $j = 1, 2, 3$.

When $p = 2$, it can be calculated that $a_1 = 0.3300$, $a_2 = 0.2325$, $a_3 = 0.3112$, $b_1 = 0.1537$, $b_2 = 0.1913$, $b_3 = 0.1837$, $a = 0.2325 > b = 0.1913$. Choosing $\delta = 2 > 1$, $t_k - t_{k-1} = 5 > 3$, and the initial condition of the system are $\varphi_1(s) = 0.2$, $\varphi_2(s) = -0.6$, $\varphi_3(s) = -0.3$, $\phi_1(s) = -0.5$, $\phi_2(s) = 0.4$, $\phi_3(s) = 0.1$, where $s \in [-1.5, 0]$.

The state trajectory of system (20) and system (21) are shown in Fig. 1, and the error curves of (20) and (21) are shown in Fig, 2(a). According to Theorem 1, (20) and (21) can reach synchronization under impulsive control, and their exponential convergence rate is 0.0159. If the error system is not affected by impulsive control, then system (20) and system (21) are not synchronized, which is shown in Fig. 2(b).

Example 2: Choosing the 2-dimensional CGNN in Example 1 of [30].
$a_{11} = 1.8$, $a_{12} = -0.1$, $a_{21} = -2$, $a_{22} = 0.4$, $d_{11} = -1.7$, $d_{12} = -0.6$, $d_{21} = 0.5$, $d_{22} = -2.5$, $u_1(t) = u_2(t) = 0$, $h_1(\cdot) = 1.4(\cdot)$, $h_2(\cdot) = 0.4(\cdot)$, $\alpha_j(\cdot) = 0.7 + \frac{0.1}{1+(\cdot)^2}$, $f_j(\cdot) = 0.4\tanh(\cdot)$, $I_j(t) = -\kappa_j(z_j(t) - w_j(t))$, $\kappa_1 = 3.68$, $\kappa_2 = 2.56$, $\rho(t) = \frac{e^t}{1+e^t}$, $j = 1, 2$. So, one can obtain

$$0.7 \le \alpha_j(\cdot) \le 0.8, \quad |\dot{\alpha}_j(\cdot)| \le 0.1,$$
$$0.84 \le \dot{g}_1(\cdot) = 1.4\Big(0.7 + 0.1\frac{1-(\cdot)^2}{(1+(\cdot)^2)^2}\Big) \le 1.12,$$

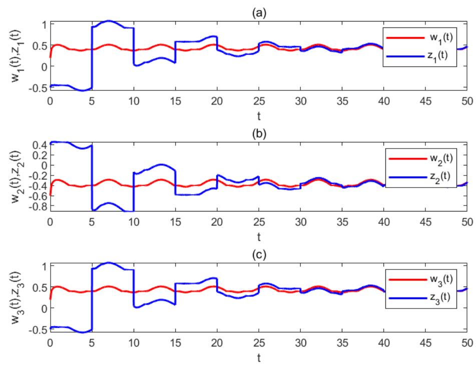

FIGURE 1. (a). Trajectories of $w_1(t)$ and $z_1(t)$; (b).Trajectories of $w_2(t)$ and $z_2(t)$; (c). Trajectories of $w_3(t)$ and $z_3(t)$.

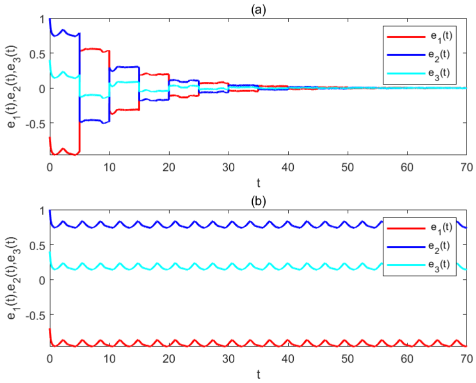

FIGURE 2. (a). Time responses of error states $e_1(t), e_2(t), e_3(t)$ with the controller; (b). Time responses of error states $e_1(t), e_2(t), e_3(t)$ without the controller.

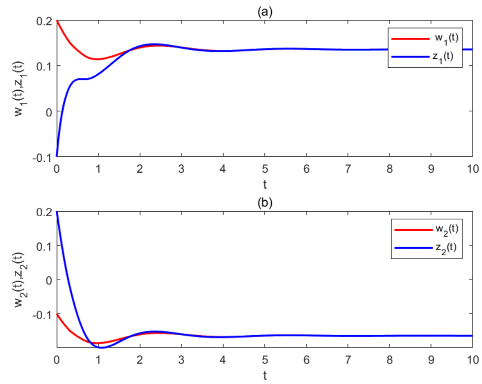

FIGURE 3. (a). Trajectories of $w_1(t)$ and $z_1(t)$; (b).Trajectories of $w_2(t)$ and $z_2(t)$.

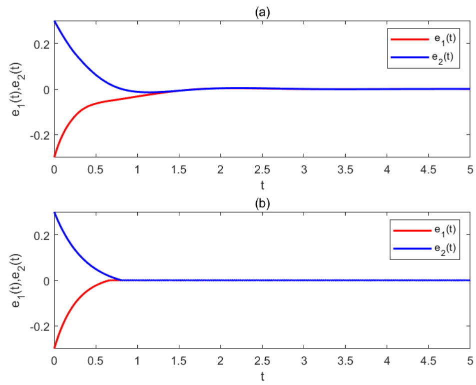

FIGURE 4. (a). Time responses of error states $e_1(t), e_2(t)$ in this paper; (b). Time responses of error states $e_1(t), e_2(t)$ in Example 1 of [30].

$$0.24 \le \dot{g}_2(\cdot) = 0.4\Big(0.7 + 0.1\frac{1-(\cdot)^2}{(1+(\cdot)^2)^2}\Big) \le 0.32,$$

$$|f_j(\mu) - f_j(\nu)| \le 0.4|\mu - \nu|.$$

Then, $l_j = 0.4$, $\overline{f}_j = 0.4$, $\underline{\alpha}_j = 0.7$, $\overline{\alpha}_j = 0.8$, $\widetilde{\alpha}_j = 0.1$, $\underset{\sim}{g}_1 = 0.84$, $\underset{\sim}{g}_2 = 0.24$, $\widetilde{g}_1 = 1.12$, $\widetilde{g}_2 = 0.32$, $\rho = 1$, $\widetilde{\rho} = 0.25$, $j = 1, 2$.

Selecting $\varphi_1(s) = 0.2$, $\varphi_2(s) = -0.1$, $\phi_1(s) = -0.1$, $\phi_2(s) = 0.2$, where $s \in [-1, 0]$ as the initial value of the system. When $p = 1$, let $q_{11} = q_{12} = q_{21} = q_{22} = 1$ and $\varepsilon = 0.1$. It can be obtained through simple calculation

$$\varepsilon - \underset{\sim}{g}_1 - \kappa_1 + \sum_{r=1}^{2}|a_{r1}|l_1\overline{\alpha}_r + \sum_{r=1}^{2}|q_{1r}| + \sum_{r=1}^{2}(|a_{1r}| + |d_{1r}|)\widetilde{\alpha}_1\overline{f}_r = -1.0360,$$

$$\varepsilon - \underset{\sim}{g}_2 - \kappa_2 + \sum_{r=1}^{2}|a_{r2}|l_2\overline{\alpha}_r + \sum_{r=1}^{2}|q_{2r}| + \sum_{r=1}^{2}(|a_{2r}| + |d_{2r}|)\widetilde{\alpha}_2\overline{f}_r = -0.3240,$$

$$\sum_{r=1}^{2}\Big(|d_{r1}|l_1\overline{\alpha}_r - (1-\widetilde{\rho})|q_{1r}|e^{-\varepsilon\rho}\Big) = -0.6533,$$

$$\sum_{r=1}^{2}\Big(|d_{r2}|l_2\overline{\alpha}_r - (1-\widetilde{\rho})|q_{2r}|e^{-\varepsilon\rho}\Big) = -0.3653.$$

The synchronization trajectories of (20) and (21) are shown in Fig. 3, and the error curves are described in Fig. 4(a). Under the condition of Theorem 2, (20) and (21) are exponentially synchronized with exponential convergence rate is 0.1.

Remark 7: Peng et al. explore the finite-time synchronization of the error system using the feedback controllers in [30]. It is worth noting that in Example 1 of [30], the delay in $(H5)$ satisfies $\dot{\tau}(t) = \frac{e^t}{(1+e^t)^2} \le \frac{1}{4} = \mu$ instead of selecting $\mu = 0$. Based on this, if the activation function is $f_j(t) = 0.4\tanh(t)$, the conditions of Example 1 in [30] is satisfied.

Remark 8: Let $f_j(t) = 0.4\tanh(t)$. Fig. 3(b) shows that Example 1 in [30] can be stabilized in finite time $T = 11.5960$. As shown in Fig. 3(a), the error system here can also be stabilized within time 11.5960. Although the stability time is difficult to calculate, theoretically, the research on exponential synchronization in this paper is closer to the situation that the error system cannot completely converge to zero in practice.

V. CONCLUSION

Based on the concept of synchronization, this paper constructs the controlled response system to obtain the error system, establishes suitable Lyapunov functions, and uses the inequality technique to explore the exponential synchronization of CGNN with delays. According to the restriction of the time delay derivative, two sufficient conditions for the error system to achieve exponential synchronization under different controllers are considered. Finally, the validity of the theoretical results is given by examples.

Exponential synchronization is theoretically closer to the case where the actual error system cannot fully converge to zero, but it is difficult to obtain the convergence time of exponential synchronization. The existing literature on finite-time synchronization of nonlinear systems, such as fixed/prescribed time synchronization, is valuable for research. Therefore, the study of finite-time synchronization for nonlinear systems is our future work.

REFERENCES

[1] M. Cohen and S. Grossberg, "Absolute stability of global pattern-formation and parallel memory storage by competitive neural networks," *IEEE Trans. Syst. Man, Cybern.*, vol. SMC-13, no. 5, pp. 815–826, Sep./Oct. 1983.

[2] Z.-W. Cai and L.-H. Huang, "Finite-time synchronization by switching state-feedback control for discontinuous Cohen–Grossberg neural networks with mixed delays," *Int. J. Mach. Learn. Cybern.*, vol. 9, no. 10, pp. 1683–1695, Oct. 2018.

[3] W. Xiong, D. Ma, and J. Liang, "Robust convergence of Cohen–Grossberg neural networks with time-varying delays," *Chaos, Solitons Fractals*, vol. 40, no. 3, pp. 1176–1184, May 2009.

[4] Y. Zhou, C. Li, L. Chen, and T. Huang, "Global exponential stability of memristive Cohen–Grossberg neural networks with mixed delays and impulse time window," *Neurocomputing*, vol. 275, pp. 2384–2391, Jan. 2018.

[5] J. Xiao, Z. Zeng, A. Wu, and S. Wen, "Fixed-time synchronization of delayed Cohen–Grossberg neural networks based on a novel sliding mode," *Neural Netw.*, vol. 128, pp. 1–12, Aug. 2020.

[6] H. Li, C. Li, T. Huang, and W. Zhang, "Fixed-time stabilization of impulsive Cohen–Grossberg BAM neural networks," *Neural Netw.*, vol. 98, pp. 203–211, Feb. 2018.

[7] C. Maharajan, R. Raja, J. Cao, G. Rajchakit, and A. Alsaedi, "Impulsive Cohen–Grossberg BAM neural networks with mixed time-delays: An exponential stability analysis issue," *Neurocomputing*, vol. 275, pp. 2588–2602, Jan. 2018.

[8] W. Yao, F. Yu, J. Zhang, and L. Zhou, "Asymptotic synchronization of memristive Cohen–Grossberg neural networks with time-varying delays via event-triggered control scheme," *Micromachines*, vol. 13, no. 5, p. 726, Apr. 2022.

[9] H. Qiu and F. Kong, "Global exponential stability of inertial Cohen–Grossberg neural networks with parameter uncertainties and time-varying delays," *Int. J. Control*, vol. 95, no. 8, pp. 2126–2140, Aug. 2022.

[10] Y. Chai, J. Feng, S. Qin, and X. Pan, "Global exponential stability of periodic solution of delayed discontinuous Cohen–Grossberg neural networks and its applications," *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 24, no. 1, pp. 245–264, Jun. 2021.

[11] X. Hu, L. Wang, C.-K. Zhang, X. Wan, and Y. He, "Fixed-time stabilization of discontinuous spatiotemporal neural networks with time-varying coefficients via aperiodically switching control," *Sci. China Inf. Sci.*, vol. 66, no. 5, pp. 1–14, May 2023, doi: 10.1007/s11432-022-3633-9.

[12] L. Yao, Z. Wang, X. Huang, Y. Li, Q. Ma, and H. Shen, "Stochastic sampled-data exponential synchronization of Markovian jump neural networks with time-varying delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 34, no. 2, pp. 909–920, Feb. 2023.

[13] X.-Z. Liu, K.-N. Wu, and W. Zhang, "Intermittent boundary stabilization of stochastic reaction–diffusion Cohen–Grossberg neural networks," *Neural Netw.*, vol. 131, pp. 1–13, Nov. 2020.

[14] O. Faydasicok, "An improved Lyapunov functional with application to stability of Cohen–Grossberg neural networks of neutral-type with multiple delays," *Neural Netw.*, vol. 132, pp. 532–539, Dec. 2020.

[15] C. Aouiti and F. Dridi, "New results on impulsive Cohen–Grossberg neural networks," *Neural Process. Lett.*, vol. 49, no. 3, pp. 1459–1483, Jun. 2019.

[16] P. Anbalagan, R. Ramachandran, J. Cao, G. Rajchakit, and C. P. Lim, "Global robust synchronization of fractional order complex valued neural networks with mixed time varying delays and impulses," *Int. J. Control, Autom. Syst.*, vol. 17, no. 2, pp. 509–520, Feb. 2019.

[17] Q. Tang and J. Jian, "Global exponential convergence for impulsive inertial complex-valued neural networks with time-varying delays," *Math. Comput. Simul.*, vol. 159, pp. 39–56, May 2019.

[18] X. Li, J. Fang, H. Li, and W. Duan, "Exponential stabilization of time-varying delayed complex-valued memristor-based neural networks via impulsive control," *Asian J. Control*, vol. 20, no. 6, pp. 2290–2301, Nov. 2018.

[19] Z. Wang and X. Liu, "Exponential stability of impulsive complex-valued neural networks with time delay," *Math. Comput. Simul.*, vol. 156, pp. 143–157, Feb. 2019.

[20] G. Rajchakit and R. Sriraman, "Robust passivity and stability analysis of uncertain complex-valued impulsive neural networks with time-varying delays," *Neural Process. Lett.*, vol. 53, no. 1, pp. 581–606, Feb. 2021.

[21] Y. Wang, Y. Tian, and X. Li, "Global exponential synchronization of interval neural networks with mixed delays via delayed impulsive control," *Neurocomputing*, vol. 420, pp. 290–298, Jan. 2021.

[22] D. Yang, X. Li, and S. Song, "Finite-time synchronization for delayed complex dynamical networks with synchronizing or desynchronizing impulses," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 33, no. 2, pp. 736–746, Feb. 2022.

[23] Y. Wang, J. Lu, X. Li, and J. Liang, "Synchronization of coupled neural networks under mixed impulsive effects: A novel delay inequality approach," *Neural Netw.*, vol. 127, pp. 38–46, Jul. 2020.

[24] K. Zou, X. Li, N. Wang, J. Lou, and J. Lu, "Stability and stabilization of delayed neural networks with hybrid impulses," *Complexity*, vol. 2020, pp. 1–9, Nov. 2020.

[25] L. Wang, H. He, and Z. Zeng, "Global synchronization of fuzzy memristive neural networks with discrete and distributed delays," *IEEE Trans. Fuzzy Syst.*, vol. 28, no. 9, pp. 2022–2034, Sep. 2020.

[26] Y. Ni, Z. Wang, Y. Fan, X. Huang, and H. Shen, "Memory-based event-triggered control for global synchronization of chaotic Lur'e systems and its application," *IEEE Trans. Syst. Man, Cybern. Syst.*, vol. 53, no. 3, pp. 1920–1931, Mar. 2023.

[27] C. Zhang, F. Deng, X. Zhao, and B. Zhang, "*P*-th exponential synchronization of Cohen–Grossberg neural network with mixed time-varying delays and unknown parameters using impulsive control method," *Neurocomputing*, vol. 218, pp. 432–438, Dec. 2016.

[28] K. Liang and L. Wanli, "Exponential synchronization in inertial Cohen–Grossberg neural networks with time delays," *J. Franklin Inst.*, vol. 356, no. 18, pp. 11285–11304, Dec. 2019.

[29] Y. Shi and J. Cao, "Finite-time synchronization of memristive Cohen–Grossberg neural networks with time delays," *Neurocomputing*, vol. 377, pp. 159–167, Feb. 2020.

[30] D. Peng, X. Li, C. Aouiti, and F. Miaadi, "Finite-time synchronization for Cohen–Grossberg neural networks with mixed time-delays," *Neurocomputing*, vol. 294, pp. 39–47, Jun. 2018.

[31] H. Jia, D. Luo, J. Wang, and H. Shen, "Fixed-time synchronization for inertial Cohen–Grossberg delayed neural networks: An event-triggered approach," *Knowledge-Based Syst.*, vol. 250, Aug. 2022, Art. no. 109104.

[32] D. W. C. Ho, J. Liang, and J. Lam, "Global exponential stability of impulsive high-order BAM neural networks with time-varying delays," *Neural Netw.*, vol. 19, no. 10, pp. 1581–1590, Dec. 2006.

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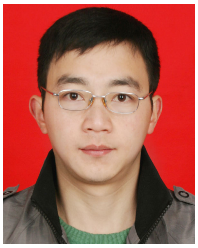

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