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RESEARCH ARTICLE

Novel Zero Circular Convolution Sequences for Detection and Channel Estimations

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ABSTRACT Multiuser communication systems or multiple access scheme systems favor sequences possessing the ideal periodic cross-correlation function (PCCF) property. In comparison, channel estimation, equalization, and synchronization applications favor sequences possessing the ideal periodic auto-correlation function (PACF) property. However, there is no set of sequences possessing both the ideal PCCF and PACF properties simultaneously, where auto-correlation and cross-correlation balance each other. In this work, sequences possessing the ideal PACF property are used as the base sequences. Then, a modulation technique is applied upon these base sequences to construct a set of zero circular convolution (ZCC) sequences within which an arbitrary pair of two sequences possesses the ideal PCCF property. Compared with least squares (LS) and minimum mean squared error (MMSE) algorithm, the simulation results show that the channel estimation performance of ZCC is better than MMSE and LS algorithms, and the computational complexity of the algorithm is the same as LS algorithm, but far lower than MMSE algorithm. This is the first study on ZCC sequences reported in the literature, in which their fundamental theorems, properties, construction, and applications are investigated. The advantage of possessing the desired PACF and the ideal PCCF properties allows the ZCC sequences to be used in a broader range of applications than other sets of sequences.

INDEX TERMS FSC, PACF, PCCF, PGIS, TSC, ZCC sequences.

I. INTRODUCTION

The theoretical correlation limit of sequences was addressed in [1], [2], [3], and [4]. Among them, Sarwate [1], [2] derived limits on periodic auto-correlation and cross-correlation and provided a tradeoff between the periodic auto-correlation and cross-correlation property. In this paper, it is demonstrated that no set of sequences possessing both the ideal periodic cross-correlation function (PCCF) and periodic auto-correlation function (PACF) property simultaneously exists. On one hand, sequences with impulse-like auto-correlation functions can be applied to various applications such as linear system parameter identification [4], real-time channel estimation [4], [5], [6], [7], equalization [8], synchronization [9], [10], [11], [12], and peak-to-average power ratio (PAPR) reduction in orthogonal frequency division multiplexing (OFDM) systems [13], [14]. On the other hand, various multiple-access schemes, such as Code Division Multiple Access [15], [16], [17], [18] and Orthogonal Frequency Division Multiplexing Access, require a desired cross-correlation property [19].

In recent years, construction and application of zero correlation or low correlation sequences have attracted numerous research interests. Lung-Sheng Tsai et al. presented a transform domain approach for generating families of sequences whose periodic AC functions have nonzero values at some subperiodic delays and whose periodic CC functions are identically zero in [20], and derived the theoretical bounds on the AC and CC for OFDM sequences with constraints of spectral nulls [21]. By taking the advantage of the Kronecker sequence

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property and a waveform set which satisfies spectrum hole constraint, Su Hu et al. presented a novel family of quasi-ZCZ CR sequences which possess zero cross-correlation and near-zero auto-correlation zone property [22]. Zilong Liu et al. investigate spectrally-constrained sequences(SCSs), and these sequences are applicable to the communication and radar systems operating over non-contiguous carriers or frequency slots [23].

The perfect Zadoff-Chu (ZC) sequences have been employed as random access preambles, primary synchronization sequences for cell search, and reference sequences for channel estimation in 3GPP LTE [11], [12], [24]. For both CDMA and OFDMA multiple schemes, the synchronization procedure is a crucial step for the communication devices to proceed with further operations such as detection and communication. Normal detection involves correlating the received signal with known reference sequences and comparing them with a detection threshold to extract the corresponding peaks and hence, the synchronization signal symbol location. The performance metrics for detecting the peaks are the detection probability (hit-rate) and false alarm rate, whose ZC sequences possessing the ideal PACF property render them the ideal option. Additionally, although the ZC sequences bear the same PAPR-property concept as the *m*-sequences and Gold sequences, the undesired cross-correlation property of the ZC sequences and the complexity of implementation, which results from their irrational coefficients, prevent them from being selected as pseudo noise (PN) codes for channelization in a CDMA scheme.

Given that the implementation of a system with irrational-valued parameters is more complex than a system based on integer parameters, the construction of a perfect Gaussian integer sequence (PGIS) has increasingly become an important research topic [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35]. A general form of even-length PGIS was proposed in [25], in which PGIS was constructed by linearly combining four sequences or their cyclic shift equivalents using Gaussian integer coefficients of equal magnitude. At the same time, an odd prime length PGIS was built using the order-2 and order-4 cyclotomic classes with respect to $\mathbf{GF}(p)$, where p is an odd prime [26]. The first odd composite length PGIS construction appeared in [27], in which Ma et al. used Whiteman's generalized order-2 cyclotomy $\mathbb{Z}_{pq}$ to construct the PGIS of length pq, where p and q are twin primes. In [28], Chang et al. initialized the *degree* concept of a sequence and constructed degree-2 and degree-3 PGISs of prime length p, which were then upsampled by an m factor and filled with new coefficients to build degree-3 and degree-4 PGISs of arbitrary composite length $N = pm$. Lee et al. [29] focused on the construction of a degree-2 PGIS of length $N = 2^m - 1$. Algorithms capable of generating a PGIS of arbitrary length were also developed by Pei et al. [30]. Such algorithms are considered a breakthrough in successfully constructing degree-5 PGIS of prime length by adapting the generalized Legendre sequences (GLS) instead of using the cyclotomic order-4 class. A systematic method for constructing sparse PGISs, in which most of the elements are zero, appeared in [31]. Lee et al. constructed the families of PGISs with high-energy efficiency. A short PGIS was used together with the polynomial or trace computation over an extension field to construct a family of long PGISs [32], [33]. PGISs of period p^k with degrees equal to or less than $k+1$ and those of period qp with degrees equal to or larger than four were proposed in [34] and [35].

From the above-mentioned PGIS examples, it can be said that the construction of PGIS has gradually been well developed. The next step should be exploring the PGIS applications. Currently there are four PGIS applications can be traced in the literature [13], [14], [18], [36]. In [13], a PGIS was applied to OFDM systems for PAPR reduction. Subsequently, the PGIS was used to construct the transform matrix for the associated precoded OFDM systems to achieve full frequency diversity and obtain optimal bit-error rate [14]. A CDMA scheme based on PGISs, called PGIS-CDMA system, was developed by Chang [18], where a set of PGISs can function as the PN codes (e.g., *m*-sequences, Gold sequences, Kasami sequences, and bent sequences) in a direct sequence (DS) CDMA system. In [36], a hybrid public/private key cryptography scheme based on PGIS of period $N = pq$ is proposed. This hybrid cryptosystem can take the advantages of public and private-key systems, and is with implementation simplicity for easy adaptation to an IoT platform.

The applications of sequences are critically determined by the inherent desired auto-correlation or cross-correlation function property these sequences must possess. However, no set of sequences possessing both the ideal auto- and cross-correlation property exists to meet the different requirements of a communication system from all aspects. In this paper, we aim at constructing a set of sequences, in which an arbitrary pair of two sequences possesses the ZCC property among sequences within the same set. The ZCC sequences can be constructed by selecting the sequences possessing the desired auto-correlation function property (e.g., PN codes, PSs, or PGISs) as the *base sequences* for *repeating* or *upsampling* to generate a new set of sequences. Subsequently, a *modulation technique* is applied so that this new set of sequences possesses the ZCC property. In this way, the advantage of possessing the ideal PCCF and the desired PACF properties allows ZCC sequences to be used in a broader range of applications than the original set of base sequences, especially in multiuser communication systems.

This study aims at two contributions, which are summarized below:

1) To the best of the authors' knowledge, this is the first study reported in the literature regarding the application of a modulation technique to construct ZCC sequences, in which the property, theorem, sequence size, construction, and applications of ZCC sequences are investigated.
2) The best binary-sequence set of period N that can be achieved in terms of correlation properties is the set of sequences with the following correlation

properties [4]:

$$R_r[\tau] = \begin{cases} N, & \tau = 0 \\ 0, & \tau \neq 0, \end{cases} \quad (1)$$

$$R_{r,s}[\tau] = \sqrt{N}, \quad \forall \tau, r \neq s. \quad (2)$$

The correlation properties of the ZCC sequence set of period mN, which is constructed based on the base sequences of period N with a sequence energy E, can be further improved to

$$R_r[\tau] = \begin{cases} mE \cdot e^{\frac{j2\pi nN}{mN}}, & \tau = nN, n=0, 1, \ldots, m-1 \\ 0, & otherwise, \end{cases} \quad (3)$$

$$R_{r,s}[\tau] = 0, \quad \forall \tau, r \neq s. \quad (4)$$

In (1) and (3), $R_r[\tau] = N$ and $|R_r[\tau]| = mE$, respectively; however, in (2) $R_{r,s}[\tau] = \sqrt{N} \neq 0$, but $R_{r,s}[\tau] = 0$ in (4). Here the cost of achieving the ideal PCCF property, $R_{r,s}[\tau] = 0$, is the additional $m - 1$ nonzero terms appeared in $R_r[\tau] = mE \cdot e^{\frac{j2\pi nN}{mN}}$, $\tau = nN, n = 0, 1, \ldots, m-1$. These additional $m - 1$ nonzero terms deviate the *ideal* PACF property definition. However, the ZCC sequences can compete the performance of the sequences possessing the ideal PACF property, which is demonstrated in the Sections IV and V. As defined in (3), the ZCC sequences possess the desired PACF property.

This paper is organized as follows. The notations and definitions are presented in Section II. The properties and related theorems of ZCC sequences are presented in Section III, where parts of these results have appeared in the conference version of this paper ICECCME (2022) [37]. Based on these theorems, four different construction types of such ZCC sequences are presented in Section IV. The applications of ZCC sequences to frequency-selective channel (FSC) and time-selective channel (TSC) estimation are presented in Sections V and VI, respectively. The simulation results are presented in Section VII. Finally, the conclusions are drawn in Section VIII.

II. NOTATIONS AND DEFINITIONS

A. NOTATIONS

The boldface character $\mathbf{s}$ denotes either a vector of size $N \times 1$ or a sequence of period N, which is expressed as $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$. $\mathbf{s}_{-}^{*} = \{s^*[(-n)_N]\}_{n=0}^{N-1}$ is defined, where the superscript * and $(\cdot)_N$ stand for complex conjugate and modulo N operation, respectively. δ_N denotes the Kronecker delta function of period N. Let $\mathbf{s}^{(-m)} = \{s[(n+m)_N]\}_{n=0}^{N-1}$ and $\mathbf{s}^{(m)} = \{s[(n-m)_N]\}_{n=0}^{N-1}$ represent the circular shift of $\mathbf{s}$ to the left and right, respectively, by m places, where $0 \leq m \leq N-1$. A set of N different sequences or vectors is expressed as $\{\mathbf{s}_m\}_{m=0}^{N-1}$. The capital boldface character $\mathbf{S}$ denotes the discrete Fourier transform (DFT) of $\mathbf{s}$. Italic notations represent scalars, ordinary bold notations represent vectors or matrices. $\hat{\mathbf{H}}_{LS}$ denotes channel estimation of least-squares. For other notations in the text, see Appendix (A).

B. DEFINITIONS

1) DEGREE

The *degree* of a sequence is defined as the number of distinct nonzero elements within one period of the sequence [28].

2) CIRCULAR CONVOLUTION

Let $\mathbf{s}_1 = \{s_1[n]\}_{n=0}^{N-1}$ and $\mathbf{s}_2 = \{s_2[n]\}_{n=0}^{N-1}$ denote two sequences of length N, respectively. The *circular convolution* of $\mathbf{s}_1$ and $\mathbf{s}_2$ is represented by $\mathbf{s}_1 \otimes_c \mathbf{s}_2 = \{s_{1,2}[n]\}_{n=0}^{N-1}$, where $\otimes_c$ denotes the circular convolution operation and

$$s_{1,2}[n] = \sum_{\tau=0}^{N-1} s_1[\tau] s_2[(n-\tau)_N].$$

3) ZCC SEQUENCES

A set of sequences $\{\mathbf{s}_m\}_{m=0}^{N-1}$ is called *ZCC sequences* set with cardinality N if and only if $\mathbf{s}_m \otimes_c \mathbf{s}_n = \mathbf{0}$, $\forall m, n$, $0 \leq m, n \leq N-1$, $m \neq n$.

4) BASE SEQUENCES

A set of sequences, which are applied for constructing a set of ZCC sequences, is defined as *base sequences* set.

5) PACF

$\mathbf{R}_s = \mathbf{s} \otimes_c \mathbf{s}_{-}^{*} = \{R_s[\tau]\}_{\tau=0}^{N-1}$ denotes the PACF of $\mathbf{s}$, i.e.,

$$R_s[\tau] = \sum_{n=0}^{N-1} s[n] s^*[(n-\tau)_N].$$

The sequence $\mathbf{s}$ is called *perfect* if and only if $\mathbf{R}_s = E \cdot \delta_N$, where $E = \sum_{n=0}^{N-1} |s[n]|^2$ is the energy of $\mathbf{s}$. $\mathbf{S}$ is denoted to be the discrete Fourier transform (DFT) of $\mathbf{s}$. The DFT of $\mathbf{R}_s = \mathbf{s} \otimes_c \mathbf{s}_{-}^{*}$ is $\mathbf{S} \circ \mathbf{S}^* = \{|S[n]|^2\}_{n=0}^{N-1}$, where $\mathbf{S} \circ \mathbf{S}^*$ is the component-wise product of $\mathbf{S}$ and $\mathbf{S}^*$, and $|\cdot|$ denotes the absolute value of the argument. The DFT pair of $E \cdot \delta_N$ and $\{|S[n]|^2\}_{n=0}^{N-1} = \{E\}_{n=0}^{N-1}$ demonstrates that the sequence $\mathbf{s}$ is perfect if and only if it satisfies the flat magnitude spectrum property, which is given by $|S[n]| = \sqrt{E}$, $\forall n$.

6) PCCF

The PCCF of $\mathbf{s}_1 = \{s_1[\tau]\}_{\tau=0}^{N-1}$ and $\mathbf{s}_2 = \{s_2[\tau]\}_{\tau=0}^{N-1}$, which is denoted by $\mathbf{s}_1 \otimes_c \mathbf{s}_{-2}^{*} = \{R_{s_1,s_2}[\tau]\}_{\tau=0}^{N-1}$, is defined as

$$R_{s_1,s_2}[\tau] = \sum_{n=0}^{N-1} s_1[n] s_2^*[(n-\tau)_N].$$

The sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ have an ideal PCCF property if and only if

$$R_{s_1,s_2}[\tau] = 0, \quad \forall \tau.$$

The DFT of $\mathbf{s}_1 \otimes_c \mathbf{s}_{-2}^{*}$ is $\mathbf{S}_1 \circ \mathbf{S}_2^* = \{S_1[n] S_2^*[n]\}_{n=0}^{N-1}$. When the sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ have an ideal PCCF property, it is evident that $\{S_1[n] S_2^*[n]\}_{n=0}^{N-1}$ is an N-tuple zero vector. Conversely, if $\{S_1[n] S_2^*[n]\}_{n=0}^{N-1}$ is an N-tuple zero vector, then $R_{s_1,s_2}[\tau] = 0$, $\forall \tau$.

7) ORTHOGONALITY

Two sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ are *orthogonal* to each other if the inner product $\mathbf{s}_1 \cdot \mathbf{s}_2^H = 0$, where $(\cdot)^H$ denotes the transpose and conjugate operation of the argument, i.e.,

$$\sum_{n=0}^{N-1} s_1[n]s_2^*[n] = 0.$$

It is noted that the ideal PCCF property maintains orthogonality between two sequences regardless of the number of circular shifts relative to each other.

III. PROPERTIES OF ZCC SEQUENCES

Let $\mathbf{s}_1 = \{s_1[\tau]\}_{\tau=0}^{N-1}$ and $\mathbf{s}_2 = \{s_2[\tau]\}_{\tau=0}^{N-1}$ be two sequences of period N, and the associative DFTs are given by $\mathbf{S}_1 = \{S_1[n]\}_{n=0}^{N-1}$ and $\mathbf{S}_2 = \{S_2[n]\}_{n=0}^{N-1}$, respectively.

Theorem 1: Sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ have the ZCC property in the time domain, that is, $\mathbf{s}_1 \otimes_c \mathbf{s}_2 = \mathbf{0}$ if and only if $\mathbf{S}_1 \circ \mathbf{S}_2 = \mathbf{0}$.

Proof: The DFT of $\mathbf{s}_1 \otimes_c \mathbf{s}_2$ is $\mathbf{S}_1 \circ \mathbf{S}_2 = \{S_1[n] S_2[n]\}_{n=0}^{N-1}$. When sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ possess the ZCC property in the time domain, $\mathbf{s}_1 \otimes_c \mathbf{s}_2 = \mathbf{0} \Rightarrow \mathbf{S}_1 \circ \mathbf{S}_2 = \{S_1[n]S_2[n]\}_{n=0}^{N-1} = \mathbf{0}$ which is an N-tuple zero vector. Conversely, if $\mathbf{S}_1 \circ \mathbf{S}_2 = \mathbf{0}$ is an N-tuple zero vector, the IDFT derives that $\mathbf{s}_1 \otimes_c \mathbf{s}_2 = \mathbf{0}$ in the time domain. ■

Theorem 2: Sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ have the ZCC property in the frequency domain, that is, $\mathbf{S}_1 \otimes_c \mathbf{S}_2 = \mathbf{0}$ if and only if $\mathbf{s}_1 \circ \mathbf{s}_2 = \mathbf{0}$.

Proof: The DFT of $\mathbf{s}_1 \circ \mathbf{s}_2$ is $\mathbf{S}_1 \otimes_c \mathbf{S}_2$. When $\mathbf{s}_1 \circ \mathbf{s}_2 = \mathbf{0}$, the DFT of an N-tuple zero vector is also a zero vector, which is expressed as $\mathbf{S}_1 \otimes_c \mathbf{S}_2 = \mathbf{0}$. Conversely, if $\mathbf{S}_1 \otimes_c \mathbf{S}_2$ is an N-tuple zero vector, its IDFT demonstrates that $\mathbf{s}_1 \circ \mathbf{s}_2$ is a zero vector as well. ■

Theorem 1 and *Theorem 2* present the time and frequency duality property with respect to Fourier transformation theory, respectively. For a sequence of period N, it might not be necessary to make distinction whether these N coefficients are the time or frequency parameters. However, to gain insights on the ZCC properties and to address different applications of ZCC sequences, we would define and distinguish ZCC sequences that possess the ZCC property either in time or frequency domain.

Corollary 1: Sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ of period N have the ZCC property in the time domain, and $\mathbf{s}_3$ and $\mathbf{s}_4$ are two arbitrary nonzero sequences of the same period. The following pairs of two sequences possess the ZCC property in the time domain as well:

1) $a \cdot \mathbf{s}_1$ and $b \cdot \mathbf{s}_2$,
2) $\mathbf{s}_1^*$ and $\mathbf{s}_2^*$,
3) $\mathbf{s}_1$ and $\mathbf{s}^*_{-2}$,
4) $\{s_1[(n-m)_N]\}$ and $\{s_2[(n-k)_N]\}$,
5) $\{s_1[(-n)_N]\}$ and $\{s_2[(-n)_N]\}$, and
6) $\mathbf{s}_1 \otimes_c \mathbf{s}_3$ and $\mathbf{s}_2 \otimes_c \mathbf{s}_4$,

where a and b are the two nonzero scalars, and m and k are integers.

Proof: Given that $\mathbf{s}_1$ and $\mathbf{s}_2$ have the ZCC property in the time domain, there exist no overlapping nonzero elements between $\mathbf{S}_1$ and $\mathbf{S}_2$ based on *Theorem 1*. When these two sequences are processed by multiplying with a scalar, taking complex conjugate, circular shift or reflection operation, as well as circular convolution with other sequences, the relative locations of the non-zero elements of the associative DFTs do not change. Hence, these six pairs of two sequences still possess the ZCC property in the time domain as well. ■

Corollary 2: Sequences $\mathbf{s}_1$ and $\mathbf{s}_2$ of period N have the ZCC property in the frequency domain, and $\mathbf{s}_3$ and $\mathbf{s}_4$ are two arbitrary nonzero sequences with DFTs $\mathbf{S}_3$ and $\mathbf{S}_4$. The following pairs of two sequences possess the ZCC property in the frequency domain:

1) $a \cdot \mathbf{S}_1$ and $b \cdot \mathbf{S}_2$,
2) $\mathbf{S}_1^*$ and $\mathbf{S}_2^*$,
3) $\mathbf{S}_1$ and $\mathbf{S}_2^*$,
4) $\{S_1[n]e^{j2\pi mn/N}\}$ and $\{S_2[n]e^{j2\pi kn/N}\}$,
5) $\{S_1[(-n)_N]\}$ and $\{S_2[(-n)_N]\}$, and
6) $\mathbf{S}_1 \circ \mathbf{S}_3$ and $\mathbf{S}_2 \circ \mathbf{S}_4$,

where a and b are nonzero scalars, and m and k are integers.

Proof: Given that $\mathbf{s}_1$ and $\mathbf{s}_2$ have the ZCC property in the frequency domain, there exist no overlapping nonzero elements between $\mathbf{s}_1$ and $\mathbf{s}_2$ by *Theorem 2*. These pairs of two sequences, which are modified from $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively, make the changes only to the amplitude rather than the relative locations of nonzero elements to both $\mathbf{s}_1$ and $\mathbf{s}_2$. All six pairs of two sequences still bear the requirement of having no overlapping nonzero elements between each other so as to possess the ZCC property in the frequency domain. ■

Theorem 3: There exists no pair of two perfect sequences that possesses the ZCC property.

Proof: The spectrum of a perfect sequence is magnitude flat, and the component-wise product between two spectrum with magnitude flat results in also a magnitude flat spectrum. This derives that the circular convolution of two perfect sequences also generates a perfect sequence. However, according to *Theorem 1* and *Corollary 1*, the DFT of the circular convolution of the two sequences should be a zero vector for these two sequences to possess the ZCC property. Hence, there exists no pair of two perfect sequences that can possess the ZCC property. ■

Let $\{\mathbf{s}_n\}_{n=0}^{m-1}$ be a set of sequences of period N, where $\mathbf{s}_n = \{s_n[k]\}_{k=0}^{N-1}$. Let $\{\mathbf{s}_n^u\}_{n=0}^{m-1}$ be a set obtained from upsampling $\{\mathbf{s}_n\}_{n=0}^{m-1}$, which $\mathbf{s}_n^u = \{s_n^u[k]\}_{k=0}^{N-1}$ is given by

$$s_n^u[k] = \begin{cases} s_n\left[\frac{k}{m}\right], & k = 0, m, \ldots, (N-1)m \\ 0, & \text{otherwise}. \end{cases} \tag{5}$$

Theorem 4: Let $\{\mathbf{s}_n\}_{n=0}^{m-1}$ be a set of ZCC sequences with cardinality m. The upsampled set of sequences $\{\mathbf{s}_n^u\}_{n=0}^{m-1}$ is also a set of ZCC sequences with the same cardinality.

Proof: Let $\mathbf{S}_n$ denote the DFT of $\mathbf{s}_n$. The DFT of $\mathbf{s}_n^u$ is given by

$$\mathbf{S}_n^u = (\underbrace{\mathbf{S}_n, \mathbf{S}_n, \ldots, \mathbf{S}_n}_{m \text{ times}}). \tag{6}$$

Because $\{\mathbf{s}_n\}_{n=0}^{m-1}$ is a set of ZCC sequences, nonzero elements of all $\mathbf{S}_n$ will be disjoint. From (6), $\mathbf{S}_n^u$ is obtained from repeating $\mathbf{S}_n$ m times such that the locations of nonzero elements of all $\mathbf{S}_n^u$ will be disjoint as well. By *Theorem 1* $\{\mathbf{s}_n^u\}_{n=0}^{m-1}$ is a set of ZCC sequences with the same cardinality. ■

A set of sequences $\{\mathbf{t}_n\}_{n=0}^{m-1}$ of period mN is obtained from $\{\mathbf{s}_n\}_{n=0}^{m-1}$ by repeating each sequence m times, that is,

$$\mathbf{t}_n = (\underbrace{\mathbf{s}_n, \mathbf{s}_n, \ldots, \mathbf{s}_n}_{m \text{ times}}). \tag{7}$$

Theorem 5: Let $\{\mathbf{s}_n\}_{n=0}^{m-1}$ be a set of ZCC sequences. The set of sequences $\{\mathbf{t}_n\}_{n=0}^{m-1}$ constructed based on $\{\mathbf{s}_n\}_{n=0}^{m-1}$ is also a set of ZCC sequences with the same cardinality.

Proof: Let $\mathbf{S}_n = \{S_n[\tau]\}_{\tau=0}^{N-1}$ be the DFT of $\mathbf{s}_n$. The DFT of $\mathbf{t}_n$ is given by

$$\mathbf{T}_n = m(\underbrace{S_n[0], 0, \cdots, 0}_{m}, \cdots, \underbrace{S_n[N-1], 0, \cdots, 0}_{m}). \tag{8}$$

From (8), the number of nonzero elements of $\mathbf{T}_n$ is the same as that of $\mathbf{S}_n$, and the locations of these nonzero elements $\mathbf{S}_n$ are not overlapped. This implies that $\{\mathbf{t}_n\}_{n=0}^{m-1}$ is a set of ZCC sequences with the same cardinality. ■

When the ZCC sequences are applied, the cardinality of a set of ZCC sequences determines the maximum number of users and the capacity of a communication system. In the following subsection, the cardinality issue is addressed.

Consider an N-tuple vector space $\mathcal{C}^N$ over the field $\mathcal{C}$ of complex number. The standard basis of $\mathcal{C}^N$ is a set of Kronecker delta function of period N and its circular shifts $\{\delta_N, \delta_N^{(1)}, \ldots, \delta_N^{(N-1)}\}$. Let $\mathbf{c}_n = \{e^{j\frac{2\pi nk}{N}}\}_{k=0}^{N-1}$ denote the DFT of $\delta_N^{(n)}$. The set $\{\frac{\mathbf{c}_0}{N}, \frac{\mathbf{c}_1}{N}, \ldots, \frac{\mathbf{c}_{N-1}}{N}\}$, which is isomorphic to $\{\delta_N, \delta_N^{(1)}, \ldots, \delta_N^{(N-1)}\}$, can also serve as a basis for vector space $\mathcal{C}^N$. Clearly, we can see that $\mathbf{c}_n \otimes_c \mathbf{c}_m = \mathbf{0}$, for all $0 \le n, m \le N-1$. This implies that $\{\frac{\mathbf{c}_0}{N}, \frac{\mathbf{c}_1}{N}, \ldots, \frac{\mathbf{c}_{N-1}}{N}\}$ is also a set of ZCC sequences.

Lemma 1: Two sequences possess the ZCC property in the time domain if and only if there are no overlapping nonzero elements between the *coordinate vectors* of these two sequences relative to the ordered basis $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$.

Proof: As can be clearly seen, the coordinate vector of a sequence relative to the ordered basis $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$ is the DFT of this sequence. Given that these two sequences possess the ZCC property, the locations of nonzero elements of their associative DFTs are disjoint, thus proving this lemma. ■

For demonstration, the coordinate vectors of sequences $\mathbf{s}_n = (1, e^{j2\pi n/N}, e^{j2\pi(2n)/N}, \ldots, e^{j2\pi(N-1)n/N})$ and $\mathbf{s}_k = (1, e^{j2\pi k/N}, e^{j2\pi(2k)/N}, \ldots, e^{j2\pi(N-1)k/N})$ of period N relative to the ordered basis $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$ are $\delta_N^{(n)}$ and $\delta_N^{(k)}$, respectively. Given that $\delta_N^{(n)} \circ \delta_N^{(k)} = \mathbf{0}$, this infers that $\mathbf{s}_n \otimes_c \mathbf{s}_k = \mathbf{0}$. Meanwhile, the coordinate vector of sequence $\mathbf{s}_1 = (0, 1, 0, \ldots, 0)$ relative to the ordered basis $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$ is $[1\ e^{j2\pi/N}\ e^{j2\pi(2)/N}\ \cdots\ e^{j2\pi(N-1)/N}]^T$. As all entries of this N-tuple coordinate vector are filled in with nonzero elements, there exists no sequence of period N that can achieve the ZCC property in the time domain.

Lemma 2: A set of ZCC sequences is a set of independent sequences.

Proof: As can be clearly seen, the coordinate vector of a sequence relative to the ordered basis $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$ is the DFT of this sequence. Given that these two sequences possess the ZCC property, the locations of the nonzero elements of their associated DFTs are disjoint, thus proving this lemma. ■

The result of *Lemma 2* can be further developed to derive *Lemma 3*.

Lemma 3: Let $\{\mathbf{s}_n\}_{n=0}^{K-1}$ be a set of distinct K sequences of period N, where $K \ge N$. $\{\mathbf{s}_n\}_{n=0}^{K-1}$ cannot be a set of ZCC sequences.

Let $\mathbf{V_B(s)}$ denote the coordinate vector of sequence $\mathbf{s}$ of period N relative to base matrix $\mathbf{B} = \{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$. Define $Dim(\mathbf{V_B(s)})$ as the number of nonzero elements of coordinate vector $\mathbf{V_B(s)}$. The cardinality of a set of ZCC sequences is determined and described in *Theorem 6*.

Theorem 6: Let $\{\mathbf{s}_n\}$ be a set of ZCC sequences of period N. The set of coordinate vectors relative to base matrix $\mathbf{B}$ is denoted by $\{\mathbf{V_B(s_n)}\}$. The cardinality $m \ge 2$ of a set of ZCC sequences $\{\mathbf{s}_n\}_{n=0}^{m-1}$ is constrained by equation given by

$$Dim(\mathbf{V_B(s_0)}) + Dim(\mathbf{V_B(s_1)}) + \cdots + Dim(\mathbf{V_B(s_{m-1})}) = N.$$

Proof: First, given that the coordinate vector is an N-tuple vector, the maximum number of nonzero elements is N. Based on *Lemma 1* the nonzero elements of $\{\mathbf{V_B(s_n)}\}$ cannot overlap, and *Lemma 2* indicates that $\{\mathbf{s}_n\}$ is a set of independent sequences. These two properties show that the above constraint equation is true. ■

Example 1: Let $\mathbf{s} = (1, 0, 1, 0)$. The coordinate vector of $\mathbf{s}$ relative to $\mathbf{B} = \{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is $\mathbf{V_B(s)}$=$[\frac{1}{2}\ \ 0\ \ \frac{1}{2}\ \ 0]^T$, where $Dim(\mathbf{V_B(s)}) = 2$. Given that $\mathbf{s}$ is fixed, there exist two sets of ZCC sequences, $\mathbf{A}_1$ and $\mathbf{A}_2$, with cardinality $|\mathbf{A}_1| = 2$ and $|\mathbf{A}_2| = 3$, respectively.

$$\mathbf{A}_1 = \{\mathbf{s}, (1, 0, -1, 0)\},$$
$$\mathbf{A}_2 = \{\mathbf{s}, (1, j, -1, -j), (1, -j, -1, j)\}.$$

When including $\mathbf{s}$ as an element, the other sets of ZCC sequences with cardinalities 2 and 3 are isomorphic to $\mathbf{A}_1$ and $\mathbf{A}_2$.

IV. CONSTRUCTION OF ZCC SEQUENCES

A. TIME DOMAIN ZCC SEQUENCES

We can apply *Theorem 1* to construct a set of ZCC sequences presenting the ZCC property in the time domain by adjusting these sequences' DFTs, which do not have the overlapped nonzero elements between each other throughout all sequences within the same set. Two kinds of construction are introduced.

Construction I: Time domain ZCC sequences based on single base sequence

Let $\mathbf{s}_t$ be a base sequence of period N, and the associative DFT is denoted by $\mathbf{S}_t$, where $\mathbf{s}_t$ and $\mathbf{S}_t$ are expressed respectively as

$$\mathbf{s}_t = (s_t[0], s_t[1], \ldots, s_t[N-1]), \tag{9}$$
$$\mathbf{S}_t = (S_t[0], S_t[1], \ldots, S_t[N-1]). \tag{10}$$

A sequence $\mathbf{z}_0$ of period mN is constructed by repeating and cascading sequence $\mathbf{s}_t$ m times, which is

$$\begin{aligned}\mathbf{z}_0 &= (\mathbf{s}_t, \mathbf{s}_t, \ldots, \mathbf{s}_t) \\ &= (\underbrace{s_t[0], \ldots, s_t[N-1]}_{N}, \underbrace{\ldots, s_t[0], \ldots, s_t[N-1]}_{(m-1)N}).\end{aligned} \tag{11}$$

The DFT of $\mathbf{z}_0$, denoted by $\mathbf{Z}_0$, is expressed as follows

$$\mathbf{Z}_0 = m(\underbrace{S_t[0], 0, \cdots, 0}_{m}, \cdots, \underbrace{S_t[N-1], 0, \cdots, 0}_{m}). \tag{12}$$

The $\mathbf{Z}_0$ in (12) is an mN-tuple sparse vector with N nonzero elements $S_t[k]$, $k = 0, 1, \ldots, N-1$, located at the entries of $0, m, \ldots, (N-1)m$, respectively. The number of zero elements between two adjacent $S_t[k]$ is $m-1$, thus, it has m distinct circular shifts of $\mathbf{Z}_0$, denoted by $\{\mathbf{Z}_0^{(k)}\}_{k=0}^{m-1}$, which can match the requirement of no overlapping nonzero elements among the whole set of m vectors.

Theorem 7: Let $\mathbf{z}_k$ denote the IDFT of $\mathbf{Z}_0^{(k)}$, $k = 0, 1, \ldots, m-1$. $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}\}$ is a set of ZCC sequences with cardinality m.

Proof: There exists no overlapping nonzero elements to all vectors in set $\{\mathbf{Z}_0^{(k)}\}_{k=0}^{m-1}$; hence, the component-wise product between arbitrary two different vectors from this set generates a zero vector. This implies that $\mathbf{z}_k \otimes_c \mathbf{z}_n = \mathbf{0}$, $0 \le k, n \le m-1$ is true, where $k \neq n$. ∎

Let $\mathbf{c}_k = \{c_k[n]\}_{n=0}^{mN-1} \in \mathcal{C}^{mN}$ be an mN-tuple vector. With elements $c_k[n] = e^{j\frac{2\pi nk}{mN}}$, $k = 0, 1, \ldots, m-1$, $\{\mathbf{c}_k\}_{k=0}^{m-1}$ is a set of m digital carriers.

Corollary 3: The set of ZCC sequences $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}\}$ can be generated by modulation with digital carrier $\mathbf{z}_k = \mathbf{z}_0 \circ \mathbf{c}_k$, $k = 0, 1, \ldots, m-1$, respectively.

Proof: $\delta_{mN}^{(k)}$ denotes the circular shift of a Kronecker delta function of period mN to the right by k places. The DFT of $\delta_{mN}^{(k)}$ is given by $\mathbf{c}_k = \{c_k[n]\}_{n=0}^{mN-1}$. As $\mathbf{z}_k$ is the IDFT of $\mathbf{Z}_0^{(k)}$, where $\mathbf{Z}_0^{(k)} = \mathbf{Z}_0 \otimes_c \delta_{mN}^{(k)}$, this implies that $\mathbf{z}_k = \mathbf{z}_0 \circ \mathbf{c}_k$, $k = 0, 1, \ldots, m-1$. ∎

Theorem 8: Let $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}\}$ be a set of ZCC sequences. $\mathbf{R}_0 = \{R_0[\tau]\}_{\tau=0}^{mN-1}$ denotes the PACF of $\mathbf{z}_0$, then the PACFs of the rest of other $\mathbf{z}_k$ are given by $\mathbf{R}_k = \{R_0[\tau]e^{j2\pi\tau k/mN}\}_{\tau=0}^{mN-1}$, $k = 1, \ldots, m-1$, respectively.

Proof: Let $\mathbf{z}_k = \{z_k[n]\}_{n=0}^{mN-1}$. Given that $\mathbf{z}_k = \mathbf{z}_0 \circ \mathbf{c}_k$, it has $z_k[n] = z_0[n]e^{j2\pi nk/mN}$. We have

$$\begin{aligned}R_k[\tau] &= \sum_{n=0}^{mN-1} z_k[n]z_k^*[(n-\tau)_{mN}] \\ &= \sum_{n=0}^{mN-1} z_0[n]e^{\frac{j2\pi nk}{mN}} z_0^*[(n-\tau)_{mN}]e^{\frac{-j2\pi(n-\tau)k}{mN}} \\ &= e^{j2\pi\tau k/mN}\sum_{n=0}^{mN-1} z_0[n]z_0^*[(n-\tau)_{mN}] \\ &= R_0[\tau]e^{j2\pi\tau k/mN}.\end{aligned}$$

∎

Let $\mathbf{R}_t = \{R_t[\tau]\}_{\tau=0}^{N-1}$ denote the PACF of base sequence $\mathbf{s}_t$, where $R_t[\tau] = \sum_{n=0}^{N-1} s_t[n]s_t^*[(n-\tau)_N]$. The relationship between $R_0[\tau]$ and $R_t[\tau]$ is described in *Corollary 4* below.

Corollary 4: $R_0[\tau + lN] = R_0[\tau] = m \cdot r_t[\tau]$, where $0 \le \tau \le N-1$ and $0 \le l \le m-1$.

Proof: First, since $z_0[\tau + lN] = s_t[\tau]$, $0 \le \tau \le N-1$ and $0 \le l \le m-1$, it has

$$\begin{aligned}R_0[\tau] &= \sum_{n=0}^{mN-1} z_0[n]z_0^*[(n-\tau)_{mN}] \\ &= \sum_{l=0}^{m-1}\sum_{n=0}^{N-1} z_0[n+lN]z_0^*[(n+lN-\tau)_{mN}] \\ &= m \cdot \sum_{n=0}^{N-1} s_t[n]s_t^*[(n-\tau)_N] \\ &= m \cdot R_t[\tau].\end{aligned}$$

Second, because $z_0[\tau + lN] = z_0[\tau]$, it is straightforward to show that $R_0[\tau + lN] = R_0[\tau]$ is true, where $0 \le l \le m-1$. ∎

Note that when the base sequence $\mathbf{s}_t$ is a PS or PGIS with energy $E_N = \sum_{n=0}^{N-1} |s_t[n]|^2$, then the PACF of $\mathbf{s}_t$ is $\mathbf{R}_t = E_N\delta_{\mathbf{N}}$ and the absolute auto-correlation values of all other PACFs $\mathbf{R}_k = \{R_0[\tau]e^{j2\pi\tau k/mN}\}_{\tau=0}^{mN-1}$ are the same, which can be expressed as $|R_0[\tau]e^{j2\pi\tau k/mN}| = mE_N$.

Let us present the extreme case $\mathbf{s}_t = 1$ for demonstration. By coping and cascading $\mathbf{s}_t = 1$ m times, we obtain a sequence $\mathbf{z}_0 = (1, 1, \ldots, 1)$ of period m. By *Corollary 3*, the sequences in set $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}\}$ are presented in *Example2*.

Example 2:

$$\mathbf{z}_k = (1, e^{j\frac{2\pi k}{m}}, e^{j\frac{2\pi 2k}{m}}, \ldots, e^{j\frac{2\pi k(m-1)}{m}}),\ k = 0, 1, \ldots, m-1.$$

The set of sequences in *Example2* is a set of ZCC sequences, where $\mathbf{z}_k \otimes_c \mathbf{z}_n = \mathbf{0}$, $0 \le k, n \le m-1$, $k \neq n$.

Construction II: Time domain ZCC sequences based on a set of base sequences

Let $\{\mathbf{s}_n\}_{n=0}^{m-1}$ be a set of base sequences of period N. All of these sequences are repeated m times as in the case of a single-base sequence in **Construction type I**. This process results in a set of m sequences of period mN denoted as $\{\mathbf{s}'_\mathbf{n}\}_{n=0}^{m-1}$, where

$$\mathbf{s}'_\mathbf{n} = (\underbrace{\mathbf{s}_n, \mathbf{s}_n, \ldots, \mathbf{s}_n}_{m \text{ times}}),\quad n = 0, 1, \ldots, m-1.$$

The sequences $\mathbf{s}'_\mathbf{n}$ are then used to modulate a set of digital carriers $\{\mathbf{c}_k\}_{k=0}^{m-1}$, and generate a set of ZCC sequences.

The set of two ZCC sequences $\mathbf{A}_1 = \{\mathbf{s}, \mathbf{s}_1\} = \{(1, 0, 1, 0), (1, 0, -1, 0)\}$ of period $N = 4$ can be applied as the base sequences to construct a set of ZCC sequences of period $2N = 8$ with cardinality 4. This is shown in **Example 3**. Let $\mathbf{c}_1 = \{e^{j2\pi n/8}\}_{n=0}^{7}$. It has

$$\begin{aligned} \mathbf{s}'_0 &= (\mathbf{s}, \mathbf{s}) = (1, 0, 1, 0, 1, 0, 1, 0), \\ \mathbf{s}'_1 &= (\mathbf{s}_1, \mathbf{s}_1) = (1, 0, -1, 0, 1, 0, -1, 0), \\ \mathbf{s}'_2 &= \mathbf{s}'_0 \circ \mathbf{c}_1 = (1, 0, j, 0, -1, 0, -j, 0), \\ \mathbf{s}'_3 &= \mathbf{s}'_1 \circ \mathbf{c}_1 = (1, 0, -j, 0, -1, 0, j, 0). \end{aligned}$$

Example 3: $\mathbf{A}'_1 = \{\mathbf{s}'_0, \mathbf{s}'_1, \mathbf{s}'_2, \mathbf{s}'_3\}$ is a set of ZCC sequences with cardinality 4, where $Dim(\mathbf{V_B}(\mathbf{s}'_n)) = 2, n = 0, 1, 2, 3$, and $\sum_{n=0}^{3} Dim(\mathbf{V_B}(\mathbf{s}'_n)) = 8$.

B. FREQUENCY DOMAIN ZCC SEQUENCES

Construction III: Frequency domain ZCC sequences based on single base sequence

The frequency domain ZCC sequences construction is based on *Theorem 2*. The base sequence $\mathbf{s}_t$ in (9) is first upsampled by a factor of m to generate a sequence of period mN, denoted by $\mathbf{s}_0$. Sequence $\mathbf{s}_0$ and its DFT $\mathbf{S}_0$ are expressed as follows:

$$\mathbf{s}_0 = (\underbrace{s_t[0], 0, \ldots, 0}_{m}, \underbrace{s_t[1], 0, \ldots, s_t[N-1], 0, \ldots, 0}_{(N-1)m}), \tag{13}$$

$$\mathbf{S}_0 = (\underbrace{S_t[0], \ldots, S_t[N-1]}_{N}, \underbrace{\ldots, S_t[0], \ldots, S_t[N-1]}_{(m-1)N}). \tag{14}$$

In (13), the number of zero elements between two adjacent $s_t[k]$ is $m-1$, thus it has m distinct circular shifts of $\mathbf{s}_t$, denoted by $\{\mathbf{s}_0^{(k)}\}_{k=0}^{m-1}$, which can match the requirement of no overlapping nonzero elements among themselves in this set to make $\{\mathbf{S}_k\}_{k=0}^{m-1}$ be ZCC sequences with cardinality m.

Theorem 9: Let $\mathbf{S}_k$ denote the DFT of $\mathbf{s}_0^{(k)}$, $k = 0, 1, \ldots, m-1$. $\{\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_{m-1}\}$ is a set of ZCC sequences in the frequency domain with cardinality m.

Proof: As the arbitrary pair of two different sequences in the set $\{\mathbf{S}_0, \mathbf{S}_1, \ldots, \mathbf{S}_{m-1}\}$ possesses the ZCC property, that is, $\mathbf{S}_k \otimes_c \mathbf{S}_n = \mathbf{0}, 0 \le k, n \le m-1, k \ne n$. This gives the proof of *Theorem 9*. ■

Example 4: Let $\mathbf{s}_k = \delta_m^{(k)}, k = 0, 1, \ldots, m-1$.

$$\mathbf{S}_k = (1, e^{j\frac{2\pi k}{m}}, \ldots, e^{j\frac{2\pi k(m-1)}{m}}), k = 0, 1, \ldots, m-1,$$

where $\mathbf{S}_k \otimes_c \mathbf{S}_n = \mathbf{0}, 0 \le k, n \le m-1, k \ne n$. *Example*4 is the *time-frequency duality* example of that shown in *Example*2.

Construction IV: Frequency domain ZCC sequences based on a set of base sequences

Let $\{\mathbf{s}_n\}_{n=0}^{m-1}$ be a set of base sequences of period N. These m sequences are first upsampled by a factor of m to generate a set of m sequences $\{\mathbf{s}_n^u\}_{n=0}^{m-1}$ of period mN. Let $\mathbf{s}_n^{u(n)}$ be the circular shift of $\mathbf{s}_n^u$ to the right by n places. The DFTs of set of sequences $\{\mathbf{s}_n^{u(n)}\}_{n=0}^{m-1}$, denoted by $\{\mathbf{S}_n^u\}_{n=0}^{m-1}$, possess the ZCC property in the frequency domain, that is, $\mathbf{S}_n^u \otimes_c \mathbf{S}_k^u = \mathbf{0}$, $k \ne n$.

For comparison with *Example 3*, $\mathbf{A}_1 = \{\mathbf{s}, (1, 0, -1, 0)\}$ from *Example 1* is applied to construct ZCC sequences. It has $\mathbf{s}_0^u = (1, 0, 0, 0, 1, 0, 0, 0)$, $\mathbf{s}_1^{u(1)} = (0, 1, 0, 0, 0, -1, 0, 0)$, $\mathbf{s}_2^{u(2)} = (0, 0, 1, 0, 0, 0, 1, 0)$ and $\mathbf{s}_3^{u(3)} = (0, 0, 0, 1, 0, 0, 0, -1)$. The associative DFTs are $\mathbf{S}_0^u = 2(1, 0, 1, 0, 1, 0, 1, 0)$, $\mathbf{S}_1^u = \sqrt{2}(0, 1-j, 0, -1-j, 0, -1+j, 0, 1+j)$, $\mathbf{S}_2^u = 2(1, 0, -1, 0, 1, 0, -1, 0)$, and $\mathbf{S}_3^u = \sqrt{2}(0, -1-j, 0, 1-j, 0, 1+j, 0, -1+j)$, respectively.

Example 5: $\mathbf{A}'_2 = \{\mathbf{S}_0^u, \mathbf{S}_1^u, \mathbf{S}_2^u, \mathbf{S}_3^u\}$ is a set of ZCC sequences, where $Dim(\mathbf{V_B}(\mathbf{S_n^u})) = 2$, $n = 0, 1, 2, 3$.

A set of ZCC sequences $\mathbf{A}'_1$ from *Example 3* and $\mathbf{A}'_2$ are isomorphic to each other.

V. APPLICATION OF ZCC SEQUENCES TO FSC ESTIMATION

A. FSC MODEL

Let us consider a single-input single-output (SISO) FSC with bandwidth W_0. The baseband sampled (at $T_s = 1/W_0$ intervals) channel impulse response is represented by $h[l](l = 0, 1, 2, \ldots, N-1)$. The response $\mathbf{y} = \{y[n]\}$ of this channel to an input sequence $\mathbf{z} = \{z[n]\}$ is given by

$$y[n] = \sum_{k=0}^{N-1} h[k]z[n-k] + v[n],$$

where the sequence $\{v[n]\}$ represents the additive noise and interference. It is convenient to model the noise $v[n]$ as a zero mean, complex-valued, circularly symmetric stationary white Gaussian process with variance $E[|v[n]|^2] = N_0 W_0 = \sigma^2$, where W_0 is the underlying system bandwidth.

B. APPLICATION OF ZCC SEQUENCES TO FSC ESTIMATION

The combination of the ZCC sequences and cyclic prefix (CP) technique is used to estimate the channel parameters $\{h[l]\}$. The CP comprises the last $N-1$ elements of $\mathbf{z}_0$, denoted as $\mathbf{z}_{c0}$. Given that the input is a vector of size $((m+1)N-1) \times 1$, $\mathbf{z} = [\mathbf{z}_{c0}\ \mathbf{z}_0]_{((m+1)N-1)\times 1}$, where $\mathbf{z}_0$ can be calculated from (11), the response of FSC in vector form can be written as follows:

$$\mathbf{y} = \mathbf{h}_0 \otimes_l \mathbf{z} + \mathbf{v} = \mathbf{H}_y \mathbf{z} + \mathbf{v}. \tag{15}$$

In (15), $\otimes_l$ denotes linear convolution, $\mathbf{h}_0 = [h[0]\ h[1]\ \cdots\ h[N-1]]^T$, the channel matrix $\mathbf{H}_y$, and associated elements $\mathbf{H}_0, \mathbf{H}_1, \mathbf{H}_2, \mathbf{H}_3$ and $\mathbf{H}_4$ are given in (17)-(22), respectively. In this scheme, the base sequence $\mathbf{s}_t$ in (9) is a PS or PGIS with energy $E_N = \sum_{n=0}^{N-1} |s_t[n]|^2$. The PACF of $\mathbf{z}_0$ is expressed as

$$\begin{aligned} &\mathbf{z}_0 \otimes_c \mathbf{z}^*_{-0} \\ &= mE_N(\underbrace{1, 0, \cdots, 0}_{N}, \underbrace{1, 0, \ldots, 0 \ldots, 1, 0, \ldots, 0}_{(m-1)N}). \end{aligned} \tag{16}$$

$$\mathbf{H}_y = \begin{bmatrix} \mathbf{H}_0 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_1 & \mathbf{H}_2 & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_3 & \ddots & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \mathbf{H}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_3 & \mathbf{H}_2 \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_4 \end{bmatrix}_{((m+2)N-2)\times ((m+1)N-1)} \tag{17}$$

$$\mathbf{H}_0 = \begin{bmatrix} h[0] & 0 & \cdots & 0 & 0 \\ h[1] & h[0] & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ h[N-3] & h[N-4] & \cdots & h[0] & 0 \\ h[N-2] & h[N-3] & \cdots & h[1] & h[0] \end{bmatrix}_{(N-1)\times(N-1)} \tag{18}$$

$$\mathbf{H}_1 = \begin{bmatrix} h[N-1] & h[N-2] & \cdots & h[1] \\ 0 & h[N-1] & \cdots & h[2] \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & h[N-1] & h[N-2] \\ 0 & \cdots & 0 & h[N-1] \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times(N-1)} \tag{19}$$

$$\mathbf{H}_2 = \begin{bmatrix} h[0] & 0 & \cdots & 0 & 0 \\ h[1] & h[0] & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ h[N-2] & h[N-3] & \cdots & h[0] & 0 \\ h[N-1] & h[N-2] & \cdots & h[1] & h[0] \end{bmatrix}_{N\times N} \tag{20}$$

$$\mathbf{H}_3 = \begin{bmatrix} 0 & h[N-1] & h[N-2] & \cdots & h[1] \\ 0 & 0 & h[N-1] & \ddots & h[2] \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & h[N-1] & h[N-2] \\ 0 & 0 & \cdots & 0 & h[N-1] \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N} \tag{21}$$

$$\mathbf{H}_4 = \begin{bmatrix} 0 & h[N-1] & h[N-2] & \cdots & h[1] \\ 0 & 0 & h[N-1] & \ddots & h[2] \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & h[N-1] & h[N-2] \\ 0 & 0 & \cdots & 0 & h[N-1] \end{bmatrix}_{(N-1)\times N} \tag{22}$$

$$\mathbf{H}_{y'} = \begin{bmatrix} \mathbf{H}_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_3 \\ \mathbf{H}_3 & \mathbf{H}_2 & \ddots & \vdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_3 & \ddots & \mathbf{0} & \vdots \\ \vdots & \vdots & \ddots & \mathbf{H}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_3 & \mathbf{H}_2 \end{bmatrix}_{mN\times mN} \tag{23}$$

The receiver receives a vector $\mathbf{y} = \{y[n]\}_{n=-(N-1)}^{(m+1)N-2}$ of length $(m+2)N-2$, where the number of these elements is calculated from $\mathbf{z}$ of length $(m+1)N-1$ convoluted with the channel $\mathbf{h}_0$ of length N. To process the channel estimation, the receiver removes the first $N-1$ and collects the next mN samples from the received signal $\mathbf{y}$ to obtain a data vector $\mathbf{y}' = [y[0]y[1]\cdots y[mN-1]]^T$ of size $mN\times 1$. By performing circular convolution with $\mathbf{z}^*_{-0}$, the data vector $\mathbf{y}'$ can be used to estimate the channel vector $\mathbf{h} = [\mathbf{h}_0^T\ \mathbf{h}_0^T \cdots \mathbf{h}_0^T]^T$ of size $mN\times 1$, where $\mathbf{h}_0 = [h[0]\ h[1]\ \cdots\ h[N-1]]$. The estimation of $\mathbf{h}$ is given as follows:

$$\begin{aligned} \hat{\mathbf{h}}_{ZCC-E} &= \mathbf{y}' \otimes_c \mathbf{z}^*_{-0} \\ &= [\mathbf{y}'\ \mathbf{y}'^{(1)}\ \cdots\ \mathbf{y}'^{(mN-1)}]\mathbf{z}^*_{-0} \\ &= \mathbf{H}_{y'}[\mathbf{z}_0\ \mathbf{z}_0^{(1)}\ \cdots\ \mathbf{z}_0^{(mN-1)}]\mathbf{z}^*_{-0} + \mathbf{v}' \\ &= \mathbf{H}_{y'}(\mathbf{z}_0 \otimes_c \mathbf{z}^*_{-0}) + \mathbf{v}' \\ &= \mathbf{H}_{y'}[\underbrace{mE_N\ 0\ \cdots\ 0}_{N} \\ &\quad \underbrace{mE_N\ 0\cdots 0 \cdots mE_N\ 0\cdots 0}_{(m-1)N}]^T + \mathbf{v}' \\ &= mE_N\mathbf{h} + \mathbf{v}', \end{aligned} \tag{24}$$

where the channel matrix $\mathbf{H}_{y'}$ is given in (23), and $\mathbf{v}' = [v[0]\ v[1]\ \cdots\ v[mN-1]]^T \otimes_c \mathbf{z}^*_{-0}$ is the estimation error vector. The component of $\mathbf{v}' = \{v'[n]\}_{n=0}^{mN-1}$ is given as follows:

$$v'[n] = \sum_{i=0}^{mN-1} v[i]z^*[(i-n)_{mN}]. \tag{25}$$

The estimation error has zero mean, which means $E(\sum_{i=0}^{mN-1} v[i]z^*[(i-n)_{mN}]) = \sum_{i=0}^{mN-1} E(v[i])z^*[(i-n)_{mN}] = 0$, and the power (variance) of the estimation error is derived as

$$\begin{aligned} var(v'[n]) &= E(\sum_{i=0}^{mN-1} v[i]z^*[(i-n)_{mN}])^2 \\ &= E(\sum_{k=0}^{mN-1} v[(n+k)_{mN}]z^*[k])^2 \\ &= \sum_{k=0}^{mN-1} E(|v[(n+k)_{mN}]|^2)|z[k]|^2 \\ &= \sigma^2 mE_N. \end{aligned} \tag{26}$$

The signal to noise ratio (SNR) gain, which can be achieved by applying ZCC to the estimation of each channel parameter $\{h[l]\}$ is mE_N. This is given as follows:

$$SNR = \frac{(mE_N)^2|h[l]|^2}{\sigma^2 mE_N} = mE_N\frac{|h[l]|^2}{\sigma^2}.$$

C. OPTIMAL JOINT SYMBOL DETECTION AND FSC ESTIMATION

Orthogonality between the symbol stream and the training sequence is the requirement for optimal joint symbol detection and channel estimation. Before applying the ZCC sequences to achieve this goal, three related theorems are addressed, which are *Lemma 2* and *5*, and *Theorem 10*. The set of mN independent vectors $\{\mathbf{c}_i\}_{i=0}^{mN-1}$, where $\mathbf{c}_i = \{e^{j2\pi in/mN}\}_{n=0}^{mN-1}$, can serve as a basis for the ordered mN-tuples of complex numbers $\mathcal{C}^{mN}$. A base matrix $\mathbf{B}' = [\mathbf{c}_0 \;\; \mathbf{c}_1 \;\; \cdots \;\; \mathbf{c}_{mN-1}]_{mN\times mN}$, is defined, and the coordinate vector of the sequence $\mathbf{z}$ based on $\mathbf{B}'$ is expressed as $\mathbf{V}_{\mathbf{B}'}(\mathbf{z})$.

Lemma 4: Based on the base matrix $\mathbf{B}'$, the spanning set of sequence $\mathbf{z}_0$ in (11), denoted by $span(\mathbf{z}_0)$, can be written as $span(\mathbf{z}_0) = \{\mathbf{c}_0, \mathbf{c}_m, \ldots, \mathbf{c}_{m(N-1)}\} = \{\mathbf{c}_{nm}\}_{n=0}^{N-1}$.

Proof: The DFT of $\mathbf{z}_0$ is given in (12), in which $\mathbf{Z}_0 = m(\underbrace{S_t[0], 0, \ldots, 0}_{m}, \underbrace{S_t[1], 0, \ldots, 0}_{m}, \ldots, \underbrace{S_t[N-1], \ldots, 0}_{m})$. When applying the IDFT operation upon $\mathbf{Z}_0$, a new expression of $\mathbf{z}_0$, which is different from that shown in (11), i.e., $\mathbf{z}_0 = \frac{1}{N}(\sum_{n=0}^{N-1} S_t[n]\mathbf{c}_{nm}) = \mathbf{B}'\mathbf{V}_{\mathbf{B}'}(\mathbf{z}_0)$, can be derived. Here $\mathbf{V}_{\mathbf{B}'}(\mathbf{z}_0) = \frac{1}{mN}\mathbf{Z}_0$ is the coordinate vector of $\mathbf{z}_0$ with respect to the base matrix $\mathbf{B}'$. Since $\mathbf{z}_0$ can be represented by the linear combination of vectors in the subset of vectors $\{\mathbf{c}_{nm}\}_{n=0}^{N-1}$ of the base matrix $\mathbf{B}'$, it can be derived that $span(\mathbf{z}_0) = \{\mathbf{c}_{nm}\}_{n=0}^{N-1}$. ■

First, let us define a Toeplitz matrix $\mathbf{T}$ using the sequence $\mathbf{z}_0$, which is defined in (11). Given that the dimension of $span(\mathbf{z}_0)$ is N, the nullity of $\mathbf{T}$ is $(m-1)N$. From this perspective, a joint symbol detection and channel estimation scheme can be applied when using $\mathbf{z}_0$ as the training sequence, which was analyzed in Section V-B. In this scheme, the maximum number of symbols in the data stream $\{d[n]\}$ which can be transmitted simultaneously with the training sequence $\mathbf{z}_0$ of period mN is $(m-1)N$. This scheme can be operated by first grouping the data stream into a set of $m-1$ data sequences of length N. These data sequences are $\mathbf{d}_i = \{d[n]\}_{n=(i-1)N}^{iN-1}$, $i = 1, 2, \ldots, m-1$, and are then used to form a set of $m-1$ sequences of length mN denoted as $\mathbf{D}_i = \{D_i[n]\}_{n=0}^{mN-1}$, respectively, by coping and cascading $\mathbf{d}_i$ m times.

Second, to meet the requirement of orthogonality between the data stream and training sequence for optimal joint symbol detection and channel estimation performance, all transmitted symbol sequences, which are denoted as $\{\mathbf{x}_i\}$, should belong to the null space of the Toeplitz matrix $\mathbf{T}$, which is generated using $\mathbf{z}_0$. **Construction type II** can be applied to achieve this goal, which is described in *Lemma 5* and *Theorem 10* shown below.

Lemma 5: Given that $\mathbf{x}_i = \{D_i[n]e^{j2\pi ni/mN}\}_{n=0}^{mN-1}$, $i = 1, 2, \ldots, m-1$, the spanning set of $\mathbf{x}_i$ is $span(\mathbf{x}_i) = \{\mathbf{c}_{nm+i}\}_{n=0}^{N-1}$ and $\mathbf{x}_i = \mathbf{x}_i \otimes_c (\frac{1}{mN}\sum_{n=0}^{N-1}\mathbf{c}_{nm+i})$, $i = 1, 2, \ldots, m-1$.

Proof: The proof of $span(\mathbf{x}_i) = \{\mathbf{c}_{nm+i}\}_{n=0}^{N-1}$ is similar to the proof of *Lemma 4*. Hence, for brevity, it is omitted here. Given that $span(\mathbf{x}_i) = \{\mathbf{c}_{nm+i}\}_{n=0}^{N-1}$, $\mathbf{x}_i$ is a linear combination of $\{\mathbf{c}_{nm+i}\}_{n=0}^{N-1}$, i.e., $\mathbf{x}_i = \sum_{n=0}^{N-1} a_n\mathbf{c}_{nm+i}$. Since $\mathbf{c}_n \otimes_c \mathbf{c}_k = \mathbf{0}$ and $\mathbf{c}_n \otimes_c \mathbf{c}_n = mN \cdot \mathbf{c}_n$, it can be derived that $\mathbf{x}_i \otimes_c (\frac{1}{mN}\sum_{k=0}^{N-1}\mathbf{c}_{km+i}) = (\sum_{n=0}^{N-1} a_n\mathbf{c}_{nm+i}) \otimes_c (\frac{1}{mN}\sum_{k=0}^{N-1}\mathbf{c}_{km+i}) = \sum_{n=0}^{N-1} a_n\mathbf{c}_{nm+i} = \mathbf{x}_i$. ■

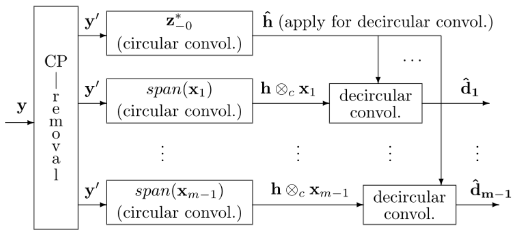


FIGURE 1. ZCC-based joint symbol detection and channel estimation.

Theorem 10: $\{\mathbf{z}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{m-1}\}$ is a set of ZCC sequences.

Proof: Since $\mathbf{B}' = \{\mathbf{c}_i\}_{i=0}^{mN-1}$ is an orthogonal basis for $\mathcal{C}^{mN}$ and $\mathbf{B}' = span(\mathbf{z}_0) \oplus span(\mathbf{x}_1) \oplus span(\mathbf{x}_2) \oplus \cdots \oplus span(\mathbf{x}_{m-1})$, where $\oplus$ denotes the *directsum* operation, the DFTs of any two different transmitted symbols $\mathbf{x}_i$ and $\mathbf{x}_k$ contain no overlapping elements between each other for which $\mathbf{x}_i \otimes_c \mathbf{x}_k = \mathbf{0}$, for $1 \le i, k \le m-1$. It is also true that $\mathbf{x}_i \otimes_c \mathbf{z}_0 = \mathbf{0}$, $i = 1, 2, \ldots, m-1$. It can be concluded that $\{\mathbf{z}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{m-1}\}$ is a set of m ZCC sequences. ■

Since the channel has N taps, the last $N-1$ elements of all data sequences $\mathbf{x}_i$ denoted as $\mathbf{x}_{ci} = \{x_i[n]\}_{n=(m-1)N+1}^{mN-1}$, should be inserted into the associated $\mathbf{x}_i$ as the CP part, $i = 1, 2, \ldots, m-1$. The final transmitted symbol sequences are denoted as $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_{m-1}\}$ where $\mathbf{X}_i = [\mathbf{x}_{ci} \;\; \mathbf{x}_i]_{((m+1)N-1)\times 1}$. The purpose of inserting $\mathbf{x}_{ci}$ as CPs is to transform the linear convolution operation encountered at the FSC channel into a circular convolution operation at the receiver end for easy symbol detection processing. The FSC response in vector form is expressed as follows:

$$\begin{aligned}\mathbf{y} &= \mathbf{h} \otimes_l (\mathbf{z} + \sum_{i=1}^{m-1}\mathbf{X}_i) + \mathbf{v}\\ &= \mathbf{h} \otimes_l \mathbf{x} + \mathbf{v},\end{aligned} \tag{27}$$

The structure of this ZCC-based joint symbol detection and channel estimation scheme is presented in Fig. 1. In this figure m branches are used to process both the channel estimate $\hat{\mathbf{h}}$ and symbol detection $\hat{\mathbf{d}}_i = \{\hat{d}[n]\}_{n=(i-1)N}^{iN-1}$, $i = 1, 2, \ldots, m-1$, respectively. At the receiver end, the CP is first removed from $\mathbf{y}$ to obtain $\mathbf{y}' = \{y[n]\}_{n=0}^{mN-1}$. Assuming orthogonality between the training sequence $\mathbf{z}_0$ and $\mathbf{x}_i$, the channel parameter estimation can be easily obtained from the top branch of Fig. 1 by performing circular convolution of the $\mathbf{y}'$ and $\mathbf{z}^*_{-0}$. Next, the derived $\hat{\mathbf{h}}$ can be substituted into the rest of the $m-1$ branches to serve as the unknown $\mathbf{h}$ for circular deconvolution operation to derive $\hat{\mathbf{d}}_i$, $i = 1, 2, \ldots, m-1$, respectively. The process of deriving the unknown $\hat{\mathbf{d}}_i$ at the i^{th} branch is analyzed below. When the data vector $\mathbf{y}'$ is circularly convoluted with all sequences $\mathbf{c}_{nm+i} \in span(\mathbf{x}_i)$, the interference from other undesired data streams $\mathbf{x}_n$, for

$\forall n \neq i$, will not appear at the output of this branch. Let us define $\mathbf{C}_i = \sum_{n=0}^{N-1} \mathbf{c}_{nm+i}$. This implies that

$$\begin{aligned}
&\mathbf{y}' \otimes_c \mathbf{C}_i \\
&= [\mathbf{y}' \ \mathbf{y}'^{(1)} \ \cdots \ \mathbf{y}'^{(mN-1)}]\mathbf{C}_i + \mathbf{v}' \otimes_c \mathbf{C}_i \\
&= \mathbf{H}_{y'}[\mathbf{x}' \ \mathbf{x}'^{(1)} \ \cdots \ \mathbf{x}'^{(mN-1)}]\mathbf{C}_i + \mathbf{v}' \otimes_c \mathbf{C}_i \\
&= \mathbf{H}_{y'}(\mathbf{x} \otimes_c \mathbf{C}_i) + \mathbf{v}' \otimes_c \mathbf{C}_i \\
&= \mathbf{H}_{y'}(\mathbf{x}_i \otimes_c \mathbf{C}_i) + \mathbf{v}' \otimes_c \mathbf{C}_i \\
&= mN \cdot \mathbf{H}_{y'}\mathbf{x}_i + \mathbf{v}' \otimes_c \mathbf{C}_i. \qquad (28)
\end{aligned}$$

The detection of $\mathbf{x}_i$ and $\mathbf{d}_i$ using (28) can be derived from the following two equations:

$$\begin{aligned}
\hat{\mathbf{x}}_i &= \hat{\mathbf{H}}_{y'}^{-1}(\mathbf{y}' \otimes_c \mathbf{C}_i) \\
&= mN \cdot \mathbf{x}_i + \hat{\mathbf{H}}_{y'}^{-1}(\mathbf{v}' \otimes_c \mathbf{C}_i), \qquad (29)\\
\hat{\mathbf{d}}_i &= \hat{\mathbf{x}}_i \circ \mathbf{c}_i^*, \qquad (30)
\end{aligned}$$

where $\hat{\mathbf{H}}_{y'}$ is obtained from $\mathbf{H}_{y'}$ by substituting $\mathbf{h}$ with $\hat{\mathbf{h}}$ in (24).

When the $N-1$ CP bits do not count, the total number of transmitted symbols is mN, which consists of N training symbols and $(m-1)N$ data sequences $\{d[n]\}$. Then, the bandwidth efficiency can be calculated from $\frac{(m-1)N}{mN} = \frac{(m-1)}{m}$. When the CP part counts, the efficiency becomes $\frac{(m-1)N}{(m+1)N-1} \approx \frac{m-1}{m+1}$. The efficiency approaches a high value, when m is large in which $\lim_{m\to\infty} \frac{m-1}{m} \to 1$ and $\lim_{m\to\infty} \frac{m-1}{m+1} \to 1$.

D. APPLICATION OF ZCC SEQUENCES TO MULTIUSER FSC ESTIMATION

The number of independent ZCC sequences in the set $\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}\}$ is m. These ZCC sequences can be applied to estimate m different FSCs simultaneously in a multiuser scenario or a MISO system, where the unknown channel impulse responses are denoted as $h_i[l](l = 0, 1, 2, \ldots, N-1)$, $i = 0, 1, \ldots, m-1$. The similar CP technique addressed in Section 5.2 can still be applied where the last number of the $N-1$ bits from the $\mathbf{z}_i$ is inserted into the training ZCC sequence $\mathbf{z}_i$ of the i user. The overall received signal becomes as follows:

$$\mathbf{y} = \sum_{k=0}^{m-1} \mathbf{h}_k \otimes_l \mathbf{z}_{k'} + \mathbf{v}, \qquad (31)$$

where $\mathbf{z}_{k'} = [\mathbf{s}_{c'}^T \ \mathbf{z}_{k'}^T]^T$, in which $\mathbf{s}_{c'}^T$ is the associated CP of the k^{th} user $\mathbf{z}_{k'}$. First, the receiver removes the CP to obtain $\mathbf{y}'$ as a single user does. Then, this $\mathbf{y}'$ is circularly convoluted with $\mathbf{z}_{-i}^* \circ \mathbf{c}_k^*$ at m different branches, and the individual channel estimate $\hat{\mathbf{h}}_i$, $i = 0, 1, \ldots, m-1$, is obtained. Let us present only the processing of the i^{th} user as follows. First, it can be derived that

$$\begin{aligned}
\mathbf{y}' \otimes_c \mathbf{z}_{-i}^* &= \sum_{k=0}^{m-1}(\mathbf{h}_k \otimes_l \mathbf{z}_{k'}) \otimes_c \mathbf{z}_{-i}^* + \mathbf{v}' \\
&= \sum_{k=0}^{m-1} \mathbf{H}_{ky'}[\mathbf{z}_k \ \mathbf{z}_k^{(1)} \ \cdots \ \mathbf{z}_k^{(mN-1)}]\mathbf{z}_{-i}^* + \mathbf{v}' \\
&= \sum_{k=0}^{m-1} \mathbf{H}_{ky'}(\mathbf{z}_k \otimes_c \mathbf{z}_{-i}^*) + \mathbf{v}' \\
&= \mathbf{H}_{iy'}(\mathbf{z}_i \otimes_c \mathbf{z}_{-i}^*) + \mathbf{v}', \qquad (32)
\end{aligned}$$

where $\mathbf{H}_{iy'} = [\mathbf{h}_i \ \mathbf{h}_i^{(1)} \ \cdots \ \mathbf{h}_i^{(mN-1)}]$ and $\mathbf{v}' = [v[0] \ v[1] \ \cdots \ v[mN-1]] \otimes_c \mathbf{z}_{-i}^*$. Second, from *Theorem 8* it can be derived

$$\mathbf{z}_i \otimes_c \mathbf{z}_{-i}^* = mE_N[\underbrace{1 \ 0 \ \cdots \ 0}_{N} \ \underbrace{e^{\frac{j2\pi iN}{mN}} \ 0 \cdots 0 \cdots e^{\frac{j2\pi(m-1)N}{mN}} \ 0 \cdots 0}_{(m-1)N}]^T. \qquad (33)$$

Thus, the estimate of $\mathbf{h}_i$, denoted as $\hat{\mathbf{h}}_i$, is given as follows:

$$\begin{aligned}
\hat{\mathbf{h}}_i &= (\mathbf{y}' \otimes_c \mathbf{z}_{-i}^*) \circ \mathbf{c}_k^* \\
&= mE_N\mathbf{h}_i + \mathbf{v}' \circ \mathbf{c}_k^*, \qquad (34)
\end{aligned}$$

where $\mathbf{v}' \circ \mathbf{c}_k^*$ is the estimation vector, which has also zero mean and the same variance as that shown in (26).

VI. APPLICATION OF ZCC SEQUENCES TO TSC ESTIMATION

A. TSC MODEL

Since the environmental medium between the transmitter and receiver varies, especially in wireless mobile communications, the channel response varies with time, giving rise to a TSC with the (normalized) Doppler-spread of the channel limited to the interval $[-f_{max}, f_{max}]$. Let us select the time instants $n = 0, T_s, 2T_s, \ldots, (N-1)T_s$, where T_s is the sampling period, as an observation window and collect all the channel gains $h_t[n]$ within this window in the vector $\mathbf{h}_t = [h_t[0] \ h_t[1] \ \cdots \ h_t[N-1]]^T$. The response $\mathbf{y} = \{y[n]\}$ of the TSC to the input sequences $\mathbf{s} = \{s[n]\}_{n=0}^{N-1}$ is given by $y[n] = s[n] \cdot h_t[n] + v[n], n = 0, 1, \ldots, N-1$, which in vector form can be expressed as follows:

$$\mathbf{y} = \mathbf{h}_t \circ \mathbf{s} + \nu. \qquad (35)$$

Equation (35) indicates that the number of channel gains $h_t[n]$, which can be unknown and might not be estimated in advance due to the time variation of the channel, is the same as the available number of measurements $y[n]$. A joint symbol detection and channel estimation scheme, in which the input sequences $\{s[n]\}$ are constructed by multiplexing both training and data symbols in the time domain, is of practical importance. However, channel coefficient identification in this scheme is challenging because this is considered as an underdetermined problem. This problem can be overcome by applying the parsimonious basis expansion models (BEMs) to approximate $h_t[n]$ during any time interval of the observation window. Examples of such BEMs are the complex exponential BEM (CE-BEM) [38], [39], the polynomial BEM (P-BEM) [40], the discrete prolate spheroidal BEM (DPS-BEM) [41], and so on. By selecting a proper scale $Q(= 2\lceil f_{max}NT_s\rceil \ll N)$, where $\lceil\cdot\rceil$ is the integer ceiling and

$Q+1$ is the number of bases for the BEM, the TSC estimation of the unknown channel gains $\{h_t[n]\}_{n=0}^{N-1}$ is reduced to estimate the $Q+1$ BEM coefficients. From these coefficients, the true channel gains $h_t[n]$ for symbol detection within the same observation window can be approximately built. The $h_t[n]$ model based on the complex DFT bases (CE-BEM) is given as [38], [39]

$$h_t[n] = \sum_{q=0}^{Q} \lambda_q e^{j2\pi(q-Q/2)n/N} + \varepsilon[n], \tag{36}$$

where $0 \le n \le N-1$, $\varepsilon[n]$ is the model error, and λ_q's are the CE-BEM coefficients, which remain invariant within the observation window but change in the next interval.

B. JOINT SYMBOL DETECTION AND TSC ESTIMATION

In this paper, frequency domain ZCC sequences are used as the training sequences, which are interleaved with the transmitted information symbols, for estimating the channel gains $h_t[n]$ directly rather than the CE-BEM coefficients of (36), as described in the sequel.

Let $\mathbf{s}_p$ be a PGIS of period p, which is upsampled by a factor m to generate a PGIS of period $N = mp$, denoted as $\mathbf{s}_u$. The set of vectors $\{\mathbf{s}_u^{(k)}\}_{k=0}^{m-1}$ represents the m distinct circular shifts of $\mathbf{s}_u$, which these vectors are also PGISs. In this scheme, the training sequence consists of $\mathbf{s}_t = \sum_{k\in D_t} \mathbf{s}_u^{(k)}$, where $D_t = \{l\frac{m}{n}\}_{l=0}^{n-1}$ and $\frac{m}{n} = k \ge 2$ is an integer. The information symbols $\{s_s[n]\}_{k\in D_s}$ are inserted into the rest of the entries of an N-tuple vector indexed by the set D_s, where D_s and D_t are disjoint and $D_s \bigcup D_t = \{0, 1, \ldots, N-1\}$. Therefore, the transmitted sequence is $\mathbf{s} = \mathbf{s}_t + \mathbf{s}_s$, in which there is no overlap between the nonzero elements of the $\mathbf{s}_t$ and $\mathbf{s}_s$. Given that the nonzero elements of the training symbols in $\mathbf{s}_t = \{s_t[n]\}_{n\in D_t}$ are known at the receiver end, the channel gains $\{h_t[n]\}_{n\in D_t}$ can be estimated from the received symbols $\{y[n]\}_{n\in D_t}$ by evaluating $\hat{h}_t[n] = \frac{y[n]}{s_t[n]}, n \in D_t$. The rest of the unknowns $\{h_t[n]\}_{n\in D_s}$, which carry the information symbols $\{s_s[n]\cdot h_t[n]\}_{n\in D_s}$, can be obtained from the derived $\hat{h}_t[n]$ using interpolation technique. The characteristics of this scheme are summarized below.

1) The element-wise non-overlapping between $\mathbf{s}_t$ and $\mathbf{s}_s$ results in two sequences possessing the ZCC property in the frequency domain. Thus, an optimal performance of symbol detection can be obtained, achieving the optimal MMSE performance, when the training symbols are equi-powered and equi-spaced over the observation window [39].
2) Both the computing loads for evaluating $\{h_t[n]\}_{n\in D_t} = \frac{y[n]}{s_t[n]}$ and deriving $\{h_t[n]\}_{n\in D_s}$ using interpolation from $\{\hat{h}_t[n]\}_{n\in D_t}$ are not as heavy as that for evaluating the CE-BEM coefficients λ_q using a projection technique or iteration algorithms [41], [42], [43].

VII. SIMULATION

Yong Soo Cho et al. simulated the channel estimation of LS and MMSE in the OFDM transmission environment in Reference [44], and obtained performance comparisons of MSE and BER, as well as the approximate comparison between the estimated channel and the true channel in db value. Based on this simulation, we added ZCC sequence as the training sequence and the cyclic prefix CP to the time domain signal, and compared the above performance indicators, reflecting that ZCC channel estimation is superior to LS and MMSE. In addition, we discuss the computational complexity of the three algorithms to illustrate the value of ZCC channel estimation in practical applications. Finally, we use the same group of three ZCC sequence to represent three users. Identify different users and channel parameters based on the two properties of ZCC sequence.

In addition to OFDM system, ZCC sequence is also applicable to other wireless transmission systems. This simulation only reflects the basic effects of the two important properties of the ZCC sequence.

A. SIMULATION MODEL

This simulation model is based on the channel estimation of LS and MMSE by Soo et al. in the reference [44]. For model parameter settings, the subcarrier, pilot interval, pilot number, and cyclic prefix are Nfft=512, Nps=32, Np=16, and CP=Nfft/8 respectively. SNR ranges from 1 to 20. The channel $\mathbf{h}$ was modelled as 4-tap channel with each the real and imaginary part of each tap being an independent Gaussian random variable. The two ZCC sequences are $\mathbf{s}_{ZCC-4}$=[1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0] and $\mathbf{s}_{ZCC-5}$ =[1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0], which length and mE_N of the sequences are 16 and 20, $mE_N = 4$ and $mE_N = 5$. The simulation model is as follows:

1) Generate random binary sequence and perform 16-QAM modulation.
2) Generate pilot position and insert pilot.
3) Perform ifft conversion on the data.
4) Add ZCC sequence and cyclic prefix to the data.
5) Perform channel and signal convolution.
6) Perform the training sequence of the received signal and the ZCC circular convolution to obtain the channel estimation of ZCC.
7) Remove the cyclic prefix and ZCC training sequence from the received signal.
8) Extract the pilot to calculate the channel estimates of LS and MMSE and the MSE of the three algorithms.
9) Demodulate 16-QAM and calculate the bit error rate.

B. COMPARISON BETWEEN ESTIMATED CHANNEL AND TRUE CHANNEL

For the conventional LS and MMSE channel estimations in OFDM systems, firstly, we insert pilot into the frequency domain in the data. Secondly, the received pilot signal is processed by the corresponding algorithm, so that we can obtain the part of the channel estimation parameters. Since the pilot is only a part of the OFDM symbol, in order to obtain complete channel estimation parameters, interpolation processing is required. Here, We discuss three algorithms and show the effect of the three channel estimation algorithms by comparing which power of channel estimation is closer to that of the true channel.

The LS channel estimation $\hat{\mathbf{H}}_{LS}$ [44] is obtained by minimizing the cost function $||\mathbf{Y}-\mathbf{X}\hat{\mathbf{H}}||^2$, which gives the solution to the LS channel estimation as

$$\hat{\mathbf{H}}_{LS} = \mathbf{X}^{-1}\mathbf{Y}. \tag{37}$$

Let us denote each component of the LS channel estimation $\hat{\mathbf{H}}_{LS}$ by $\hat{H}_{LS}[k]$, $k = 0, 1, 2, \ldots, N-1$. Since $\mathbf{X}$ is assumed to be diagonal due to the ICI-free condition, the LS channel estimate $\hat{\mathbf{H}}_{LS}$ can be written for each subcarrier as

$$\hat{\mathbf{H}}_{LS} = \{\hat{H}_{LS}[k]\}_{k=0}^{N-1} = \{\frac{Y[k]}{X[k]}\}_{k=0}^{N-1}. \tag{38}$$

We define $\hat{\mathbf{H}} \triangleq \mathbf{W}\hat{\mathbf{H}}_{LS}$, where $\mathbf{W}$ is weight matrix. Similarly, the minimum value of the cost function $E\{||\mathbf{H}-\hat{\mathbf{H}}||^2\}$ is used to obtain the channel estimation of MMSE. The orthogonality principle states that the estimation error vector $\mathbf{H}-\hat{\mathbf{H}}$ is orthogonal to $\hat{\mathbf{H}}_{LS}$, such that $E\{(\mathbf{H}-\hat{\mathbf{H}})\hat{\mathbf{H}}_{LS}^H\} = 0$, and Solving Equation for $\mathbf{W}$, we can get $\mathbf{W} = \mathbf{R}_{\mathbf{H}\hat{\mathbf{H}}_{LS}}\mathbf{R}^{-1}_{\hat{\mathbf{H}}_{LS}\hat{\mathbf{H}}_{LS}}$, where $\mathbf{R}_{\hat{\mathbf{H}}_{LS}\hat{\mathbf{H}}_{LS}}$ is the auto-correlation matrix of $\hat{\mathbf{H}}_{LS}$ given as $\mathbf{R}_{\mathbf{HH}}+\frac{\sigma_z^2}{\sigma_x^2}\mathbf{I}$, and $\mathbf{R}_{\mathbf{H}\hat{\mathbf{H}}_{LS}}$ is the cross-correlation matrix between the true channel vector and temporary channel estimate vector in the frequency domain. Therefore, the channel estimation of MMSE follows as [44]

$$\begin{aligned}\hat{\mathbf{H}}_{MMSE} &= \mathbf{W}\hat{\mathbf{H}}_{LS} \\ &= \mathbf{R}_{\mathbf{H}\hat{\mathbf{H}}_{LS}}(\mathbf{R}_{\mathbf{HH}}+\frac{\sigma_z^2}{\sigma_x^2}\mathbf{I})^{-1}\hat{\mathbf{H}}_{LS}.\end{aligned} \tag{39}$$

It can be seen from (24) that $\hat{\mathbf{h}}_{ZCC-E}$ is the channel estimation of periodic auto-correlation operation of ZCC sequences, and the result of the periodic auto-correlation operation of ZCC sequences is its energy mE_N which gave in (16), so its channel estimation is multiplied by mE_N times. Therefore, in order to obtain the channel estimation of ZCC, we need to divide it by mE_N, and the channel estimation of ZCC can be expressed as

$$\hat{\mathbf{h}}_{ZCC} = \frac{\mathbf{y}' \otimes_c \mathbf{z}^*_{-0}}{mE_N} = \mathbf{h} + \frac{\mathbf{v}'}{mE_N}. \tag{40}$$

According to (38) and (39), the $\hat{\mathbf{H}}_{LS}$ and $\hat{\mathbf{H}}_{MMSE}$ [44] are channel estimations of 512 subcarriers in frequency domain. The true channel $\mathbf{h}$ of this simulation is a complex parameter in the time domain. In order to compare LS and MMSE channel estimations with true channel estimation, we use the formula $10log10(abs(\hat{\mathbf{H}} \circ \hat{\mathbf{H}}^*))$ to convert the component values of LS and MMSE into dB values. The true channel needs to perform the Nfft transform of n-points first, and then use the same conversion formula to convert to dB value for comparison. Similarly, the channel estimation of ZCC needs to be operated in the same way. As shown in Fig. 2, the channel estimation of ZCC is closer to true channel than LS and MMSE channel estimations.

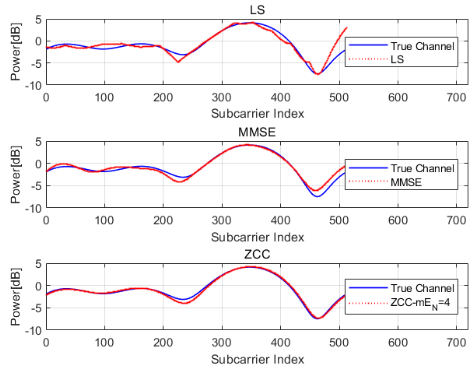


FIGURE 2. Comparison between channel estimation and true channel of LS, MMSE and ZCC-$mE_N = 4$.

C. COMPLEXITY ANALYSIS

In this part, we show the analysis of computational complexity of our proposed method and compare it with that of the conventional LS and MMSE estimations in Table 1.

According to (38), the computational complexity of LS channel estimation comes from the calculation of $Y[k]$ divided by $X[k]$, and k is the pilot number. Therefore, its main computational complexity is 16 times division, and the complexity of LS channel estimation is $O(n)$.

From (39), the most complex part of conventional MMSE estimation comes from getting inverse of matrix $(\mathbf{R}_{\mathbf{HH}} + \frac{\sigma_z^2}{\sigma_x^2}\mathbf{I})$, the dimension of the matrix is the pilot number, so its main calculation amount is to calculate the inverse of 16 * 16 matrix, and the complexity of MMSE channel estimation is $O(n^3)$.

For the proposed method, we can see from (40) that the computational complexity of ZCC channel estimation comes from the circular convolution of ZCC sequence. Because the length of ZCC sequence in this simulation is 16, its main calculation amount is 16 times multiplication, and the complexity of ZCC channel estimation is $O(n)$.

Through the comparison of the computational complexity of the three channel estimates, it can be seen that ZCC algorithm has lower computational complexity, which is more conducive to the realization of the terminal.

D. MSE PERFORMANCE

The MSE performance curves of the three algorithms are shown in Fig. 3. The MSE of LS channel estimation is given as [44]

$$MSE_{LS} = E\{(\mathbf{H}-\hat{\mathbf{H}}_{LS})^H(\mathbf{H}-\hat{\mathbf{H}}_{LS})\} = \frac{\sigma_z^2}{\sigma_x^2}. \tag{41}$$

The MSE of MMSE channel estimation is given as [45]

$$\begin{aligned}&MSE_{MMSE} \\ &= E\{(\mathbf{H}-\hat{\mathbf{H}}_{MMSE})^H(\mathbf{H}-\hat{\mathbf{H}}_{MMSE})\} \\ &= \sigma_z^2 tr\{\mathbf{R}_{\mathbf{HH}}(\sigma_x^2\mathbf{R}_{\mathbf{HH}}+\sigma_z^2\mathbf{I})^{-1}\}.\end{aligned} \tag{42}$$

TABLE 1. Complexity Comparison of LS, MMSE and ZCC Algorithms.

algorithm	equation	Main calculation amount	complexity
LS	$\hat{\mathbf{H}}_{LS} = \{\frac{Y[k]}{X[k]}\}_{k=0}^{N-1}$	16 times division	$T(n) = O(n)$
MMSE	$\hat{\mathbf{H}}_{MMSE} = \mathbf{R}_{\mathbf{H}\hat{\mathbf{H}}_{LS}}(\mathbf{R}_{\mathbf{HH}} + \frac{\sigma_z^2}{\sigma_x^2}\mathbf{I})^{-1}\hat{\mathbf{H}}_{LS}$	16*16 matrix inversion	$T(n) = O(n^3)$
ZCC	$\hat{\mathbf{h}}_{ZCC} = \frac{\mathbf{y}' \otimes_c \mathbf{z}^*_{-0}}{mE_N}$	16 times multiplication	$T(n) = O(n)$

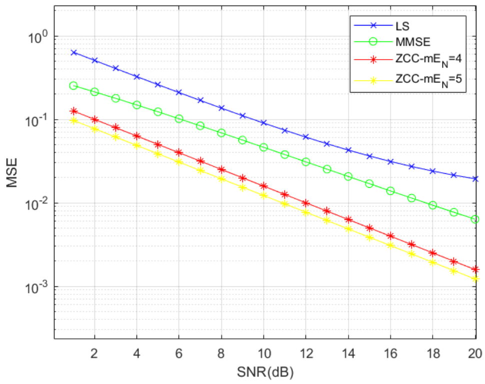


FIGURE 3. MSE performance of LS, MMSE, ZCC-mE_N=4 and ZCC-mE_N=5.

From (40), we can see that the estimation error of ZCC algorithm is affected by noise. $\hat{\mathbf{h}}_{ZCC-E}$ and $var(v'[n])$ are given in (24) and (26) and MSE of channel estimation of ZCC in time domain is expressed as

$$\begin{aligned} MSE_{ZCC} &= E[|\hat{h}(i) - h(i)|^2] \\ &= E[|\frac{mE_N h(i) + v'(i)}{mE_N} - h(i)|^2] \\ &= \frac{E[|v'(i)|^2]}{(mE_N)^2} \\ &= \frac{\sigma_z^2 mE_N}{(mE_N)^2} \\ &= \frac{\sigma_z^2}{mE_N}. \end{aligned} \tag{43}$$

The mean square error estimate obtained in this paper is $\frac{\sigma_z^2}{mE_N}$. The parameter mE_N can be changed. Only the ZCC sequence with larger mE_N to be selected as the training sequence, which can reduce the mean square error and achieve the performance that the error is less than MMSE channel estimation method, but the cost increases.

E. BER PERFORMANCE

As shown in Fig. 4, BER curves are the result of equalization of three channel estimation algorithms and true channel. The calculation method of BER is that the received data is equalized by the channel estimation parameter $\hat{\mathbf{H}}$, and then demodulated by 16-QAM. The demodulated data is compared with the data before transmission, and the accumulated error data is used to calculate the bit error rate. The equation used for channel equalization is given as

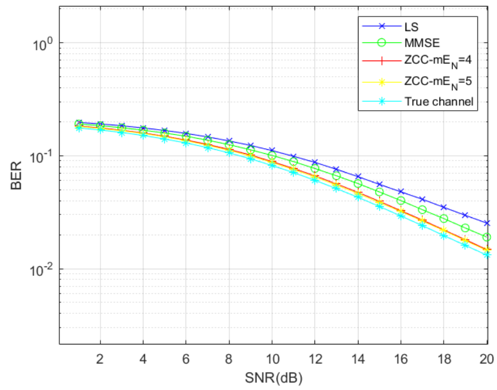
$$\mathbf{Y}_{est} = \{\frac{Y[k]}{\hat{H}[k]}\}_{k=0}^{N-1}. \tag{44}$$

Because the $\mathbf{h}_{ZCC}$ and true channel $\mathbf{h}$ are time-domain parameters, the two channel parameters need to be processed by Nfft point FFT before they can be used for equalization.

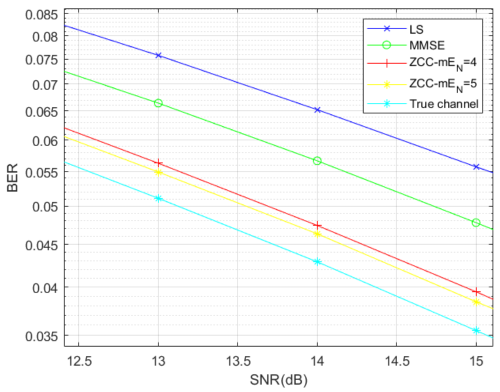
We compare the BER of the true channel and the four channel estimations, where the four channel estimations are LS, MMSE, ZCC-mE_N=4, ZCC-mE_N=5. The curves in Fig. 4 show that the order of error rate is as follows: $BER_{True\ channel} < BER_{ZCC-mE_N=5} < BER_{ZCC-mE_N=4} < BER_{MMSE} < BER_{LS}$. The BER of equalization through true channel parameters is the lowest.The bit error rate of zcc algorithm is lower than that of MMSE and LS algorithms. From the comparison of MSE performance in Fig. 3, the performance of MSE is consistent with that of BER. Different calculation methods have consistent performance results, which also verifies the correctness of simulation results.

F. MULTIUSER CHANNEL ESTIMATION

According to (3), (4), (24) and *Corollary 1*, we can get two important properties of ZCC sequences. The first property is that the absolute value of the periodic auto-correlation function of the ZCC sequences is $|R_{ZCC}[\tau]| = mE_N$. The second property is that the value of the cross-correlation function of the same set of ZCC sequences is $R_{ZCC1,ZCC2}[\tau] = 0$. We use these two properties to perform multiuser channel estimation. Assuming that three ZCC sequences of the same group are selected as training sequences of three users, which are given as $\mathbf{s}_{ZCC1}$=[1,0,0,0,1,0,0,0,1,0,0,0,1,0,0,0], $\mathbf{s}_{ZCC2}$=[1,0,0,0,−1,0,0,0,1, 0,0,0,−1,0,0,0], $\mathbf{s}_{ZCC3}$=[1,0,0,0,−j,0,0,0,−1,0,0,0,j,0,0,0]. Each user passes through different channels **h1**, **h2** and **h3**, and the three channels are 4-tap channels which follow the Rayleigh distribution. According to the property that the value of the periodic cross-correlation function of the same group of ZCC sequences is zero, there will be $\mathbf{s}_{ZCC1} \otimes_c \mathbf{s}_{ZCC2}{}^*_{-} = \mathbf{s}_{ZCC1} \otimes_c \mathbf{s}_{ZCC3}{}^*_{-} = 0$. But in fact, as shown in Fig. 5, due to the influence of additive Gaussian noise, the result of periodic cross-correlation calculation is not zero, that is, $\mathbf{R}_{\mathbf{y}_{ZCC1sig},\mathbf{s}_{ZCC2}} = \mathbf{y}_{ZCC1sig} \otimes_c \mathbf{s}_{ZCC2}{}^*_{-} \neq 0$. However, it is far less than the value of auto-correlation function. After



(a) SNR:1-20



(b) SNR:12-15

FIGURE 4. BER performance of LS, MMSE, ZCC-mE_N=4, ZCC-mE_N=5 and True channel.

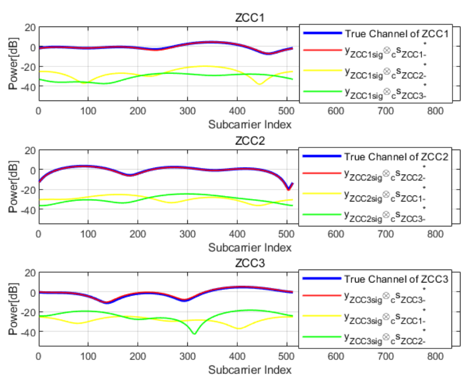


FIGURE 5. Multiuser channel estimation of $\mathbf{s}_{ZCC1}$, $\mathbf{s}_{ZCC2}$ and $\mathbf{s}_{ZCC3}$ with different channels.

setting the appropriate threshold value, it is easy to identify different users and their estimated channel parameters.

VIII. CONCLUSION

In this paper, a set of base sequences of period N are initially either repeated m times or upsampled by a factor m to generate a new set of sequences of period mN for further modulation by a set of m digital carriers to construct a set of ZCC sequences with cardinality m. Possessing the ideal PCCF property, the ZCC sequences are capable of achieving zero CCI or MAI performance in a multiuser communication system or a multiple access scheme system. Additionally, when the base sequences applied for constructing ZCC sequences possess the ideal PACF property, the auto-correlation function of the resultant ZCC sequences can still preserve the desired PACF property. Because of this property, the ZCC sequences can be applied to perform channel estimation so that optimal joint symbol detection and channel estimation similar to those sequences possessing the ideal PACF property can be achieved. Based on these results, it is evident that the ZCC sequences have broader applications compared with other sets of sequences.

APPENDIX A
NOTATION TABLE

Notation	Description
$\mathbf{B}$	base matrix $\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{N-1}\}$
$\mathbf{c}_n$	the DFT of $\delta_N^{(n)}$
$\mathcal{C}^N$	a set of Kronecker delta function of period N its circular shifts $\{\delta_N, \delta_N^{(1)}, \ldots, \delta_N^{(N-1)}\}$
D_t	an integer vector, $D_t = \{l\frac{m}{n}\}_{l=0}^{n-1}$ and $\frac{m}{n} = k \geq 2$
D_s	the rest of the entries of an N-tuple vector of D_t
E_N	energy of the base sequence $\mathbf{s}_t$
$\mathbf{h}$	true channel
$\hat{\mathbf{h}}_{ZCC-E}$	channel estimation of ZCC sequence periodic auto-correlation operation, mE_N times of $\hat{\mathbf{h}}_{ZCC}$
$\hat{\mathbf{h}}_{ZCC}$	channel estimation of ZCC
$\mathbf{h}_0$	channel parameters $\mathbf{h}_0 = [h[0]\ h[1]\ \cdots\ h[N-1]]^T$
$\mathbf{H}_0$	the DFT of $\mathbf{h}_0$
$\mathbf{H}_y$	channel matrix and associated elements $\mathbf{H}_0$, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$ and $\mathbf{H}_4$
$\hat{\mathbf{H}}_{LS}$	channel estimation of least-squares
$\hat{\mathbf{H}}_{MMSE}$	channel estimation of minimum-mean-squared-error
$\hat{\mathbf{H}}$	channel estimation
$\hat{H}_{LS[k]}$	each component of the LS channel estimation
$\hat{H}_{[k]}$	each component of the channel estimation
$R_r[\tau]$	auto-correlation function
$R_{r,s}[\tau]$	cross-correlation function
$\mathbf{R}_s$	the periodic auto-correlation function of $\mathbf{s}$
$\mathbf{R}_{\hat{\mathbf{H}}_{LS}\hat{\mathbf{H}}_{LS}}$	the auto-correlation matrix of $\hat{\mathbf{H}}_{LS}$
$\mathbf{R}_{\mathbf{H}\hat{\mathbf{H}}_{LS}}$	the cross-correlation matrix between the true channel vector and temporary channel estimate vector in the frequency domain
$\lvert R_{ZCC}[\tau] \rvert$	the absolute value of the periodic auto-correlation function of the ZCC sequence
$\mathbf{s}$	a vector of size $N \times 1$ or a sequence of period N
$\mathbf{s}_{ZCC}$	ZCC training sequence used in simulation
$\mathbf{s}_{ZCC-4}$	ZCC training sequence with $mE_N = 4$ used in simulation

$\mathbf{S}$	the DFT of $\mathbf{s}$
$\mathbf{s}^{(m)}$	the circular shift of $\mathbf{s}$ to right
$\mathbf{s}^{(-m)}$	the circular shift of $\mathbf{s}$ to left
$\mathbf{s}_n^u$	Upsampling of $\mathbf{s}_n$
$\mathbf{S}_n^u$	the DFT of $\mathbf{s}_n^u$
$\mathbf{s}_t$	a base sequence of period N
$\mathbf{S}_t$	the DFT of $\mathbf{S}_t$
$\mathbf{s}_u^{(k)}$	distinct circular shifts of $\mathbf{s}_u$
$\mathbf{t}_n$	repeating $\mathbf{s}_n$ m times
$\mathbf{T}_n$	the DFT of $\mathbf{t}_n$
$\mathbf{T}$	Toeplitz matrix
$\mathbf{V_B}(\mathbf{s})$	the coordinate vector of sequence $\mathbf{s}$ of period N relative to base matrix $\mathbf{B}$
$\mathbf{v}$	the vector of noise
$\mathbf{v}'$	he estimation error vector
$v'[n]$	The component of $\mathbf{v}' = \{v'[n]\}_{n=0}^{mN-1}$
$\mathbf{W}$	weight matrix of MMSE estimation
$\mathbf{y}$	vector of received signal
$\mathbf{y}_{ZCC1sig}$	the part of the received signal containing ZCC1 training sequence
$\mathbf{Y}$	The DFT of y
$\mathbf{Y}_{est}$	The DFT of the received signal after channel estimation equalization
$\mathbf{z}_0$	A sequence of period mN is constructed by epeating and cascading sequence $\mathbf{s}_t$ m times
$\mathbf{Z}_0$	the DFT of $\mathbf{z}_0$
$\{\mathbf{Z}_0^{(k)}\}$	distinct circular shifts of $\mathbf{Z}_0$
δ_N	Kronecker delta function of period N
$\varepsilon[n]$	the model error
λ_q	the CE-BEM coefficient
σ_z^2	Variance of signal
σ_x^2	Variance of noise
$\otimes_c$	circular convolution
$\otimes_l$	linear convolution
$\circ$	component-wise product
$\lvert \cdot \rvert$	the absolute value of the argument
$\lvert\lvert \cdot \rvert\rvert$	the norm of the argument
$(\cdot)^*$	complex conjugate of the argument
$(\cdot)'$	transposition of the argument
$(\cdot)_N$	modulo N operation
$(\cdot)^H$	conjugate transpose operation
$tr\{\}$	traccs of matrix
$\triangleq$	definition

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REFERENCES

[1] D. V. Sarwate, "Bounds on cross-correlation and auto-correlation of sequences," *IEEE Trans. Inf. Theory*, vol. IT-25, no. 6, pp. 720–724, Nov. 1979.

[2] D. V. Sarwate, "An upper bounds on the aperidoc auto-correlation for a maximal-length sequence," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 4, pp. 685–687, Jul. 1984.

[3] L. R. Welch, "Lower bounds on the maximum cross correlation of signals (Corresp.)," *IEEE Trans. Inf. Theory*, vol. IT-20, no. 3, pp. 397–399, May 1974.

[4] P. Z. Fan and M. Darnell, *Sequences Design for Communications Applications*, New York, NY, USA: Wiley, 1996.

[5] S.-H. Choi, J.-S. Baek, J.-S. Han, and J.-S. Seo, "Channel estimations using extended orthogonal codes for AF multiple-relay networks over frequency-selective fading channels," *IEEE Trans. Veh. Technol.*, vol. 63, no. 1, pp. 417–423, Jan. 2014.

[6] S. Qureshi, "Fast start-up equalization with periodic training sequences," *IEEE Trans. Inf. Theory*, vol. IT-23, no. 5, pp. 553–563, Sep. 1977.

[7] G. Gong, F. Huo, and Y. Yang, "Large zero autocorrelation zones of Golay sequences and their applications," *IEEE Trans. Commun.*, vol. 61, no. 9, pp. 3967–3979, Sep. 2013.

[8] A. Milewski, "Periodic sequences with optimal properties for channel estimation and fast start-up equalization," *IBM J. Res. Develop.*, vol. 27, no. 5, pp. 426–431, Sep. 1983.

[9] D. Chu, "Polyphase codes with good periodic correlation properties (Corresp.)," *IEEE Trans. Inf. Theory*, vol. IT-18, no. 4, pp. 531–532, Jul. 1972.

[10] J.-C. Lin, "Initial synchronization assisted by inherent diversity over time-varying frequency-selective fading channels," *IEEE Trans. Wireless Commun.*, vol. 13, no. 5, pp. 2518–2529, May 2014.

[11] B. M. Popovic, "Generalized chirp-like polyphase sequences with optimum correlation properties," *IEEE Trans. Inf. Theory*, vol. 38, no. 4, pp. 1406–1409, Jul. 1992.

[12] I. Kim, Y. Han, Y. Kim, and S. Bang, "Sequence hopping cell search scheme for OFDM cellular systems," *IEEE Trans. Wireless Commun.*, vol. 7, no. 5, pp. 1483–1489, May 2008.

[13] C.-P. Li, S.-H. Wang, and C.-L. Wang, "Novel low-complexity SLM schemes for PAPR reduction in OFDM systems," *IEEE Trans. Signal Process.*, vol. 58, no. 5, pp. 2916–2921, May 2010.

[14] S. Wang, C. Li, K. Lee, and H. Su, "A novel low-complexity precoded OFDM system with reduced PAPR," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1366–1376, Mar. 2015.

[15] J. O. M. Amok and N. M. Saad, "Error rate performance analysis of multiantenna MC DS-CDMA system η–μ over frequency selective fading channels with arbitrary parameters," *AEU-Int. J. Electron. Commun.*, vol. 70, no. 5, pp. 517–529, May 2016.

[16] M. A. C. Fernandes and D. S. Arantes, "Spatial and temporal adaptive receiver for DS-CDMA systems," *AEU-Int. J. Electron. Commun.*, vol. 68, no. 3, pp. 216–226, Mar. 2014.

[17] Y. T. Wu, S. H. Leung, W. K. Wong, and Y. S. Zhu, "A constrained minimum mean square error code tracking loop for synchronous DS-CDMA systems," *AEU-Int. J. Electron. Commun.*, vol. 64, no. 6, pp. 558–569, Jun. 2010.

[18] H.-H. Chang, S.-C. Lin, and C.-D. Lee, "A CDMA scheme based on perfect Gaussian integer sequences," *AEU-Int. J. Electron. Commun.*, vol. 75, pp. 70–81, May 2017.

[19] K.-C. Lee, S.-H. Wang, C.-P. Li, H.-H. Chang, and H.-J. Li, "Adaptive resource allocation algorithm based on cross-entropy method for OFDMA systems," *IEEE Trans. Broadcast.*, vol. 60, no. 3, pp. 524–531, Sep. 2014.

[20] L.-S. Tsai and Y. T. Su, "Transform domain approach for sequence design and its applications," *IEEE J. Sel. Areas Commun.*, vol. 24, no. 1, pp. 75–83, Jan. 2006, doi: 10.1109/JSAC.2005.858888.

[21] L.-S. Tsai, W.-H. Chung, and D.-S. Shiu, "Lower bounds on the correlation property for OFDM sequences with spectral-null constraints," *IEEE Trans. Wireless Commun.*, vol. 10, no. 8, pp. 2652–2659, Aug. 2011, doi: 10.1109/TWC.2011.060711.100626.

[22] S. Hu, Z. Liu, Y. L. Guan, W. Xiong, G. Bi, and S. Li, "Sequence design for cognitive CDMA communications under arbitrary spectrum hole constraint," *IEEE J. Sel. Areas Commun.*, vol. 32, no. 11, pp. 1974–1986, Nov. 2014, doi: 10.1109/JSAC.2014.141103.

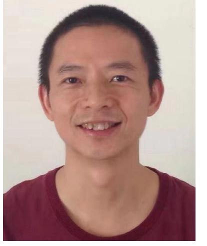
[23] Z. Liu, Y. L. Guan, U. Parampalli, and S. Hu, "Spectrally-constrained sequences: Bounds and constructions," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2571–2582, Apr. 2018, doi: 10.1109/TIT.2018.2800012.

[24] *3GPP LTE Specifications*, Jun. 2011. [Online]. Available: http://www.3gpp.org/Highlights/LTE/LTE.htm

[25] W.-W. Hu, S.-H. Wang, and C.-P. Li, "Gaussian integer sequences with ideal periodic autocorrelation functions," *IEEE Trans. Signal Process.*, vol. 60, no. 11, pp. 6074–6079, Nov. 2012.

[26] Y. Yang, X. Tang, and Z. Zhou, "Perfect Gaussian integer sequences of odd prime length," *IEEE Signal Process. Lett.*, vol. 19, no. 10, pp. 615–618, Oct. 2012.

[27] X. Ma, Q. Wen, J. Zhang, and H. Zuo, "New perfect Gaussian integer sequences of period pq," *IEICE Trans. Fundam. Electron., Commun. Comput. Sci.*, vol. 96, no. 11, pp. 2290–2293, 2013.
[28] H.-H. Chang, C.-P. Li, C.-D. Lee, S.-H. Wang, and T.-C. Wu, "Perfect Gaussian integer sequences of arbitrary composite length," *IEEE Trans. Inf. Theory*, vol. 61, no. 7, pp. 4107–4115, Jul. 2015.
[29] C.-D. Lee, Y.-P. Huang, Y. Chang, and H.-H. Chang, "Perfect Gaussian integer sequences of odd period 2^m-1," *IEEE Signal Process. Lett.*, vol. 22, no. 7, pp. 881–885, Jul. 2015.
[30] S.-C. Pei and K.-W. Chang, "Perfect Gaussian integer sequences of arbitrary length," *IEEE Signal Process. Lett.*, vol. 22, no. 8, pp. 1040–1044, Aug. 2015.
[31] S.-H. Wang, C.-P. Li, H.-H. Chang, and C.-D. Lee, "A systematic method for constructing sparse Gaussian integer sequences with ideal periodic autocorrelation functions," *IEEE Trans. Commun.*, vol. 64, no. 1, pp. 365–376, Jan. 2016.
[32] C.-D. Lee and S.-H. Hong, "Generation of long perfect Gaussian integer sequences," *IEEE Signal Process. Lett.*, vol. 24, no. 4, pp. 515–519, Apr. 2017.
[33] C. Lee and Y. Chen, "Families of Gaussian integer sequences with high energy efficiency," *IET Commun.*, vol. 10, no. 17, pp. 2416–2421, Nov. 2016.
[34] K.-J. Chang and H.-H. Chang, "Perfect Gaussian integer sequences of period p^k with degrees equal to or less than $k+1$," *IEEE Trans. Commun.*, vol. 65, no. 9, pp. 3723–3733, Sep. 2017.
[35] H.-H. Chang, K.-J. Chang, and C.-P. Li, "Construction of period qp PGISs with degrees equal to or larger than four," *IEEE Access*, vol. 6, pp. 64790–64800, 2018.
[36] C.-H. Hsia, S.-J. Lou, H.-H. Chang, and D. Xuan, "Novel hybrid public/private key cryptography based on perfect Gaussian integer sequences," *IEEE Access*, vol. 9, pp. 145045–145059, 2021.
[37] D. Xuan, H. Chang, C. Li, and W. Xie, "Construction of zero circular convolution sequences," in *Proc. Int. Conf. Electr., Comput., Commun. Mechatronics Eng. (ICECCME)*, Nov. 2022, pp. 1–5, doi: 10.1109/ICECCME55909.2022.9988686.
[38] G. B. Giannakis and C. Tepedelenlioglu, "Basis expansion models and diversity techniques for blind identification and equalization of time-varying channels," *Proc. IEEE*, vol. 86, no. 10, pp. 1969–1986, Oct. 1998.
[39] X. Ma, G. Giannakis, and S. Ohno, "Optimal training for block transmissions over doubly-selective fading channels," *IEEE Trans. Signal Process.*, vol. 51, no. 5, pp. 1351–1366, May 2003.
[40] D. K. Borah and B. T. Hart, "Frequency-selective fading channel estimation with a polynomial time-varying channel model," *IEEE Trans. Commun.*, vol. 47, no. 6, pp. 862–873, Jun. 1999.
[41] T. Zemen and C. F. Mecklenbrauker, "Time-variant channel estimation using discrete prolate spheroidal sequences," *IEEE Trans. Signal Process.*, vol. 53, no. 9, pp. 3597–3607, Sep. 2005.
[42] E.-S. Jeon, J. Seo, J. Yang, and D. Kim, "Iterative joint detection, ICI cancelation and estimation of multiple CFOs and channels for DVB-T2 in MISO transmission mode," *IEEE Trans. Broadcast.*, vol. 60, no. 1, pp. 29–37, Mar. 2014.
[43] X. Zeng and A. Ghrayeb, "Joint CFO and channel estimation for OFDMA uplink: An application of the variable projection method," *IEEE Trans. Wireless Commun.*, vol. 8, no. 5, pp. 2306–2311, May 2009.
[44] Y. S. Cho, J. Kim, W. Y. Yang, and C. G. Kang, *MIMO-OFDM Wireless Communications With MATLAB*. Singapore: Wiley, 2010, pp. 190–193.
[45] A. Hjorungnes and D. Gesbert, "Complex-valued matrix differentiation: Techniques and key results," *IEEE Trans. Signal Process.*, vol. 55, no. 6, pp. 2740–2746, Jun. 2007, doi: 10.1109/TSP.2007.893762.



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