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RESEARCH ARTICLE

New Variable Precision Reduction Algorithm for Decision Tables

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ABSTRACT Variable precision reduction (VPR) and positive region reduction (PRR) are common definitions in attribute reduction. The compacted decision table is an extension of a decision table. In this paper, we propose another extension, called the weighted decision table. In both types of decision tables, VPR is defined, and the corresponding discernibility matrices for the PRR are proposed. Then, algorithms for obtaining the PRR from the discernibility matrices are presented. In both types of decision tables, the relationship between VPR and PRR is established by comparing the corresponding discernibility matrices. If the precision of the VPR meets the given conditions, then the PRR algorithms can be used to obtain the results after modifying some decision values in the decision tables. An analysis of the modification process of the decision values and the compression process of decision tables is used to propose a new algorithm for VPR in decision tables that ensures credibility. The effectiveness of the proposed algorithm was evaluated by an experimental comparison with existing VPR algorithms.

INDEX TERMS Variable precision reduction, positive region reduction, decision table, compacted decision table, weighted decision table.

I. INTRODUCTION

As the number of data features increases, the cost of analyzing and processing multi-dimensional data increases, and hence the research on attribute reduction has become a popular topic in recent years. Attribute reduction is an important task in rough sets [1] which were proposed by Pawlak in the 1980s and are used to handle problems involving uncertainty [2]. In recent years, the research on rough sets has combined rough sets with fuzzy sets [3], [4], [5] evidence theory [6], [7] information entropy [8], [9], [10], [11] and other fields, and has made great progress. Attribute reduction removes redundant features and retains the subset with the minimum number of attributes to improve the efficiency of the algorithm. In its initial stages, attribute reduction research mainly focused on the definitions of positive region reduction (PRR) [12], [13], [14], [15] and absolute reduction [16]. As rough set research has further developed, some concepts such as variable precision reduction (VPR) [17], [18], [19] and assignment reduction [20] have been proposed according to practical needs, and they have been successfully applied in power systems [21] bioinformatics [22] text categorization [23] and industrial applications [24].

In recent years, attribute reduction methods have been used in granular computing [25], [26] formal concept analysis [27], three-way decision analysis [28], [29] and many other fields. Attribute reduction in classical rough sets has been studied thoroughly. The indiscernibility relations between any object have been constructed to study the relationship between the positive region and indiscernibility relation sets in depth [30], [31], [32]. Because some information in information tables will be missing, as the binary relation is equivalence, the reduction method research is limited.

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To improve the practicability of the reduction algorithm, the equivalence relation has been extended to non-equivalent binary relations such as tolerance relations [33], [34], [35] and similarity relations [17], [36], [37] and the definition of reduction was extended again after the theory of rough sets was generalized. Reduction methods based on information view [38] or the discernibility matrix [13], [14], [15], [18] and heuristic methods based on attribute importance [37] have been proposed. In addition, considering the dynamic changes of objects in a universe, incremental attribute reduction [39], [40] has also made new progress.

On the basis of decision tables, compacted decision tables [41] are described, and the weighted decision table, which is obtained after the decision table has been compressed, is proposed. In both types of decision table, the relationship between VPR and PRR is studied from the point of view of discernibility matrix construction, and an optimization algorithm for VPR is proposed for each type of decision table. Based on the above research, the construction processes of transforming the decision table into each type of decision table are compared, and the VPR in the decision table is transformed into the PRR in the weighted decision table for calculation, thus improving the efficiency of the VPR algorithm.The core of the VPR algorithm, which is based on a discernibility matrix, is to construct the discernibility matrix. However, the time complexity of constructing the discernibility matrix in existing reduction algorithms is $O(n^2)$. The main contribution of this paper is to develop the PRR algorithm for decision tables. The research framework is shown in Figure 1.

The structure of the paper is as follows. Section II introduces the basic concepts of the positive region and variable precision in decision tables. In Section III, the compacted decision table is introduced, the weighted decision table is proposed, and VPR for both types of decision tables is proposed. In Section IV, in the compacted decision table and the weighted decision table, the PRR corresponding discernibility matrices are proposed, and then the corresponding algorithms are proposed. Based on the discernibility matrix, the relationship between VPR and PRR is analyzed in two types of decision tables. In Section V, the optimization algorithms of VPR are proposed for two types of decision tables, and the optimization algorithm of VPR in decision tables is also proposed. The Section VI verifies the proposed algorithms through experiments. Finally, the conclusion summarizes the paper.

II. PRELIMINARIES

The tuple $S = (U, A\ T, \{V_a \mid a \in AT\}, \{f(x,a) \mid x \in U, a \in AT\})$ is an information table, where U is a universe set, AT is a finite nonempty set of attributes, V_a is a nonempty set of values for $a \in AT$, and $f(x,a) : U \to AT$ is a function, where $f(x,a)$ takes a value on attribute a.

When A is an nonempty subset of AT, $f(x,A)$ is denoted as a value on attribute set A. An equivalence relation is defined by $\mathrm{R_A} = \{(x,y) \mid (x,y) \in U \times U, f(x,a) = f(y,a), \forall a \in A\}$, where $f(x,a)$ and $f(y,a)$ are the attribute values of x and y on a, respectively. Class $[x]_A$ is the equivalence class determined with respect to A and is denoted by $[x]_A = \{y \mid (x,y) \in R_A\}$ or can also be denoted by $[x]_A = \{y \mid (x,y) \in R_a, a \in A\}$.

For the tuple S, if $AT = C \cup D$, and $C \cap D = \emptyset$, C is the condition attribute set, D is the decision attribute set. Then, tuple S is called a decision table, briefly written as $(U, C \cup D)$.

Definition 1 ([1], [2]): Let $X \subseteq U, B \subseteq C$, and $x \in U$. Then, the lower and upper approximations of X are defined as

$$\underline{R}_B(X) = \{x \mid [x]_B \subseteq X\}, \tag{1}$$

$$\overline{R}_B(X) = \{x \mid [x]_B \cap X \neq \emptyset\} \tag{2}$$

Definition 2 ([1], [2]): Let $(U, C \cup D)$ be a decision table. Then, $U/R_D = \{D_1, D_2, \ldots, D_t\}$ is the quotient set determined by D. The positive region is defined as

$$\mathrm{Pos}_C\, D = \bigcup_{i=1}^{r} \underline{R}_C\,(D_i) \tag{3}$$

For a decision table $(U, C \cup D)$, if $\mathrm{Pos}_C\, D = U$, then the decision table is called consistent; otherwise, it is inconsistent.

Definition 3: Given $X \subseteq U$, for each $x \in U$, the characteristic function $\lambda_X(x)$ is defined as follows:

$$\lambda_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X \end{cases} \tag{4}$$

Lemma 1 [18]: For $X \subseteq U$ and $[x_i]_R \subseteq U$, $[x_i]_R$ is an equivalence class on the equivalence relation R. Then, $W_R\lambda_X$ is as follows:

$$W_R\lambda_X = \left[\frac{|[x_1]_R \cap X|}{|[x_1]_R|}, \frac{|[x_2]_R \cap X|}{|[x_2]_R|}, \ldots, \frac{|[x_n]_R \cap X|}{|[x_n]_R|}\right]^T \tag{5}$$

where T denotes the transpose. The Boolean column vector $\lambda_X = (\lambda_X(x_1), \lambda_X(x_2), \ldots, \lambda_X(x_n))^T$ for $x_i \in U$. In addition, W_R is denoted as

$$W_R = \begin{pmatrix} \frac{\lambda_R(x_1,x_1)}{|[x_1]_R|} & \frac{\lambda_R(x_1,x_2)}{|[x_1]_R|} & \cdots & \frac{\lambda_R(x_1,x_n)}{|[x_1]_R|} \\ \frac{\lambda_R(x_2,x_1)}{|[x_2]_R|} & \frac{\lambda_R(x_2,x_2)}{|[x_2]_R|} & \cdots & \frac{\lambda_R(x_2,x_n)}{|[x_2]_R|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_R(x_n,x_1)}{|[x_2]_R|} & \frac{\lambda_R(x_n,x_2)}{|[x_2]_R|} & \cdots & \frac{\lambda_R(x_n,x_n)}{|[x_2]_R|} \end{pmatrix} \tag{6}$$

Definition 4 [18]: Let R be an equivalence relation on U and $\beta \in (0,1]$. Then, the β-approximation of X is defined as

$$R^\beta(X) = \{x \mid P(X \mid [x]_R) \geq \beta\} \tag{7}$$

where $P(X \mid [x]_R) = \frac{|[x]_R \cap X|}{|[x]_R|}$.

Using the β-approximation and the quotient set, a fuzzy matrix can be constructed.

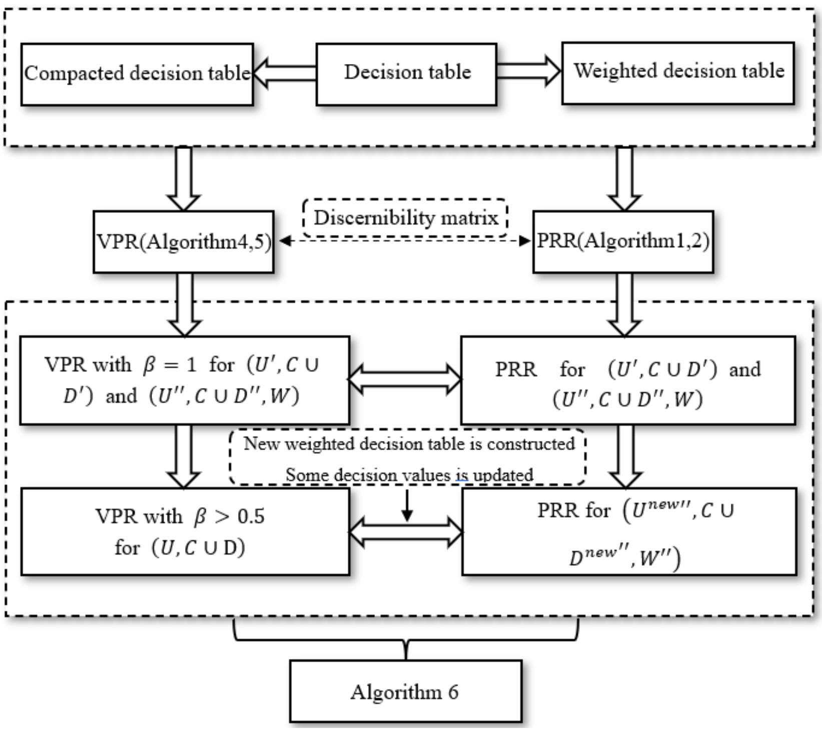


FIGURE 1. Research framework.

Theorem 1: For a decision table $(U, C \cup D)$, where $\beta \in (0, 1]$, for $x \in U$, we have $(\mu_{CD}(x))_\beta = \left(\lambda_{(R_C)^{(\beta)}(D_1)}(x), \lambda_{(R_C)^{(\beta)}(D_2)}(x), \ldots, \lambda_{(R_C)^{(\beta)}(D_t)}(x)\right)$.

According to the above definition of a fuzzy matrix, VPR can be defined in the decision table as follows.

Definition 5 ([17], [18]): Let $(U, C \cup D)$ be a decision table, $\beta \in (0, 1]$, and $B \subseteq C$. Then, B is called the VPR of C if it satisfies the following conditions:

1) $\forall x \in U, (\mu_{CD}(x))_\beta = (\mu_{BD}(x))_\beta$
2) $\exists x \in U$, for any $B' \neq \emptyset$, and $B' \subseteq B, (\mu_{CD}(x))_\beta \neq (\mu_{B'D}(x))_\beta$

To summarize, the definition of VPR and its discernibility matrix in a decision table were introduced in this section. Objective data can be described by decision tables. However, changes in some data sets may cause the form of the decision tables to also change. In this paper, the research framework is shown in Figure 1.

III. VPR IN COMPACTED AND WEIGHTED DECISION TABLES

On the basis of retaining all decision table information, this section describes how the number of objects (rows) in the decision table is compressed to change the form of the decision table, thus forming two types of decision tables. Compacted decision tables are formed by summing the number of identical decision attribute values in any equivalence class and then adding the number in the decision tables [41]. By contrast, the weighted decision table is formed by determining the decision attribute values of any two objects in any equivalence class and adding weights.

TABLE 1. Example decision table.

U	C			D
	a_1	a_2	a_3	
x_1	1	0	0	0
x_2	1	0	0	0
x_3	1	0	0	0
x_4	1	0	0	1
x_5	1	0	0	2
x_6	0	1	0	2
x_7	0	1	0	1
x_8	0	1	0	1
x_9	0	1	1	0
x_{10}	0	1	1	0

Definition 6 [41]: Let $(U, C \cup D)$ be a decision table, and let $U/R_C = \{[x_1]_C, [x_2]_C, \ldots, [x_m]_C\}$ be the quotient set determined by condition set C, where $U = \bigcup_{j=1}^{m} [x_j]_C$. Set $U/R_D = \{D_1, D_2, \ldots, D_t\}$ is the quotient set determined by decision set D, and the decision values are $v_{d_i} \in V_D$. Then, $(U', C \cup D')$ is a compacted decision table, where $U' = \{x_1, x_2, \ldots, x_m\}$ and $f(x_k, d_i) = |\ \{x \mid f(x, d) = v_{d_i}$ for $x \in [x_k]_C\}\ |$. Operator $|\cdot|$ is used to denote the cardinality of a set.

The following is an example to illustrate the transformation of a decision table into a compressed decision table. Given $(U, C \cup D)$ (Table 1), the compacted table $(U', C \cup D')$ (Table 2) is obtained by calculating the number of identical decision values in any equivalence class.

TABLE 2. Compacted decision table.

U'	C			D'		
	a_1	a_2	a_3	d_1	d_2	d_3
u_1	1	0	0	3	1	1
u_2	0	1	0	0	2	1
u_3	0	1	1	2	0	0

TABLE 3. Weighted decision table.

U''	W	C			D''
		a_1	a_2	a_3	
u_1	3	1	0	0	0
u_2	1	1	0	0	1
u_3	1	1	0	0	2
u_4	1	0	1	0	2
u_5	2	0	1	0	1
u_6	2	0	1	1	0

The process of the weighted decision table construction is as follows. In any equivalence class, for the objects with the same decision value, only one object is retained and the other object(s) are deleted, and the number of identical decision values is used to determine the weight, forming the weighted decision table.

Definition 7: Let $(U, C \cup D)$ be a decision table. For each $x'' \in U''$, which satisfies $x'' \in [x]_C$, we have $f(x'', W) = \left|\left\{x'' \mid x'' \in [x]_C \cap D_i, x \in U\right\}\right|$, where $D_i \in U/R_D$, Then, $\left(U'', C \cup D'', W\right)$ is defined as a weighted decision table.

In the weighted decision table, if $x''_1 \in [x'']_C$ and $x''_2 \in [x'']_C$, then $f(x''_1, D) \neq f(x''_2, D)$. Obviously, $\left|U''\right| \geq |U/R_C|$.

The following is an example to illustrate the transformation of a decision table into a weighted decision table. Given $(U, C \cup D)$ (Table 1), the weighted decision table is obtained by Definition 7.

In compacted and weighted decision tables, the definitions of VPR are as follows.

Definition 8: Let $\left(U', C \cup D'\right)$ be a compacted decision table. For $\beta \in (0, 1]$, $(\mu_{CD'}(x))_\beta$ is defined as

$$(\mu_{CD'}(x))_\beta = \left(\frac{f(x, d_1)}{\sum_{d_i \in D'} f(x, d_i)}, \frac{f(x, d_2)}{\sum_{d_i \in D'} f(x, d_i)}, \ldots, \frac{f(x, d_t)}{\sum_{d_i \in D'} f(x, d_i)}\right)_\beta \quad (8)$$

where $f(x, d_i)$ is the number of objects with decision value v_{d_i} in $[x]_C$, and t is the number of decision values in the compacted decision table.

For example, in Table 2, $u_1 \in U'$ because $f(x, d_1) = 3$, $f(x, d_2) = 1$, and $f(x, d_3) = 1$, and hence $(\mu_{CD'}(u_1))_\beta = \left(\frac{3}{5}, \frac{1}{5}, \frac{1}{5}\right)_\beta$.

Definition 9: Let $\left(U', C \cup D'\right)$ be a compacted decision table, where $\beta \in (0, 1]$, and $B \subseteq C$. Then, B is called the VPR of C if it satisfies the following conditions:

1) $\forall x \in U$, $(\mu_{CD'}(x))_\beta = (\mu_{BD'}(x))_\beta$
2) $\exists x \in U$, for any $B' \neq \emptyset$ and $B' \subseteq B$, $(\mu_{CD'}(x))_\beta \neq (\mu_{B'D'}(x))_\beta$

Similarly, to define VPR in a weighted decision table, $(\mu_{CD''}(x))_\beta$ is defined in a weighted decision table as follows.

Definition 10: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table, and $U/R_{D''} = \left\{D''_1, D''_2, \ldots, D''_t\right\}$, where W is the weight. $\beta \in (0, 1]$. Then, $(\mu_{CD''}(x))_\beta$ is defined as

$$(\mu_{CD''}(x))_\beta = \left(\frac{\sum_{x \in [x]_C \cap D''_1} f(x, W)}{\sum_{x_i \in [x]_C} f(x_i, W)}, \frac{\sum_{x \in [x]_C \cap D''_2} f(x, W)}{\sum_{x_i \in [x]_C} f(x_i, W)}, \ldots, \frac{\sum_{x \in [x]_C \cap D''_t} f(x, W)}{\sum_{x_i \in [x]_C} f(x_i, W)}\right)_\beta \quad (9)$$

Definition 11: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table. Then, $B \subseteq C$, B is called the VPR of C if it satisfies the following conditions:

1) $\forall x \in U$, $(\mu_{CD''}(x))_\beta = (\mu_{BD''}(x))_\beta$
2) $\exists x \in U$, for any $B' \neq \emptyset$ and $B' \subseteq B$, $(\mu_{CD''}(x))_\beta \neq (\mu_{B'D''}(x))_\beta$

Compacted and weighted decision tables are similar in that both reduce the number of objects (rows) in a decision table. The difference is that the cardinality of any equivalence class in the compacted decision table is 1, whereas an equivalence class may have several objects in the weighted decision table.

IV. ATTRIBUTE REDUCTION IN COMPACTED AND WEIGHTED DECISION TABLES

Because the form of the compacted and weighted decision tables have changed, the method for calculating the positive region changes accordingly. In this section, the corresponding discernibility matrices for the PRR in both types of decision tables are proposed. In addition, these discernibility matrices are derived so that the PPR algorithms for both types of decision tables can be developed.

A. PRR IN COMPACTED DECISION TABLES

For $x \in \mathrm{Pos}_C D'$, x has only one decision value in the compacted decision table. Therefore, the positive region is calculated as follows.

Definition 12: Let $\left(U', C \cup D'\right)$ be a compacted decision table. The positive region is defined as follows:

$$\mathrm{Pos}_C D' = \left\{x \mid \frac{f(x, d_h)}{\sum_{d_k \in D'} f(x, d_k)} = 1\right\} \quad (10)$$

Given $\left(U', C \cup D'\right)$, where $x \in U'$, if there exists only one d_i such that $f(x, d_i) \neq 0$, and $f(x, d_x) = 0 (i \neq k)$, then $x \in Pos_C D'$. Otherwise, $x \notin Pos_C D'$.

Using Definition 12, the corresponding matrix $M' = (m'_{ij})_{n \times n}$ is proposed as follows:

$$m'_{ij} = \begin{cases} \left\{a \mid a \in C, \left(x_i, x_j\right) \notin R_a\right\}, \\ \quad \text{if } \dfrac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1, \left(x_i, x_j\right) \notin R_{D'} \\ \emptyset, \quad \text{otherwise} \end{cases} \quad (11)$$

where n is the number of objects.

In matrix M', for $(x_i, x_j) \notin R_{D'}$, becasuse $x_i \in Pos_C D'$, then there exists d_k such that $f(x_j, d_k) \neq 0$ and $f(x_i, d_k) = 0$.

Lemma 2: Let $\left(U', C \cup D'\right)$ be a compacted decision table. If $\frac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1$ and $\left(x_i, x_j\right) \notin R_{D'}$, then $m'_{ij} \neq \emptyset$.

Proof: For $x_i \in U$, there exists $d_h \in D'$, $\frac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1$. If $\left(x_i, x_j\right) \notin R_{D'}$, then $\left(x_i, x_j\right) \notin R_C$. Thus, $\exists a \in C$ such that $\left(x_i, x_j\right) \notin R_a$, and therefore $m'_{ij} \neq \emptyset$.

The following theorem can be obtained from Lemma 2.

Theorem 2: Let $\left(U', C \cup D'\right)$ be a compacted decision table. If $B \subseteq C$, then $\mathrm{Pos}_C\, D' = \mathrm{Pos}_B\, D'$ if and only if $m'_{ij} \cap B \neq \emptyset$ for $m'_{ij} \neq \emptyset$.

Proof: $(\Rightarrow)$ For $[x_i]_C \subseteq U'$, if $\frac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1$ and $\left(x_i, x_j\right) \notin R_{D'}$, then $m'_{ij} \neq \emptyset$ by Lemma 2, which means $\left(x_i, x_j\right) \notin R_C$. Because $\mathrm{Pos}_C\, D' = \mathrm{Pos}_B\, D'$, for $[x_i]_B \subseteq U'$, $\frac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1$. Hence, $\exists l \in B$ such that $\left(x_i, x_j\right) \notin R_l$. Therefore, $m'_{ij} \cap B \neq \emptyset$.

$(\Leftarrow)$ Because $B \subseteq C$, we have $\mathrm{Pos}_B\, D' \subseteq \mathrm{Pos}_C\, D'$. We now show that $\mathrm{Pos}_C\, D' \subseteq \mathrm{Pos}_B\, D'$. This implies that if $x_i \in \mathrm{Pos}_C\, D'$, then $x_i \in \mathrm{Pos}_B\, D'$.

For $x_i \in \mathrm{Pos}_C\, D'$, there exists $\frac{f(x_i, d_h)}{\sum_{d_k \in D'} f(x_i, d_k)} = 1$. Using proof by contradiction, suppose that $x_j \notin [x_i]_D$. Then, $m'_{ij} \neq \emptyset$, and then $m'_{ij} \cap B \neq \emptyset$ by the condition. Let $l \in m_{ij} \cap B$. Then, $\left(x_i, x_j\right) \notin R_l$ and $x_j \notin [x_i]_B$. That is, $[x_i]_B \subseteq [x_i]_D$. Therefore, $x_i \in \mathrm{Pos}_B\, D'$ holds.

From Theorem 2, we have the following corollary.

Corollary 1: Let $\left(U', C \cup D'\right)$ be a compacted decision table, and let $B \neq \emptyset$ and $B \subseteq C$. Set B is a reduct of C if and only if it is a minimal subset satisfying $m'_{ij} \cap B \neq \emptyset$ for $m'_{ij} \neq \emptyset$.

According to Corollary 1, for a compacted decision table, the corresponding discernibility matrix for PRR is constructed as described in Algorithm 1.

Algorithm 1 Discernibility Matrix Construction for PRR

Input: compacted decision table $\left(U', C \cup D'\right)$
Output: matrix M'

```
m'_ij = ∅, B = ∅;
for all x in U' do
  compute Pos_C D';
end for
for all x in Pos_C D' do
  for all x in U' do
    if (x_i, x_i) ∉ R_D' then
      m'_ij = m'_ij ∪ a_i, a_i ∈ C;
    end if
  end for
end for // the discernibility matrix is constructed
return M';
```

The form of the decision table has been changed in the compacted decision table. Hence, the discernibility matrix is constructed to obtain the reduction results by calculating $f(x, d)$.

B. PRR IN WEIGHTED DECISION TABLES

The weighted decision table is another form of decision table, and we calculate the positive region according to the weight. Simultaneously, the weight is used to construct the discernibility matrix. The positive region in the weighted decision table is defined as follows.

Definition 13: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table, where W is the weight and $U/R_{D''} = \left\{D''_1, D''_2, \ldots, D''{}_t\right\}$. The positive region is defined as

$$\mathrm{Pos}_C\, D'' = \left\{ x \mid x \in U'', \frac{f(x, W)}{\sum_{x_i \in [x]_C} f(x_i, W)} = 1 \right\} \tag{12}$$

According to Definition 13, the corresponding discernibility matrix $M'' = (m''_{ij})_{n \times n}$ is expressed as follows:

$$m''_{ij} = \begin{cases} \left\{a \mid a \in C, \left(x_i, x_j\right) \notin R_a\right\}, \\ \dfrac{f(x_i, W)}{\sum_{x \in [x_i]C} f(x, W)} = 1 and \left(x_i, x_j\right) \notin R_{D''} \\ \emptyset, \quad otherwise \end{cases} \tag{13}$$

where n is the number of objects.

Lemma 3: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table. If $\frac{f(x_i, W)}{\sum_{x \in [x_j]C} f(x, W)} = 1$ and $\left(x_i, x_j\right) \notin R_{D''}$, then $m''{}_{ij} \neq \emptyset$.

The proof is similar to that of Lemma 2.

The following can be obtained from Lemma 3.

Theorem 3: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table, if $B \subseteq C$, then $\mathrm{Pos}_C\, D'' = \mathrm{Pos}_B\, D''$ if and only if it is a minimal subset satisfying $m''{}_{ij} \cap B \neq \emptyset$ for $m''{}_{ij} \neq \emptyset$.

The proof is similar to that of Theorem 2.

From Theorem 3, we have the following corollary.

Corollary 2: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table, and let $B \neq \emptyset$ and $B \subseteq C$. Set B is a reduct of C if and only if it is a minimal subset satisfying $m''_{ij} \cap B \neq \emptyset$ for $m''_{ij} \neq \emptyset$.

According to Corollary 2, the corresponding discernibility matrix of the PPR is constructed for the weighted decision table as described in Algorithm 2.

The method of transforming from the conjunctive normal form (CNF) to the disjunctive normal form (DNF) is an NP-hard problem. To improve efficiency, a binary programming algorithm can be used to quickly obtain the result [25]. The pseudocode of this algorithm is given in Algorithm 3.

Algorithm 1 is suitable for the compacted decision table, and Algorithm 2 is suitable for the weighted decision table. In the process of constructing the discernibility matrix, the former obtains the positive region by calculating the $f(x, d)$, whereas the latter obtains the positive region by calculating the weight.

C. RELATIONSHIP BETWEEN VPR AND PRR

Using Definitions 9 and 11, this section analyzes the relationship between VPR and PRR in decision tables from the perspective of constructing discernibility matrix. Moreover, it

Algorithm 2 Discernibility Matrix Construction Algorithm for PPR

Input: weighted decision table $\left(U'', C \cup D'', W\right)$.
Output: matrix M''.

$m''_{ij} = \emptyset, B = \emptyset$;
for all x in U'' **do**
  compute $\mathrm{Pos}_C\, D''$;
end for
for all x in $\mathrm{Pos}_C\, D''$ **do**
  for all x in U'' **do**
    if $\left(x_i, x_j\right) \notin R_{D''}$ **then**
      $m''_{ij} = m''_{ij} \cup a_i, a_i \in C$
    end if
  end for
end for // the discernibility matrix is constructed
return M'';

Algorithm 3 Obtain the Reduct of a Discernibility Matrix

Input: matrix M.
Output: reduct of C.

for all m_{ij} in M **do** $//m_{ij} \in M$
  if $m_{ij} \subseteq m_{pq}$ **then** $//m_{ij}, m_{pq} \in M$
    m_{pq} be deleted;
  end if
  $T_{\min} = \left\{m_{ij}\right\}$; $//T_{\min}$ be the minimal element set;
end for
for m_{ij} in $T_{\min}$ **do**
  if $a \in m_{ij}$ **then**
    $a = 1$;
  else
    $a = 0$;
  end if
end for
minimize $\sum_{a \in m_{ij}} a > 0$ such that $m_{ij} \in T_{\min}$;
return B; // A reduct of C

proves that VPR with $\beta = 1$ is the PRR in both types of decision tables.

Theorem 4: Let $\left(U', C \cup D'\right)$ be a compacted decision table. If $B \subseteq C$, then $\mathrm{Pos}_C\, D' = \mathrm{Pos}\,_B D'$ if and only if $(\mu_{CD'}(x))_1 = (\mu_{BD'}(x))_1$ for $\forall x \in U$.

Proof: ($\Rightarrow$) If $\mathrm{Pos}_C\, D' = \mathrm{Pos}_B\, D', x \in U$. Then, $[x]_C \subseteq [x]_{D'}$ if and only if $[x]_B \subseteq [x]_{D'}$. That is, for $[x]_B$ and $[x]_C$, there exist only $d_k \in V_{D'}$ such that, $f\,(x, d_k) \neq 0$ and $f\,(x, d_i) = 0 (i \neq k)$. Thus, $(\mu_{CD'}(x))_1 = (\mu_{BD'}(x))_1$.

($\Leftarrow$) Note that $(\mu_{CD'}(x))_1 = (\mu_{BD'}(x))_1$. There are two cases: if $\frac{f(x,d_k)}{\sum_{d_i \in D'} f(x,d_i)} = 1$ for $[x]_C$ and $\frac{f(x,d_k)}{\sum_{d_i \in D'} f(x,d_i)} = 1$ for $[x]_B$, then $[x]_C \subseteq [x]_{D'}$ if and only if $[x]_B \subseteq [x]_{D'}$. Thus, $\mathrm{Pos}_C\, D' = \mathrm{Pos}_B\, D'$. $\forall d_k \in V_{D'}$. If $\frac{f(x,d_k)}{\sum_{d_i \in D'} f(x,d_i)} = 0$ for $[x]_C$ and $[x]_B$, $x \notin \mathrm{Pos}_C\, D'$ and $x \notin \mathrm{Pos}_B\, D'$ by Definition 12. Therefore, $\mathrm{Pos}_C\, D' = \mathrm{Pos}_B\, D'$.

It can also be shown that a VPR with $\beta = 1$ is equivalent to the PRR in the compacted decision table. Next, the relation between VPR with $\beta = 1$ and the PRR is constructed in the weighted decision table.

Theorem 5: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table. If $B \subseteq C$, then $\mathrm{Pos}_C\, D'' = \mathrm{Pos}_B\, D''$ if and only if $(\mu_{CD''}(x))_1 = (\mu_{BD''}(x))_1$.

The proof is similar to that of Theorem 4.

Using the discernibility matrix, Theorems 4 and 5 show that the result of VPR with $\beta = 1$ is the result of PRR. The time complexity of VPR for constructing the discernibility matrix is $O(|U|^2|C|)$, but the time complexity of PRR is $O(|\mathrm{Pos}_C\, D||U||U/D||C|)$, which is lower.

V. PROPOSED VPR ALGORITHMS

When calculating VPR, the decision values or weights of some objects in equivalence classes that meet the condition $\beta > 0.5$ are modified. For both types of decision tables, because the relationship between VPR and PRR is established based on the discernibility matrix, the PPR can be used for the calculation to optimize the calculation of VPR. The construction processes of transforming the decision table into each type of decision table are compared, a new VPR algorithm for decision table is hence proposed to improve the efficiency of the VPR algorithm.

A. PROPOSED VPR ALGORITHM FOR COMPACTED DECISION TABLES

By establishing the relation between VPR and PRR, an optimized algorithm for obtaining the VPR in a compacted decision table is proposed in this subsection. If $\beta > 0.5$, then a new compacted decision table is formed by updating some decision values in the compacted decision table.

Definition 14: Let $\left(U', C \cup D'\right)$ be a compacted decision table. For $x \in U'$, if $\frac{f(x,d_h)}{\sum_{d_k \in D'} f(x,d_k)} \geq \beta(\beta > 0.5)$, let $f\,(x, d_k) = 0$ when $d_k \neq d_h$. Then $\left(U', C \cup D^{new'}\right)$ is a new compacted decision table, where $V_{D'}^{new}$ is a nonempty set of values for $D^{new'}$.

Note that for $(U', C \cup D')$, because of the possibility that $f(x, d_k) = 0$ for $\forall x$, the range $V_{D'}^{new}$ may change.

Theorem 6: Let $\left(U', C \cup D'\right)$ be a compacted decision table for each $\mu_{CD'}(x)$, where $(\mu_{CD'}(x))_\beta = \left(\frac{f(x,d_1)}{\sum_{d_k \in D'} f(x,d_k)}, \frac{f(x,d_2)}{\sum_{d_k \in D'} f(x,d_k)}, \ldots, \frac{f(x,d_t)}{\sum_{d_k \in D'} f(x,d_k)}\right)_\beta$. For $\beta > 0.5$, if $\left(U', C \cup D^{new'}\right)$ is a new compacted decision table, where $\left(\mu_{CD'}^{new}(x)\right)_1 = \left(\frac{f(x,d_1)}{\sum_{d_k \in D^{new'}} f(x,d_k)}, \frac{f(x,d_2)}{\sum_{d_k \in D^{new'}} f(x,d_k)}, \ldots, \frac{f(x,d_t)}{\sum_{d_k \in D^{new'}} f(x,d_k)}\right)_1$ for each $x \in U$. Then, $(\mu_{CD'}(x))_\beta = \left(\mu_{CD'}^{new}(x)\right)_1$.

Proof: There are two cases: For $(\mu_{CD'}(x))_\beta$ in a compacted decision table, if there exist $d_h \in D'$ such that $\frac{f(x,d_h)}{\sum_{d_k \in D'} f(x,d_k)} \geq \beta$, then $\frac{f(x,d_i)}{\sum_{d_k \in D'} f(x,d_k)} < \beta\,(d_i \neq d_h)$. In $\left(U', C \cup D^{new'}\right)$, because $f\,(x, d_h) \neq 0$ and

Algorithm 4 Optimization Algorithm for VPR

Input: compacted decision table $\left(U', C \cup D'\right), \beta > 0.5$.
Output: matrix M'.

```
m'_ij = ∅, B = ∅;
for all x in U' do
  if f(x,d_h) / Σ_{d_k ∈ V_D} f(x,d_k) ≥ β then
    f(x,d_i) = 0 (d_i ≠ d_h);
  end if
end for
compute Pos_C D';
for all x in Pos_C D' do
  for all x in U' do
    if (x_i, x_j) ∉ R_D' then
      m'_ij = m'_ij ∪ a_i, a_i ∈ C;
    end if
  end for
end for // the discernibility matrix is constructed
return M';
run Algorithm 3;
return B;
```

$f(x, d_i) = 0\,(d_i \neq d_h)$, we have $\frac{f(x,d_h)}{\sum_{d_k \in D^{new'}} f(x,d_k)} = 1$ and $\frac{f(x,d_i)}{\sum_{d_k \in D^{new'}} f(x,d_k)} = 0\,(d_i \neq d_h)$. Thus, $(\mu_{CD'}(x))_\beta = \left(\mu_{CD'}^{new}(x)\right)_1$.

For $\frac{f(x,d_h)}{\sum_{d_k \in D'} f(x,d_k)}$ in $(\mu_{CD'}(x))_\beta$, if each $\frac{f(x,d_h)}{\sum_{d_k \in D'} f(x,d_k)} < \beta$, then $\frac{f(x,d_h)}{\sum_{d_k \in D^{new'}} f(x,d_k)} = 0$ in $\left(\mu_{CD'}^{new}(x)\right)_1$. Therefore, $(\mu_{CD'}(x))_\beta = \left(\mu_{CD'}^{new}(x)\right)_1$.

According to Theorem 4, VPR with $\beta = 1$ can be transformed into PPR for calculation in a compacted decision table. The new algorithm for constructing the VPR discernibility matrix in the compacted decision table is given in Algorithm 4.

An example based on Table 2 is given to illustrate the feasibility of the proposed Algorithm 4 given $\beta = 0.65$.

1) In Table 2, for $[u_2]_C$, $\frac{f(u_2,d_2)}{f(u_2,d_1)+f(u_2,d_2)+f(u_2,d_3)} = \frac{2}{0+2+1} \geq \beta$. Hence, let $f(u_2, d_3) = 0$ such that $f(u_2, d_2) = 3$. For $[u_1]_C$, because $\frac{f(u_1,d_1)}{f(u_1,d_1)+f(u_1,d_2)+f(u_1,d_3)} = \frac{3}{3+1+1} < \beta$, no decision values in $[u_1]_C$ are modified. Similarly, $[u_3]_C$ does not meet the condition for modifying the decision value, so they are not modified. The new compacted decision table is given in Table 4.
2) In Table 4, $\mathrm{Pos}_C D' = \{u_2, u_3\}$.
3) The corresponding matrix M' is as follows:

$$\begin{bmatrix} \{a_1, a_2\} & \emptyset & \{a_3\} \\ C & \{a_3\} & \emptyset \end{bmatrix}$$

4) The result is $\{a_1, a_2\}$ or $\{a_1, a_3\}$.

TABLE 4. New compacted decision table.

U'	C			D'		
	a_1	a_2	a_3	d_1	d_2	d_3
u_1	1	0	0	3	1	1
u_2	0	1	0	0	3	0
u_3	0	1	1	2	0	0

B. PROPOSED VPR ALGORITHM FOR WEIGHTED DECISION TABLES

Similar to the VPR optimization method for a compacted decision table, in the VPR optimization method for a weighted decision table, the VPR is calculated by modifying the weights of some objects and using the PPR. The new weighted decision table is defined as follows.

Definition 15: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table. If $\frac{f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)} \geq \beta(\beta > 0.5)$, for $[x]_C \subseteq U''$, let $f(x, W) = \sum_{x_i \in [x]_C} f(x_i, W)$. Further, let any information of x_i be deleted when $\frac{f(x_i,W)}{\sum_{x \in [x_i]_C} f(x,W)} < \beta$. For $[x]_C \subseteq U''$, if $\frac{f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)} < \beta$, let any information of x_i be unchanged. Then, $(U^{new''}, C \cup D^{new''}, W'')$ is the new weighted decision table.

Note that for $(U', C \cup D'', W)$, where $D_i'' \in U/R_{D''}$, $f(x, d)$ may not exist for each x because of the process for modifying the decision values.

Theorem 7: Let $\left(U'', C \cup D'', W\right)$ be a weighted decision table, $\forall [x]_C \subseteq U''$, where $(\mu_{CD''}(x))_\beta = \left(\frac{\sum_{x \in [x]_C \cap D_1''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)}, \frac{\sum_{x \in [x]_C \cap D_2''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)}, \ldots, \frac{\sum_{x \in [x]_C \cap D_t''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)}\right)_\beta$. For $\beta > 0.5$, if $\left(U^{new''}, C \cup D^{new''}, W''\right)$ is a new weighted decision table, where each $x'' \in U^{new''}$ corresponds to $x \in U''$, then $(\mu_{CD''}(x))_\beta = \left(\mu_{CD''}^{new}(x'')\right)_1$, where $\left(\mu_{CD''}^{new}(x'')\right)_1 = \left(\frac{\sum_{x'' \in [x'']_C \cap D_1^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')}, \frac{\sum_{x'' \in [x'']_C \cap D_2^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')}, \ldots, \frac{\sum_{x'' \in [x'']_C \cap D_t^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')}\right)_1$.

Proof: There are two cases: For $(\mu_{CD''}(x))_\beta$ and $D_i'' \in U/R_{D''}$, if $\frac{\sum_{x \in [x]_C \cap D_i''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)} \geq \beta$, and $\beta > 0.5$, then $\frac{\sum_{x \in [x]_C \cap D_j''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)} < \beta(j \neq i)$. For x'' in $U^{new''}$, which corresponds to $x \in U''$, $\frac{\sum_{x'' \in [x'']_C \cap D_i^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')} = 1$ and $\frac{\sum_{x'' \in [x'']_C \cap D_j^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')} = 0(j \neq i)$. Thus, $(\mu_{CD''}(x))_\beta = \left(\mu_{CD''}^{new}(x'')\right)_1$.

For each $D_i'' \in U/R_{D''}$, if $\frac{\sum_{x \in [x]_C \cap D_i''} f(x,W)}{\sum_{x_i \in [x]_C} f(x_i,W)} < \beta$, x'' in $U^{new''}$ corresponds to $x \in U''$, then $\frac{\sum_{x'' \in [x'']_C \cap D_i^{new''}} f(x'',W'')}{\sum_{y \in [x'']_C} f(y,W'')} = 0$ in $\left(\mu_{CD''}^{new}(x'')\right)_1$. Therefore, $(u_{CD''}(x))_\beta = \left(\mu_{CD''}^{new}(x)\right)_1$.

Algorithm 5 Optimized VPR Calculation

Input: weighted decision table $\left(U'', C \cup D'', W\right)$, $\beta > 0.5$.
Output: matrix M''.

$m''_{ij} = \emptyset, B = \emptyset$;
for all x in U'' **do**
  if $\frac{\sum_{x\in[x]_C \cap D''_j} f(x,W)}{\sum_{x_i\in[x]_C} f(x_i,W)} \geq \beta$ **then**
    $f(x, W) = \sum_{x_i\in[x]_C} f(x_i, W)$;
    x_j be deleted; $//i \neq j$;
  end if
end for $//(U'', C \cup D'', W)$ is updated as $(U^{new''}, C \cup D^{new''}, W'')$
compute $\mathrm{Pos}_C\, D^{new''}$;
for all x in $\mathrm{Pos}_C\, D^{new''}$ **do**
  for all x in U'' **do**
    if $\left(x_i, x_j\right) \notin R_{D''}$ **then**
      $m''_{ij} = m''_{ij} \cup a_i$; $//a_i \in C$;
    end if
  end for
end for // the discernibility matrix is constructed
return M'';
run Algorithm 3;
return B;

TABLE 5. New weighted decision table.

U''	W''	C			$D^{new''}$
		a_1	a_2	a_3	
u_1	3	1	0	0	0
u_2	1	1	0	0	1
u_3	1	1	0	0	2
u_5	3	0	1	0	1
u_6	2	0	1	1	0

Using Theorems 5 and 7, the new algorithm for constructing the VPR discernibility matrix for a weighted decision table is given in Algorithm 5.

An example based on Table 3 is given to illustrate the feasibility of the proposed Algorithm 5, where $[u_1]_C = \{u_1, u_2, u_3\}$,$[u_4]_C = \{u_4, u_5\}$, and$[u_6]_C = \{u_6\}$, given $\beta = 0.65$.

1) For $[u_4]_C$ in Table 3, $\frac{f(u_5,w)}{f(u_4,w)+f(u_5,w)} = \frac{2}{3} \geq \beta$, let $f(u_4, \mathrm{d}) = 1$. For $[u_1]_C$, because $\frac{f(u_1,W)}{f(u_1,W)+f(u_2,W)+f(u_3,W)} = \frac{3}{3+1+1} < \beta$, no decision values in $[u_1]_C$ are modified. Therefore, the new weighted decision table (Table 5) is updated.
2) In Table 5, $\mathrm{Pos}_C\, D' = \{u_5, u_6\}$.
3) The corresponding matrix M' is as follows:

$$\begin{bmatrix} \{a_1, a_2\} & \{a_1, a_2\} & \{a_1, a_2\} & \emptyset & \{a_3\} \\ C & C & C & \{a_3\} & \emptyset \end{bmatrix}$$

4) The result is $\{a_1, a_2\}$ or $\{a_1, a_3\}$.

The main reason for setting $\beta > 0.5$ is the high confidence of the reduction. In a VPR process, when the precision is greater than 0.5, the new compacted/weighted decision table is formed after modifying some decision values of the compacted/weighted decision table. It has been proved that VPR with $\beta > 0.5$ in the two types of decision tables is equal to the VPR with $\beta = 1$. Subsequently, it was proved that VPR with $\beta = 1$ is the PRR based on the discernibility matrix. Therefore, after changing the form of the decision table, optimized VPR algorithms were proposed.

Algorithm 6 New VPR Algorithm for Decision Tables

Input: decision table $(U, C \cup D)$, $\beta > 0.5$.
Output: A reduct of C.

$m_{ij} = \emptyset, B = \emptyset$;
for all x in U **do**
  if $\frac{|[x]_C \cap D_i|}{|[x]_C|} > \beta$ **then**
    modify all the decision values of $x_i//x_i$ in $[x]_C$;
  end if
end for
while $|U| > 0$ **do**
  for all x in U **do**
    any information of x_i be deleted; $//x_i = x$
  end for
end while // the weighted decision table is constructed
run Algorithm 2;
run Algorithm 3;
return B;

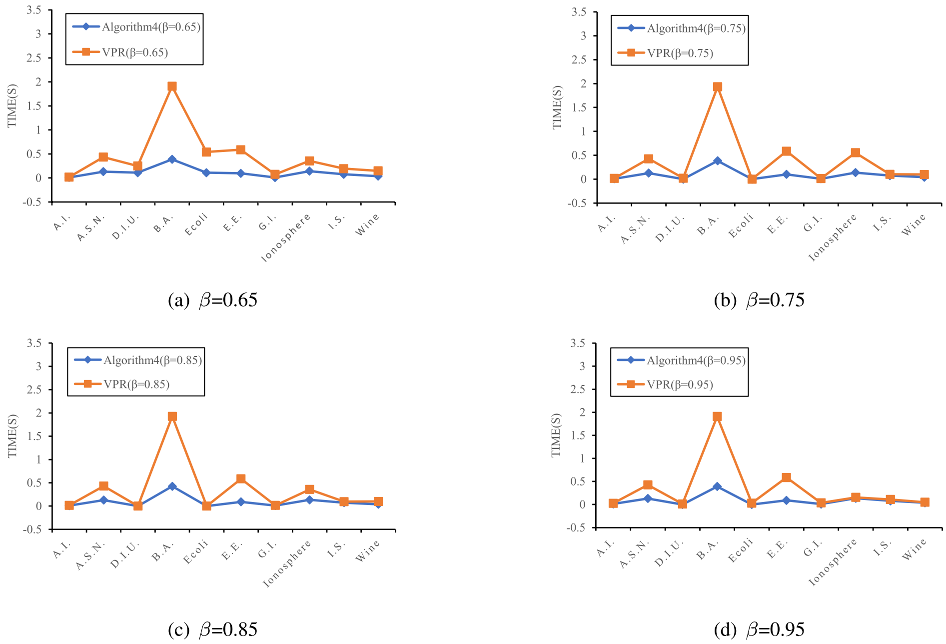
TABLE 6. Description of the datasets.

Datasets	Abbr.	$\lvert U\rvert$	$\lvert C\rvert$	$\lvert U/D\rvert$
Acute Inflammations	A.I.	120	6	4
Airfoit_Self_Neise	A.S.N.	1503	5	38
Dishonest Internet Users	D.I.U.	322	4	2
Banknote_Authentication	B.A.	1372	4	2
Ecoli	–	336	6	8
Energy Efficiency	E.E.	768	8	160
Glass Identification	G.I.	214	9	7
Ionosphere	–	351	34	2
Image Segmentation	I.S.	210	19	7
Wine	–	178	13	3

For VPR in a decision table, considering the number of modified decision values and the number of objects in equivalence class, in contrast to the time complexity of the process of constructing the weighted decision table, the time complexity of constructing d_i is $O(|U/D|)(U/C)\,(|[x]_C|)$ when constructing the compacted decision table. Therefore, the method of transforming the decision table into a weighted decision table is adopted after modifying the decision values, and the VPR is calculated using Algorithms 2 and 3. Compared with the existing VPR algorithms [17], [19] the time complexity of discernibility matrix construction is $O\left(|U|^2|C|\right)$, whereas the time complexity of Algorithm 6 to construct the discernibility matrix is $O\left(|\mathrm{Pos}_C\, D''||U''||C|\right)$, then $|\mathrm{Pos}_C\, D''||U''| < |U|^2$. Hence, Algorithm 6 has relatively high computational efficiency.

VI. EXPERIMENTAL ANALYSIS

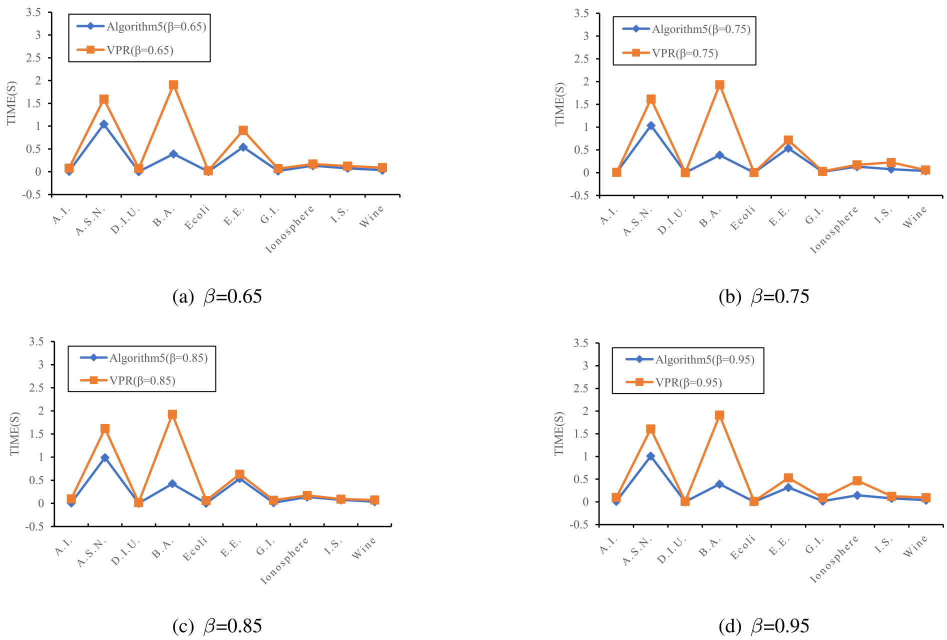
To evaluate the performance of the algorithms, we selected 10 datasets from the UCI datasets and compared the proposed



(a) β=0.65 (b) β=0.75

(c) β=0.85 (d) β=0.95

FIGURE 2. **Runtime results for two discernibility matrix construction algorithms on compacted decision tables.**



(a) β=0.65 (b) β=0.75

(c) β=0.85 (d) β=0.95

FIGURE 3. **Runtime results for two discernibility matrix construction algorithms on weighted decision tables.**

algorithms with existing algorithms VPR-DM [17] and KNR-DM [19]. Three classifiers including fine Gaussian naïve Bayes (NB), decision trees (DT), and support vector classification (SVC) were used to test the classification accuracy for different reduction results, in which ten-fold cross-validation was used. All experiments were coded in Python 3.7 and were tested on a Lenovo R7000 PC (early 2020s) with an AMD Ryzen 5 CPU at 3.0 GHz and Radeon Graphics 4600H GPU.

Table 6 summarizes the details of the selected datasets, in which $|U|$ indicates the number of objects, $|C|$ indicates the number of condition attributes, and $|U/D|$ indicates the number of classes. To ensure high confidence, the precision should be greater than 0.5 in the experiment.

The compacted and weighted decision tables were constructed for each dataset. Figures 2 and 3 show the runtimes for constructing the discernibility matrices using VPR and

TABLE 7. Number of rows of the discernibility matrix results.

	0.65			0.75			0.85			0.95		
	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6
A.I.	120	120	99	120	120	99	120	120	99	120	120	99
A.S.N.	1503	1503	131	1503	1503	120	1503	1503	119	1503	1503	119
D.I.U.	322	322	20	322	322	20	322	322	20	322	322	20
B.A.	1372	1372	693	1372	1372	693	1372	1372	693	1372	1372	693
Ecoli	336	336	29	336	336	23	336	336	18	336	336	15
E.E.	768	768	49	768	768	48	768	768	48	768	768	48
G.I.	214	214	85	214	214	81	214	214	77	214	214	76
Ionosphere	351	351	258	351	351	257	351	351	257	351	351	257
I.S.	210	210	194	210	210	194	210	210	194	210	210	194
Wine	178	178	170	178	178	170	178	178	170	178	178	170

TABLE 8. Reduction length results of the algorithms.

	0.65			0.75			0.85			0.95		
	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6	VPR-DM	KNR-DM	Algorithm6
A.I.	3	3	3	3	3	2	3	3	2	3	3	3
A.S.N.	3	3	3	3	3	3	3	3	3	3	3	3
D.I.U.	3	3	3	3	3	3	3	3	3	3	3	3
B.A.	3	3	3	3	3	3	3	3	3	3	3	3
Ecoli	6	6	6	6	6	6	6	6	6	6	6	6
E.E.	3	3	3	3	3	3	3	3	3	3	3	3
G.I.	7	7	7	7	7	7	7	7	6	7	7	7
Ionosphere	12	12	12	14	14	16	14	14	13	14	14	13
I.S.	4	4	4	4	4	4	4	4	4	4	4	4
Wine	6	6	4	6	6	6	6	6	4	6	6	4

the optimized algorithms for both types of decision tables. Figure 2 compares the runtimes for VPR and Algorithm 4 at different precisions for the compacted decision tables. For example, when $\beta = 0.65$, the number of objects in compacted decision tables are 131, 691, respectively, for A.S.N. dataset and B.A. dataset. It should be noted that the heuristic VPR algorithms may be efficient, but the focus of this paper is to study the reduction algorithms based on the discernibility matrix.

Figure 3 compares the runtimes of the VPR and Algorithm 5 at different precisions on the weighted decision tables. Obviously, the runtimes of Algorithms 4 and 5 are shorter than those of the VPR algorithms for both types of decision tables.

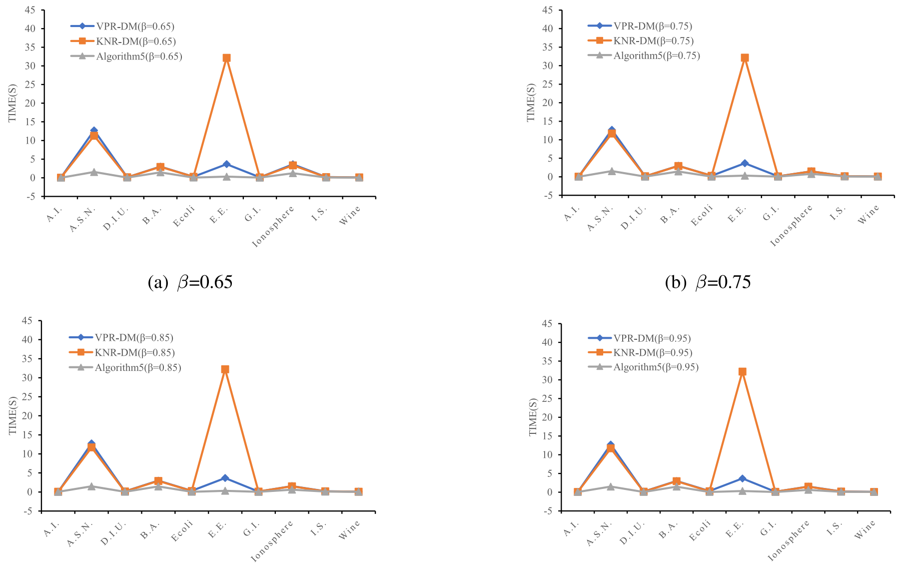
Three algorithms, VPR-DM, KNR-DM, and Algorithm 6, were compared with respect to parameter $|U|$ for different precisions. The results are listed in Table 7. The numbers of rows of the discernability matrices constructed by Algorithm 6 are smaller than those of the other algorithms. The lengths of the reducts of the three algorithms are presented in Table 8. Because all three algorithms are based on the discernibility matrix, the reduct results are obtained by binary programming, and hence the reduction lengths differ.

In terms of runtime (Figure 4), Algorithm 6 is obviously faster than VPR-DM and KNR-DM. For example, on the A.S.N. dataset, when $\beta = 0.65$, the runtimes are 12.72 s, 11.28 s, and 1.556 s, respectively, for VPR-DM, KNR-DM, and Algorithm 6. When $\beta = 0.95$, the runtimes are 12.72 s, 11.69 s, and 1.48 s, respectively. When the precision is smaller, the positive region is larger. For example, on the Ecoli dataset, when $\beta = 0.65$ and $|\mathrm{Pos}_C D''| = 29$, size of the constructed discernibility matrix is 29×39, and when $\beta = 0.95$ and $|\mathrm{Pos}_C D''| = 15$, its size is 15×55. When the number of modified decision values in the data set is small, Algorithm 6 has a slight performance advantage. However, when the number of modified decision values is large, the performance advantage of Algorithm 6 is clear. Although there is little difference in the lengths of the reducts of the three algorithms, the run time of Algorithm 6 is shorter than that of the other algorithms, which is consistent with the time complexity analysis.

We evaluated the reduction quality using the fine Gaussian NB, DT, and SVC classifiers. The training accuracies (as a percentage) and runtimes (in milliseconds) of the reduction results obtained by Algorithm 6 at different precisions are shown Figures. 5-7, in which the solid lines represent the training time and the dashed lines represent classification accuracy. The original datasets are the datasets before reduction. It can be observed that higher precisions lead to higher classification accuracies for the reduction results.

When the precision of Algorithm 6 is higher, the classification accuracy is not much different from that of the original dataset and the training time is reduced. For example, for the DT classifier on the D.I.U. dataset, when $\beta = 0.95$, the training accuracy is 87.5%, whereas it is 81.25% on the original dataset. Moreover, the training time on the data after reduction is 3.2 ms, whereas the training time on the original dataset is 4.5 ms.

When the precision is 0.65, the accuracies of the classifiers on the reduction results are lower. For example, on the E.E. dataset, the training accuracy of the fine Gaussian NB is 73.02%, whereas it is 80.51% for the original dataset; the training accuracy of the DT classifier is 69.85%, whereas it is 72.9% for the original dataset; the training accuracy of the SVC classifier is 59.85%, whereas it is 72.89% for the original dataset.



(a) β=0.65 (b) β=0.75 (c) β=0.85 (d) β=0.95

FIGURE 4. Runtime results for three algorithms.

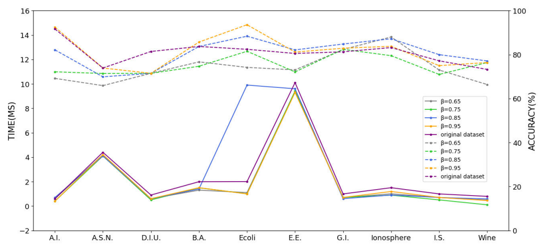


FIGURE 5. Classification accuracy and runtime results for the reduction results at different precisions using the fine Gaussian NB classifier.

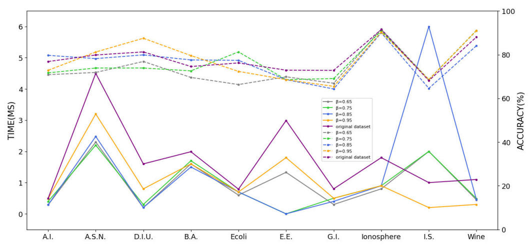


FIGURE 6. Classification accuracy and runtime results for the reduction results at different precisions for the DT classifier.

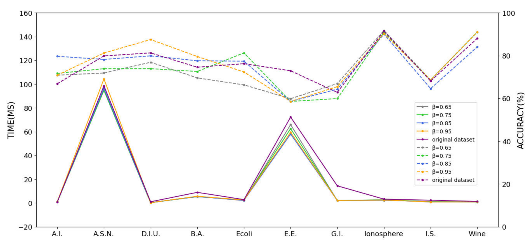


FIGURE 7. Classification accuracy and runtime results for the reduction results at different precisions for the SVC classifier.

The experiment first compared the VPR algorithms with their optimized algorithms on both types of decision tables, then the proposed VPR algorithm (Algorithm 6) was compared with existing algorithms based on the discernibility matrix. The experimental results show that Algorithm 6 has better operating efficiency, especially when the datasets are large. Moreover, the reduction results of Algorithm 6 at different precisions were compared using different classifiers to verify that the proposed algorithm is feasible.

VII. CONCLUSION

Compacted and weighted decision tables are two extended forms of the decision table. The relationship between VPR and PRR was established for both types of decision table, and the VPR algorithm was optimized by modifying the decision values of objects that satisfy the given condition and then using the PRR algorithm for the calculation. Furthermore, by comparing the modification process of the decision values in both types of decision tables, a new VPR algorithm was proposed that updates the decision table into a weighted decision table and uses PRR to calculate the VPR. Finally, this proposed algorithm was verified by experiments. In future, we will attempt to remove the restriction of equivalence relation and further study problems such as VPR.

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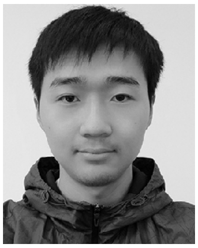

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