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RESEARCH ARTICLE

Analysis and Application of *p, q*-Quasirung Orthopair Fuzzy Aczel–Alsina Aggregation Operators in Multiple Criteria Decision-Making

JAWAD ALI [1] **AND MUHAMMAD NAEEM** [2]

[1]Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat 26000, Pakistan
[2]Department of Mathematics, Deanship of Applied Sciences, Umm Al-Qura University, Mecca 21421, Saudi Arabia

Corresponding author: Jawad Ali (jawadali@math.qau.edu.pk)

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ABSTRACT The p,q-quasirung orthopair fuzzy set (p, q-ROFS) is a potent tool for representing fuzziness and uncertainty compared to the q-rung orthopair fuzzy set (q-ROFS). This research aims to introduce the aggregation technique of p, q-ROFSs with the help of Aczel–Alsina (AA) operations. We first propound several novel AA operations of p, q-ROFSs such as AA sum, AA product, AA scalar multiplication, and AA exponentiation. Following these operations, we develop a range of p, q-quasirung orthopair fuzzy averaging and geometric aggregation operators to efficiently aggregate p, q-quasirung fuzzy data. Additionally, we construct different features of these operators, discuss certain special cases, and study their fundamental results. Afterward, we use these operators to design a method for dealing with multi-criteria decision-making with p, q-quasirung orthopair fuzzy information. Finally, we provide a case study to demonstrate the suggested method's practicality followed by parameter analysis and comparison study.

INDEX TERMS Aczel–Alsina t-norms, p, q-quasirung orthopair fuzzy set, aggregation operators, MCDM.

Symbols	Meanings
$\mathcal{P}$	p,q-ROFS.
Ш	Element of universal set.
X	Universal set.
$\mu_{\mathcal{P}}$	Membership grade.
$\nu_{\mathcal{P}}$	Non-membership grade.
T_A^{Λ}	Aczel-Alsina t-norm.
S_A^{Λ}	Aczel-Alsina t-conorm.
Λ	Parameter.
ι	Positive integer.

I. INTRODUCTION

The multi-criteria decision-making (MCDM) process is a deft technique for dealing with difficult and intricate data in real-world scenarios. MCDM is a technique that generates rankings for the available alternatives that correspond to the distinct options' characteristic objects and is a critical component of decision-making sciences [1], [2], [3], [4], [5], [6]. In real-world decision-making scenarios, it is critical to bear the characteristic object largely well and accurately. Owing to the convolution and incapacity of decision-making circumstances, carrying the distinctive objects of alternatives requires more than exact objects. Zadeh [7] suggested a theory of fuzzy sets (FS) to do this, in which only the truth grade is constrained to the unit interval. The FS theory has garnered widespread attention from eminent professionals and has been employed in various circumstances in a variety of domains [8], [9], [10], [11], [12]. However, the FS theory has failed to work precisely on numerous occasions. For instance, it is challenging to apply FS to information expressed in terms of truthfulness (μ) and falsehood (ν) grades. To address these issues, Atanassov [13] dispatched the theory of intuitionistic FS (IFS), a revamped version of the FS that enables the effective management of awkward and unreliable information. IFS covers the grades μ and ν with the rule $0 \leq \mu + \nu \leq 1$. As a result, the IFS theory has garnered considerable research attention and has been incorporated into numerous works. Li [14] investigated various linear programming models to

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derive optimal weights for criteria and the corresponding decision-making techniques. Based on the Jensen-Shannon divergence, Xiao [15] devised a distance measure between IFSs and discussed its applicability to model classification challenges. Burillo and Bustince [16] propound a measure of entropy for IFs to assess the intuitionism of an IFS. Recently, Alcantud et al. [17] came up with a method for aggregating infinite sequences of IFS. However, this FS has drawbacks in the context of decision-making information description since these drawbacks put constraints on the depiction of truthfulness and falsehood grades, resulting in the total of the two parameters being less than or equal to 1.

Due to the aforesaid shortcomings of IFSs, investigators created a more complete fuzzy set known as Pythagorean fuzzy sets (PFS). Notably, the notion of PFSs is derived from IFSs, but with further generality, [18]. PFSs are defined by the sum of the squares of truthfulness and falsehood grades, which are real values less than or equal to one [19]. As illustrated in this example, the constrain of PFSs is superior to that of IFSs: $0.5^2 + 0.7^2 = 0.25 + 0.49 = 0.74$. PFSs have garnered substantial interest from scientists because of their ability to resolve increasing grades of ambiguity [20], [21], [22], [23]. In practice, DMs are constrained by PFS limitations since they cannot properly assign values to truthfulness and falsehood grades based on their own selections [19]. Owed to the inherent limits of PFSs, a precise and successive FS is required to discourse the constraints faced by DMs.

Yager [24] introduced an innovative fuzzy idea anointed the q-rung orthopair fuzzy set (q-ROFS) to address the shortcomings of classic fuzzy sets' data representation (i.e., IFSs and PFSs). The limitation imposed by other fuzzy sets is abolished with q-ROFSs, and the total of the q powers of truthfulness and falsehood grades are real values between [0, 1]. Thus, DMs are free to choose any grades [25]. For instance, when DM is questioned about a certain case, he or she provides a value of 0.9 to truthfulness grade and 0.8 to falsehood grade. In this situation, the IFS and PFS requirements cannot be met due to their restrictions. However, the truthfulness and falsehood grades shown above may be expressed using a q-ROFS and increasing the q value parameter to a value equal to or higher than 4. When q equals 1, the q-ROFS is converted to an IFS. When q equals 2, the q-ROFS transforms into a PFS.

According to the structural illustration, the q-ROFS restriction is deemed superior to the others since it gives more room and flexibility under unknown situations and allows DMs to freely pick truthfulness and falsehood grades [26]. Since its inception, several scholars have thoroughly investigated and applied it to the resolution of unwieldy and perplexing fuzzy issues from a variety of angles. Several aggregation operators in the context of q-ROFSs have been tried to present [27]. These include q-quasirung orthopair fuzzy Einstein ordered weighted geometric, q-quasirung orthopair fuzzy Einstein weighted geometric, q-quasirung orthopair fuzzy Einstein weighted averaging, and q-quasirung orthopair fuzzy Einstein ordered weighted averaging. The authors in [28] coupled the Bonferroni mean (BM) operator with q-quasirung orthopair fuzzy numbers (q-ROFNs) to offer up the q-quasirung orthopair fuzzy BM (q-ROFBM) operator, the q-quasirung orthopair fuzzy weighted BM (q-ROFWBM) operator, the q-quasirung orthopair fuzzy geometric BM operator, and the q-ROFWG BM operator; and then formed the MCDM mechanisms using all these operators. Wei et al. [29] presented the q-quasirung orthopair fuzzy generalized Heronian mean (q-ROFGHM) operator, the q-quasirung orthopair fuzzy geometric Heronian mean (q-ROFGHM) operator, the q-quasirung orthopair fuzzy generalized weighted Heronian mean (q-ROFWG Heronian mean) operator, and the q-ROFWG Heronian mean operator. Another research [30] examined the scheme for MCDM problems based on q-quasirung orthopair fuzzy Hamy mean (HM) operators. Apart from these aggregation-based methods, some other kinds of MCDM approaches are also explored by the authors [20], [31], [32], [33] under the q-quasirung orthopair fuzzy setting. Deveci et al. [34] presented a q-quasirung orthopair Fuzzy Einstein based weighted aggregated sum product assessment (WASPAS) approach. In [20], the entropy measure and order of preference by similarity to ideal solution (TOPSIS) based on the correlation coefficient was investigated. Accordingly, the performance of green suppliers with experts' subjective evaluations was measured with an effective and applicable MCDM method and q-ROFSs-based TOPSIS method [32]. Liang and Cao [35] put forward the projection-based distance for computing the positive ideal solution and loss functions, respectively. A study by [33] critically analyzed the available ranking techniques for q-quasirung orthopair fuzzy values and proposed a new graphical ranking method based on hesitancy index and entropy. Very recently, Seikh and Mandal [36] introduced p, q-quasirung orthopair fuzzy set (p, q-ROFS), which is an expansion of the q-ROFS. In p, q-ROFS, where p and q are natural integers, the sum of the pth power of membership grade and the qth power of nonmembership grade is less than or equal to 1. Due to the inclusion of the parameter p, the p, q-ROFS is able to describe incomplete information in a more flexible and comprehensive manner.

Menger introduced the concept of triangle norms (t-norms) in his hypothesis of probabilistic metric spaces [37]. It has been discovered that t-norms and their related t-conorms are critical operations in FSs and systems, such as the Einstein t-norm and t-conorm [38], the product t-norm and probabilistic sum t-conorm [39], the Lukasiewicz t-norm and t-conorm [40], and the Hamacher t-norm and t-conorm [41]. Klement et al. [42] conducted a thorough examination of the characteristics and related elements of t-norms in recent years. In 1982, Aczel and Alsina [43] introduced new procedures named AA t-norm and AA t-conorm, which places a high premium on parameter changeability. Based on the AA triangular norm (AA t-norm), Wang et al. [44] devised a score level fusion technique that simultaneously increases the distance between imposters and decreases the distance between real. Senapati et al. [45] recently introduced the intuitionistic fuzzy AA aggregation operators and demonstrated their usage in the MCDM approach.

In the aforesaid analysis of literature, we found that a number of researches presented various sorts of operators, such as averaging, geometric, and AA aggregation operators for FSs, IFSs, PFSs, and q-ROFSs. The notion of p, q- ROFSs can be deemed very reliable because it is the reformed version of all existent ideas. It is quite challenging for scholars to define the theory of AA operational laws and their aggregation operators in the presence of the p, q- ROFS. As a result, it has not previously been investigated where the proposed aggregation operators are more significantly modified than the existent operators. Although the area has benefited from the establishment of ideas based on IFS, such as the AA aggregation operators and the WASPAS technique, the task of effectively applying or combining theories (including q-ROFS, aggregation operators, AA t-norm and t-conorm) in order to develop a theory of the MCDM method and AA aggregation operators based on p, q-ROFS theory remains. The theory of averaging, geometric operators, AA aggregation operators, and the aggregation approach based on FSs, IFSs, PFSs, and q-ROFSs are specific cases of the AA aggregation operators and the aggregation method for p,q-ROFS. Motivated by the AA operators' work under IFSs [45], and by considering the importance of p, q-ROFS, this study concentrates on the following pioneering contributions:

- **i).** To investigate novel p, q-quasirung orthopair fuzzy operational laws based on the AA t-norm and t-conorm and their accompanying results.
- **ii).** To explore the p, q-quasirung orthopair fuzzy Aszel-Alsina weighted averaging (p, q-ROFAAWA) operator, p, q-quasirung orthopair fuzzy AA ordered weighted averaging (p,q-ROFAAOWA) operator, p, q-quasirung orthopair fuzzy AA hybrid averaging (p, q-ROFAAHA) operator, p, q-quasirung orthopair fuzzy AA weighted geometric (p, q-ROFAAWG) operator, p, q-quasirung orthopair fuzzy AA ordered weighted geometric (p, q-ROFAAOWG) operator, and p, q-quasirung orthopair fuzzy AA hybrid geometric (p, q-ROFAAHG) operator, using the investigated operational laws and investigate some of their fundamental results.
- **iii).** To construct an MCDM approach utilizing the explored operators to find the optimal alternative using p, q-ROFSs.
- **iv).** To demonstrate a practical example for analyzing the validity and capability of the proposed operators.
- **v).** To conduct comparisons between the presented operators and certain preexisting operators.

The flowchart of the developed work is depicted in Fig. 1.

This paper's structure is as follows: Section II will go through the fundamental ideas of p, q-ROFSs and AA triangular norms. Section III describes the AA operational laws for p,q-ROFNs. Section IV defines q-quasirung orthopair fuzzy AA averaging and geometric operators and proves some of their desirable properties and special cases. Section V uses the suggested operators to build a decision-making framework for dealing with MCDM problems using q-ROFNs as characteristic values. Section VI provides a case study concerning corruption intensity to demonstrate how the suggested method might be implemented. This section also investigates how a parameter influences decision-making results. Section VII provides a comparative examination of various acceptable approaches to demonstrate the adequacy of the provided technique. Section VIII includes concluding comments and forthcoming study topics.

II. SOME BASIC CONCEPTS

In this part, we will present t-norm, t-conorm, AA t-norm, AA t-conorms, and some core concepts of p,q-ROFSs to help readers comprehend the work.

Definition 1 ([37]): A t-norm is a function $T : [0,1]^2 \longrightarrow [0,1]$ that meets

T1. $T(\text{Ш}_1, \text{Ш}_2) = T(\text{Ш}_2, \text{Ш}_1)\ \forall\, \text{Ш}_1, \text{Ш}_2 \in [0,1]$;

T2. $T(\text{Ш}_1, \text{Ш}_2) \leq T(\text{Ш}_3, \text{Ш}_4)$ if $\text{Ш}_1 \leq \text{Ш}_3, \text{Ш}_2 \leq \text{Ш}_4\ \forall\, \text{Ш}_1, \text{Ш}_2, \text{Ш}_3, \text{Ш}_4 \in [0,1]$;

T3. $T(\text{Ш}, 1) = \text{Ш}\ \forall\, \text{Ш} \in [0,1]$;

T4. $T(\text{Ш}_1, T(\text{Ш}_2, \text{Ш}_3)) = T(T(\text{Ш}_1, \text{Ш}_2), \text{Ш}_3)$.

Some examples of t-norms

1). $T_P(\text{Ш}_1, \text{Ш}_2) = \text{Ш}_1\text{Ш}_2$ (product t-norm),

2). $T_M(\text{Ш}_1, \text{Ш}_2) = \min(\text{Ш}_1, \text{Ш}_2)$ (minimum t-norm),

3). $T_L(\text{Ш}_1, \text{Ш}_2) = \max(\text{Ш}_1 + \text{Ш}_2 - 1, 0)$ (Lukasiewicz t-norm),

4). $$T_D(\text{Ш}_1, \text{Ш}_2) = \begin{cases} \text{Ш}_1, & \text{if } \text{Ш}_2 = 1 \\ \text{Ш}_2, & \text{if } \text{Ш}_1 = 1 \\ & \text{(drastic t-norm)} \\ 0, & \text{otherwise.} \end{cases}$$

$\forall\, \text{Ш}_1, \text{Ш}_2 \in [0,1]$.

Definition 2 ([42]): A t-conorm is a function $S : [0,1]^2 \longrightarrow [0,1]$ that meets

S1. $S(\text{Ш}_1, \text{Ш}_2) = S(\text{Ш}_2, \text{Ш}_1)\ \forall\, \text{Ш}_1, \text{Ш}_2 \in [0,1]$;

S2. $S(\text{Ш}_1, \text{Ш}_2) \leq S(\text{Ш}_3, \text{Ш}_4)$ if $\text{Ш}_1 \leq \text{Ш}_3, \text{Ш}_2 \leq \text{Ш}_4\ \forall\, \text{Ш}_1, \text{Ш}_2, \text{Ш}_3, \text{Ш}_4 \in [0,1]$;

S3. $S(\text{Ш}, 0) = \text{Ш}\ \forall\, \text{Ш} \in [0,1]$;

S4. $S(\text{Ш}_1, S(\text{Ш}_2, \text{Ш}_3)) = S(S(\text{Ш}_1, \text{Ш}_2), \text{Ш}_3)$.

Some examples of t-conorms

1). $S_P(\text{Ш}_1, \text{Ш}_2) = \text{Ш}_1 + \text{Ш}_2 - \text{Ш}_1\text{Ш}_2$ (probabilistic sum),

2). $S_M(\text{Ш}_1, \text{Ш}_2) = \max(\text{Ш}_1, \text{Ш}_2)$ (maximum t-conorm),

3). $S_L(\text{Ш}_1, \text{Ш}_2) = \min(\text{Ш}_1 + \text{Ш}_2, 1)$ (Lukasiewicz t-conorm),

4). $$S_D(\text{Ш}_1, \text{Ш}_2) = \begin{cases} \text{Ш}_1, & \text{if } \text{Ш}_2 = 0 \\ \text{Ш}_2, & \text{if } \text{Ш}_1 = 0 \\ & \text{(drastic t-conorm)} \\ 1, & \text{otherwise.} \end{cases}$$

$\forall\, \text{Ш}_1, \text{Ш}_2 \in [0,1]$.

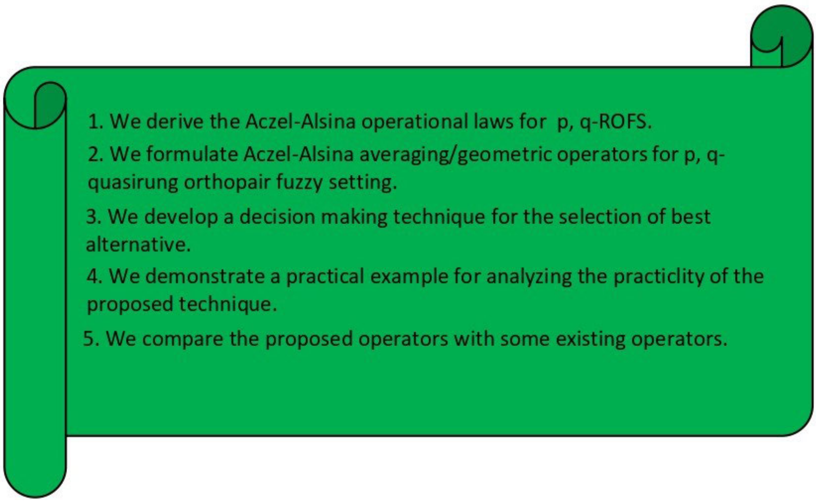

FIGURE 1. **Geometrical illustration of the proposed work.**

Additionally, it established the fact [42] that when T is a t-norm and S is a t-conorm, then $T(\amalg_1, \amalg_2) \leq \min\{\amalg_1, \amalg_2\}$ and $S(\amalg_1, \amalg_2) \geq \max\{\amalg_1, \amalg_2\}$ $\forall$ $\amalg_1, \amalg_2 \in [0, 1]$.

Definition 3 ([43]): The AA t-norm $\left(T_A^{\Lambda}\right)_{\Lambda\in[0,\infty]}$ is postulated as

$$T_A^{\Lambda}(\amalg_1, \amalg_2) = \begin{cases} T_D(\amalg_1, \amalg_2), & \text{if } \Lambda = 0 \\ \min\{\amalg_1, \amalg_2\}, & \text{if } \Lambda = \infty \\ e^{-\left((-\ln \amalg_1)^{\Lambda} + (-\ln \amalg_2)^{\Lambda}\right)^{1/\Lambda}}, & \text{otherwise.} \end{cases} \tag{1}$$

Some special cases: $T_A^{\infty} = \min$, $T_A^0 = T_D$, $T_A^1 = T_P$.

Definition 4 ([46]): The AA t-conorm $\left(S_A^{\Lambda}\right)_{\Lambda\in[0,\infty]}$ is postulated as

$$S_A^{\Lambda}(\amalg_1, \amalg_2) = \begin{cases} S_D(\amalg_1, \amalg_2), & \text{if } \Lambda = 0 \\ \max\{\amalg_1, \amalg_2\}, & \text{if } \Lambda = \infty \\ 1 - e^{-\left((-\ln(1-\amalg_1))^{\Lambda} + (-\ln(1-\amalg_2))^{\Lambda}\right)^{1/\Lambda}}, & \text{otherwise} \end{cases} \tag{2}$$

Some special cases: $S_A^{\infty} = \max$, $S_A^0 = S_D$, $S_A^1 = S_P$.

The t-norm T_A^{Λ} and t-conorm S_A^{Λ} are dual with respect to each other $\forall$ $\Lambda \in [0, \infty]$. Further, T_A^{Λ} and S_A^{Λ} are strictly increasing and strictly decreasing, respectively.

It is worthy to note that the AA category of t-norms are the only ones that meet the equivalence $T_A^{\Lambda}\left(\amalg_1^{\lambda}, \amalg_2^{\lambda}\right) = T_A^{\Lambda}(\amalg_1, \amalg_2)^{\lambda}$ $\forall$ $\lambda > 0$ and $\amalg_1, \amalg_2 \in [0, 1]$.

In this section, we present a concise overview of q-ROFSs.

Definition 5 ([36]): Let X be a fixed set. A p,q-ROFS $\mathcal{P}$ on X is described as

$$\mathcal{P} = \{(\amalg, \mu_{\mathcal{P}}(\amalg), \nu_{\mathcal{P}}(\amalg)) \,|\, \amalg \in X\}, \quad p, q \geq 1, \tag{3}$$

where $\mu_{\mathcal{P}}(\amalg)$ $\nu_{\mathcal{P}}(\amalg) \in [0, 1]$ denote the membership and non-membership grades of $\amalg \in X$, respectively, accorded that $0 \leq (\mu_{\mathcal{P}}(\amalg))^p + (\nu_{\mathcal{P}}(\amalg))^q \leq 1$. The degree of indeterminacy is $(\pi_{\mathcal{P}}(\amalg))^{\ell} = 1 - (\mu_{\mathcal{P}}(\amalg))^p + (\nu_{\mathcal{P}}(\amalg))^q$, where ℓ is the least common multiple (lcm) of p and q. For convince, Seikh and Mandal [36] termed $\mathcal{P} = (\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ a p,q-ROFN.

Remark 1: Consider the scenario in which we must determine the minimal value of $p, q \geq 1$ for a given orthopair $(\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ such that $\mu_{\mathcal{P}}^p + \nu_{\mathcal{P}}^q \leq 1$ is satisfied. Even though there is no closed-form solution, it is always feasible to develop a unique solution to these problems using iterative computing methods. The minimal values of p and q fulfilling $\mu_{\mathcal{P}}^p + \nu_{\mathcal{P}}^q \leq 1$ will be referred to as the p, q-niche of $(\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$. If p', q' is the p, q-niche of $(\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$, then $(\mu_{\mathcal{P}}, \nu_{\mathcal{P}})$ is valid for all $p \geq p'$ and $q \geq q'$.

Let $X = \{\amalg_1, \amalg_2, \ldots, \amalg_{\eth}\}$ be some supplied data and $\mathscr{F}$ be a fuzzy notion. Assume an expert offers his choice for each $\amalg_\iota \in X$ as an orthopair $(\mu_{\mathcal{P}}(\amalg_\iota), \nu_{\mathcal{P}}(\amalg_\iota))$. Now, the problem is to estimate the proper values of p and q to accurately reflect the data. We may now continue as follows:

- For each p, q-orthopair $(\mu_{\mathcal{P}}(\amalg_\iota), \nu_{\mathcal{P}}(\amalg_\iota))$ find its p, q-niche, say p_ι, q_ι.
- Set out the p^*, q^*-niche such that $p^* = \max_\iota p_\iota$ and $q^* = \max_\iota q_\iota$.
- Then we can represent $\mathscr{F}$ as p^*, q^*-ROFS.

Remark 2:
- The Definition 5 reduced to IFS if we set $p = q = 1$.
- The Definition 5 reduced to PyFS if we set $p = q = 2$.
- The Definition 5 reduced to FFS if we set $p = q = 3$.
- The Definition 5 reduced to q-ROFS if we set $p = q$.
- The Definition 5 reduced to 3,4-quasirung fuzzy set if we set $p = 3, q = 4$.

Definition 6 ([36]): Let $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be any three p,q-ROFNs and $\eta > 0$, then the basic rules of operation on them are listed as

1) $\mathcal{P}_1 \oplus \mathcal{P}_2 = \left(\begin{array}{c} \left(\mu_{\mathcal{P}_1}^{p^*} + \mu_{\mathcal{P}_2}^{p^*} - \mu_{\mathcal{P}_1}^{p^*}\mu_{\mathcal{P}_2}^{p^*}\right)^{1/p^*}, \\ \nu_{\mathcal{P}_1}\nu_{\mathcal{P}_2} \end{array} \right);$

2) $\mathcal{P}_1 \otimes \mathcal{P}_2 = \left(\begin{array}{c} \mu_{\mathcal{P}_1}\mu_{\mathcal{P}_2}, \\ \left(\nu_{\mathcal{P}_1}^{q^*} + \nu_{\mathcal{P}_2}^{q^*} - \nu_{\mathcal{P}_1}^{q^*}\nu_{\mathcal{P}_2}^{q^*}\right)^{1/p^*} \end{array} \right);$

3) $\mathcal{P}^\eta = \left(\mu_{\mathcal{P}}^\eta, \left(1-\left(1-\nu_{\mathcal{P}}^q\right)^\eta\right)^{1/q}\right);$

4) $\eta\mathcal{P} = \left(\left(1-\left(1-\mu_{\mathcal{P}}^p\right)^\eta\right)^{1/p}, \nu_{\mathcal{P}}^\eta\right);$

5) $\mathcal{P}^c = (\nu_{\mathcal{P}}, \mu_{\mathcal{P}})$.

Definition 7 ([36]): Let $\mathcal{P}$ be a p,q-ROFN, then the score function is characterized by:

$$S(\mathcal{P}) = \frac{1 + \mu_{\mathcal{P}}^p - \nu_{\mathcal{P}}^q}{2}, \tag{4}$$

where $p, q \in [1, \infty)$, $S(\mathcal{P}) \in [0, 1]$. The larger the value of $S(\mathcal{P})$, the larger the p,q-ROFN $\mathcal{P}$.

Definition 8 ([36]): Let $\mathcal{P}$ be a p,q-ROFN, then the degree of accuracy is defined in the following manner:

$$A(\mathcal{P}) = (\mu_{\mathcal{P}})^p + (\nu_{\mathcal{P}})^q; \quad A(\mathcal{P}) \in [0, 1]. \tag{5}$$

When the computed score values are similar, the larger the degree of accuracy $A(\mathcal{P})$, the larger the p,q-ROFN.

III. *p,q*-QUASIRUNG ORTHOPAIR FUZZY AA OPERATIONAL LAWS

In view of the Definitions 3 and 4, in what follows, we put forward some generalized operational rules of p,q-ROFNs and their relevant characteristics.

Definition 9: Let $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be any three p,q-ROFNs, $\Lambda \geq 1$ and $\lambda > 0$, then the AA t-norm and t-conorm operations on them are given by

1)

$$\mathcal{P}_1 \oplus \mathcal{P}_2 = \left(\sqrt[p^*]{1 - e^{-\left(\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^\Lambda + \left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\left(-\ln \nu_{\mathcal{P}_1}^{q^*}\right)^\Lambda + \left(-\ln \nu_{\mathcal{P}_2}^{q^*}\right)^\Lambda\right)^{1/\Lambda}}} \right);$$

2)

$$\mathcal{P}_1 \otimes \mathcal{P}_2 = \left(\sqrt[p^*]{e^{-\left(\left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^\Lambda + \left(-\ln \mu_{\mathcal{P}_2}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1 - e^{-\left(\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^\Lambda + \left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right);$$

3)

$$\mathcal{P}^\lambda = \left(\sqrt[q]{e^{-\left(\lambda\left(-\ln \mu_{\mathcal{P}}^q\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q]{1 - e^{-\left(\lambda\left(-\ln\left(1-\nu_{\mathcal{P}}^q\right)\right)^\Lambda\right)^{1/\Lambda}}} \right);$$

4)

$$\lambda\mathcal{P} = \left(\sqrt[q]{1 - e^{-\left(\lambda\left(-\ln\left(1-\mu_{\mathcal{P}}^q\right)\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q]{e^{-\left(\lambda\left(-\ln \nu_{\mathcal{P}}^q\right)^\Lambda\right)^{1/\Lambda}}} \right).$$

Theorem 1: Let $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ be any three p,q-ROFNs, then we have

1) $\mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_2 \oplus \mathcal{P}_1$;
2) $\mathcal{P}_1 \otimes \mathcal{P}_2 = \mathcal{P}_2 \otimes \mathcal{P}_1$;
3) $\lambda(\mathcal{P}_1 \oplus \mathcal{P}_2) = \lambda\mathcal{P}_1 \oplus \lambda\mathcal{P}_2, \ \lambda > 0$;
4) $(\lambda_1 + \lambda_2)\mathcal{P} = \lambda_1\mathcal{P} \oplus \lambda_2\mathcal{P}, \ \lambda_1, \lambda_2 > 0$;
5) $(\mathcal{P}_1 \otimes \mathcal{P}_2)^\lambda = \mathcal{P}_1^\lambda \otimes \mathcal{P}_2^\lambda, \ \lambda > 0$;
6) $\mathcal{P}^{\lambda_1} \otimes \mathcal{P}^{\lambda_2} = \mathcal{P}^{(\lambda_1+\lambda_2)}, \ \lambda_1, \lambda_2 > 0$.

Proof:

1)

$$\begin{aligned} \mathcal{P}_1 \oplus \mathcal{P}_2 &= \left(\sqrt[p^*]{1 - e^{-\left(\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^\Lambda + \left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\left(-\ln \nu_{\mathcal{P}_1}^{q^*}\right)^\Lambda + \left(-\ln \nu_{\mathcal{P}_2}^{q^*}\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{1 - e^{-\left(\left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^\Lambda + \left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\left(-\ln \nu_{\mathcal{P}_2}^{q^*}\right)^\Lambda + \left(-\ln \nu_{\mathcal{P}_1}^{q^*}\right)^\Lambda\right)^{1/\Lambda}}} \right) = \mathcal{P}_2 \oplus \mathcal{P}_1. \end{aligned}$$

2)

$$\begin{aligned} \mathcal{P}_1 \otimes \mathcal{P}_2 &= \left(\sqrt[p^*]{e^{-\left(\left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^\Lambda + \left(-\ln \mu_{\mathcal{P}_2}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1 - e^{-\left(\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^\Lambda + \left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{e^{-\left(\left(-\ln \mu_{\mathcal{P}_2}^{p^*}\right)^\Lambda + \left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \right. \end{aligned}$$

$$\left.\sqrt[q^*]{1-e^{-\left(\left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^{\Lambda}+\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$= \mathcal{P}_2 \otimes \mathcal{P}_1.$$

3)

$$\lambda\left(\mathcal{P}_1 \oplus \mathcal{P}_2\right) = \left(\sqrt[p^*]{1-e^{-\left(\lambda\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^{\Lambda}+\left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\lambda\left(-\ln \nu_{\mathcal{P}_1}^{q^*}\right)^{\Lambda}+\left(-\ln \nu_{\mathcal{P}_2}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$= \left(\sqrt[p^*]{1-e^{-\left(\lambda\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\lambda\left(-\ln \nu_{\mathcal{P}_1}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right) \oplus \left(\sqrt[p^*]{1-e^{-\left(\lambda\left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\lambda\left(-\ln \nu_{\mathcal{P}_2}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$= \lambda\mathcal{P}_1 \oplus \lambda\mathcal{P}_2.$$

4)

$$\lambda_1\mathcal{P} \oplus \lambda_2\mathcal{P} = \left(\sqrt[p]{1-e^{-\left(\lambda_1\left(-\ln\left(1-\mu_{\mathcal{P}}^{p}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q]{e^{-\left(\lambda_1\left(-\ln \nu_{\mathcal{P}}^{q}\right)^{\Lambda}\right)^{1/\Lambda}}}\right) \oplus \left(\sqrt[p]{1-e^{-\left(\lambda_2\left(-\ln\left(1-\mu_{\mathcal{P}}^{p}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q]{e^{-\left(\lambda_2\left(-\ln \nu_{\mathcal{P}}^{q}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$= \left(\sqrt[p]{1-e^{-\left((\lambda_1+\lambda_2)\left(-\ln\left(1-\mu_{\mathcal{P}}^{p}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q]{e^{-\left((\lambda_1+\lambda_2)\left(-\ln \nu_{\mathcal{P}}^{q}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$= (\lambda_1 + \lambda_2)\,\mathcal{P}.$$

5)

$$\left(\mathcal{P}_1 \otimes \mathcal{P}_2\right)^{\lambda} = \left(\sqrt[p^*]{e^{-\left(\left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^{\Lambda}+\left(-\ln \mu_{\mathcal{P}_2}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^{\Lambda}+\left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}\right)^{\lambda}$$

$$= \left(\sqrt[p^*]{e^{-\left(\lambda\left(\left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^{\Lambda}+\left(-\ln \mu_{\mathcal{P}_2}^{p^*}\right)^{\Lambda}\right)\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\lambda\left(\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^{\Lambda}+\left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^{\Lambda}\right)\right)^{1/\Lambda}}}\right)$$

$$= \left(\sqrt[p^*]{e^{-\left(\lambda\left(\left(-\ln \mu_{\mathcal{P}_1}^{p^*}\right)^{\Lambda}\right)\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\lambda\left(\left(-\ln\left(1-\nu_{\mathcal{P}_1}^{q^*}\right)\right)^{\Lambda}\right)\right)^{1/\Lambda}}}\right) \otimes \left(\sqrt[p^*]{e^{-\left(\lambda\left(\left(-\ln \mu_{\mathcal{P}_2}^{p^*}\right)^{\Lambda}\right)\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\lambda\left(\left(-\ln\left(1-\nu_{\mathcal{P}_2}^{q^*}\right)\right)^{\Lambda}\right)\right)^{1/\Lambda}}}\right)$$

$$= \mathcal{P}_1^{\lambda} \otimes \mathcal{P}_2^{\lambda}.$$

6)

$$\mathcal{P}^{\lambda_1} \otimes \mathcal{P}^{\lambda_2} = \left(\sqrt[p^*]{e^{-\left(\lambda_1\left(-\ln \mu_{\mathcal{P}}^{p^*}\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\lambda_1\left(-\ln\left(1-\nu_{\mathcal{P}}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}\right) \otimes \left(\sqrt[p^*]{e^{-\left(\lambda_2\left(-\ln \mu_{\mathcal{P}}^{p^*}\right)^{\Lambda}\right)^{1/\Lambda}}},\right.$$

$$\left.\sqrt[q^*]{1-e^{-\left(\lambda_2\left(-\ln\left(1-\nu_{\mathcal{P}}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$=\left(\sqrt[p^*]{e^{-\left((\lambda_1+\lambda_2)\left(-\ln\mu_{\mathcal{P}}^{p^*}\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{1-e^{-\left((\lambda_1+\lambda_2)\left(-\ln\left(1-\nu_{\mathcal{P}}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$=\mathcal{P}^{(\lambda_1+\lambda_2)}.$$

■

IV. p,q-QUASIRUNG ORTHOPAIR FUZZY AA AGGREGATION OPERATORS

This segment presents various p,q-quasirung orthopair fuzzy AA aggregation operators based on the arithmetic average operator and the geometric average operator.

A. p,q-QUASIRUNG ORTHOPAIR FUZZY AA AVERAGING AGGREGATION OPERATORS

Based on the proposed operations, in this section, we introduce some novel averaging aggregation operators, including p,q-quasirung orthopair fuzzy AA average (p,q-ROFAAA) operator, p,q-quasirung orthopair fuzzy AA weighted averaging (p,q-ROFAAWA) operator, p,q-quasirung orthopair fuzzy AA ordered weighted averaging (p,q-ROFAAOWA) operator, p,q-quasirung orthopair fuzzy AA ordered weighted averaging (p,q-ROFAAOWA) operator, and p,q-quasirung orthopair fuzzy AA hybrid averaging (p,q-ROFAAHA) operator. In addition, we investigate some special cases and properties of these operators.

Definition 10: Let $\mathcal{P}_\iota=\left(\mu_{\mathcal{P}_\iota},\nu_{\mathcal{P}_\iota}\right)$ $(\iota=1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAWA operator is:

$$p,q-ROFAAWA\left(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\right)=\oplus_{\iota=1}^{\eth}\left(w_\iota\mathcal{P}_\iota\right), \quad (6)$$

where $w=(w_1,w_2,\ldots,w_\eth)^T$ is the weight vector of $\mathcal{P}_\iota$ $(\iota=1,2,\ldots,\eth)$ such that $w_\iota>0$ and $\sum_{\iota=1}^{\eth}w_\iota=1$. Especially, if $w=\left(\frac{1}{\eth},\frac{1}{\eth},\ldots,\frac{1}{\eth}\right)^T$, then the p,q-ROFAAWA operator reduces to p,q-ROFAAA operator of dimension $\eth$, which is described as follows:

$$p,q-ROFAAA\left(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\right)=\frac{1}{\eth}\oplus_{\iota=1}^{\eth}\left(\mathcal{P}_\iota\right). \quad (7)$$

Theorem 2: Let $\mathcal{P}_\iota=\left(\mu_{\mathcal{P}_\iota},\nu_{\mathcal{P}_\iota}\right)$ $(\iota=1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAWA operator is still a p,q-ROFN, and

$$p,q-ROFAAWA\left(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\right)$$

$$=\left(\sqrt[p^*]{1-e^{-\left(\sum_{\iota=1}^{\eth}w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth}w_\iota\left(-\ln\nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right). \quad (8)$$

Proof: We can prove Theorem 2 with the help of the mathematical induction method in the following way:

For $\eth=2$, we have

$$p,q-ROFAAWA\left(\mathcal{P}_1,\mathcal{P}_2\right)=w_1\mathcal{P}_1\oplus w_2\mathcal{P}_2$$

$$=\left(\sqrt[p^*]{1-e^{-\left(w_1\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{e^{-\left(w_1\left(-\ln\nu_{\mathcal{P}_1}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$\oplus\left(\sqrt[p^*]{1-e^{-\left(w_2\left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{e^{-\left(w_2\left(-\ln\nu_{\mathcal{P}_2}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$=\left(\sqrt[p^*]{1-e^{-\left(w_1\left(-\ln\left(1-\mu_{\mathcal{P}_1}^{p^*}\right)\right)^{\Lambda}+w_2\left(-\ln\left(1-\mu_{\mathcal{P}_2}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{e^{-\left(w_1\left(-\ln\nu_{\mathcal{P}_1}^{q^*}\right)^{\Lambda}+w_2\left(-\ln\nu_{\mathcal{P}_2}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right)$$

$$=\left(\sqrt[p^*]{1-e^{-\left(\sum_{\iota=1}^{2}w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}},\ \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{2}w_\iota\left(-\ln\nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}}\right).$$

Hence, the result is true for $\eth=2$.

Suppose that Eq. (8) is true for $\eth=k$, then we have

$$p,q-ROFAAWA\left(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_k\right)=\oplus_{\iota=1}^{k}\left(w_\iota\mathcal{P}_\iota\right)$$

$$= \left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right)$$

Now for $\eth = k+1$, we have

$$\begin{aligned} &p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k, \mathcal{P}_{k+1}\right) \\ &= \oplus_{\iota=1}^{k}\left(w_\iota \mathcal{P}_\iota\right) \oplus \left(w_{k+1}\mathcal{P}_{k+1}\right) \\ &= \left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{k} w_\iota\left(-\ln \nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \\ &\oplus \left(\sqrt[p^*]{1 - e^{-\left(w_{k+1}\left(-\ln\left(1-\mu_{\mathcal{P}_{k+1}}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(w_{k+1}\left(-\ln \nu_{\mathcal{P}_{k+1}}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{k+1} w_\iota\left(-\ln \nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right). \end{aligned}$$

Thus, Eq. (8) is legitimate for $\eth = k+1$, and hence, by the principle of mathematical induction, the result given in Eq. (8) is true for all positive integer $\eth$. ■

Theorem 3 (Idempotency): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1, 2, \ldots, \eth)$ be a family of *p*,*q*-ROFNs, if $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, then

$$p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \mathcal{P}. \tag{9}$$

Proof: Since $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, and $\sum_{\iota=1}^{\eth} w_\iota = 1$ so by Theorem 2, we have

$$\begin{aligned} &p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &= \left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln \nu_{\mathcal{P}}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{1 - e^{-\left(\left(-\ln\left(1-\mu_{\mathcal{P}}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\left(-\ln \nu_{\mathcal{P}}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \\ &= \left(\mu_{\mathcal{P}}, \nu_{\mathcal{P}}\right) = \mathcal{P}. \end{aligned}$$

■

Theorem 4 (Monotonicity): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1, 2, \ldots, \eth)$ and $\dot{\mathcal{P}}_\iota = \left(\dot{\mu}_{\mathcal{P}_\iota}, \dot{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be two families of *p*,*q*-ROFNs, such that $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$, then

$$\begin{aligned} &p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &\quad \geq p, q - ROFAAWA\left(\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2, \ldots, \dot{\mathcal{P}}_\eth\right). \end{aligned} \tag{10}$$

Proof: Since $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$. Based on these, we have the subsequent inequalities,

$$\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}} \geq \sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\dot{\mu}_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}$$

and

$$\sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\nu_{\mathcal{P}_\iota}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}} \leq \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\dot{\nu}_{\mathcal{P}_\iota}^{q^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}$$

which implies that

$$\begin{aligned} &\left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln \nu_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \\ &\geq \left(\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\dot{\mu}_{\mathcal{P}_\iota}^{p^*}\right)\right)^{\Lambda}\right)^{1/\Lambda}}}, \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln \dot{\nu}_{\mathcal{P}_\iota}^{q^*}\right)^{\Lambda}\right)^{1/\Lambda}}} \right) \end{aligned}$$

Hence $p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \geq p, q - ROFAAWA\left(\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2, \ldots, \dot{\mathcal{P}}_\eth\right)$. ■

Theorem 5 (Boundedness): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1, 2, \ldots, \eth)$ be a family of *p*,*q*-ROFNs, and let $\mathcal{P}^- = \min\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\}$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\}$, then

$$\mathcal{P}^- \leq p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \leq \mathcal{P}^+. \tag{11}$$

Proof: As given that $\mathcal{P}^- = \min\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\} = \left(\mu_{\mathcal{P}}^-, \nu_{\mathcal{P}}^-\right)$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\} = \left(\mu_{\mathcal{P}}^+, \nu_{\mathcal{P}}^+\right)$, where $\mu_{\mathcal{P}}^- = \min\left\{\mu_{\mathcal{P}1}, \mu_{\mathcal{P}2}, \ldots, \mu_{\mathcal{P}\eth}\right\}$, $\nu_{\mathcal{P}}^- = \max\{\nu_{\mathcal{P}1}, \nu_{\mathcal{P}2}, \ldots, \nu_{\mathcal{P}\eth}\}$, $\mu_{\mathcal{P}}^+ = \max\{\mu_{\mathcal{P}1}, \mu_{\mathcal{P}2}, \ldots, \mu_{\mathcal{P}\eth}\}$, and $\nu_{\mathcal{P}}^+ = \min\{\nu_{\mathcal{P}1}, \nu_{\mathcal{P}2}, \ldots, \nu_{\mathcal{P}\eth}\}$. As a result, there are ongoing inequities:

$$\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu^{-p^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu^{p^*}_{\mathcal{P}_\iota}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\mu^{+p^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

$$\sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\nu^{+q^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\nu^{q^*}_{\mathcal{P}_\iota}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\nu^{-q^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}}.$$

Thereby, $\mathcal{P}^- \leq p, q-ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \leq \mathcal{P}^+$. ∎

Theorem 6 (Symmetry): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs. Then, if $\breve{\mathcal{P}}_\iota = \left(\breve{\mu}_{\mathcal{P}_\iota}, \breve{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be any permutation of $\mathcal{P}_\iota$, then we have

$$p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = p, q - ROFAAWA\left(\breve{\mathcal{P}}_1, \breve{\mathcal{P}}_2, \ldots, \breve{\mathcal{P}}_\eth\right). \tag{12}$$

Proof: The proof is obvious and thus omitted. ∎

Next, we introduce p,q-quasirung orthopair fuzzy AA ordered weighted averaging (p,q-ROFAAOWA) operator.

Definition 11: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAOWA operator is:

$$p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \oplus_{\iota=1}^{\eth}\left(\varpi_\iota \mathcal{P}_{\delta(\iota)}\right), \tag{13}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_\eth)^T$ is the position weights of $\mathcal{P}_\iota$ $(\iota = 1, 2, \ldots, \eth)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\eth} \varpi_\iota = 1$. $(\delta(1), \delta(2), \ldots, \delta(\eth))$ is a permutation of $(1, 2, \ldots, \eth)$ such that $\mathcal{P}_{\delta(\iota-1)} \geq \mathcal{P}_{\delta(\iota)}$ for $\iota = 1, 2, \ldots, \eth$.

Theorem 7: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAOWA operator is still a p,q-ROFN, and

$$p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \left(\begin{array}{c}\sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} \varpi_\iota\left(-\ln\left(1-\mu^{q^*}_{\mathcal{P}_{\delta(\iota)}}\right)\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} \varpi_\iota\left(-\ln \nu^{q^*}_{\mathcal{P}_{\delta(\iota)}}\right)^\Lambda\right)^{1/\Lambda}}}\end{array}\right). \tag{14}$$

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 2. ∎

The following features may be efficiently shown by using the p,q-ROFAAOWA operator.

Theorem 8 (Idempotency): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, if $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, then

$$p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \mathcal{P}. \tag{15}$$

Theorem 9 (Monotonicity): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ and $\dot{\mathcal{P}}_\iota = \left(\dot{\mu}_{\mathcal{P}_\iota}, \dot{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be two families of p,q-ROFNs, such that $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$, then

$$p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \geq p, q - ROFAAOWA\left(\dot{\mathcal{P}}_1, \dot{\mathcal{P}}_2, \ldots, \dot{\mathcal{P}}_\eth\right). \tag{16}$$

Theorem 10 (Boundedness): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, and let $\mathcal{P}^- = \min\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\}$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\}$, then

$$\mathcal{P}^- \leq p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \leq \mathcal{P}^+. \tag{17}$$

Theorem 11 (Symmetry): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs. Then, if $\breve{\mathcal{P}}_\iota = \left(\breve{\mu}_{\mathcal{P}_\iota}, \breve{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be any permutation of $\mathcal{P}_\iota$, then we have

$$p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = p, q - ROFAAWA\left(\breve{\mathcal{P}}_1, \breve{\mathcal{P}}_2, \ldots, \breve{\mathcal{P}}_\eth\right). \tag{18}$$

The p,q-ROFAAWA operator weights just the p,q-ROFNs, as defined by Definition 10, while the p,q-ROFAAOWA operator weights only the ordered locations of the p,q-ROFNs, as defined by Definition 11. As a result, weights represent various aspects of the p,q-ROFAAWA and p,q-ROFAAOWA operators. Nonetheless, one of the operators, as well as the other operators, consider just one of them. To address this shortcoming, we study the p,q-quasirung orthopair fuzzy AA hybrid averaging (p,q-ROFAAHA) operator, which weights all of the provided p,q-ROFN and their appropriate ordered position.

Definition 12: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAHA operator is:

$$p, q - ROFAAHA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \oplus_{\iota=1}^{\eth}\left(\varpi_\iota \hat{\mathcal{P}}_{\delta(\iota)}\right), \tag{19}$$

where $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_\eth)^T$ is the weight vector associated with p,q-ROFAAHA $\mathcal{P}_\iota$ $(\iota = 1, 2, \ldots, \eth)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\eth} \varpi_\iota = 1$, $w = (w_1, w_2, \ldots, w_\eth)^T$ is the weight vector of $\mathcal{P}_\iota$ $(\iota = 1, 2, \ldots, \eth)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\eth} w_\iota = 1$. $\hat{\mathcal{P}}_{\delta(\iota)}$ is the ιth largest of the weighted p,q-ROFNs $\hat{\mathcal{P}}_\iota$ $\left(\hat{\mathcal{P}}_\iota = (\eth w_\iota)\, \mathcal{P}_\iota\right)$, $(\iota = 1, 2, \ldots, \eth)$ and $\eth$ is the balancing coefficient.

Theorem 12: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAHA operator is still a p,q-ROFN, and

$$p, q - ROFAAHA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \left(\begin{array}{c} \sqrt[p^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} \varpi_\iota \left(-\ln\left(1-\hat{\mu}^{p^*}_{\mathcal{P}_{\delta(\iota)}}\right)\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} \varpi_\iota \left(-\ln \hat{\nu}^{q^*}_{\mathcal{P}_{\delta(\iota)}}\right)^\Lambda\right)^{1/\Lambda}}} \end{array}\right). \tag{20}$$

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 2. ∎

As analogous to those of p,q-ROFAAWA operator and p,q-ROFAAOWA, the p,q-ROFAAHA operator also follows the idempotency, monotonicity, boundedness and symmetry properties. Besides the aforesaid characteristics, the p,q-ROFAAHA operator has the following special cases.

Corollary 1: p,q-ROFAAWA operator is a special case of the p,q-ROFAAHA operator.

Proof: Let $\varpi = \left(\frac{1}{\eth}, \frac{1}{\eth}, \ldots, \frac{1}{\eth}\right)^T$, then

$$\begin{aligned} &p, q - ROFAAHA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &= \varpi_1 \hat{\mathcal{P}}_{\delta(1)} \oplus \varpi_2 \hat{\mathcal{P}}_{\delta(2)} \oplus \ldots \oplus \varpi_\eth \hat{\mathcal{P}}_{\delta(\eth)} \\ &= \frac{1}{\eth}\left(\hat{\mathcal{P}}_{\delta(1)} \oplus \hat{\mathcal{P}}_{\delta(2)} \oplus \ldots \oplus \hat{\mathcal{P}}_{\delta(\eth)}\right) \\ &= w_1 \mathcal{P}_1 \oplus w_2 \mathcal{P}_2 \oplus \ldots \oplus w_\eth \mathcal{P}_\eth \\ &= p, q - ROFAAWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right). \end{aligned}$$

∎

Corollary 2: p,q-ROFAAOWA operator is a special case of the p,q-ROFAAHA operator.

Proof: Let $w = \left(\frac{1}{\eth}, \frac{1}{\eth}, \ldots, \frac{1}{\eth}\right)^T$, then

$$\begin{aligned} &p, q - ROFAAHA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &= \varpi_1 \hat{\mathcal{P}}_{\delta(1)} \oplus \varpi_2 \hat{\mathcal{P}}_{\delta(2)} \oplus \ldots \oplus \varpi_\eth \hat{\mathcal{P}}_{\delta(\eth)} \\ &= \varpi_1 \mathcal{P}_{\delta(1)} \oplus \varpi_2 \mathcal{P}_{\delta(2)} \oplus \ldots \oplus \varpi_\eth \mathcal{P}_{\delta(\eth)} \\ &= p, q - ROFAAOWA\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right). \end{aligned}$$

∎

B. p,q-QUASIRUNG ORTHOPAIR FUZZY AA GEOMETRIC AGGREGATION OPERATORS

Based on the designed operations, in this section, we put forward some novel geometric aggregation operators including p,q-quasirung orthopair fuzzy AA geometric (p,q-ROFAAG) operator, p,q-quasirung orthopair fuzzy AA weighted geometric (p,q-ROFAAWG) operator, p,q-quasirung orthopair fuzzy AA ordered weighted averaging (p,q-ROFAAOWA) operator, p,q-quasirung orthopair fuzzy AA ordered weighted geometric (p,q-ROFAAOWG) operator, and p,q-quasirung orthopair fuzzy AA hybrid geometric (p,q-ROFAAHG) operator. In addition, we investigate some special cases and properties of these operators.

Definition 13: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAWG operator is:

$$p, q - ROFAAWG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \otimes_{\iota=1}^{\eth} \left(\mathcal{P}_\iota\right)^{w_\iota}, \tag{21}$$

where $w = (w_1, w_2, \ldots, w_\eth)^T$ is the weight vector of $\mathcal{P}_\iota$ $(\iota = 1, 2, \ldots, \eth)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\eth} w_\iota = 1$. Especially, if $w = \left(\frac{1}{\eth}, \frac{1}{\eth}, \ldots, \frac{1}{\eth}\right)^T$, then the p,q-ROFAAWG operator reduces to p,q-ROFAAG operator of dimension $\eth$, which is described as follows:

$$p, q - ROFAAG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \otimes_{\iota=1}^{\eth} \left(\mathcal{P}_\iota\right)^{\frac{1}{\eth}}. \tag{22}$$

Theorem 13: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1, 2, \ldots, \eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAWG operator is still a p,q-ROFN, and

$$p, q - ROFAAWG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) = \left(\begin{array}{c} \sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln \mu^{p^*}_{\mathcal{P}_\iota}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1 - e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_\iota}\right)\right)^\Lambda\right)^{1/\Lambda}}} \end{array}\right). \tag{23}$$

Proof: We can prove Theorem 13 with the help of the mathematical induction method in the following way:

For $\eth = 2$, we have

$$\begin{aligned} &p, q - ROFAAWG\left(\mathcal{P}_1, \mathcal{P}_2\right) = \mathcal{P}_1^{w_1} \otimes \mathcal{P}_2^{w_2} \\ &= \left(\begin{array}{c} \sqrt[p^*]{e^{-\left(w_1\left(-\ln \mu^{p^*}_{\mathcal{P}_1}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1 - e^{-\left(w_1\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_1}\right)\right)^\Lambda\right)^{1/\Lambda}}} \end{array}\right) \\ &\quad \otimes \left(\begin{array}{c} \sqrt[p^*]{e^{-\left(w_2\left(-\ln \mu^{p^*}_{\mathcal{P}_2}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1 - e^{-\left(w_2\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_2}\right)\right)^\Lambda\right)^{1/\Lambda}}} \end{array}\right) \\ &= \left(\begin{array}{c} \sqrt[p^*]{e^{-\left(w_1\left(-\ln \mu^{p^*}_{\mathcal{P}_1}\right)^\Lambda + w_2\left(-\ln \mu^{p^*}_{\mathcal{P}_2}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1 - e^{-\left(\begin{array}{c} w_1\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_1}\right)\right)^\Lambda + \\ w_2\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_2}\right)\right)^\Lambda \end{array}\right)^{1/\Lambda}}} \end{array}\right) \\ &= \left(\begin{array}{c} \sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln \mu^{p^*}_{\mathcal{P}_\iota}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1 - e^{-\left(\sum_{\iota=1}^{2} w_\iota\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_\iota}\right)\right)^\Lambda\right)^{1/\Lambda}}} \end{array}\right). \end{aligned}$$

Hence, the result is true for $\eth = 2$.

Suppose that Eq. (23) is true for $\eth = k$, then we have

$$p, q - ROFAAWG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k\right) = \otimes_{\iota=1}^{k} \left(\mathcal{P}_\iota\right)^{w_\iota}$$

$$= \left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{k} w_\iota \left(-\ln \mu_{\mathcal{P}_\iota}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{k} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right)$$

Now for $\eth = k+1$, we have

$$\begin{aligned} &p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_k,\mathcal{P}_{k+1}) \\ &= \otimes_{\iota=1}^{k}(\mathcal{P}_\iota)^{w_\iota} \otimes (\mathcal{P}_{k+1})^{w_{k+1}} \\ &= \left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{k} w_\iota \left(-\ln \mu_{\mathcal{P}_\iota}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{k} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &\otimes \left(\sqrt[p^*]{e^{-\left(w_{k+1} \left(-\ln \mu_{\mathcal{P}_{k+1}}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(w_{k+1} \left(-\ln\left(1-\nu_{\mathcal{P}_{k+1}}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{k+1} w_\iota \left(-\ln \mu_{\mathcal{P}_\iota}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{k+1} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right). \end{aligned}$$

Thus, Eq. (23) is legitimate for $\eth = k+1$ and hence, by the principle of mathematical induction, result given in Eq. (23) is true for all positive integer $\eth$. ■

Theorem 14 (Idempotency): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, if $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, then

$$p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \mathcal{P}. \quad (24)$$

Proof: Since $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, and $\sum_{\iota=1}^{\eth} w_\iota = 1$ so by Theorem 13, we have

$$\begin{aligned} &p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \\ &= \left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln \mu_{\mathcal{P}}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &= \left(\sqrt[p^*]{e^{-\left(\left(-\ln \mu_{\mathcal{P}}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\left(-\ln\left(1-\nu_{\mathcal{P}}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &= (\mu_{\mathcal{P}}, \nu_{\mathcal{P}}) = \mathcal{P}. \end{aligned}$$

■

Theorem 15 (Monotonicity): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1,2,\ldots,\eth)$ and $\dot{\mathcal{P}}_\iota = (\dot{\mu}_{\mathcal{P}_\iota}, \dot{\nu}_{\mathcal{P}_\iota})$ $(\iota = 1,2,\ldots,\eth)$ be two families of p,q-ROFNs, such that $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$, then

$$\begin{aligned} &p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \\ &\quad \geq p,q-ROFAAWG(\dot{\mathcal{P}}_1,\dot{\mathcal{P}}_2,\ldots,\dot{\mathcal{P}}_\eth). \end{aligned} \quad (25)$$

Proof: Since $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$. Based on these, we have the subsequent inequalities,

$$\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \geq \sqrt[q^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(\dot{\mu}_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

and

$$\sqrt[p^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}_\iota}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\dot{\nu}_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

which implies that

$$\begin{aligned} &\left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln \mu_{\mathcal{P}_\iota}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\nu_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \\ &\geq \left(\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln \dot{\mu}_{\mathcal{P}_\iota}^{p^*}\right)^\Lambda\right)^{1/\Lambda}}}, \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(1-\dot{\nu}_{\mathcal{P}_\iota}^{q^*}\right)\right)^\Lambda\right)^{1/\Lambda}}} \right) \end{aligned}$$

Hence $p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \geq p,q-ROFAAWG(\dot{\mathcal{P}}_1,\dot{\mathcal{P}}_2,\ldots,\dot{\mathcal{P}}_\eth)$. ■

Theorem 16 (Boundedness): Let $\mathcal{P}_\iota = (\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota})$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, and let $\mathcal{P}^- = \min\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\}$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\}$, then

$$\mathcal{P}^- \leq p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \leq \mathcal{P}^+. \quad (26)$$

Proof: As given that $\mathcal{P}^- = \min\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\} = (\mu_{\mathcal{P}}^-, \nu_{\mathcal{P}}^-)$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\} = (\mu_{\mathcal{P}}^+, \nu_{\mathcal{P}}^+)$, where $\mu_{\mathcal{P}}^- = \min\{\mu_{\mathcal{P}1},\mu_{\mathcal{P}2},\ldots,\mu_{\mathcal{P}\eth}\}$, $\nu_{\mathcal{P}}^- = \max\{\nu_{\mathcal{P}1},\nu_{\mathcal{P}2},\ldots,\nu_{\mathcal{P}\eth}\}$, $\mu_{\mathcal{P}}^+ = \max\{\mu_{\mathcal{P}1},\mu_{\mathcal{P}2},\ldots,\mu_{\mathcal{P}\eth}\}$, and $\nu_{\mathcal{P}}^+ = \min\{\nu_{\mathcal{P}1},\nu_{\mathcal{P}2},\ldots,\nu_{\mathcal{P}\eth}\}$. As a result, there are ongoing inequities:

$$\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(\mu^{-p^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}} \leq \sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota \left(-\ln\left(\mu_{\mathcal{P}_\iota}^{p^*}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

$$\leq \sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(\mu^{+p^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}},$$

$$\sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\nu^{+q^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

$$\leq \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_\iota}\right)\right)^\Lambda\right)^{1/\Lambda}}}$$

$$\leq \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth} w_\iota\left(-\ln\left(1-\nu^{-q^*}_{\mathcal{P}}\right)\right)^\Lambda\right)^{1/\Lambda}}}.$$

Thereby,

$\mathcal{P}^- \leq p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \leq \mathcal{P}^+$. ■

Theorem 17 (Symmetry): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs. Then, if $\breve{\mathcal{P}}_\iota = \left(\breve{\mu}_{\mathcal{P}_\iota}, \breve{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be any permutation of $\mathcal{P}_\iota$, then we have

$$p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = p,q-ROFAAWG\left(\breve{\mathcal{P}}_1,\breve{\mathcal{P}}_2,\ldots,\breve{\mathcal{P}}_\eth\right). \quad (27)$$

Proof: The proof is obvious and thus omitted. ■

Next, we introduce p,q-quasirung orthopair fuzzy AA ordered weighted geometric (p,q-ROFAAOWG) operator.

Definition 14: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAOWG operator is:

$$p,q-ROFAAOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \otimes_{\iota=1}^{\eth}\left(\mathcal{P}_{\delta(\iota)}\right)^{\varpi_\iota}, \quad (28)$$

where $\varpi = (\varpi_1,\varpi_2,\ldots,\varpi_\eth)^T$ is the position weights of $\mathcal{P}_\iota$ $(\iota = 1,2,\ldots,\eth)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\eth}\varpi_\iota = 1$. $(\delta(1),\delta(2),\ldots,\delta(\eth))$ is a permutation of $(1,2,\ldots,\eth)$ such that $\mathcal{P}_{\delta(\iota-1)} \geq \mathcal{P}_{\delta(\iota)}$ for $\iota = 1,2,\ldots,\eth$.

Theorem 18: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAOWG operator is still a p,q-ROFN, and

$$p,q-ROFAAOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \left(\begin{array}{c}\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth}\varpi_\iota\left(-\ln\mu^{p^*}_{\mathcal{P}_{\delta(\iota)}}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth}\varpi_\iota\left(-\ln\left(1-\nu^{q^*}_{\mathcal{P}_{\delta(\iota)}}\right)\right)^\Lambda\right)^{1/\Lambda}}}\end{array}\right). \quad (29)$$

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 13. ■

The following features may be efficiently shown by using the p,q-ROFAAOWG operator.

Theorem 19 (Idempotency): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, if $\mathcal{P}_\iota = \mathcal{P}\ \forall\ \iota$, then

$$p,q-ROFAAOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \mathcal{P}. \quad (30)$$

Theorem 20 (Monotonicity): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ and $\dot{\mathcal{P}}_\iota = \left(\dot{\mu}_{\mathcal{P}_\iota}, \dot{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be two families of p,q-ROFNs, such that $\mu_{\mathcal{P}_\iota} \geq \dot{\mu}_{\mathcal{P}_\iota}$ and $\nu_{\mathcal{P}_\iota} \leq \dot{\nu}_{\mathcal{P}_\iota}\ \forall\ \iota$, then

$$p,q-ROFAAOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \geq p,q-ROFAAOWG\left(\dot{\mathcal{P}}_1,\dot{\mathcal{P}}_2,\ldots,\dot{\mathcal{P}}_\eth\right). \quad (31)$$

Theorem 21 (Boundedness): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, and let $\mathcal{P}^- = \min\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\}$ and $\mathcal{P}^+ = \max\{\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth\}$, then

$$\mathcal{P}^- \leq p,q-ROFAAOWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) \leq \mathcal{P}^+. \quad (32)$$

Theorem 22 (Symmetry): Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs. Then, if $\breve{\mathcal{P}}_\iota = \left(\breve{\mu}_{\mathcal{P}_\iota}, \breve{\nu}_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be any permutation of $\mathcal{P}_\iota$, then we have

$$p,q-ROFAAWG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = p,q-ROFAAWG\left(\breve{\mathcal{P}}_1,\breve{\mathcal{P}}_2,\ldots,\breve{\mathcal{P}}_\eth\right). \quad (33)$$

The p,q-ROFAAWG operator weights just the p,q-ROFNs, as defined by Definition 13, while the p,q-ROFAAOWG operator weights only the ordered locations of the p,q-ROFNs, as defined by Definition 14. As a result, weights represent various aspects of the p,q-ROFAAWG and p,q-ROFAAOWG operators. Nonetheless, one of the operators and the other operators consider just one of them. To address this shortcoming, we study the p,q-quasirung orthopair fuzzy AA hybrid geometric (p,q-ROFAAHG) operator, which weights all of the provided p,q-ROFN and their appropriate ordered position.

Definition 15: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the p,q-ROFAAHG operator is:

$$p,q-ROFAAHG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \otimes_{\iota=1}^{\eth}\left(\hat{\mathcal{P}}_{\delta(\iota)}\right)^{\varpi_\iota}, \quad (34)$$

where $\varpi = (\varpi_1,\varpi_2,\ldots,\varpi_\eth)^T$ is the weight vector associated with p,q-ROFAAHG $\mathcal{P}_\iota$ $(\iota = 1,2,\ldots,\eth)$ such that $\varpi_\iota > 0$ and $\sum_{\iota=1}^{\eth}\varpi_\iota = 1$, $w = (w_1,w_2,\ldots,w_\eth)^T$ is the weight vector of $\mathcal{P}_\iota$ $(\iota = 1,2,\ldots,\eth)$ such that $w_\iota > 0$ and $\sum_{\iota=1}^{\eth} w_\iota = 1$. $\hat{\mathcal{P}}_{\delta(\iota)}$ is the ιth largest of the weighted p,q-ROFNs $\hat{\mathcal{P}}_\iota \left(\hat{\mathcal{P}}_\iota = \mathcal{P}_\iota^{(\eth w_\iota)}\right)$, $(\iota = 1,2,\ldots,\eth)$ and $\eth$ is the balancing coefficient.

Theorem 23: Let $\mathcal{P}_\iota = \left(\mu_{\mathcal{P}_\iota}, \nu_{\mathcal{P}_\iota}\right)$ $(\iota = 1,2,\ldots,\eth)$ be a family of p,q-ROFNs, then the result obtained by utilizing p,q-ROFAAHG operator is still a p,q-ROFN, and

$$p,q-ROFAAHG(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_\eth) = \left(\begin{array}{c}\sqrt[p^*]{e^{-\left(\sum_{\iota=1}^{\eth}\varpi_\iota\left(-\ln\hat{\mu}^{p^*}_{\mathcal{P}_{\delta(\iota)}}\right)^\Lambda\right)^{1/\Lambda}}}, \\ \sqrt[q^*]{1-e^{-\left(\sum_{\iota=1}^{\eth}\varpi_\iota\left(-\ln\left(1-\hat{\nu}^{q^*}_{\mathcal{P}_{\delta(\iota)}}\right)\right)^\Lambda\right)^{1/\Lambda}}}\end{array}\right). \quad (35)$$

Proof: We skip the proof of this theorem since it is analogous to that of Theorem 13. ∎

As analogous to those of p,q-ROFAAWG operator and p,q-ROFAAOWG, the p,q-ROFAAHG operator also follows the idempotency, monotonicity, boundedness and symmetry properties. Besides the aforesaid characteristics, the p,q-ROFAAHG operator has the following special cases.

Corollary 3: p,q-ROFAAWG operator is a special case of the p,q-ROFAAHG operator.

Proof: Let $\varpi = \left(\frac{1}{\eth}, \frac{1}{\eth}, \ldots, \frac{1}{\eth}\right)^T$, then

$$\begin{aligned} & p,q-ROFAAHG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &= \hat{\mathcal{P}}_{\delta(1)}^{\varpi_1} \otimes \hat{\mathcal{P}}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \hat{\mathcal{P}}_{\delta(\eth)}^{\varpi_\eth} \\ &= \left(\hat{\mathcal{P}}_{\delta(1)} \otimes \hat{\mathcal{P}}_{\delta(2)} \otimes \ldots \otimes \hat{\mathcal{P}}_{\delta(\eth)}\right)^{\frac{1}{\eth}} \\ &= \mathcal{P}_1^{w_1} \otimes \mathcal{P}_2^{w_2} \otimes \ldots \otimes \mathcal{P}_\eth^{w_\eth} \\ &= p,q-ROFAAWG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right). \end{aligned}$$

∎

Corollary 4: p,q-ROFAAOWG operator is a special case of the p,q-ROFAAHG operator.

Proof: Let $w = \left(\frac{1}{\eth}, \frac{1}{\eth}, \ldots, \frac{1}{\eth}\right)^T$, then

$$\begin{aligned} & p,q-ROFAAHG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right) \\ &= \hat{\mathcal{P}}_{\delta(1)}^{\varpi_1} \otimes \hat{\mathcal{P}}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \hat{\mathcal{P}}_{\delta(\eth)}^{\varpi_\eth} \\ &= \mathcal{P}_{\delta(1)}^{\varpi_1} \otimes \mathcal{P}_{\delta(2)}^{\varpi_2} \otimes \ldots \otimes \mathcal{P}_{\delta(\eth)}^{\varpi_\eth} \\ &= p,q-ROFAAOWG\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_\eth\right). \end{aligned}$$

∎

V. MCDM APPROACH

In this part, we use our proposed p,q-ROFAAHA and p,q-ROFAAHG operators to provide an MCDM method for dealing with MCDM problems in p,q-quasirung orthopair fuzzy situations.

Let $o = \{o_1, o_2, \ldots, o_m\}$ be a discrete set of alternatives and $\kappa = \{\kappa_1, \kappa_2, \ldots, \kappa_\eth\}$ be the corresponding set of criteria with weight vector $w = \{w_1, w_2, \ldots, w_\eth\}$ where $w_\iota \in [0,1]$ such that $\sum_{\iota=1}^{\eth} w_\iota = 1$. A team of experts is assembled to evaluate each alternative o_i $(i = 1, 2, \ldots m)$ in relation to the relevant criteria κ_ι $(\iota = 1, 2, \ldots, \eth)$. The experts provide the evaluation information in the form of p,q-ROFNs marked by $\mathcal{P}_{i\iota} = \left(\mu_{\mathcal{P}_{i\iota}}, \nu_{\mathcal{P}_{i\iota}}\right)$ where according to experts $\mu_{\mathcal{P}_{i\iota}}$ denotes membership, and $\nu_{\mathcal{P}_{i\iota}}$ denotes non-membership grades to which alternative o_i meets that the criteria κ_ι having the constraint that $0 \le \mu_{\mathcal{P}_{i\iota}}^p + \nu_{\mathcal{P}_{i\iota}}^q \le 1$ for $p, q \ge 1$.

Algorithm

In the subsequent steps, we outline the suggested model's decision process.

Step 1 From the preceding analysis gather the expert's evaluation information provided for each alternative to their corresponding criteria and then build a decision matrix as shown in the equation at the bottom of the page.

Step 2 Build the normalized decision matrix $N = \left(\widetilde{\mathcal{P}}_{i\iota}\right)_{m\times\eth}$ by use of the following transformation

$$\widetilde{\mathcal{P}}_{i\iota} = \begin{cases} \mathcal{P}_{i\iota}, & \kappa_\iota \text{ is benefit criteria,} \\ \mathcal{P}_{i\iota}^c, & \kappa_\iota \text{ is cost criteria.} \end{cases} \tag{36}$$

where $\mathcal{P}_{i\iota}^c = \left(\nu_{\mathcal{P}_{i\iota}}, \mu_{\mathcal{P}_{i\iota}}\right)$ is the complement of $\mathcal{P}_{i\iota}$.

Step 3 Use the newly designed p,q-ROFAAHA or p,q-ROFAAHG operator to obtain the overall aggregated result from matrix N row-wise for each alternative o_i.

Step 4 Employ Eq. (4) to determine the score value of each aggregated result derived in Step 2.

Step 5 Rank the alternatives o_i $(i = 1, 2, \ldots, m)$ in descending order according to their score values and get the optimal one.

VI. AN ILLUSTRATIVE EXAMPLE

This section contends for the evaluation of the intensity of corruption to show the implication of the proposed framework.

A. BACKGROUND DESCRIPTION

Corruption is now the most prevalent of all societal ills. Every society in today's world suffers from the ills of corruption in some way. The degree of corruption may differ from one country to the next or from one social system to the other, but its presence cannot be ignored. The causes of corruption vary and are influenced by variables ranging from historical to economic. In a recent study, we attempted to mathematically attach/link the causes of corruption using fuzzy mathematics techniques. Our research intends to create an application for assessing the severity of this social hazard based on its causes. Corruption erodes a society's good standards, and the social structure soon degenerates. It is consequently critical not just to detect corruption and its causes but also

$$M_{m\times\eth} = \begin{array}{c} \\ o_1 \\ \vdots \\ o_i \\ \vdots \\ o_m \end{array} \begin{array}{c} \begin{array}{ccccc} \kappa_1 & \cdots & \kappa_\iota & \cdots & \kappa_\eth \end{array} \\ \begin{pmatrix} \left(\mu_{\mathcal{P}_{11}}, \nu_{\mathcal{P}_{11}}\right) & \cdots & \left(\mu_{\mathcal{P}_{1\iota}}, \nu_{\mathcal{P}_{1\iota}}\right) & \cdots & \left(\mu_{\mathcal{P}_{1\eth}}, \nu_{\mathcal{P}_{1\eth}}\right) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(\mu_{\mathcal{P}_{i1}}, \nu_{\mathcal{P}_{i1}}\right) & \cdots & \left(\mu_{\mathcal{P}_{i\iota}}, \nu_{\mathcal{P}_{i\iota}}\right) & \cdots & \left(\mu_{\mathcal{P}_{i\eth}}, \nu_{\mathcal{P}_{i\eth}}\right) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \left(\mu_{\mathcal{P}_{m1}}, \nu_{\mathcal{P}_{m1}}\right) & \cdots & \left(\mu_{\mathcal{P}_{m\iota}}, \nu_{\mathcal{P}_{m\iota}}\right) & \cdots & \left(\mu_{\mathcal{P}_{m\eth}}, \nu_{\mathcal{P}_{m\eth}}\right) \end{pmatrix} \end{array}$$

TABLE 1. Expert's evaluation matrix M.

	κ_1	κ_2	κ_3	κ_4
o_1	$(0.6, 0.4)$	$(0.5, 0.6)$	$(0.9, 0.3)$	$(0.7, 0.5)$
o_2	$(0.6, 0.7)$	$(0.8, 0.6)$	$(0.5, 0.5)$	$(0.4, 0.6)$
o_3	$(0.7, 0.3)$	$(0.6, 0.5)$	$(0.4, 0.8)$	$(0.4, 0.5)$
o_4	$(0.5, 0.6)$	$(0.4, 0.7)$	$(0.8, 0.5)$	$(0.6, 0.3)$

to take action to combat it. Weak economic circumstances, a weak accountability system, and a weak moral foundation are the primary causes of corruption. Consider the following example: consider o_1, o_2, o_3, o_4 to be a collection of countries, and consider κ_1, κ_2, κ_3, κ_4 to be a set of four criteria, where κ_1 refers "Weak System of Accountability", κ_2 refers "Weak Economic Conditions", κ_3 refers "Weak Moral Basis", and κ_4 refers "Unemployment." The weight vector of these criteria is $w = (0.2, 0.3, 0.2, 0.3)$, while the weight vector associated with aggregation operator is taken $\varpi = (0.3, 0.2, 0.25, 0.25)$. To provide sufficient flexibility in evaluating the values of the four criteria, experts were permitted to use p,q-ROFNs. The assessment information of experts is recorded in Table 1.

B. DECISION PROCESS BASED ON p,q-ROFAAHA AND p,q-ROFAAHG OPERATOR

Step 1: The assessment information matrix is established in Table 1.

Step 2: Because each criteria is benefit type, then normalization isn't implemented. Thus, normalized decision matrix $N = M = (\mathcal{P}_{i\iota})_{4\times4}$, provided as Table 1.

Step 3: Using the p,q-quasirung orthoair fuzzy information detailed in Table 1, the values of $\hat{\mathcal{P}}_{i\iota} = (\eth w_\iota)\,\mathcal{P}_{i\iota}$ are computed as shown follows:

$$\hat{\mathcal{P}}_{11} = (0.5195, 0.4805)\,, \hat{\mathcal{P}}_{12} = (0.5403, 0.5417)\,,$$
$$\hat{\mathcal{P}}_{13} = (0.8415, 0.3817)\,, \hat{\mathcal{P}}_{14} = (0.7643, 0.4354)\,,$$
$$\hat{\mathcal{P}}_{21} = (0.5479, 0.7518)\,, \hat{\mathcal{P}}_{22} = (0.8405, 0.5418)\,,$$
$$\hat{\mathcal{P}}_{23} = (0.4256, 0.5744)\,, \hat{\mathcal{P}}_{24} = (0.4583, 0.5417)\,,$$
$$\hat{\mathcal{P}}_{31} = (0.6183, 0.3817)\,, \hat{\mathcal{P}}_{32} = (0.6671, 0.4354)\,,$$
$$\hat{\mathcal{P}}_{33} = (0.3608, 0.8365)\,, \hat{\mathcal{P}}_{34} = (0.4583, 0.4353)\,,$$
$$\hat{\mathcal{P}}_{41} = (0.4534, 0.6646)\,, \hat{\mathcal{P}}_{42} = (0.4345, 0.6518)\,,$$
$$\hat{\mathcal{P}}_{43} = (0.7473, 0.5744)\,, \hat{\mathcal{P}}_{44} = (0.6671, 0.2357)\,.$$

Based on the score function Eq. (4), we have

$$\hat{\mathcal{P}}_{\delta(11)} = \hat{\mathcal{P}}_{13} = (0.8415, 0.3817)\,,$$
$$\hat{\mathcal{P}}_{\delta(12)} = \hat{\mathcal{P}}_{14} = (0.7643, 0.4354)\,,$$
$$\hat{\mathcal{P}}_{\delta(13)} = \hat{\mathcal{P}}_{11} = (0.5195, 0.4805)\,,$$
$$\hat{\mathcal{P}}_{\delta(14)} = \hat{\mathcal{P}}_{12} = (0.5403, 0.5417)\,,$$
$$\hat{\mathcal{P}}_{\delta(21)} = \hat{\mathcal{P}}_{22} = (0.8405, 0.5418)\,,$$
$$\hat{\mathcal{P}}_{\delta(22)} = \hat{\mathcal{P}}_{24} = (0.4583, 0.5417)\,,$$
$$\hat{\mathcal{P}}_{\delta(23)} = \hat{\mathcal{P}}_{23} = (0.4256, 0.5744)\,,$$
$$\hat{\mathcal{P}}_{\delta(24)} = \hat{\mathcal{P}}_{21} = (0.5479, 0.7518)\,,$$
$$\hat{\mathcal{P}}_{\delta(31)} = \hat{\mathcal{P}}_{32} = (0.6671, 0.4354)\,,$$
$$\hat{\mathcal{P}}_{\delta(32)} = \hat{\mathcal{P}}_{31} = (0.6183, 0.3817)\,,$$
$$\hat{\mathcal{P}}_{\delta(33)} = \hat{\mathcal{P}}_{34} = (0.4583, 0.4353)\,,$$
$$\hat{\mathcal{P}}_{\delta(34)} = \hat{\mathcal{P}}_{33} = (0.3608, 0.8365)\,,$$
$$\hat{\mathcal{P}}_{\delta(41)} = \hat{\mathcal{P}}_{44} = (0.6671, 0.2357)\,,$$
$$\hat{\mathcal{P}}_{\delta(42)} = \hat{\mathcal{P}}_{43} = (0.7473, 0.5744)\,,$$
$$\hat{\mathcal{P}}_{\delta(43)} = \hat{\mathcal{P}}_{41} = (0.4534, 0.6646)\,,$$
$$\hat{\mathcal{P}}_{\delta(44)} = \hat{\mathcal{P}}_{42} = (0.4345, 0.6518)\,.$$

Before implementing p,q-ROFAAHA operator, the values of p^* and q^* for each row are determined. In accordance with the method outlined in Section V, $p^* = q^* = 2$ for each row of the Table 1.

Now utilizing p,q-ROFAAHA operator i.e., Eq. (19) (taking $p^* = q^* = 2$ and $\Lambda = 1$), having associated weight vector $\varpi = (0.3, 0.2, 0.25, 0.25)$ to work out the overall value of each alternative o_i, shown as follows:

$$\mathcal{P}_1 = (0.7129, 0.4530)\,, \mathcal{P}_2 = (0.6522, 0.5966)\,,$$
$$\mathcal{P}_3 = (0.5542, 0.4993)\,, \mathcal{P}_4 = (0.6019, 0.4707)\,.$$

Step 4: In the light of Eq. (4), figure out the score value of each alternative o_i, derived as follows:

$$S(o_1) = 0.7538, S(o_2) = 0.5347,$$
$$S(o_3) = 0.6524, S(o_4) = 0.6902.$$

Step 5: The ranking of alternatives is $o_1 \succ o_4 \succ o_3 \succ o_2$. Hence, the top rank corrupt country is o_1.

Now, we are leveraging the p,q-ROFAAHG operator to emulate the decision-making process.

According to the p,q-ROFAAHG operator, the main steps are as follows:

Step 1-2: These are identical to above Steps 1-2.

Step 3: Using the p,q-quasirung orthoair fuzzy information detailed in Table 1, the values of $\hat{\mathcal{P}}_{i\iota} = (\eth w_\iota)\,\mathcal{P}_{i\iota}$ are computed as shown follows:

$$\hat{\mathcal{P}}_{11} = (0.6646, 0.3354)\,, \hat{\mathcal{P}}_{12} = (0.4354, 0.6671)\,,$$
$$\hat{\mathcal{P}}_{13} = (0.9191, 0.2696)\,, \hat{\mathcal{P}}_{14} = (0.6518, 0.5403)\,,$$
$$\hat{\mathcal{P}}_{21} = (0.6645, 0.6453)\,, \hat{\mathcal{P}}_{22} = (0.7650, 0.6440)\,,$$
$$\hat{\mathcal{P}}_{23} = (0.5744, 0.4256)\,, \hat{\mathcal{P}}_{24} = (0.3329, 0.6671)\,,$$
$$\hat{\mathcal{P}}_{31} = (0.7517, 0.2483)\,, \hat{\mathcal{P}}_{32} = (0.5417, 0.5403)\,,$$
$$\hat{\mathcal{P}}_{33} = (0.4804, 0.7239)\,, \hat{\mathcal{P}}_{34} = (0.3329, 0.5647)\,,$$
$$\hat{\mathcal{P}}_{41} = (0.5744, 0.5195)\,, \hat{\mathcal{P}}_{42} = (0.3329, 0.7643)\,,$$
$$\hat{\mathcal{P}}_{43} = (0.8365, 0.4534)\,, \hat{\mathcal{P}}_{44} = (0.5417, 0.3482)\,.$$

Based on the score function Eq. (4), we have

$$\hat{\mathcal{P}}_{\delta(11)} = \hat{\mathcal{P}}_{13} = (0.9191, 0.2696)\,,$$
$$\hat{\mathcal{P}}_{\delta(12)} = \hat{\mathcal{P}}_{14} = (0.6518, 0.5403)\,,$$
$$\hat{\mathcal{P}}_{\delta(13)} = \hat{\mathcal{P}}_{11} = (0.6646, 0.3354)\,,$$
$$\hat{\mathcal{P}}_{\delta(14)} = \hat{\mathcal{P}}_{12} = (0.4354, 0.6671)\,,$$
$$\hat{\mathcal{P}}_{\delta(21)} = \hat{\mathcal{P}}_{22} = (0.7650, 0.6440)\,,$$

$$\hat{\mathcal{P}}_{\delta(22)} = \hat{\mathcal{P}}_{23} = (0.5744, 0.4256),$$
$$\hat{\mathcal{P}}_{\delta(23)} = \hat{\mathcal{P}}_{21} = (0.6645, 0.6453),$$
$$\hat{\mathcal{P}}_{\delta(24)} = \hat{\mathcal{P}}_{24} = (0.3329, 0.6671),$$
$$\hat{\mathcal{P}}_{\delta(31)} = \hat{\mathcal{P}}_{31} = (0.7517, 0.2483),$$
$$\hat{\mathcal{P}}_{\delta(32)} = \hat{\mathcal{P}}_{32} = (0.5417, 0.5403),$$
$$\hat{\mathcal{P}}_{\delta(33)} = \hat{\mathcal{P}}_{34} = (0.3329, 0.5647),$$
$$\hat{\mathcal{P}}_{\delta(34)} = \hat{\mathcal{P}}_{33} = (0.4804, 0.7239),$$
$$\hat{\mathcal{P}}_{\delta(41)} = \hat{\mathcal{P}}_{43} = (0.8365, 0.4534),$$
$$\hat{\mathcal{P}}_{\delta(42)} = \hat{\mathcal{P}}_{44} = (0.5417, 0.3482),$$
$$\hat{\mathcal{P}}_{\delta(43)} = \hat{\mathcal{P}}_{41} = (0.5744, 0.5195),$$
$$\hat{\mathcal{P}}_{\delta(44)} = \hat{\mathcal{P}}_{42} = (0.3329, 0.7643).$$

Now utilizing p,q-ROFAAHG operator i.e., Eq. (34) (taking $p^* = q^* = 2$ and $\Lambda = 1$), having associated weight vector $\varpi = (0.3, 0.2, 0.25, 0.25)$ to work out the overall value of each alternative o_i, shown as follows:

$$\mathcal{P}_1 = (0.6564, 0.4854), \mathcal{P}_2 = (0.5664, 0.6187),$$
$$\mathcal{P}_3 = (0.5135, 0.5560), \mathcal{P}_4 = (0.5543, 0.5701).$$

Step 4: According to Eq. (4), compute the score value of each alternative o_i, derived as follows:

$$S(o_1) = 0.7104, S(o_2) = 0.3511,$$
$$S(o_3) = 0.3538, S(o_4) = 0.3686.$$

Step 5: The ranking of alternatives is $o_1 \succ o_4 \succ o_3 \succ o_2$. Hence, the top rank corrupt country is o_1. It is the same as p,q-ROFAAHA operator.

C. IMPACT OF VARIOUS PARAMETERS ON THE MCDDM TECHNIQUE

This section is devoted to perform a sensitivity discussion to examine the impact of various parameters on the ranking results.

1) IMPACT OF PARAMETER p^* AND q^* ON DECISION-MAKING RESULTS

To determine the dependability and consistency of the preceding example, we use the p,q-ROFAAHA and p,q-ROFAAHG operators with various values of p^* and q^* under the discussed algorithm.

To do this, we fix the value of $q^* = 2$ and take different values of p^*= 2, 3, 5, 7, 9 in p,q-ROFAAHA operator. Here, we started the value of p^* from 2 because 2 is the least possible value of p^* and q^*, for which all of the data of the DMs become p, q-ROFN. The values of the score and ranking order of the alternatives for various values of p^* attained by the suggested methodology have been depicted in Table 2. Figure 2 presents a clearer illustration of the ranking outcomes for various p^* values. Table 2 and Fig. 2 demonstrate that the greater values of the parameter p^* produce greater score values, but the ranking order remains the same. Likewise, if we fix p^*=2 and take different values of q^*= 2, 3, 5, 7, 9 in p,q-ROFAAHA operator. We can notice from Table 2 that it does not effect the score values and the alternatives are ranked in the same vein. Hence, the suggested method is stable under p,q-ROFAAHA operator concerning different values of p^* and q^*.

Next, we analyze the impact of the parameter p^* and q^* on the decision making results under the p,q-ROFAAHG in the discussed algorithm. To do this, we solve the same numerical example presented in Section VI-A first by fixing value of q^*=2 and adopt varying values of p^*. We consider q^*= 2 and p^*= 2,3, 5,7,9. The values of the score function and ranking order of the alternatives for varying values of p^* gotten by the provided method has manifested in Table 3. In additional, we interpret these experimental findings graphically, as represented in Figure 3. From Table 3, we can confirm that in this case, the decision making results are independent of the parameter p^*. Besides, if we fix p^*= 2 and q^*= 2,3, 5,7,9 the impact in light of p,q-ROFAAHG executive is tabulated in Table 3. From Table 3 it is clear that when the value of q^* rises for p,q-ROFAAHG operator, the score values of alternatives decreases progressively. And the ranking position of some alternatives also alters which shows the dependability of the formulated p,q-ROFAAHG operator on q^*. Anyhow, the best alternative is always o_1 in each case.

2) IMPACT OF PARAMETER Λ ON DECISION-MAKING RESULTS

To showcase the influence of varying magnitudes of the parameter Λ, we employ distinct parametric values of Λ within our stated method to describe the alternatives. Tables 4 and 5 display the ranking implications of the alternatives $o_i (i = 1, 2, \ldots m)$ based on the p,q-ROFAAHA and p,q-ROFAAHG operators, respectively, as shown visually in Figs. 4 and 5. It is clear that as the magnitude of Λ for the p,q-ROFAAHA operator increases, so do the score values of the available alternatives, but as the magnitude of Λ increases for p,q-ROFAAHG operator, the score values of alternatives decreases. However, the associated ranking remains constant in both cases, indicating that the proposed approach always had the isotonicity property, allowing DMs to choose the appropriate value based on their preferences. From these values and analysis, it is concluded that a DM can choose the respective value of Λ depending upon its behavior towards the decision making. For instance, if the DM is the most optimistic towards the decision, then he/she can choose the p,q-ROFAAHG operator with lower values of Λ. On the other hand, if he/she chooses p,q-ROFAAHA operator to aggregate the process, then he/she can choose larger values of the parameters Λ. Besides this, if a DM utilizes p,q-ROFAAHA operator towards the aggregation process to get the most pessimistic decision, he/she can choose smaller values of Λ. This impact of Λ values on the decision makes our suggested approach more flexible as DMs can choose the parameters according to their preferences and practical situations. Furthermore, we can observe in Figs. 4 and 5 that even when the values of Λ are varied throughout the

TABLE 2. Ranking results by ***p,q*-ROFAAHA** with various p^* and q^*.

p^*, q^*	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
$p^*=2, q^*=2$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=3, q^*=2$	0.7581	0.5457	0.6570	0.6949	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=5, q^*=2$	0.7662	0.5668	0.6655	0.7041	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=7, q^*=2$	0.7732	0.5843	0.6723	0.7120	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=9, q^*=2$	0.7789	0.5983	0.6776	0.7182	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=2$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=3$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=5$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=7$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=9$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$

TABLE 3. Ranking results by ***p,q*-ROFAAHG** with various p^* and q^*.

p^*, q^*	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
$p^*=2, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=3, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=5, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=7, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=9, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=2$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$p^*=2, q^*=3$	0.7006	0.3493	0.3456	0.3610	$o_1 \succ o_4 \succ o_2 \succ o_3$
$p^*=2, q^*=5$	0.6832	0.3460	0.3326	0.3462	$o_1 \succ o_4 \succ o_2 \succ o_3$
$p^*=2, q^*=7$	0.6701	0.3438	0.3231	0.3334	$o_1 \succ o_2 \succ o_4 \succ o_3$
$p^*=2, q^*=9$	0.6601	0.3419	0.3158	0.3233	$o_1 \succ o_2 \succ o_4 \succ o_3$

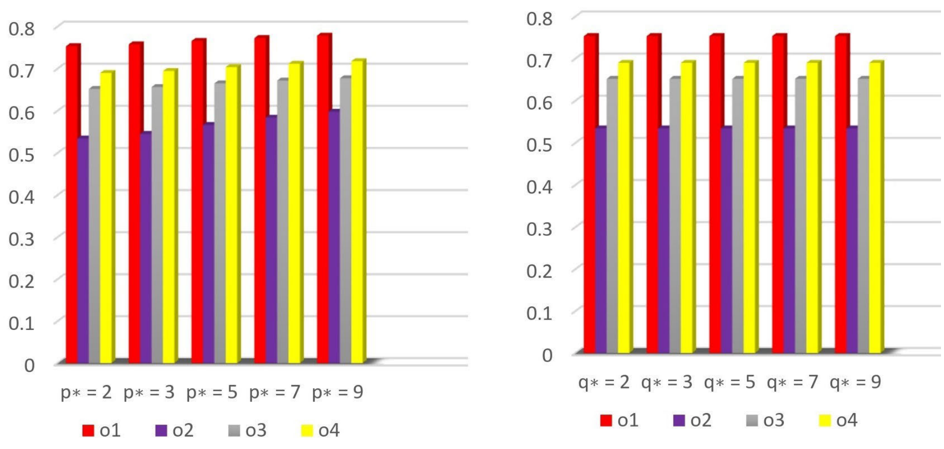

FIGURE 2. Ranking of alternatives for different values p^* and q^* by ***p,q*-ROFAAHA** operator.

demonstration, the choices' outputs seem to be the same, indicating the consistency of the suggested operators.

VII. COMPARATIVE ANALYSIS

In this chapter, we compare our suggested technique with some prevailing operators, including q-rung orthopair fuzzy weighted averaging (q-ROFWA) operator [28], q-rung orthopair fuzzy weighted geometric (q-ROFWG) operator [28], q-rung orthopair fuzzy Einstein ordered weighted averaging (q-ROFEOWA) operator [27], q-rung orthopair fuzzy Einstein weighted geometric (q-ROFEWG) operator [27], weighted q-rung orthopair

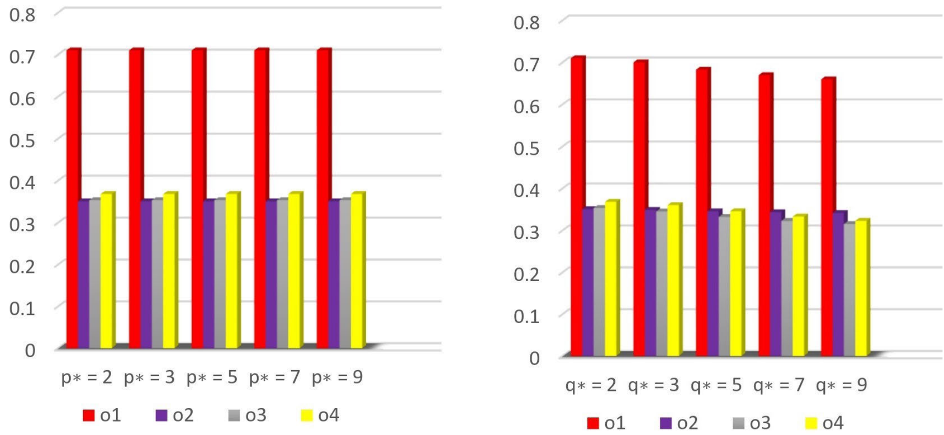

FIGURE 3. Ranking of alternatives for different values p^* and q^* by p,q-ROFAAHG operator.

TABLE 4. Ranking results by p,q-ROFAAHA with various Λ.

Λ	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
$\Lambda=1$	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=3$	0.8169	0.5913	0.6948	0.7520	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=5$	0.8467	0.6192	0.7230	0.7880	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=7$	0.8622	0.6354	0.7419	0.8069	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=9$	0.8716	0.6459	0.7550	0.8178	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=50$	0.8992	0.6858	0.7963	0.8485	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=75$	0.9012	0.6888	0.7990	0.8508	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=100$	0.9021	0.6904	0.8005	0.8519	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda=150$	0.9031	0.6924	0.8020	0.8528	$o_1 \succ o_4 \succ o_3 \succ o_2$

fuzzy Hamacher average (Wq-ROFHA) operator [47], intuitionistic fuzzy AA weighted average (IFAAWA) operator [45] and intuitionistic fuzzy AA hybrid average (IFAAHA) operator [45]. Table 6 contains the comparison results, which are graphically depicted in Fig. 6.

From the listed results in Table 6, it is evident that the ranking results derived by the q-ROFWA operator [28] totally match with the proposed aggregation operators' results. Next to utilizing q-ROFWG operator [28], we get $o_1 \succ o_4 \succ o_2 \succ o_3$. Thus, the ranking result varies little with respect to the q-ROFWG operator. Both q-ROFWA and q-ROFWG operators weights just the q-ROFNs and neglect the ordered positions of the q-ROFNs, whereas the proposed operators encounter both the q-ROFNs as well as their ordered positions weights. What's more, the experimental results obtained on the basis of Einstein operators [27] i.e., q-ROFEOWA and q-ROFEOWG operators are the same as the ones acquired by our suggested operators. Albeit these Einstein operators suffer from the drawback of paying no attention to q-ROFNs weights and dealing with only ordered positions weights. In addition, the result based on the Wq-ROFHA operator [47] is little change, i.e., the alternative o_2 and o_3 have swapped their positions. This can be attributed to the fact that likewise q-ROFWA and q-ROFWG operators Wq-ROFHA operator neglects the ordered positions of q-ROFNs. And this shortfall is overcome by the suggested q-ROFAAHA and q-ROFAAHG operators. For good measure, likewise the proposed aggregation operators, the Wq-ROFHA operator also permits the DMs to choose their preferences with respect to the different values of the parameter. Furthermore, Senapati et al. presented work [45] is limited notion and can work only with intuitionistic fuzzy data i.e., $\mu_{\mathcal{P}}+\nu_{\mathcal{P}} \leq 1$. From this restriction, it is clear that Senapati et al. [45] operators are not capable of handling the data given in Table 6. From Table 6, it is easy to note that $\left(\mu_{\mathcal{P}_{12}}, \nu_{\mathcal{P}_{12}}\right)=(0.5,0.6)$ is not intuitionistic fuzzy data because $0.5+0.6 \nleq 1$. Due to this reason, the data provided in Table 6 cannot be handled by [45], and only the proposed AA operators are able to resolve all such data where $\mu_{\mathcal{P}}^q+\nu_{\mathcal{P}}^q \leq 1$; $q \geq 1$. This implies that the expounded work is dominant

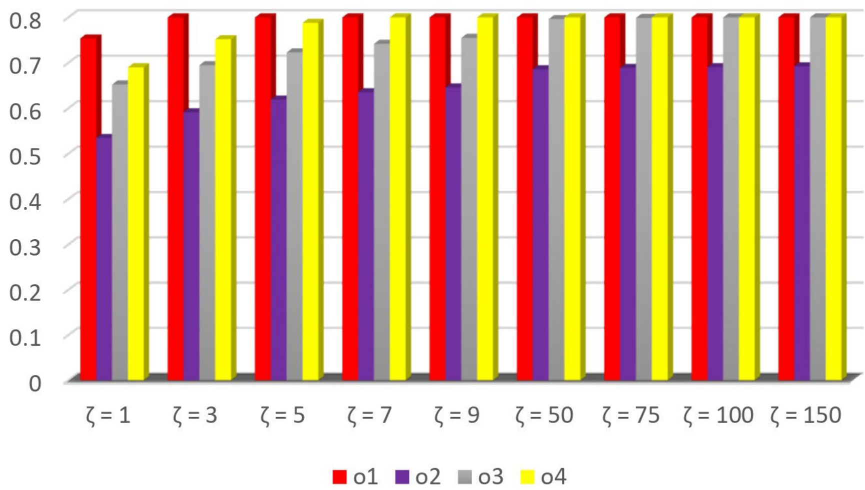

FIGURE 4. Ranking of alternatives for different values Λ by *p,q*-ROFAAHA operator.

TABLE 5. Ranking results by *p,q*-ROFAAHG with various Λ.

Λ	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
$\Lambda = 1$	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 3$	0.6574	0.3145	0.2764	0.5020	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 5$	0.6324	0.2934	0.2409	0.2875	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 7$	0.3715	0.2799	0.2236	0.2735	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 9$	0.3622	0.2707	0.2138	0.2646	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 50$	0.3316	0.2375	0.1860	0.2362	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 75$	0.3296	0.2350	0.1840	0.2342	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 100$	0.3284	0.2337	0.1830	0.2331	$o_1 \succ o_4 \succ o_3 \succ o_2$
$\Lambda = 150$	0.3275	0.2326	0.1820	0.2322	$o_1 \succ o_4 \succ o_3 \succ o_2$

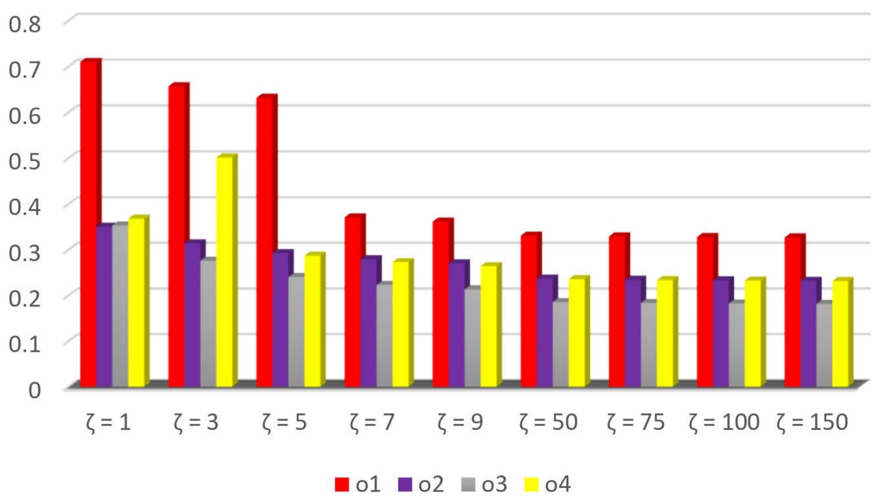

FIGURE 5. Ranking of alternatives for different values Λ by *p,q*-ROFAAHG operator.

than [45] and provides more space to DMs to make their decisions.

We summarise the advantages of the proposed approach as follows:

TABLE 6. Ranking results based on different aggregation operators.

Operator	$S(o_1)$	$S(o_2)$	$S(o_3)$	$S(o_4)$	Ranking
q-ROFWA [28]	0.2992	0.0401	0.0509	0.1170	$o_1 \succ o_4 \succ o_3 \succ o_2$
q-ROFWG [28]	0.1752	−0.0581	−0.0737	−0.0216	$o_1 \succ o_4 \succ o_2 \succ o_3$
q-ROFEOWA [27]	−0.2363	−0.3191	−0.2989	−0.2919	$o_1 \succ o_4 \succ o_3 \succ o_2$
q-ROFEOWG [27]	0.3656	0.2463	0.2938	0.3185	$o_1 \succ o_4 \succ o_3 \succ o_2$
Wq-ROFHA [47]	0.3828	0.1999	0.1576	0.2169	$o_1 \succ o_4 \succ o_2 \succ o_3$
IFAAWA [45]	...	...	...	...	Failed
IFAAHA [45]	...	...	...	...	Failed
PyFAAHA [48]	0.3256	0.0013	0.0542	0.1172	$o_1 \succ o_4 \succ o_3 \succ o_2$
PyFAAHG [48]	0.1915	−0.0584	−0.0561	−0.0123	$o_1 \succ o_4 \succ o_3 \succ o_2$
Proposed p,q-ROFAAHA	0.7538	0.5347	0.6524	0.6902	$o_1 \succ o_4 \succ o_3 \succ o_2$
Proposed p,q-ROFAAHG	0.7104	0.3511	0.3538	0.3686	$o_1 \succ o_4 \succ o_3 \succ o_2$

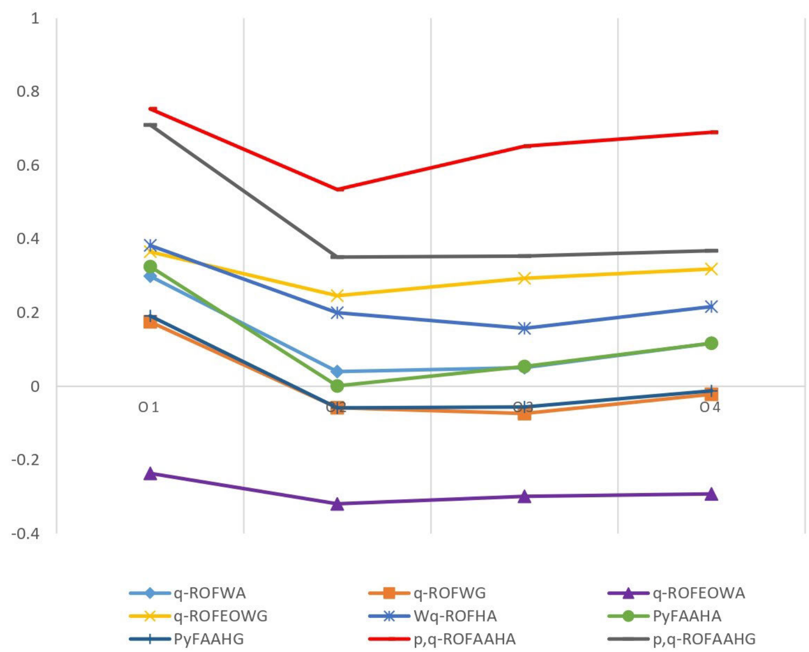

FIGURE 6. Comparative analysis with existing aggregation operators.

1). Generality: The aggregation operators utilized in the proposed method are the generalization of certain prevailing aggregation operators. For instance, q-ROFWA and q-ROFWG operators are special cases of the suggested p,q-ROFAAHA and p,q-ROFAAHG operators, respectively. And this happens when $\Lambda = 1$ and $\varpi = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)^T$. So our methodology is more comprehensive and more reasonable.

2). Parameter Λ: The devised aggregation operators include a parameter Λ that allows DMs to alter the aggregate value based on real-world decision demands, and they capture numerous current p,q-quasirung orthopair aggregation operators. Along these lines, the benefit is that the proposed operators have a better level of consensus and flexibility.

3). Property of isotonicity: In contrast to the Wq-ROFHA operator [47], the provided operators adhere to the isotonicity property. The p,q-ROFAAHA (p,q-ROFAAHG) operator values grow (reduce) monotonically with the increase of parameter Λ, allowing DMs to select the right value based on their risk preferences. If the DMs are risk preference, they may take the parameter value as low as fairly practicable; if the DMs are risk aversion, they can take the parameter value as high as reasonably achievable in the case of the p,q-ROFAAHA operator, and vice versa for the p,q-ROFAAHG operator. Thus, the DMs can use the appropriate parameter value based on their risk tolerance and real demands.

VIII. CONCLUSION AND RECOMMENDATIONS

This manuscript defined some novel p,q-quasirung orthopair fuzzy operation rules based on AA t-norm and t-conorm and investigated their relevant characteristics. In view of these AA operational rules, we explored some novel p,q-quasirung orthopair fuzzy weighted, ordered weighted and hybrid averaging as well as geometric aggregation operators. We also examined various alluring properties and special cases of the presented operators in depth. Based on the explored operators, an MCDM approach was outlined to access the optimal alternative(s) according to their ranking order. An example was demonstrated to justify the implication of the presented work in real life. Also, the influence of the parameter Λ employed in the suggested aggregation operators on the score values of alternatives was detailed. According to the experimental findings, it was observed that the suggested technique is stable and adheres to the isotonicity condition. At last, a comparative study of the suggested MCDM technique with various earlier approaches was provided in order to Scrutinize the validity, advantages, and reliability of the proposed generalized aggregation operators and MCDM approach.

Moving forward, we will continue to work on the following aspects:

i. The suggested AA aggregation operators have the drawback that they cannot capture interrelationships between criteria [49]. To widen the scope of application, we will define other functional aggregation operators such as power Maclaurin symmetric mean operators in terms of AA operations.

ii. Based on the suggested operational rules, we will redefine the operational rules for some other advanced fuzzy sets, including the T-spherical fuzzy set [50], and link them to fuzzy graph theory and fuzzy-regression approaches [51].

iii. We will further enhance the presented study by considering the relative weights of DMs. To this end, we will put forward a mathematical model [52] to acquire the weights.

ABBREVIATIONS

q-rung orthopair fuzzy set: q-ROFS. p,q-quasirung orthopair fuzzy set: p,q-ROFS. Aczel-Alsina: AA. Multi-criteria decision-making: MCDM. Fuzzy set: FS. Intuitionistic FS: IFS. Pythagorean fuzzy set: PFS. q-quasirung orthopair fuzzy numbers: q-ROFNs. p,q-quasirung orthopair fuzzy AA weighted averaging: p,q-ROFAAWA. p,q-quasirung orthopair fuzzy AA ordered weighted averaging: p,q-ROFAAOWA. p,q-quasirung orthopair fuzzy AA hybrid averaging: p,q-ROFAAHA. p,q-quasirung orthopair fuzzy AA weighted geometric: p,q-ROFAAWG. p,q-quasirung orthopair fuzzy AA ordered weighted geometric: p, q-ROFAAOWG. p, q-quasirung orthopair fuzzy AA hybrid geometric: p, q-ROFAAHG. least common multiple: lcm. Decision maker: DM. Bonferroni mean: BM. Weighted aggregated sum product assessment: WASPAS. Technique for order of preference by similarity to ideal solution: TOPSIS. Triangle norms: t-norms.

AVAILABILITY OF DATA AND MATERIALS

All data generated or analysed during this study are included in this published article.

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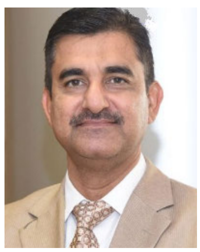
JAWAD ALI received the Ph.D. degree in mathematics from Quaid-i-Azam University Islamabad, Pakistan, in 2021. He is currently a Lecturer with the Institute of Numerical Sciences, Kohat University of Science and Technology. He was involved in fixed point theory and its applications. He has published several research articles in well-reputed journals. His research interests include generalizations of fuzzy sets, multi-criteria decision-making, aggregation operators, cluster analysis, and artificial intelligence.



MUHAMMAD NAEEM received the M.Phil. degree from Peshawar University, in 1996, and the Ph.D. degree from the Ghulam Ishaq Khan Institute Engineering Sciences and Technology, Swabi, Pakistan, in 2009.

He is currently an Associate Professor of mathematics with Umm Al-Qura University, Mecca, Saudi Arabia. As a former Assistant Professor with the Higher Education Department (KPK), Government of Pakistan, and the Army Public Education Institution, Peshawar Branch, he has experience in teaching mathematics from lower level to master's level with fulfilling administrative responsibilities as the Assistant Vice Principal and HOD (mathematics). He is working on several research projects supported by the Scientific Research Department, Umm Al-Qura University. His research was primarily focused on asymptotic properties of statistics based on a sum of functions of non-overlapping higher order spacing, which was the topic of study during research for his Ph.D. degree. He has published several papers in international journals of repute. His research interests include fuzzy set theory and fuzzy fractional calculus and contributed to several papers published in internationally reputable journals in this new area. He has experience as a reviewer of research articles for several reputable research journals.