IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 31 March 2023, accepted 24 April 2023, date of publication 28 April 2023, date of current version 8 May 2023.
Digital Object Identifier 10.1109/ACCESS.2023.3271628

RESEARCH ARTICLE

Qubit Quasi-Probability Coherence Induced by a Nonlinear Coherent Cavity Filled With a Kerr-Like Medium Under Dissipation Effect

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This work was supported by Prince Sattam bin Abdulaziz University under Project PSAU/2023/R/1444.

ABSTRACT The work considers a qubit interacting off-resonantly a nonlinear Kerr-like quantum-harmonic-oscillator cavity field through nonlinear intensity-dependent and one-photon interactions. The analytical solution for the master equation is obtained when the qubit starts with an excited pure state while the harmonic-oscillator field starts with a coherent state. The dynamics of the phase space Husimi-distribution and its Wehrl-Husimi entropy entanglement/mixedness is explored under the effects of the atom-field detuning, Kerr-like nonlinearity as well as atomic spontaneous-emission dissipation. For resonant case, the Wehrl-Husimi entropy qubit-oscillator entanglement and atomic mixedness are generated (due to the unitary nonlinear intensity-dependent evolution) with a regular oscillatory behavior. For off-resonant case, the quantum coherence is generated partially with a high-frequency irregular oscillatory behavior. The Kerr-like nonlinearity and the atomic spontaneous dissipation lead to enhancing the generated atomic mixedness Wehrl-Husimi entropy enhances and stabilizing the atomic state in a maximally mixed state. The phase space Husimi-distribution information dynamics of the corresponding the generated atomic mixed states confirms the vital link between the formed interference Husimi-distributions and the generated atomic Wehrl-Husimi entropy mixedness. It is found that the dynamics of the Husimi-distribution information and its Wehrl-Husimi entropy is highly sensitive to the qubit-cavity detuning, Kerr-like nonlinearity as well as the dissipation.

INDEX TERMS Master equation, Husimi-distribution, spontaneous decay.

I. INTRODUCTION

The interactions of qubits (two-level atomic systems) with field-cavity modes have potential applications in quantum optics and quantum information [1]. One of the most common resources for generating quantum qubits-effects (as coherence, correlation, entanglement, mixedness, non-classicality, and …) is the closed and open qubit-cavity interactions [2], [3]. More investigations demonstrate the quantum effects in several physical two-qubit systems [4], [5], [6]. Therefore, the qubit-cavity interactions have been realized experimentally in different real systems as trapped ions [7], [8], quantum dots [9], [10], and superconducting circuits [11], [12]. Qubit-cavity interactions were described by Jaynes-Cummings (JC) model [13], which is generalized to including nonlinear interactions (of the intensity-dependent coupling [14], two-photon transitions, Kerr-like medium [15]), multi-level atomic system [16], atomic motion [17], Stark-shift [18], driven JC model [19], and multi-qubit-cavity Tavis-Cummings model [20]. Kerr-like nonlinearity can be realized with superconducting quantum interference device [21] and with pertubing potential barrier [22]. Quantum coherence can be enhanced by considering the cavity-field Kerr nonlinearity [23], [24]. Recently, the exploring cavity-atom state dynamics under the Kerr nonlinearity effects [25] in the presence of dissipation is limited [26].

By using the JC models, the qubit-cavity system and its sub-system dynamics are explored by different equations of

The associate editor coordinating the review of this manuscript and approving it for publication was Zeev Zalevsky.

motion. For closed interactions, Schrödinger equation and intrinsic-decoherence master equation [27]. While for open interactions, that are other master equations [28], [29], which is used to investigate the effects of dissipation (which is resulted from cavity damping and qubit spontaneous decay, and the energy is not conserved) and phase decoherence (which is resulted from field and qubit phase damping, and the energy is conserved). The effects of surrounding environments (dissipation and decoherence) on the quantum effect (as coherence and entanglement) is a central topic of quantum optics and quantum information [30], [31], [32]. The dissipation and decoherence induced by surrounding environments lead to the degradation of quantum correlation [33] and the growth of the mixedness [34], [35], [36].

Phase-space quantum effects (as non-classicality, mixedness, and entanglement) based on quasi-probability Husimi and Wigner distributions [37], [38] are important tools to quantum information [39]. The positivity and negativity of the Wigner distribution are good indicators to explore the non-classicality and entanglement [40], [41], [42], [43], [44]. Atomic Husimi-distribution is used to explore phase-space quantum effects (by using Wehrl-Husimi entropy [45]), which are useful quantum information resources [46].

The motivation behind this publication is that (1) previous investigations of the Husimi-distribution information and its Wehrl-Husimi entropy have been explored exclusively in closed systems. (2) The investigations of the Husimi-distribution information and entanglement/mixedness's Wehrl-Husimi entropy for open systems remain limited, particularly with the Kerr nonlinearity and the off-resonant interaction effects. Therefore, the main objective of this paper is to explore the dynamics of the atomic Husimi-distribution information, entanglement, and mixedness's Wehrl-Husimi entropy, which are induced by the off-resonant interaction of a dissipative qubit (by considering a spontaneous-emission dissipation) with a nonlinear Kerr-like quantum-harmonic-oscillator cavity through nonlinear intensity-dependent and one-photon interactions.

The paper is arranged as follows: Section II is devoted to introduce the dissipative qubit-oscillator model and its analytical solution. In section III, we offer the Husimi-function distribution and its Wehrl entropy, then the dynamics of the phase space Husimi-distribution information loss as well as Wehrl-Husimi entropy entanglement and mixedness. Finally, our conclusion is presented in section IV.

II. DISSIPATIVE QUBIT-OSCILLATOR MODEL

To explore the phase space quasi-probability information of a qubit interacting nonlinearity with a harmonic-oscillator field mode under atomic spontaneous-emission damping, we consider the generalized JC model that describes a qubit, having the two-level excited $|1\rangle$ and the ground states $|0\rangle$ as well as the frequency ω_0, coupling off-resonantly to a nonlinear Kerr-like quantum-harmonic-oscillator cavity, having the creation $\hat{a}$ and annihilation $\hat{a}^\dagger$ operators as well as the frequency ω, through the intensity-dependent interaction. If we consider the rotating wave approximation and the dissipation sources (which are results due to interacting the qubit-oscillator with a harmonic oscillator reservoir) acting only on the qubit [47], [48], then by using the master equation, the time-dependent qubit-mode density matrix $R(t)$ evolves as

$$\frac{\partial}{\partial t}\hat{R}(t) = -i[\hat{H},\hat{R}] + \kappa_A([|0\rangle\langle 1|,\hat{R}|1\rangle\langle 0|] + [|0\rangle\langle 1|\hat{R},|1\rangle\langle 0|]), \quad (1)$$

where κ_A is atomic spontaneous-emission rate and the nonlinear qubit-mode Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\omega_o\hat{\sigma}_z + \chi\hat{a}^{\dagger 2}\hat{a}^2 + \lambda(\hat{a}\sqrt{\hat{a}^\dagger\hat{a}}|1\rangle\langle 0| + h.c, \quad (2)$$

λ is the qubit-mode coupling, $\hat{\sigma}_z$ represents the population inversion operator. The qubit-mode detuning is given by: $2D = \omega_o - \omega$. $\chi > 0$ is the Kerr-like nonlinearity. i.e. In the qubit-mode basis states: $\{|B_1\rangle = |1,n\rangle, |B_2\rangle = |0,n+1\rangle\}$, the eigenstates $|V_n^\pm\rangle$ and eigenvalues $V_n^\pm$ of the nonlinear qubit-mode Hamiltonian (2) are respectively given by

$$|V_n^\pm\rangle = X_n^\pm|B_1\rangle \pm X_n^\mp|B_2\rangle, \quad \forall n \geq 0 \quad (3)$$

$$V_n^\pm = \omega(n+\frac{1}{2}) + \frac{1}{2}(M_n + M_{n+1}) \pm \mu_n. \quad (4)$$

With

$$\mu_n = \sqrt{(D + M_n - M_{n+1})^2 + \lambda^2(n^2+2n+1)} \quad (5)$$

$$X_n^\pm = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{D + M_n - M_{n+1}}{\mu_n}}, \quad (6)$$

and the Kerr-like nonlinearity coupling is controlled by the function: $M_n = n^2\chi - n\chi$. To find an analytical solution for Eq.(1), we work in a high$-Q$ cavity limit ($\kappa_A \ll \lambda$) and the eigen-states representation [49], i.e. all the atomic operators appearing in Eq.(1) will be represented in terms of the eigenstates $|V_n^\pm\rangle$, for example,

$$|1\rangle\langle 0| = \sum_{n=0}^{\infty}\{X_n^+X_{n-1}^-|V_n^+\rangle\langle V_{n-1}^+| - X_n^+X_{n-1}^+|V_n^+\rangle\langle V_{n-1}^-| + X_n^-X_{n-k}^-|V_n^-\rangle\langle V_{n-1}^+| - X_n^-X_{n-1}^+|V_n^-\rangle\langle V_{n-1}^-|\},$$

$$|1\rangle\langle 1| = \sum_{n=0}^{\infty} X_n^{+2}|V_n^+\rangle\langle V_n^+| + X_n^{-2}|V_n^-\rangle\langle V_n^-| + X_n^+X_n^-(|V_n^+\rangle\langle V_n^-| + |V_n^-\rangle\langle V_n^+|). \quad (7)$$

Then the following canonical atom-field-Hamiltonian transform is used,

$$\frac{\partial\hat{S}(t)}{\partial t} = e^{i\hat{H}t}\frac{\partial\hat{R}(t)}{\partial t}e^{-i\hat{H}t} + i[\hat{H}, R(t)], \quad (8)$$

According to the canonical atom-field-Hamiltonian transform of Eq.(8), the master equation (1) becomes

$$\begin{aligned}\frac{\partial \hat{S}(t)}{\partial t} &= 2\kappa_A \sum_{n=k}^{\infty}(X_n^{+2}X_{n-k}^{-2}Y_{nn}^{+} + X_n^{-2}X_{n-1}^{-2}Y_{nn}^{-})\hat{M}_{n-1}^{+} \\ &\quad + (X_n^{+2}X_{n-1}^{+2}Y_{nn}^{+} + X_n^{-2}X_{n-1}^{+2}Y_{nn}^{-})\hat{M}_{n-1}^{-} \\ &\quad - \kappa_A\gamma \sum_{n=0}^{\infty} X_n^{+2}(\hat{M}_n^{+}S + S\hat{M}_n^{+}) \\ &\quad + X_n^{-2}(\hat{M}_n^{-}\hat{S} + \hat{S}\hat{M}_n^{-}),\end{aligned} \tag{9}$$

where $Y_{nn}^{\pm} = \langle V_n^{\pm}|S|V_n^{\pm}\rangle$ and $\hat{M}_n^{\pm} = |V_n^{\pm}\rangle\langle V_n^{\pm}|$.

The atom-mode coherence loss (or generation of atom-mode mixedness) can be achieved by the interaction of the system with an environment, which leads to neglecting the interference between system states, i.e., the disappearance of the density matrix off-diagonal elements leads to coherence loss. Therefore, we consider that the atom-mode interaction starts with an initial pure state,

$$\hat{R}(0) = \hat{R}_A(0) \otimes \hat{R}_F(0) = |1\rangle\langle 1| \otimes |\alpha\rangle\langle\alpha|, \tag{10}$$

where the atom starts with the excited state $|1\rangle$ while the harmonic oscillator starts with a coherent state that has the coherent intensity $|\alpha|^2$ and the photon number distribution,

$$q_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}. \tag{11}$$

This maximally coherent/pure state $\hat{R}(0)$ is considered to show how the atomic spontaneous-emission damping evolves its initial purity into a maximal mixedness. In the used eigen-states-representation method, the initial density operator is rewritten in terms of the eigen-states $|V_n^{\pm}\rangle$ as

$$\begin{aligned}\hat{R}(0) = \hat{S}(0) = \sum_{m,n=0} q_m q_n^*[X_m^{+}|V_m^{+}\rangle + X_m^{-}|V_m^{-}\rangle] \\ \otimes [X_m^{+}\langle V_m^{+}| + X_m^{-}\langle V_m^{-}|],\end{aligned} \tag{12}$$

Therefore, by using the canonical transform of Eq.(8) and the pervious initial states, the time dependent atom-mode density matrix of Eq.(1) (in the qubit-mode-states basis: $\{|B_i\rangle\}$) is given by

$$\hat{R}(t) = \sum_{m,n}\sum_{ij=1,2} R_{ij}(t)|B_i\rangle\langle B_j|. \tag{13}$$

The diagonal elements of the density matrix $\hat{R}(t)$ are given by

$$R_{11(22)}(t) = \begin{cases} U_{mn}, & \forall\ m \neq n; \\ Z_m, & \forall\ m = n. \end{cases} \tag{14}$$

with

$$\begin{aligned}U_{mn} &= X_m^{\pm}X_n^{\pm}Y_{l,j}^{++} + X_m^{\mp}X_n^{\mp}Y_{m,n}^{--} \pm X_m^{\mp}X_n^{\pm}Y_{m,n}^{-+} \\ &\quad + X_m^{\pm}X_j^{\mp}Y_{m,n}^{+-}, \\ Z_m &= X_m^{\pm2}K_m^{\pm} + X_m^{\mp2}K_m^{\mp} \pm 2q_m q_m^* X_m^{\pm2}X_m^{\mp2}\cos 2\mu_m,\end{aligned}$$

where the coefficients $Y_{m,n}^{\epsilon\kappa}(t)(\epsilon, \varepsilon = +, -)$ and $K_m^{\pm}(t)$ are given by

$$Y_{m,n}^{\epsilon\varepsilon}(t) = q_m q_n^* X_m^{\epsilon}X_n^{\varepsilon}e^{-i(V_m^{\epsilon}-V_n^{\varepsilon})t}\ e^{-\kappa_A[(X_m^{\epsilon})^2+(X_n^{\varepsilon})^2]t},$$

and

$$\begin{aligned}K_m^{\pm}(t) &= e^{-2X_m^{\pm2}\kappa_A t}\{K_m^{\pm}(0) \\ &\quad + 2\kappa_A\int_0^t e^{2X_m^{\pm2}\kappa_A t}[X_{m+1}^{\pm2}X_m^{\mp2}K_{m+1}^{\pm}(t) \\ &\quad + X_{m+1}^{\mp2}X_m^{\mp2}K_{m+1}^{\mp}(t)]dt\}.\end{aligned} \tag{15}$$

To produce the integral term, we take the case where the atom-mode state has at most N photons only i.e $K_{N+1}^{\pm}(t) = 0$, and in the end we take $N \to \infty$ [49].

While the off-diagonal elements of the density matrix $\hat{R}(t)$ are given by

$$\begin{aligned}R_{12(21)}(t) &= X_m^{\pm}X_n^{\mp}Y_{m,n}^{++} - X_m^{\mp}X_n^{\pm}Y_{m,n}^{--} \\ &\quad \pm X_m^{\mp}X_n^{\mp}Y_{m,n}^{-+} \mp X_l^{\pm}X_j^{\pm}Y_{m,n}^{+-}, \forall m, n.\end{aligned} \tag{16}$$

To explore the atomic quasi-probability Husimi-distribution information dynamics under the atomic spontaneous-emission damping, we take the trace of the field-mode number states, $\{|k\rangle(k = 0, 1, 2, \ldots, \infty\}$, to get the time-dependent atomic reduced matrix,

$$\hat{R}_A(t) = \mathrm{Tr}_{\mathrm{field}}\{\hat{R}(t)\} = \sum_{k=0}^{\infty}\langle k|\hat{R}(t)|k\rangle. \tag{17}$$

Now, we can investigate the sensitivity of the quasi-probability Husimi-distribution information and its Wehrl-Husimi entropy coherence under the atom-field detuning and the atomic damping for one-photon precess.

III. HUSIMI-FUNCTION DISTRIBUTION AND ITS WEHRL ENTROPY

In follows, we study the dynamics of the phase space quasi-probability Husimi-distribution information and its Wehrl-Husimi entropy coherence induced by the dissipative atomic reduced density matrix $\hat{R}_A(t)$. The atomic Wehrl entropy is a powerful tool to measure important quantum information resources [45], [50] (as purity loss/mixedness and atom-mode entanglement) and also the extraction of a phase space quantum information. For the phase space of the atomic coherent states $|x, y\rangle$ determined by the angles $x \in [0, \pi]$, and $y \in [0, 2\pi]$ [51],

$$|x, y\rangle = \sum_{k=-j}^{k=j}\frac{1}{2^j}e^{i(j-k)y}\sin^j x\ \coth^k\frac{x}{2}\sqrt{\begin{pmatrix}2j \\ j-k\end{pmatrix}}|j, k\rangle. \tag{18}$$

with the angular momentum j, the atomic Husimi-distribution information of the corresponding the atomic generated time-dependent state $R_A(t)$ is defined as [38]

$$H(x, y, t) = \frac{1}{2\pi}\langle x, y|R_A(t)|x, y\rangle = \frac{1}{4\pi}\Lambda(t), \tag{19}$$

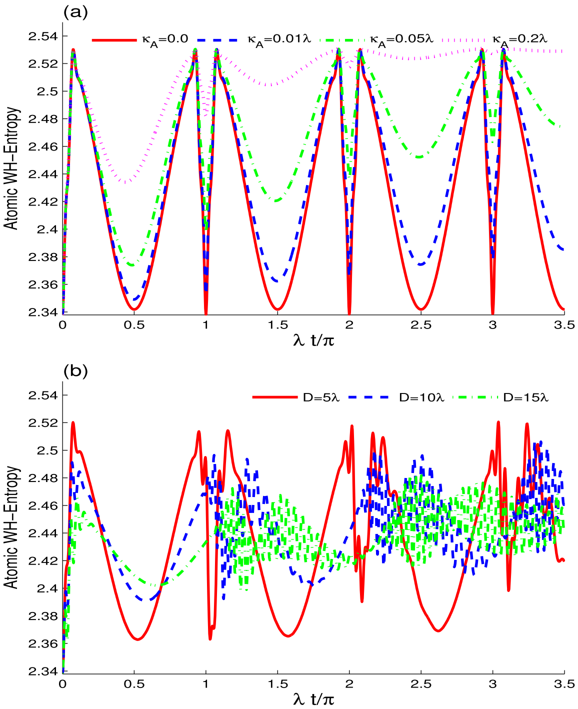

FIGURE 1. **Atomic WH-entropy dynamics for different values's atomic spontaneous emission decay with the initial coherent intensity $N = 16$ if the atom and field resonated $D = 0$ in (a). (b) shows the atom-field detuning effect.**

where $\Lambda(t) = 1 + \beta_z \cos x + (\beta_x \cos y + \beta_y \sin y)\sin x$. β_x, β_y, and β_y are expectation values of the atomic Pauli operators $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$, respectively. The atomic Husimi distribution is used as a useful measure for the phase space information loss, where the atomic Husimi distribution information is dependent of the phase space parameters (x, y).

The atomic Wehrl-Husimi entropy (atomic WH-entropy) [45] is given by

$$S(t) = \ln(4\pi) - \sum_{n=1}^{\infty} \int_0^{2\pi} \int_0^{\pi} \frac{(\Lambda(t) - 1)^n \Lambda(t)}{(4\pi n!)} \sin x \, dx \, dy. \tag{20}$$

For the initial atomic state $|1\rangle$, the atomic WH-entropy evolves with respect to the following inequality [52], [53]:

$$2.3379 \leq S(t) \leq \ln(4\pi) \approx 2.5310. \tag{21}$$

A. PHASE SPACEWH-ENTROPY DYNAMICS

Fig.(1a) illustrates the atomic WH-entropy dynamics $S(t)$ for different values's atomic spontaneous-emission decay when the harmonic oscillator coherent field state is initially with a large coherent intensity, $N = |\alpha|^2 = 4$, and the field is resonated with the two-level atomic system, $D = 0$. In this case where the atomic spontaneous emission is absent, the generated phase space atomic mixedness and atom-field entanglement of the WH-entropy, due to the unitary atom-field interaction, have the same quantum information resource. The solid curve of Fig.(1a) shows the ability of the atom-field interactions to generate entanglement

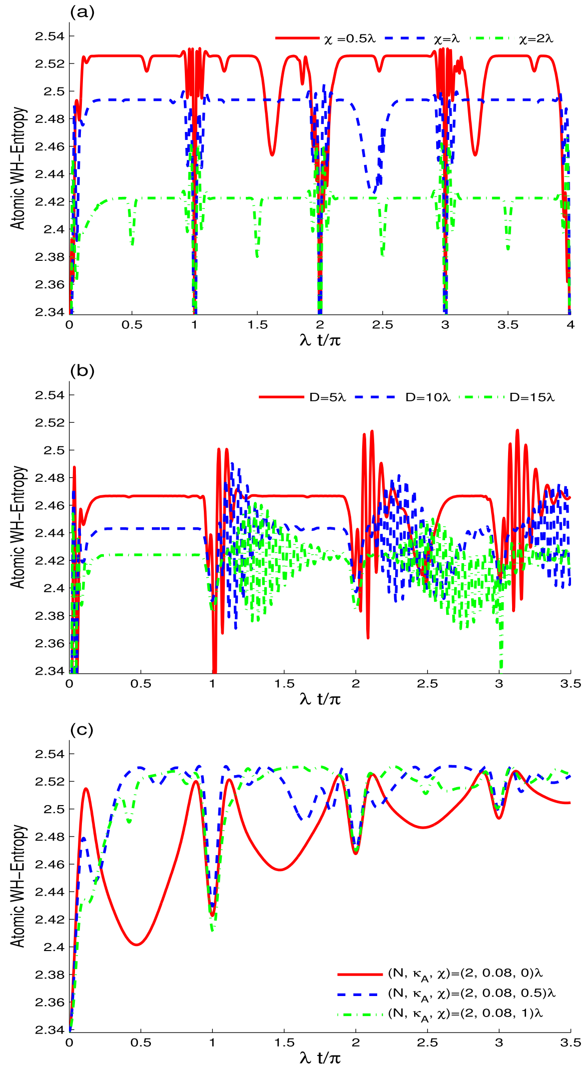

FIGURE 2. **Atomic WH-entropy dynamics of the solid curve of Fig.(1) is plotted under different values's Kerr-like nonlinearity in (a). In (b), the detuning effect on the WH-entropy dynamics of Fig.1b is shown with $\chi = \lambda$. The combined effect of the Kerr-like nonlinearity and the spontaneous-emission decay is shown in (c) with a small initial coherent intensity, $N = 2$.**

between the qubit and the harmonic-oscillator field or to generate atomic WH-entropy mixedness. The generated atomic WH-entropy entanglement grows and oscillates, periodically with regular 2π-period oscillatory behavior. Theoretically, it is proven that the one-photon Jaynes-Cummings model with intensity-dependent coupling (for a coherent state) displays regular quantum phenomena with π-period [54], [55], [56], [57] and the revivals take place at $t_{rev} = \lambda t = n\pi (n = 0, 1, 2, \ldots)$. The regular dynamics of the WH-entropy is due to that the coefficients $X_n^{\pm}$ of the periodical cosine and complex exponential functions in the basic term $Y_{m,n}^{\epsilon\kappa}(t)$ (which controls the atomic density matrix elements) are constant ($X_n^{\pm} = \frac{1}{\sqrt{2}}$ when $\delta = 0$ and $\chi = 0$). The atom-field/atom are in maximally pure states at $t_{Ini} = t_{rev} = n\pi (n = 0, 1, 2, \ldots)$ and are in partially pure states at $t_{Par} = \lambda t = \frac{(2n+1)}{2}\pi (n = 0, 1, 2, \ldots)$. Also, the atom-field interaction has a high ability to generate maximally entangled/mixed states at different times: $t_{Max} = \lambda t = (n.073)\pi$ and $t_{Max} = (n.9263)\pi, (n = 0, 1, 2, \ldots)$. The atomic WH-entropy reaches its maxima and minima,instantaneously. Therefore, the time intervals of the generated partial atomic WH-entropy entanglement/mixedness are extensive.

Dashed, dash-doted, and dot curves of Fig.(1a) show the effect of the increase of the atomic spontaneous-emission dissipation on the phase space WH-entropy dynamics. According to Eq.(15), the m, n-elements of the density matrix of Eq.(13) have the atomic spontaneous emission dissipation term: $T_D^{mn} = e^{-\kappa_A[(X_m^\epsilon)^2+(X_n^\epsilon)^2]t}$. For the case of $\kappa_A \neq 0$, the WH-entropy quantities only the phase space mixedness between the atomic upper $|1\rangle$ and lower $|0\rangle$ states. And the atomic dissipation term T_D^{mn} leads to increasing the regular π-period-oscillatory amplitudes of the WH-entropy mixedness. For a large value of the atomic spontaneous-emission dissipation, the WH-entropy minima shifted up to reaches and stables at its maximum value $S_{\max}(t) = \ln 4\pi$, which is time dependent. Therefore, the time intervals of the generated phase space stationary maximally atomic mixed state are very large.

Fig.(1b) displays the atomic WH-entropy dynamics $S(t)$ of the solid curve of Fig.(1a) but for different values's atom-field detuning in the absence of the atomic spontaneous-emission decay with the intensity of the coherent states, $|\alpha|^2 = 4$. By comparing the results of the solid curves of Fig.(1a) and Fig.(1b), we find that, for a small value of the atom-field detuning, the generated atomic WH-entropy entanglement and atomic mixedness grow with irregular oscillatory behavior. The atomic entanglement and atomic mixedness enhance due to increasing atom-field detuning. The atom-oscillator and the qubit do not return to their initial quantum information resources. The dashed and dotted-dashed plots show that the increase of the atom-field detuning has a high ability to generating partial entanglement and mixedness with irregular oscillatory behavior with high fluctuations. Comparing by the resonant case, we can deduce that the off-resonant case leads to degrading the amplitudes and increasing the frequency of the generated irregular entanglement and mixedness oscillatory behavior.

Fig.(2a) shows the atomic WH-entropy dynamics for different values's harmonic-oscillator field Kerr-like nonlinearity, in the absence of the atomic-spontaneous dissipation with the initial intensity coherent $N = 16$ when the atom and the harmonic-oscillator field is resonated. By comparing the solid curves of the Fig.(2a) and Fig.(2b), we find that, for a small value of the harmonic-oscillator field Kerr-like nonlinearity, the generated atomic WH-entropy entanglement/mixedness (due to the unitary atom-oscillator interaction) enhances. Most of the partial entanglement/mixedness intervals of Fig.(2a) are replaced by stationary maximal entanglement/mixedness intervals. In this case, the atom-oscillator and two-level atomic system have stable maximally atom-oscillator entanglement and qubit mixed states, which can be used to building quantum information and computation [58], [59] and quantum-channel metrology [60]. The dashed and dotted-dashed curves show that the increase of the harmonic-oscillator field Kerr-like nonlinearity leads to decreasing the stationary maximal entanglement and mixedness amplitudes. This contributes to the appearance of partial stable atom-oscillator entanglement and atomic mixedness.

Fig.(2b) illustrates the atom-oscillator detuning effect $D > 0$ on the WH-entropy dynamics of Fig.(1b) in the presence of the cavity Kerr-like nonlinearity $\chi = 0.5\lambda$. After adding the cavity Kerr-like nonlinearity, we find the Kerr-like nonlinearity leads to decreasing the amplitudes and increasing the frequency of the generated irregular atomic WH-entropy entanglement/mixedness oscillatory behavior. In addition, the partial stable atom-oscillator entanglement and atomic mixedness intervals reduce by increasing the atom-oscillator detuning. This means that the combined effect of the cavity Kerr-like nonlinearity and the atom-oscillator detuning leads to notable changes in the WH-entropy entanglement and atomic mixedness dynamics. From Fig.(1b) and Fig.(2b) of the cases of $\delta \neq 0$ and $\chi \neq 0$, we find that the WH-entropy has irregular dynamics, this is due to that the coefficients of the periodical cosine and complex exponential functions, in the basic term $Y_{m,n}^{\epsilon\kappa}(t)$ of the atomic density matrix elements, are n-dependent ($X_n^{\pm} = \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{D+M_n-M_{n+1}}{\mu_n}}$).

Fig.(2c) shows the atomic WH-entropy dynamics under the combined effects of the cavity Kerr-like nonlinearity and the atomic spontaneous-emission decay for a small initial coherent intensity. The cases of $(N, \kappa_A, \chi) = (2, 0.08, 0)\lambda$ (solid curve), $(N, \kappa_A, \chi) = (2, 0.08, 0.5)\lambda$ (dashed plot curve), $(N, \kappa_A, \chi) = (2, 0.08, 1)\lambda$ (dotted-dashed curve) are considered. From the solid-curve case, we observe that the WH-entropy atomic mixedness dynamics is susceptible to the atomic spontaneous-emission decay effect. The dashed dotted-dashed curves confirm that the cavity Kerr-like nonlinearity increases the atomic spontaneous-emission effect (here, the cavity Kerr-like nonlinearity is work as an additional dissipation resource to increase the coherence loss). The generated stable atomic mixedness is enhanced. The numerical results of the Fig.(1) and Fig.(2) show that the generated lower and upper bounds of the atomic WH-entropy entanglement/mixedness are in the line of the theoretical inequality of Eq.(21) $(2.3379 \leq S(t) \leq \ln(4\pi) \approx 2.5310)$.

B. PHASE SPACE HUSIMI-DISTRIBUTION INFORMATION LOSS

Atomic phase space Husimi distribution information loss/erasing means the atomic Husimi distribution is independent of the phase space parameters (x, y). Each atomic pure/mixed state is described by only one Husimi distribution. In Fig.(3), at $\lambda t = n\pi$, the atomic Husimi distribution $H(x, y, n\pi)$ is plotted for the phase space $x \in [0, 2\pi]$ and $y \in [0, 2\pi]$ angles and the initial intensity-coherent harmonic-oscillator field $N = 16$. The $n\pi$-atomic Husimi distributions have the same distribution of the initial atomic state $\hat{\rho}_A(0) = |1\rangle\langle 1|$, which is given by: $H(x, y) = \frac{1}{4\pi}1 + \cos x$ that

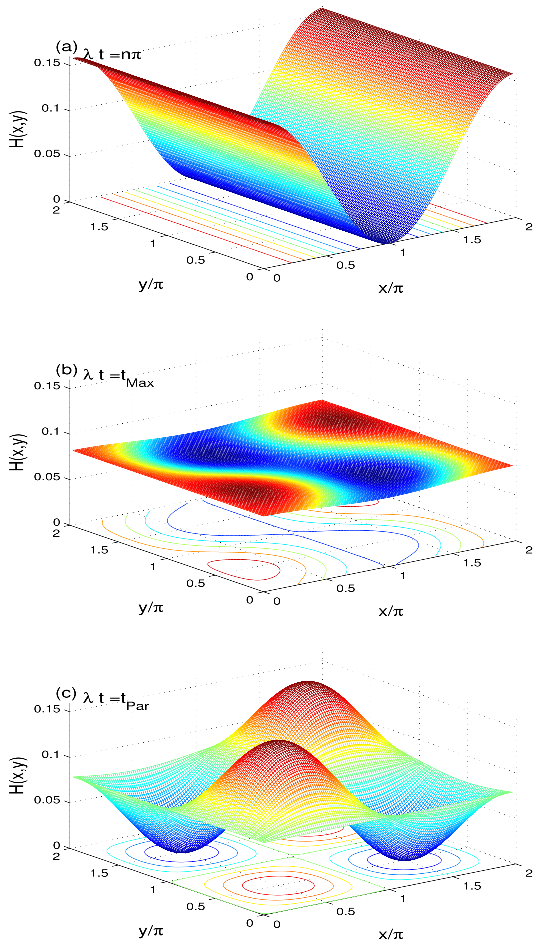

FIGURE 3. **Atomic Husimi-function distribution $H(x, y)$ is depicted at different times: $t_{Ini} = \lambda t = n\pi$, $(n = 0, 1, 2, \ldots)$ in (a), $t_{Max} = (n.9263)\pi$ in (b), and $t_{Par} = \frac{(2n+1)}{2}\pi$ with $n = 2$ in (c) for the initial intensity coherent field $N = 16$ and without the atomic spontaneous emission decay $\kappa_A = 0$.**

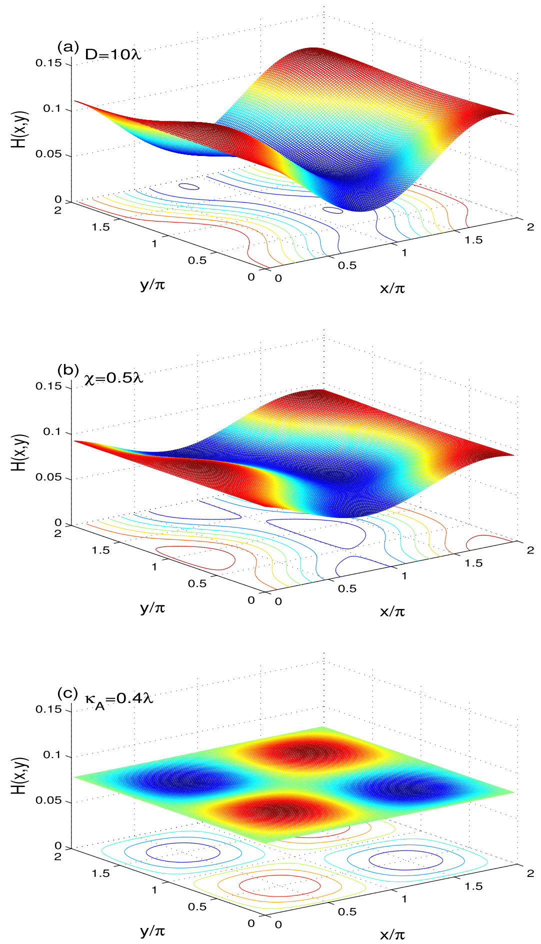

FIGURE 4. **Husimi-function distribution of the generated partially atomic pure state at the time of $t_{Par} = \frac{(2n+1)}{2}\pi$ of Fig.(3c) is depicted under the effects of the atom-field detuning $D = 10\lambda$ in (a), the cavity Kerr-like nonlinearity $\chi = 0.5\lambda$ in (b) as well as the atomic spontanous-emission dissipation $\kappa_A = 0.4\lambda$ in (c).**

depends only on the x-axis and it is y-independent. Fig.(3a) shows that the $n\pi$-atomic Husimi distribution information is only distributed with a regular oscillatory behavior with respect to the x-axis with 2π-period oscillatory surface. The maxima of the $n\pi$-atomic HD, $H_{\max} = \frac{1}{2\pi} \simeq 0.16$, are at $x = 2n\pi, (n = 1, 2, \ldots)$ whereas the $n\pi$-atomic HD minima, $H_{\min} = 0$, are at $x = (2n + 1)\pi, (n = 0, 1, 2, \ldots)$. To see the dynamics of the atomic Husimi-function (H-F) distribution information $H(x, y) = H(x, y, t)$, the atomic H-F will be depicted at different times. The choice of these times depends on atomic WH-entropy dynamics of Fig.(1). From the atomic WH-entropy mixedness dynamics of Fig.(1), we find that the two-level atomic system is in maximally pure states at $t_{Ini} = n\pi(n = 0, 1, 2, \ldots)$, in partially pure states at $t_{Par} = \frac{1}{2}\pi(2n + 1)(n = 0, 1, 2, \ldots)$, and are in maximally atomic mixed states at $t_{Max} = (n.073)\pi$ and $t_{Max} = (n.9263)\pi$, $(n = 0, 1, 2, \ldots)$.

From Fig.(3b), we find that the maxima and minima of atomic H-F distribution of the corresponding the generated maximally atomic mixed state at $t_{Max} = (n.9263)$ (for $D = 0$, $\chi = 0$, and $\kappa_A = 0$) are squeezed to form the constant atomic H-F distribution $H_{const} \simeq \frac{1}{4\pi}$ that is (x, y)-independent. It has the same shape of a two-level atomic state described by a statistically mixed state: $\frac{1}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|)$.

In Fig.(3c), the atomic H-F distribution of the generated partially atomic pure state at the time of $t_{Par} = \frac{(2n+1)}{2}\pi$ with $n = 2$ is depicted for $D = 0$, $\chi = 0$, and $\kappa_A = 0$. Due to the unitary evolution, the initial pure atomic state transform to be another partially mixed atomic state at the time of $t_{Par} = \frac{(2n+1)}{2}\pi$, which is described by a superposition of the atomic state $|1_A\rangle$ and $|0_A\rangle$. At these times of $t_{Par} = \frac{(2n+1)}{2}\pi$, the atomic H-F distribution of the corresponding the time of t_{Par} has regular oscillatory distribution. This regular oscillatory distribution has peaks (at $(x, y) = (0.5, 0.5)\pi$ and $(x, y) = (0.5, 1)\pi$) and bottoms (at $(x, y) = (1, 0.5)\pi$ and $(x, y) = (0.5, 1)\pi$), which are up and down the constant Husimi distribution H_{const}. The peak and bottom appearance is an indicator to the interference between $|1_A\rangle\langle 1_A|$ and $|0_A\rangle\langle 0_A|$ atomic states.

Fig.(4) shows H-F distribution dynamics of the generated partially atomic pure state at the time of $t_{Par} = \frac{(2n+1)}{2}\pi$ of Fig.(3c) under the effects of the atom-field detuning

$D = 10\lambda$ in (a), the cavity Kerr-like nonlinearity $\chi = 0.5\lambda$ in (b), and the atomic spontaneous-emission dissipation $\kappa_A = 0.4\lambda$ in (c). Due to that the amplitudes of the atomic WH-entropy mixedness of the case where the atom-field detuning $D = 10\lambda$ is smaller than that of the resonant case (see Fig.(1a)), the generated atomic H-F distribution is formed to be similar to the $n\pi$-atomic Husimi distributions, but their minima (which is at $x = \pi$) are shifted up. see Fig.(4a). From Fig.(4b), we find that the Kerr-like nonlinearity and the spontaneous-emission dissipation effects lead to that the generated atomic H-F distribution that formed to be similar to the constant atomic H-F distribution $H_{const} \simeq \frac{1}{4\pi}$. The atomic generated H-F distributions under the atom-field detuning, the Kerr-like nonlinearity, and the atomic spontaneous-emission dissipation effects are agreed with the generated atomic WH-entropy mixedness. Having the atomic H-F distribution $H(x, y)$ the form of $H_{const} \simeq \frac{1}{4\pi}$ (that no have peaks and bottoms) is due to that the expectation values β_x, β_y, and β_y are vanished by increasing the dissipation rate κ_A in the atomic spontaneous emission dissipation term that controls the atomic density matrix elements.

IV. CONCLUSION

This paper considers a qubit interacting nonlinearity with a harmonic-oscillator field mode. The qubit is coupled off-resonantly to a nonlinear Kerr-like quantum-harmonic oscillator cavity through intensity-dependent interaction. A particular solution for the considered master motion equation, which is considered only the atomic spontaneous-emission decay effect, is obtained when the qubit starts with an excited pure state while the harmonic oscillator starts with a coherent state. The time evolutions of the Husimi-distribution information, the phase space WH-entropy entanglement, and atomic mixedness are explored under the effects of the unitary nonlinear atom-oscillator interaction, atom-field detuning, Kerr-like nonlinearity as well as the atomic spontaneous-emission dissipation. If the atomic spontaneous-emission dissipation is absent, the regular oscillatory behavior of the WH-entropy is used to investigate the generated qubit-oscillator entanglement and the atomic mixedness dynamics. While for the off-resonant case, the entanglement and the atomic mixedness are generated with a high-frequency irregular oscillatory behavior. With the Kerr-like nonlinearity, the generated atomic WH-entropy entanglement/mixedness enhances and the stationary maximal and partial entanglement/mixedness intervals appear. The atomic spontaneous emission dissipation leads to enhancing the generated WH-entropy mixedness. The cavity Kerr-like nonlinearity affects as an additional dissipation resource and enhances the generated mixedness and its stability. The phase space Husimi-distribution information dynamics confirms the link between the formed Husimi-distributions and the atomic WH-entropy mixedness. The generated qubit quasi-probability coherence and stationary maximally atomic mixed state have several potential applications as quantum information resources. Where, the generated phase space stationary maximally mixed state is used to realize quantum computation and quantum-channel metrology.

COMPLIANCE WITH ETHICAL STANDARDS DISCLOSURES

The authors declare no conflicts of interest.

FUNDING

This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

ACKNOWLEDGMENT

The authors are very grateful to the referees for their important remarks which improve the manuscript. This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

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