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RESEARCH ARTICLE

The Inverse XLindley Distribution: Properties and Application

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ABSTRACT This work proposes a new extension to the XLindley model called an inverse XLindley distribution (IXLD). Its probability density function and hazard rate function shapes were deduced mathematically. Several statistical properties of the IXLD were derived mathematically. We use ten different approaches to calculate the parameters of the IXLD. The asymptotic behaviour of these estimators were studied thanks to a comprehensive simulation investigation. Through modelling real data sets, the effectiveness and applicability of the IXLD are examined. This proves that the IXLD better fits the real data set than competing models.

INDEX TERMS Lindley distribution, stress-strength, Anderson-Darling estimation, engineering data, maximum likelihood estimation, maximum product of spacing.

I. INTRODUCTION

One of the key methods in the study of statistics and probability is the simulation of real occurrences and natural phenomena using probability distributions. Because of these factors, researchers have concentrated on creating probability distributions, despite the fact that the data produced by natural events cannot be adequately described by existing probability distributions. This has several implications for how probability distributions are generalised and extended.

A new kind of probability distribution, generalised probability distributions, emerged as a result of the broad accessibility of additional components. The accuracy and sufficiency of data gathered from natural occurrences as well as the precision with which the distribution tail shape is described enhance when a specific parameter is added to a known probability function. We resorted to probability to lower the risk element in many sectors and productions to minimise cost and time since these events are crucial and are surrounded by complexity and hazard. An important topic of statistics is survival and reliability analysis, which has several applications in fields including engineering, economics, demography, medical, actuarial science, and life testing. In the statistical literature, many lifespan distributions have been established to provide data modelling in these applied disciplines more flexibility.

However, there is a method that may be used to increase the flexibility of the traditional distributions, such the inverse transformation (IT). Let's assume that X and T are two random variables. Many writers have used the PT, such as $X = T^{-1}$, to create inverted distributions. For example, the generalized inverse-gamma [18], the inverse power Maxwell [2], the inverse power-Lindley [8], two-parameter Burr-Hatke distribution [1], the inverse-Lindley with two-parameter [4], the reverse-Lindley [28], and inverse Power-Lomax [13]. Some of these models can be obtained as a special case from the family of distributions presented by Omair et al. [21].

The primary goal of this study is to provide a novel flexible distribution that is based on the IT approach, and that has the

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potential to be applied to a variety of phenomena that occur in real life. This novel model that has been suggested is known as the inverse XLindley distribution (IXLD). The model that has been suggested just has one parameter. Hence it can be classified as a unimodal model. For more information about papers related to Lindley distribution, see [5], [9], [12], [14], [15], and [22]. We are concerned with a one-parameter distribution similar of the one-parameter Lindley distribution presented by Chouia and Zeghdoudi [10] and defined below by its probability density function (PDF)

$$g(x) = \frac{\theta^2 e^{-\theta x}(\theta + x + 2)}{(\theta+1)^2}, \quad (1)$$

and its distribution function (CDF) taking the following form

$$G(x) = 1 - \left(\frac{\theta x}{(\theta+1)^2} + 1\right) e^{-\theta x}. \quad (2)$$

This paper is presented in the following six sections. Derivation of our proposed model was presented in Section II. Many statistical properties of our proposed model were derived in Section III such as fuzzy reliability, quantile function, moments, stochastic orders, entropy, and stress-strength. Ten different methods of estimation were used to determine proposed model parameters in Section IV and behaviour of these method were checked in Section V. Analyzing of the real data set to show the superiority and flexibility of our proposed model was presented in Section VI. Finally, in Section VII, we presented some of our concluding remarks.

II. FORMULATION OF THE IXLD

In this section, we proposed an inverted version of the XLindley distribution. If a random variable (r.v for shortly) Y has the XLindley model (XL), then the r.v $X = \frac{1}{Y}$ follows the IXLD with scale parameter θ, its PDF is defined as follows

$$f(x;\theta) = \begin{cases} \frac{\theta^2((2+\theta)x+1)}{(1+\theta)^2 x^3} e^{\frac{-\theta}{x}} & x, \theta > 0 \\ 0, & \text{otherwise.} \end{cases} \quad (3)$$

Proposition 1: $\forall \theta > 0$, the PDF (3) is an increasing-decreasing function.

Proof: The first derivative of the PDF (3) is determined as follows

$$\frac{d}{dx} f(x) = -\frac{\theta^2}{(1+\theta)^2} \frac{e^{-\frac{\theta}{x}}}{x^5}\left((2\theta+4)x^2 - (\theta^2+2\theta-3)x - \theta\right),$$

by equating last equation to zero and solve it with respect to x, we have

$$x = \frac{\theta}{4} - \frac{\sqrt{(\theta+1)^4+8}-3}{4(\theta+2)} < 0,$$
$$x = \frac{\theta}{4} + \frac{\sqrt{(\theta+1)^4+8}-3}{4(\theta+2)} > 0,$$

then, our critical point is

$$x' = \frac{\theta}{4} + \frac{\sqrt{(\theta+1)^4+8}-3}{4(\theta+2)}.$$

The second derivative of the PDF (3) is determined as follows

$$\frac{d^2}{dx^2} f(x) = \frac{\theta^2 e^{-\frac{\theta}{x}}}{x^7(\theta+1)^2}\left(6(\theta+2)x^3 - 6(\theta^2+2\theta-2)x^2 \right.$$
$$\left. \times + \theta(\theta^2+2\theta-8)x + \theta^2\right),$$
$$\frac{d^2}{dx^2} f(x)|_{x=x'} < 0,$$

then, $\forall \theta > 0, x' = \frac{\theta}{4} + \frac{\sqrt{(\theta+1)^4+8}-3}{4(\theta+2)}$ is the unique critical point which maximize the PDF (3).

Therefore, the mode of IXLD is defined as follws

$$M^* = \frac{\theta}{4} + \frac{\sqrt{(\theta+1)^4+8}-3}{4(\theta+2)}. \quad (4)$$

A. SURVIVAL AND HAZARD RATE FUNCTIONS

The CDF of the IXLD is

$$F(x;\theta) = \left(1 + \frac{\theta}{x(1+\theta)^2}\right) e^{-\frac{\theta}{x}}. \quad (5)$$

The following formulas represent the survival and hazard rate functions of the IXLD respectively

$$S(x) = 1 - \left(1 + \frac{\theta}{x(1+\theta)^2}\right) e^{-\frac{\theta}{x}}, \quad (6)$$
$$h(x) = \frac{\theta^2 e^{-\frac{\theta}{x}}((2+\theta)x+1)}{x^2((1-e^{-\frac{\theta}{x}})(1+\theta)^2 x - \theta e^{-\frac{1}{x}\theta})}. \quad (7)$$

Proposition 2: Let $f(x)$ and $h(x)$ be the PDF and hazard rate function of the IXLD respectively, $\lim_{x\to 0} f_{IXL}(x) = 0$. Then $h(x)$ is an increasing-decreasing function.

Proof: According to Glaser [11] and from the PDF (3), we have

$$\rho(x) = -\frac{f'(x;\theta)}{f(x;\theta)} = -\frac{\left(-2x^2\theta - 4x^2 + x\theta^2 + 2x\theta - 3x + \theta\right)}{x^2((2+\theta)x+1)}.$$

We have after computations

$$\rho'(x) = \frac{H(x)}{x^3(2x+x\theta+1)^2},$$

where $H(x) = ax^3 + bx^2 + cx + d$, $a = \left(-2\theta^2 - 8\theta - 8\right) < 0$, $b = \left(2\theta^3 + 8\theta^2 + 2\theta - 12\right)$, $c = \left(4\theta^2 + 8\theta - 3\right)$, and $d = 2\theta$.

In algebra, a cubic equation of the form $ax^3 + bx^2 + cx + d = 0$, has $a \neq 0, b, c, d$ are real numbers, and its discriminant Δ has three cases as the following

- If $\Delta > 0$, the cubic has three distinct real roots.
- If $\Delta < 0$, the cubic has one real root and two non-real complex conjugate roots.
- If $\Delta = 0$, the cubic has three multiple real root.

In our case

$$\Delta = -336\theta^{10} - 4032\theta^9 - 17\,696\theta^8 - 29\,120\theta^7 + 10\,900\theta^6$$
$$+ 76\,448\theta^5 + 15\,128\theta^4 - 97\,840\theta^3$$
$$- 16\,524\theta^2 + 63\,504\theta - 432.$$

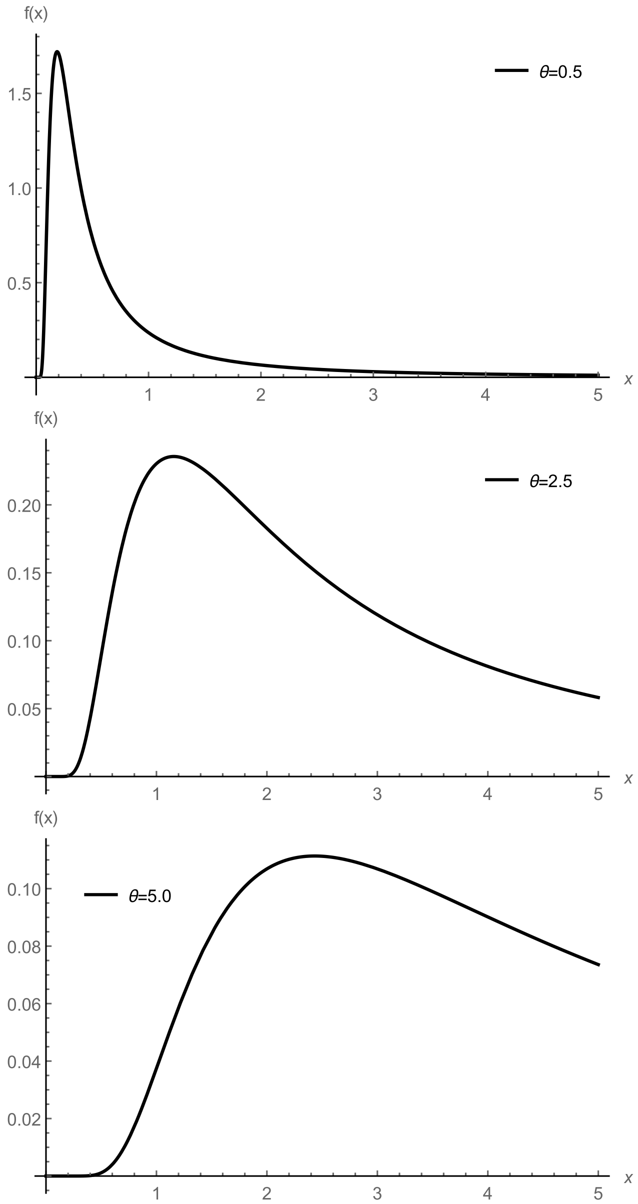

FIGURE 1. **Plots of the pdf of the IXLD.**

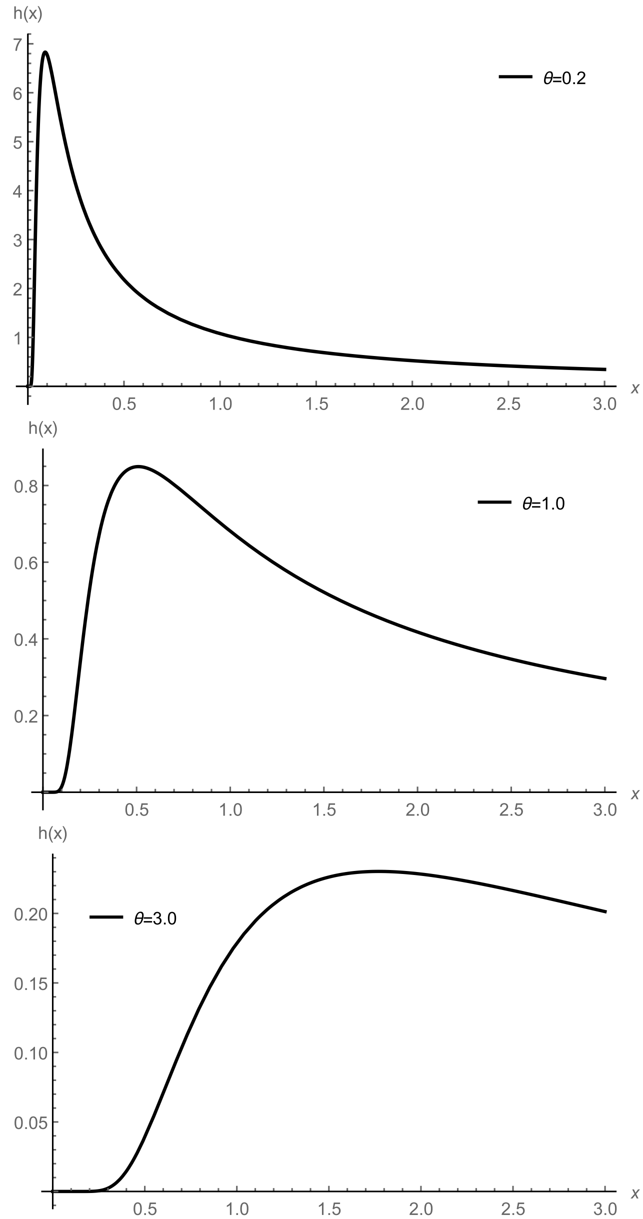

FIGURE 2. **Plots of the HF of the IXLD.**

When $\Delta > 0$, $H(x)$ has three distinct real roots $x_0 > 0$ and $x_1, x_2 < 0$.

When $\Delta < 0$, $H(x)$ has one real root and two non-real complex conjugate roots, $x_0 > 0$ and $z, \bar{z}$.

When $\Delta = 0$, $H(x)$ has three multiple real root, $x_0 = x_1 = x_2 > 0$.

Then $\rho^{'}(x) > 0,\ \forall x < x_0$, and $\rho^{'}(x) = 0$ and $\rho^{'}(x) < 0$ for all $x > x_0$, then the hazard function $h(x)$ is an increasing-decreasing function.

Different Plots of the PDF of the proposed model are presented in Figure 1 which satisfied Proposition 1. ALso, hazard function plots of proposed model are presented garphically in Figure 2 which satisfied Proposition 2.

III. STATISTICAL PROPERTIES

A. FUZZY RELIABILITY

The failure time of a system (component) will be represented by a continuous random variable T. For the calculation of fuzzy reliability using the following fuzzy probability

$$R_{FU}(t) = \int_t^{\infty} \nu(y) f(y) dy,\ 0 \leq y < \infty,$$

with $\nu(y)$ is a membership function (degree to which each element of given universe belongs to a fuzzy set). Thus, assuming that $\nu(y)$ is

$$\nu(y) = \left\{ \begin{array}{cc} 0, & y \leq t_1 \\ \frac{y-t_1}{t_2-t_1}, & t_1 < y < t_2, t_1 \geq 0 \\ 1, & y \geq t_2 \end{array} \right\}.$$

TABLE 1. Fuzzy reliability with different values of θ, t_1, t_2, I.

					R_{Fu}	R_{Fu}	R_{Fu}
θ	t_1	t_2	$S(t_1)$	$S(t_2)$	I=0.25	I=0.5	I=0.9
0.2	0.01	1	3.068829e-08	0.9324434	0.6009217	0.6026285	0.603442
0.5	0.5	2	0.5313814	0.8653342	0.1288808	0.06102	0.03547
1	0.1	3	0.0001588998	0.7762423	0.5067757	0.420571	0.38036
3	0.2	1	5.926857e-07	0.05912214	0.38660	0.4006923	0.4429233
5	0.1	1.5	4.607569e-22	0.03897714	0.6000978	0.7514442	0.8416038

Using computational analysis of fuzzy number function for $\nu(y)$ whose lifetime $y(I)$ can be obtained corresponds to some value of $I - Cut, I \in [0, 1]$ by $\nu(y) = I \rightarrow \frac{(y-t_1)}{(t_2-t_1)} = I$, then

$$\left\{ \begin{array}{cc} y(I) \leq t_1 & I = 0 \\ y(I) = t_1 + I(t_2 - t_1) & 0 < I < 1 \\ y(I) \geq t_2, & I = 1 \end{array} \right\}.$$

As a result, the fuzzy reliability values may be determined for all I values. The fuzzy dependability of the inverse X-Lindley distribution is determined by the fuzzy reliability definition. The fuzzy reliability of the inverse X-Lindley distribution can be defined as follows

$$R_{Fu}(t) = (1 + \frac{\theta}{t_1(1+\theta)^2})e^{-\frac{\theta}{t_1}} - (1 + \frac{\theta}{y(I)(1+\theta)^2})e^{-\frac{\theta}{y(I)}}.$$

Also, we obtained comparison between traditional reliability and Fuzzy reliability, where the traditional reliability is defined in Equation (6). We executed this comparison by the following steps

- initial values of a, interval time (t_1, t_2) and I where $0 < I < 1$.
- Calculate: $y(I) = t_1 + I(t_2 - t_1)$.
- Estimate IXLD parameter $\widehat{\theta}$, then calculate

$$\widehat{R}_{Fu}(t) = (1+\frac{\theta}{t_1(1+\theta)^2})e^{-\frac{\theta}{t_1}} - (1 + \frac{\theta}{y(I)(1+\theta)^2})e^{-\frac{\theta}{y(I)}}.$$

The numerical results of these comparison are presented in Table 1. The following observations are based on the comparison findings

- When the I-Cut increases, the fuzzy reliability increases and decreases according to the values of t_1 and t_2.
- The traditional reliability with t_1 is lower than the traditional reliability with t_2.

B. QUANTILE FUNCTION

Formulas for mean and variance are difficult to obtain explicitly since explicit algebraic expressions for the integrals involved are not available, quantiles are easy to evaluate. The quantile function of the IXLD is defined as follows

$$Q_X(q) = F_X^{-1}(q) = \frac{-\theta}{(\theta+1)^2 + \Omega(q)},\ q \in [0, 1],$$

where $\Omega(q) = W\left(-e^{-(\theta+1)^2}(\theta+1)^2 q\right)$ is Lambert function.

C. MOMENTS

The r^{th} moments of the IXLD is determined as follows

$$E\left(x^r\right) = \int_0^\infty x^r \frac{\theta^2((2+\theta)x+1)}{(1+\theta)^2 x^3} e^{\frac{-\theta}{x}} dx$$
$$= \frac{\theta^2}{(\theta+1)^2} \int_0^\infty (2x + x\theta + 1)\, x^{r-3} e^{-\frac{1}{x}\theta}\, dx,$$

by using the definition of the inverse gamma function $\int_0^\infty x^{-\alpha-1} e^{\frac{-\theta}{x}} = \theta^{-\alpha}\Gamma(\alpha)$, we have

$$E\left(x^r\right) = \frac{\theta^2}{(\theta+1)^2}\left(\theta^{r-2}\Gamma(2-r) + \theta^r\Gamma(1-r) + 2\theta^{r-1}\Gamma(1-r)\right),$$

for $r < 1$ which $E\left(x^r\right) = \infty$, for $r \geq 1$, which implies that all moments of the IXLD are infinite. Thus, the IXLD has no mean and no variance. On the other hand, the negative moments are useful in many domain and application (life testing problems, estimation purposes). Therefore, we discuss the r^{th} negative moments for this distribution. The r^{th} negative moment of the IXLD is determined as follows

$$E\left(x^{-r}\right) = \int_0^\infty x^{-r} \frac{\theta^2((2+\theta)x+1)}{(1+\theta)^2 x^3} e^{\frac{-\theta}{x}} dx$$
$$= \frac{\theta^2}{(\theta+1)^2} \int_0^\infty (2x + x\theta + 1)\, x^{-r-3} e^{-\frac{1}{x}\theta} dx = \frac{\theta^2}{(\theta+1)^2}$$
$$\times \left(\theta^{-2-r}\Gamma(r+2) + \theta^{-r}\Gamma(r+1) + 2\theta^{-1-r}\Gamma(r+1)\right),\ \forall r > 0.$$

D. STOCHASTIC ORDERS

Definition 1: Consider T_1 and $T2$, two random variables. Therefore, $T1$ is seen as being smaller than $T2$ in

- Stochastic order ($T_1 \prec_s T_2$, if $F_{T_1}(y) \prec F_{T_2}(y)$, $\forall y$.
- Convex order ($T_1 \leq_{ct1} T2$), if for all convex functions$\phi$and provided expectation exist, $E[\phi(T_1)] \leq E[\phi(T_2)]$.
- Hazard rate order ($T_1 \prec_{hr} T_2$), if $h_{T_1}(y) \geq h_{T_2}(y)$, $\forall y$.
- Likelihood ratio order ($T_1 \prec_{lr} T_2$), if $\frac{f_{T_1}(y)}{f_{T_2}(y)}$ is decreasing in y.

Theorem 1: Suppose that $X_i \sim IXL(\theta_i), i = 1, 2$. If $\theta_1 \leq \theta_2$, then $X_1 \prec_{lr} X_2, X_1 \prec_{hr} X_2, X_1 \prec_s X_2$ and $X_1 \leq_{cx} X_2$.

Proof: We have

$$\frac{f_{X_1}(t)}{f_{X_2}(t)} = \frac{\theta_1^2(1+\theta_2)^2((2+\theta_1)t+1)}{\theta_2^2(2+\theta_1)^2((1+\theta_2)t+1)} e^{-\frac{(\theta_1-\theta_2)}{t}}.$$

For simplify we use $\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right)$. Now, we can find

$$\frac{d}{dt}\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right) = \frac{2+\theta_1}{(\theta_1+2)t+1} - \frac{(2+\theta_2)}{(\theta_2+2)t+1} + \frac{(\theta_1-\theta_2)}{t^2}.$$

To this end, if $\theta_1 \leq \theta_2$, we have $\frac{d}{dt}\ln\left(\frac{f_{X_1}(t)}{f_{X_2}(t)}\right) \leq 0$. This means that $X_1 \prec_{lr} X_2$. proved.

E. ENTROPY

The entropy of a random variable, denoted by the letter X, is used to quantify the degree to which uncertainty varies. Rényi entropy [26] is defined by

$$J(\gamma)=\frac{1}{1-\gamma}\log\{\int f^{\gamma}(x)dx\},$$

where $\gamma > 0$ and $\gamma \neq 1$. For the IXLD, it is determined as follows

$$\begin{aligned}J(\gamma)&=\frac{1}{1-\gamma}\log\int_0^{\infty}\frac{\theta^{2\gamma}}{(1+\theta)^{2\gamma}}\frac{((2+\theta)x+1)^{\gamma}}{x^{3\gamma}}e^{\frac{-\theta\gamma}{x}}dx\\&=\frac{1}{1-\gamma}\log[\frac{\theta^{2\gamma}}{(1+\theta)^{2\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}(2+\theta)^j\int_0^{\infty}\frac{e^{\frac{-\theta\gamma}{x}}}{x^{3\gamma-j}}dx\\&=\frac{1}{1-\gamma}\log[\frac{\theta^{2\gamma}}{(1+\theta)^{2\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}(2+\theta)^j\frac{\Gamma(3\gamma-j-1)}{(\theta\gamma)^{3\gamma-j-1}}.\end{aligned}$$

F. ESTIMATION OF THE STRESS-STRENGTH PARAMETER

As it measures system performance, the stress-strength parameter (R) is crucial to the reliability analysis. Additionally, the chance of a system failure is provided by R; a system fails anytime the applied stress exceeds its capacity. Let $X \sim IXL(\theta_1)$ be the strength of a system subject to stress Y, and $Y \sim IXL(\theta_2)$, X and Y are independent of each other. In our case, the stress-strength parameter R is given by

$$\begin{aligned}R&=P(Y\prec X)\\&=\int_0^{\infty}\frac{\theta_1^2((2+\theta_1)x+1)}{(1+\theta_1)^2x^3}e^{\frac{-\theta_1}{x}}\left(1+\frac{\theta_2}{x(1+\theta_2)^2}\right)e^{-\frac{\theta_2}{x}}dx\\&=\frac{\theta_1^2}{(1+\theta_1)^2}\int_0^{\infty}(\frac{(2+\theta_1)x+1}{x^3})e^{-\frac{\theta_1+\theta_2}{x}}dx\\&\quad+\frac{\theta_1^2\theta_2(2+\theta_1)}{(1+\theta_1)^2(1+\theta_2)^2}\int_0^{\infty}x^{-3}e^{-\frac{\theta_1+\theta_2}{x}}dx+\int_0^{\infty}x^{-4}e^{-\frac{\theta_1+\theta_2}{x}}dx.\end{aligned}$$

by using $\int_0^{\infty}\frac{e^{-\frac{a}{x}}}{x^{b+1}}dx=\frac{\Gamma(b)}{a^{b2}}$, we have

$$\begin{aligned}R=&\frac{\theta_1^2}{(\theta_1+1)^2(\theta_2+1)^2(\theta_1+\theta_2)^3}\left(\theta_1^2\theta_2^2+3\theta_1^2\theta_2\right.\\&\left.+\theta_1^2+\theta_1\theta_2^3+6\theta_1\theta_2^2+9\theta_1\theta_2+3\theta_1+3\theta_2^3+10\theta_2^2+7\theta_2+2\right).\end{aligned}$$

IV. ESTIMATION OF IXLD PARAMETERS

In this section, we'll examine how to estimate IXLD parameter using standard methods. Maximizing or minimising an objective function will provide this estimator, as we'll see.

- The estimated parameter of IXLD is obtained by the maximum likelihood estimation (MLE) approach by maximizing the following definition

$$\ln L=2\ln\theta-2\ln(\theta+1)-3\ln x+\ln((2+\theta)x+1)-\frac{\theta}{x}.$$

- The estimated IXLD parameter is obtained via Anderson-Darling estimation (ADE), by minimizing the following equation

$$\begin{aligned}A(x_i)=&-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[-\frac{\theta}{x_i}+\log\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)\right.\\&\left.+\log\left(1-\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}\right)\right].\end{aligned}$$

- The estimated IXLD parameter is obtained via right-tail Anderson-Darling estimation (RADE), by minimizing the following equation

$$\begin{aligned}R(x_i)=&\frac{n}{2}-2\sum_{i=1}^{n}\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}\\&-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(1-\left(1+\frac{\theta}{x_{n+1-i}(1+\theta)^2}\right)\right.\\&\left.e^{-\frac{\theta}{x_{n+1-i}}}\right).\end{aligned}$$

- The estimated IXLD parameter is obtained via left-tailed Anderson-Darling estimation (LTADE), by minimizing the following equation

$$\begin{aligned}L(x_i)=&-\frac{3}{2}n+2\sum_{i=1}^{n}\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}\\&-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[-\frac{\theta}{x_i}+\log\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)\right].\end{aligned}$$

- The estimated IXLD parameter is obtained via Cramér-von Mises estimation (CVME), by minimizing the following equation

$$C(x_i)=-\frac{1}{12n}+\sum_{i=1}^{n}\left[\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}-\frac{2i-1}{2n}\right]^2.$$

- The estimated IXLD parameter is obtained via least-squares estimation (LSE), by minimizing the following equation

$$V(x_i)=\sum_{i=1}^{n}\left[\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}-\frac{i}{n+1}\right]^2.$$

- The estimated IXLD parameter is obtained via weighted least-squares estimation (WLSE), by minimizing the following equation

$$W(x_i)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}-\frac{i}{n+1}\right]^2.$$

- The estimated IXLD parameter is obtained via maximum product of spacing estimation (MPSE), by maximizing the following equation

$$\Psi(x_i)=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\Phi_i(x_i),$$

TABLE 2. Simulation values of BIAS, MSE and MRE of IXLD for ($\theta = 0.25$).

n	Est.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	BIAS	$0.039299^{\{2\}}$	$0.040062^{\{4\}}$	$0.042823^{\{8\}}$	$0.03805^{\{1\}}$	$0.041812^{\{6\}}$	$0.047553^{\{10\}}$	$0.041277^{\{5\}}$	$0.039606^{\{3\}}$	$0.046883^{\{9\}}$	$0.041838^{\{7\}}$
	MSE	$0.002598^{\{2\}}$	$0.00279^{\{5\}}$	$0.003255^{\{8\}}$	$0.002327^{\{1\}}$	$0.003033^{\{7\}}$	$0.004256^{\{9\}}$	$0.002947^{\{6\}}$	$0.002715^{\{3\}}$	$0.004407^{\{10\}}$	$0.002785^{\{4\}}$
	MRE	$0.157196^{\{2\}}$	$0.160249^{\{4\}}$	$0.171292^{\{8\}}$	$0.152199^{\{1\}}$	$0.167249^{\{6\}}$	$0.190213^{\{10\}}$	$0.16511^{\{5\}}$	$0.158423^{\{3\}}$	$0.187532^{\{9\}}$	$0.167351^{\{7\}}$
	$\sum Ranks$	$6^{\{2\}}$	$13^{\{4\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$19^{\{7\}}$	$29^{\{10\}}$	$16^{\{5\}}$	$9^{\{3\}}$	$28^{\{9\}}$	$18^{\{6\}}$
35	BIAS	$0.024682^{\{2\}}$	$0.025262^{\{4\}}$	$0.027614^{\{7\}}$	$0.024559^{\{1\}}$	$0.027424^{\{6\}}$	$0.029979^{\{9\}}$	$0.026341^{\{5\}}$	$0.025072^{\{3\}}$	$0.031494^{\{10\}}$	$0.028007^{\{8\}}$
	MSE	$0.001^{\{2\}}$	$0.001038^{\{4\}}$	$0.001224^{\{7\}}$	$0.000949^{\{1\}}$	$0.001221^{\{6\}}$	$0.0015^{\{9\}}$	$0.001121^{\{5\}}$	$0.001009^{\{3\}}$	$0.001641^{\{10\}}$	$0.00124^{\{8\}}$
	MRE	$0.098729^{\{2\}}$	$0.101049^{\{4\}}$	$0.110456^{\{7\}}$	$0.098235^{\{1\}}$	$0.109694^{\{6\}}$	$0.119917^{\{9\}}$	$0.105365^{\{5\}}$	$0.100287^{\{3\}}$	$0.125978^{\{10\}}$	$0.112029^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$
80	BIAS	$0.01628^{\{1\}}$	$0.016565^{\{3\}}$	$0.017778^{\{6\}}$	$0.016632^{\{4\}}$	$0.017933^{\{7\}}$	$0.019465^{\{9\}}$	$0.017234^{\{5\}}$	$0.016505^{\{2\}}$	$0.020392^{\{10\}}$	$0.01849^{\{8\}}$
	MSE	$0.000429^{\{3\}}$	$0.000436^{\{4\}}$	$0.000504^{\{6\}}$	$0.000427^{\{1\}}$	$0.000515^{\{7\}}$	$0.000614^{\{9\}}$	$0.000467^{\{5\}}$	$0.000428^{\{2\}}$	$0.000681^{\{10\}}$	$0.000535^{\{8\}}$
	MRE	$0.06512^{\{1\}}$	$0.066259^{\{3\}}$	$0.071114^{\{6\}}$	$0.066526^{\{4\}}$	$0.071732^{\{7\}}$	$0.077861^{\{9\}}$	$0.068935^{\{5\}}$	$0.06602^{\{2\}}$	$0.081568^{\{10\}}$	$0.073961^{\{8\}}$
	$\sum Ranks$	$5^{\{1\}}$	$10^{\{4\}}$	$18^{\{6\}}$	$9^{\{3\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$6^{\{2\}}$	$30^{\{10\}}$	$24^{\{8\}}$
120	BIAS	$0.013575^{\{4\}}$	$0.01354^{\{3\}}$	$0.014661^{\{7\}}$	$0.013263^{\{1\}}$	$0.014464^{\{6\}}$	$0.015661^{\{9\}}$	$0.014233^{\{5\}}$	$0.013309^{\{2\}}$	$0.017048^{\{10\}}$	$0.015552^{\{8\}}$
	MSE	$0.000288^{\{3\}}$	$0.000295^{\{4\}}$	$0.000342^{\{7\}}$	$0.000274^{\{1\}}$	$0.000338^{\{6\}}$	$0.000392^{\{9\}}$	$0.000318^{\{5\}}$	$0.000282^{\{2\}}$	$0.000463^{\{10\}}$	$0.000378^{\{8\}}$
	MRE	$0.0543^{\{4\}}$	$0.054162^{\{3\}}$	$0.058645^{\{7\}}$	$0.053054^{\{1\}}$	$0.057856^{\{6\}}$	$0.062644^{\{9\}}$	$0.056933^{\{5\}}$	$0.053237^{\{2\}}$	$0.068192^{\{10\}}$	$0.062206^{\{8\}}$
	$\sum Ranks$	$11^{\{4\}}$	$10^{\{3\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$6^{\{2\}}$	$30^{\{10\}}$	$24^{\{8\}}$
180	BIAS	$0.01073^{\{1\}}$	$0.01146^{\{5\}}$	$0.011928^{\{7\}}$	$0.010754^{\{2\}}$	$0.011809^{\{6\}}$	$0.012868^{\{9\}}$	$0.011304^{\{4\}}$	$0.010949^{\{3\}}$	$0.013535^{\{10\}}$	$0.012393^{\{8\}}$
	MSE	$0.000182^{\{2\}}$	$0.000209^{\{5\}}$	$0.000226^{\{7\}}$	$0.000181^{\{1\}}$	$0.000222^{\{6\}}$	$0.000264^{\{9\}}$	$0.000205^{\{4\}}$	$0.000191^{\{3\}}$	$0.000293^{\{10\}}$	$0.000242^{\{8\}}$
	MRE	$0.042921^{\{1\}}$	$0.04584^{\{5\}}$	$0.047711^{\{7\}}$	$0.043015^{\{2\}}$	$0.047238^{\{6\}}$	$0.051472^{\{9\}}$	$0.045216^{\{4\}}$	$0.043795^{\{3\}}$	$0.054141^{\{10\}}$	$0.049573^{\{8\}}$
	$\sum Ranks$	$4^{\{1\}}$	$15^{\{5\}}$	$21^{\{7\}}$	$5^{\{2\}}$	$18^{\{6\}}$	$27^{\{9\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$
250	BIAS	$0.009126^{\{1\}}$	$0.00979^{\{5\}}$	$0.009949^{\{6\}}$	$0.009298^{\{3\}}$	$0.010113^{\{7\}}$	$0.011016^{\{9\}}$	$0.009669^{\{4\}}$	$0.009169^{\{2\}}$	$0.011665^{\{10\}}$	$0.010438^{\{8\}}$
	MSE	$0.000131^{\{1\}}$	$0.000152^{\{5\}}$	$0.000158^{\{6\}}$	$0.000135^{\{2.5\}}$	$0.00016^{\{7\}}$	$0.000191^{\{9\}}$	$0.000146^{\{4\}}$	$0.000135^{\{2.5\}}$	$0.000215^{\{10\}}$	$0.000171^{\{8\}}$
	MRE	$0.036502^{\{1\}}$	$0.039162^{\{5\}}$	$0.039797^{\{6\}}$	$0.037192^{\{3\}}$	$0.040454^{\{7\}}$	$0.044065^{\{9\}}$	$0.038676^{\{4\}}$	$0.036676^{\{2\}}$	$0.046662^{\{10\}}$	$0.041753^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$8.5^{\{3\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$12^{\{4\}}$	$6.5^{\{2\}}$	$30^{\{10\}}$	$24^{\{8\}}$
500	BIAS	$0.006515^{\{2\}}$	$0.006725^{\{4\}}$	$0.007161^{\{7\}}$	$0.006456^{\{1\}}$	$0.007053^{\{6\}}$	$0.007678^{\{9\}}$	$0.00697^{\{5\}}$	$0.006602^{\{3\}}$	$0.008226^{\{10\}}$	$0.007544^{\{8\}}$
	MSE	$6.7e-05^{\{2\}}$	$7.2e-05^{\{4\}}$	$7.9e-05^{\{6.5\}}$	$6.5e-05^{\{1\}}$	$7.9e-05^{\{6.5\}}$	$9.4e-05^{\{9\}}$	$7.6e-05^{\{5\}}$	$6.8e-05^{\{3\}}$	$0.000106^{\{10\}}$	$9e-05^{\{8\}}$
	MRE	$0.026061^{\{2\}}$	$0.026898^{\{4\}}$	$0.028644^{\{7\}}$	$0.025824^{\{1\}}$	$0.028213^{\{6\}}$	$0.030712^{\{9\}}$	$0.027881^{\{5\}}$	$0.026407^{\{3\}}$	$0.032905^{\{10\}}$	$0.030178^{\{8\}}$
	$\sum Ranks$	$7^{\{2\}}$	$13^{\{4\}}$	$21.5^{\{8\}}$	$4^{\{1\}}$	$19.5^{\{6\}}$	$27^{\{10\}}$	$16^{\{5\}}$	$10^{\{3\}}$	$21^{\{7\}}$	$26^{\{9\}}$

where

$$\Phi_i(\xi)=\left(1+\frac{\theta}{x_i(1+\theta)^2}\right)e^{-\frac{\theta}{x_i}}-\left(1+\frac{\theta}{x_{i-1}(1+\theta)^2}\right)e^{-\frac{\theta}{x_{i-1}}}.$$

- The estimated IXLD parameter is obtained via minimum spacing absolute distance estimation (MSADE), by minimizing the following equation

$$\Delta_1(x_i)=\sum_{i=1}^{n+1}|\Phi_i-\frac{1}{n+1}|.$$

- The estimated IXLD parameter is obtained via minimum spacing absolute-log distance estimation method (MSALDE), by minimizing the following equation

$$\Delta_2(x_i)=\sum_{i=1}^{n+1}|\log\Phi_i-\log\frac{1}{n+1}|.$$

V. NUMERICAL SIMULATION

All of the estimate methods covered in Section IV will be used in this section. Now that we have these various estimation methods, we will examine how well they perform when used to estimate the IXLD's parameter. Additionally, by contrasting the numerical values of each strategy, we assess a comparison of each approach by determining average of bias (BIAS) $|Bias(\widehat{\theta})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\theta}-\theta|$, mean squared errors (MSE), $MSE = \frac{1}{M}\sum_{i=1}^{M}(\widehat{\theta}-\theta)^2$, and mean relative errors (MRE) $MRE = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\theta}-\theta|/\theta$. Selecting the best approach for model parameter estimate may be based on simulation results. The $M = 10000$ random samples from IXLD are generated using the R programming language with sample sizes of 15, 35, 80, 120, 180, 250, and 500.

Tables 2-6 provide the numerical results of the simulation, and each number's power is related to its rank in relation to other methods of estimating along the same line. Table 7 displays the partial and total ranks of our estimator. When taking random samples from the IXLD, we discover that MPSE is the most effective method for estimated IXLD parameter.

VI. REAL DATA ANALYSIS

The use of data obtained from the ''real world'' helps illustrate the adaptability of the distribution that is being discussed in this section. The first real world data set describes the

TABLE 3. Simulation values of BIAS, MSE and MRE of IXLD for ($\theta = 0.75$).

n	Est.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	BIAS	$0.131128^{\{3\}}$	$0.131178^{\{4\}}$	$0.13869^{\{8\}}$	$0.118673^{\{1\}}$	$0.136034^{\{7\}}$	$0.158192^{\{10\}}$	$0.131335^{\{5\}}$	$0.128883^{\{2\}}$	$0.149423^{\{9\}}$	$0.133231^{\{6\}}$
	MSE	$0.03363^{\{6\}}$	$0.031028^{\{3\}}$	$0.040997^{\{8\}}$	$0.025957^{\{1\}}$	$0.035288^{\{7\}}$	$0.052231^{\{10\}}$	$0.031206^{\{4\}}$	$0.030383^{\{2\}}$	$0.04323^{\{9\}}$	$0.032283^{\{5\}}$
	MRE	$0.174838^{\{3\}}$	$0.174904^{\{4\}}$	$0.18492^{\{8\}}$	$0.158231^{\{1\}}$	$0.181379^{\{7\}}$	$0.210923^{\{10\}}$	$0.175113^{\{5\}}$	$0.171844^{\{2\}}$	$0.199231^{\{9\}}$	$0.177641^{\{6\}}$
	$\sum Ranks$	$12^{\{4\}}$	$11^{\{3\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$14^{\{5\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$17^{\{6\}}$
35	BIAS	$0.082018^{\{4\}}$	$0.081115^{\{2\}}$	$0.086562^{\{7\}}$	$0.081016^{\{1\}}$	$0.086278^{\{6\}}$	$0.096531^{\{9\}}$	$0.084059^{\{5\}}$	$0.081208^{\{3\}}$	$0.098961^{\{10\}}$	$0.091099^{\{8\}}$
	MSE	$0.0152^{\{7\}}$	$0.011112^{\{1\}}$	$0.012985^{\{5\}}$	$0.014909^{\{6\}}$	$0.012709^{\{4\}}$	$0.016505^{\{8\}}$	$0.011832^{\{3\}}$	$0.011404^{\{2\}}$	$0.01738^{\{9\}}$	$0.017402^{\{10\}}$
	MRE	$0.109357^{\{4\}}$	$0.108153^{\{2\}}$	$0.115416^{\{7\}}$	$0.108021^{\{1\}}$	$0.115037^{\{6\}}$	$0.128708^{\{9\}}$	$0.112078^{\{5\}}$	$0.108278^{\{3\}}$	$0.131947^{\{10\}}$	$0.121466^{\{8\}}$
	$\sum Ranks$	$15^{\{5\}}$	$5^{\{1\}}$	$19^{\{7\}}$	$8^{\{2.5\}}$	$16^{\{6\}}$	$26^{\{8.5\}}$	$13^{\{4\}}$	$8^{\{2.5\}}$	$29^{\{10\}}$	$26^{\{8.5\}}$
80	BIAS	$0.053471^{\{4\}}$	$0.051591^{\{1\}}$	$0.055752^{\{6\}}$	$0.053077^{\{3\}}$	$0.054896^{\{5\}}$	$0.062726^{\{9\}}$	$0.060188^{\{8\}}$	$0.052123^{\{2\}}$	$0.071511^{\{10\}}$	$0.058816^{\{7\}}$
	MSE	$0.008925^{\{7\}}$	$0.004438^{\{2\}}$	$0.005152^{\{4\}}$	$0.008971^{\{8\}}$	$0.004961^{\{3\}}$	$0.006565^{\{6\}}$	$0.005918^{\{5\}}$	$0.004426^{\{1\}}$	$0.009049^{\{9\}}$	$0.010392^{\{10\}}$
	MRE	$0.071294^{\{4\}}$	$0.068788^{\{1\}}$	$0.074335^{\{6\}}$	$0.07077^{\{3\}}$	$0.073194^{\{5\}}$	$0.083634^{\{9\}}$	$0.080251^{\{8\}}$	$0.069498^{\{2\}}$	$0.095348^{\{10\}}$	$0.078421^{\{7\}}$
	$\sum Ranks$	$15^{\{5\}}$	$4^{\{1\}}$	$16^{\{6\}}$	$14^{\{4\}}$	$13^{\{3\}}$	$24^{\{8.5\}}$	$21^{\{7\}}$	$5^{\{2\}}$	$29^{\{10\}}$	$24^{\{8.5\}}$
120	BIAS	$0.039718^{\{2\}}$	$0.04144^{\{3\}}$	$0.045148^{\{6\}}$	$0.037273^{\{1\}}$	$0.044354^{\{5\}}$	$0.049786^{\{8\}}$	$0.054434^{\{9\}}$	$0.041723^{\{4\}}$	$0.066142^{\{10\}}$	$0.04653^{\{7\}}$
	MSE	$0.006476^{\{8\}}$	$0.002821^{\{1\}}$	$0.00332^{\{4\}}$	$0.005492^{\{7\}}$	$0.003195^{\{3\}}$	$0.004054^{\{5\}}$	$0.005011^{\{6\}}$	$0.002831^{\{2\}}$	$0.008372^{\{10\}}$	$0.008062^{\{9\}}$
	MRE	$0.052957^{\{2\}}$	$0.055253^{\{3\}}$	$0.060197^{\{6\}}$	$0.049697^{\{1\}}$	$0.059139^{\{5\}}$	$0.066381^{\{8\}}$	$0.072579^{\{9\}}$	$0.055631^{\{4\}}$	$0.08819^{\{10\}}$	$0.06204^{\{7\}}$
	$\sum Ranks$	$12^{\{4\}}$	$7^{\{1\}}$	$16^{\{6\}}$	$9^{\{2\}}$	$13^{\{3\}}$	$21^{\{7\}}$	$24^{\{9\}}$	$10^{\{3\}}$	$30^{\{10\}}$	$23^{\{8\}}$
180	BIAS	$0.030404^{\{1\}}$	$0.033108^{\{3\}}$	$0.036278^{\{7\}}$	$0.030737^{\{2\}}$	$0.036229^{\{5.5\}}$	$0.040245^{\{8\}}$	$0.054939^{\{9\}}$	$0.033138^{\{4\}}$	$0.062045^{\{10\}}$	$0.036229^{\{5.5\}}$
	MSE	$0.004558^{\{6\}}$	$0.001844^{\{2\}}$	$0.002116^{\{4\}}$	$0.004783^{\{7\}}$	$0.002093^{\{3\}}$	$0.002637^{\{5\}}$	$0.005107^{\{8\}}$	$0.001796^{\{1\}}$	$0.007974^{\{10\}}$	$0.006415^{\{9\}}$
	MRE	$0.040538^{\{1\}}$	$0.044143^{\{3\}}$	$0.04837^{\{7\}}$	$0.040983^{\{2\}}$	$0.048305^{\{5.5\}}$	$0.05366^{\{8\}}$	$0.073253^{\{9\}}$	$0.044184^{\{4\}}$	$0.082727^{\{10\}}$	$0.048305^{\{5.5\}}$
	$\sum Ranks$	$8^{\{1.5\}}$	$8^{\{1.5\}}$	$18^{\{6\}}$	$11^{\{4\}}$	$14^{\{3\}}$	$21^{\{8\}}$	$26^{\{9\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$20^{\{7\}}$
250	BIAS	$0.023459^{\{2\}}$	$0.027464^{\{5\}}$	$0.030601^{\{7\}}$	$0.022219^{\{1\}}$	$0.030595^{\{6\}}$	$0.033546^{\{8\}}$	$0.05742^{\{9\}}$	$0.027274^{\{4\}}$	$0.062409^{\{10\}}$	$0.025505^{\{3\}}$
	MSE	$0.003512^{\{7\}}$	$0.001266^{\{2\}}$	$0.001485^{\{3\}}$	$0.00331^{\{6\}}$	$0.001488^{\{4\}}$	$0.001815^{\{5\}}$	$0.005633^{\{9\}}$	$0.001208^{\{1\}}$	$0.008601^{\{10\}}$	$0.004053^{\{8\}}$
	MRE	$0.031279^{\{2\}}$	$0.036618^{\{5\}}$	$0.040801^{\{7\}}$	$0.029626^{\{1\}}$	$0.040793^{\{6\}}$	$0.044728^{\{8\}}$	$0.07656^{\{9\}}$	$0.036366^{\{4\}}$	$0.083212^{\{10\}}$	$0.034007^{\{3\}}$
	$\sum Ranks$	$11^{\{3\}}$	$12^{\{4\}}$	$17^{\{7\}}$	$8^{\{1\}}$	$16^{\{6\}}$	$21^{\{8\}}$	$27^{\{9\}}$	$9^{\{2\}}$	$30^{\{10\}}$	$14^{\{5\}}$
500	BIAS	$0.008102^{\{1\}}$	$0.01784^{\{4\}}$	$0.022096^{\{6\}}$	$0.008751^{\{2\}}$	$0.022214^{\{7\}}$	$0.024085^{\{8\}}$	$0.067327^{\{10\}}$	$0.018706^{\{5\}}$	$0.056483^{\{9\}}$	$0.008817^{\{3\}}$
	MSE	$0.001016^{\{6\}}$	$0.000554^{\{1\}}$	$0.000767^{\{3\}}$	$0.001133^{\{7\}}$	$0.000786^{\{4\}}$	$0.000923^{\{5\}}$	$0.006658^{\{9\}}$	$0.000578^{\{2\}}$	$0.007628^{\{10\}}$	$0.001301^{\{8\}}$
	MRE	$0.010802^{\{1\}}$	$0.023787^{\{4\}}$	$0.029461^{\{6\}}$	$0.011669^{\{2\}}$	$0.029618^{\{7\}}$	$0.032114^{\{8\}}$	$0.089769^{\{10\}}$	$0.024941^{\{5\}}$	$0.075311^{\{9\}}$	$0.011756^{\{3\}}$
	$\sum Ranks$	$8^{\{1\}}$	$9^{\{2\}}$	$15^{\{6\}}$	$11^{\{3\}}$	$18^{\{7\}}$	$21^{\{8\}}$	$29^{\{10\}}$	$12^{\{4\}}$	$28^{\{9\}}$	$14^{\{5\}}$

TABLE 4. Simulation values of BIAS, MSE and MRE of IXLD for ($\theta = 1.0$).

n	Est.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	BIAS	$0.177532^{\{3\}}$	$0.182104^{\{4\}}$	$0.195381^{\{8\}}$	$0.167736^{\{1\}}$	$0.19234^{\{7\}}$	$0.222946^{\{10\}}$	$0.191251^{\{6\}}$	$0.174685^{\{2\}}$	$0.207039^{\{9\}}$	$0.183757^{\{5\}}$
	MSE	$0.059479^{\{5\}}$	$0.058051^{\{3\}}$	$0.072198^{\{8\}}$	$0.047702^{\{1\}}$	$0.069218^{\{7\}}$	$0.104885^{\{10\}}$	$0.06877^{\{6\}}$	$0.054521^{\{2\}}$	$0.079176^{\{9\}}$	$0.058234^{\{4\}}$
	MRE	$0.177532^{\{3\}}$	$0.182104^{\{4\}}$	$0.195381^{\{8\}}$	$0.167736^{\{1\}}$	$0.19234^{\{7\}}$	$0.222946^{\{10\}}$	$0.191251^{\{6\}}$	$0.174685^{\{2\}}$	$0.207039^{\{9\}}$	$0.183757^{\{5\}}$
	$\sum Ranks$	$11^{\{3.5\}}$	$11^{\{3.5\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$27^{\{9\}}$	$14^{\{5\}}$
35	BIAS	$0.110927^{\{2\}}$	$0.115467^{\{5\}}$	$0.124646^{\{8\}}$	$0.107386^{\{1\}}$	$0.124133^{\{7\}}$	$0.136297^{\{10\}}$	$0.114304^{\{4\}}$	$0.111865^{\{3\}}$	$0.134548^{\{9\}}$	$0.122086^{\{6\}}$
	MSE	$0.023143^{\{5\}}$	$0.022043^{\{3\}}$	$0.02655^{\{7\}}$	$0.020635^{\{1\}}$	$0.026485^{\{6\}}$	$0.032156^{\{9\}}$	$0.02236^{\{4\}}$	$0.020651^{\{2\}}$	$0.032573^{\{10\}}$	$0.02717^{\{8\}}$
	MRE	$0.110927^{\{2\}}$	$0.115467^{\{5\}}$	$0.124646^{\{8\}}$	$0.107386^{\{1\}}$	$0.124133^{\{7\}}$	$0.136297^{\{10\}}$	$0.114304^{\{4\}}$	$0.111865^{\{3\}}$	$0.134548^{\{9\}}$	$0.122086^{\{6\}}$
	$\sum Ranks$	$9^{\{3\}}$	$13^{\{5\}}$	$23^{\{8\}}$	$3^{\{1\}}$	$20^{\{6.5\}}$	$29^{\{10\}}$	$12^{\{4\}}$	$8^{\{2\}}$	$28^{\{9\}}$	$20^{\{6.5\}}$
80	BIAS	$0.074552^{\{2\}}$	$0.074592^{\{3\}}$	$0.081162^{\{7\}}$	$0.074698^{\{4\}}$	$0.078519^{\{6\}}$	$0.085365^{\{9\}}$	$0.075405^{\{5\}}$	$0.073894^{\{1\}}$	$0.090041^{\{10\}}$	$0.085109^{\{8\}}$
	MSE	$0.012843^{\{7\}}$	$0.009089^{\{2\}}$	$0.01077^{\{5\}}$	$0.013801^{\{8\}}$	$0.010104^{\{4\}}$	$0.011997^{\{6\}}$	$0.009238^{\{3\}}$	$0.008741^{\{1\}}$	$0.015525^{\{9\}}$	$0.016797^{\{10\}}$
	MRE	$0.074552^{\{2\}}$	$0.074592^{\{3\}}$	$0.081162^{\{7\}}$	$0.074698^{\{4\}}$	$0.078519^{\{6\}}$	$0.085365^{\{9\}}$	$0.075405^{\{5\}}$	$0.073894^{\{1\}}$	$0.090041^{\{10\}}$	$0.085109^{\{8\}}$
	$\sum Ranks$	$11^{\{3\}}$	$8^{\{2\}}$	$19^{\{7\}}$	$16^{\{5.5\}}$	$16^{\{5.5\}}$	$24^{\{8\}}$	$13^{\{4\}}$	$3^{\{1\}}$	$29^{\{10\}}$	$26^{\{9\}}$
120	BIAS	$0.06181^{\{4\}}$	$0.060218^{\{3\}}$	$0.064196^{\{6\}}$	$0.062061^{\{5\}}$	$0.064336^{\{7\}}$	$0.070192^{\{8\}}$	$0.059553^{\{2\}}$	$0.058685^{\{1\}}$	$0.07379^{\{10\}}$	$0.070722^{\{9\}}$
	MSE	$0.010927^{\{7\}}$	$0.005894^{\{3\}}$	$0.006614^{\{4\}}$	$0.011085^{\{8\}}$	$0.006655^{\{5\}}$	$0.0081^{\{6\}}$	$0.005686^{\{2\}}$	$0.005567^{\{1\}}$	$0.012563^{\{9\}}$	$0.013699^{\{10\}}$
	MRE	$0.06181^{\{4\}}$	$0.060218^{\{3\}}$	$0.064196^{\{6\}}$	$0.062061^{\{5\}}$	$0.064336^{\{7\}}$	$0.070192^{\{8\}}$	$0.059553^{\{2\}}$	$0.058685^{\{1\}}$	$0.07379^{\{10\}}$	$0.070722^{\{9\}}$
	$\sum Ranks$	$15^{\{4\}}$	$9^{\{3\}}$	$16^{\{5\}}$	$18^{\{6\}}$	$19^{\{7\}}$	$22^{\{8\}}$	$6^{\{2\}}$	$3^{\{1\}}$	$29^{\{10\}}$	$28^{\{9\}}$
180	BIAS	$0.050405^{\{4\}}$	$0.048721^{\{2\}}$	$0.051421^{\{5\}}$	$0.052465^{\{7\}}$	$0.051927^{\{6\}}$	$0.056792^{\{8\}}$	$0.04891^{\{3\}}$	$0.047369^{\{1\}}$	$0.06483^{\{10\}}$	$0.057177^{\{9\}}$
	MSE	$0.00853^{\{7\}}$	$0.003799^{\{3\}}$	$0.004231^{\{4\}}$	$0.009686^{\{8\}}$	$0.004242^{\{5\}}$	$0.005112^{\{6\}}$	$0.003796^{\{2\}}$	$0.003613^{\{1\}}$	$0.013914^{\{10\}}$	$0.010731^{\{9\}}$
	MRE	$0.050405^{\{4\}}$	$0.048721^{\{2\}}$	$0.051421^{\{5\}}$	$0.052465^{\{7\}}$	$0.051927^{\{6\}}$	$0.056792^{\{8\}}$	$0.04891^{\{3\}}$	$0.047369^{\{1\}}$	$0.06483^{\{10\}}$	$0.057177^{\{9\}}$
	$\sum Ranks$	$15^{\{5\}}$	$7^{\{2\}}$	$14^{\{4\}}$	$22^{\{7.5\}}$	$17^{\{6\}}$	$22^{\{7.5\}}$	$8^{\{3\}}$	$3^{\{1\}}$	$30^{\{10\}}$	$27^{\{9\}}$
250	BIAS	$0.042145^{\{4\}}$	$0.040848^{\{2\}}$	$0.043785^{\{5\}}$	$0.044072^{\{6\}}$	$0.044556^{\{7\}}$	$0.048133^{\{8\}}$	$0.041793^{\{3\}}$	$0.040419^{\{1\}}$	$0.057862^{\{10\}}$	$0.049162^{\{9\}}$
	MSE	$0.007061^{\{7\}}$	$0.002706^{\{2\}}$	$0.003045^{\{4\}}$	$0.007826^{\{8\}}$	$0.00314^{\{5\}}$	$0.003668^{\{6\}}$	$0.002735^{\{3\}}$	$0.00257^{\{1\}}$	$0.015266^{\{10\}}$	$0.009857^{\{9\}}$
	MRE	$0.042145^{\{4\}}$	$0.040848^{\{2\}}$	$0.043785^{\{5\}}$	$0.044072^{\{6\}}$	$0.044556^{\{7\}}$	$0.048133^{\{8\}}$	$0.041793^{\{3\}}$	$0.040419^{\{1\}}$	$0.057862^{\{10\}}$	$0.049162^{\{9\}}$
	$\sum Ranks$	$15^{\{5\}}$	$6^{\{2\}}$	$14^{\{4\}}$	$20^{\{7\}}$	$19^{\{6\}}$	$22^{\{8\}}$	$9^{\{3\}}$	$3^{\{1\}}$	$30^{\{10\}}$	$27^{\{9\}}$
500	BIAS	$0.029359^{\{4\}}$	$0.028383^{\{2\}}$	$0.030363^{\{5\}}$	$0.029285^{\{3\}}$	$0.030896^{\{6\}}$	$0.03443^{\{9\}}$	$0.033464^{\{8\}}$	$0.028208^{\{1\}}$	$0.057602^{\{10\}}$	$0.032689^{\{7\}}$
	MSE	$0.004504^{\{7\}}$	$0.001317^{\{2\}}$	$0.001465^{\{3\}}$	$0.004629^{\{8\}}$	$0.001508^{\{4\}}$	$0.001883^{\{6\}}$	$0.001747^{\{5\}}$	$0.001254^{\{1\}}$	$0.024202^{\{10\}}$	$0.005551^{\{9\}}$
	MRE	$0.029359^{\{4\}}$	$0.028383^{\{2\}}$	$0.030363^{\{5\}}$	$0.029285^{\{3\}}$	$0.030896^{\{6\}}$	$0.03443^{\{9\}}$	$0.033464^{\{8\}}$	$0.028208^{\{1\}}$	$0.057602^{\{10\}}$	$0.032689^{\{7\}}$
	$\sum Ranks$	$15^{\{5\}}$	$6^{\{2\}}$	$13^{\{3\}}$	$14^{\{4\}}$	$16^{\{6\}}$	$24^{\{9\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$30^{\{10\}}$	$23^{\{8\}}$

TABLE 5. Simulation values of BIAS, MSE and MRE of IXLD for ($\theta = 2.5$).

n	Est.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	BIAS	$0.510314^{\{3\}}$	$0.532652^{\{5\}}$	$0.572809^{\{9\}}$	$0.47832^{\{1\}}$	$0.568873^{\{8\}}$	$0.666956^{\{10\}}$	$0.54115^{\{6\}}$	$0.507411^{\{2\}}$	$0.56729^{\{7\}}$	$0.522378^{\{4\}}$
	MSE	$0.474942^{\{4\}}$	$0.527089^{\{5\}}$	$0.641358^{\{8\}}$	$0.370274^{\{1\}}$	$0.628301^{\{7\}}$	$0.964916^{\{10\}}$	$0.545387^{\{6\}}$	$0.458388^{\{2\}}$	$0.646266^{\{9\}}$	$0.461219^{\{3\}}$
	MRE	$0.204125^{\{3\}}$	$0.213061^{\{5\}}$	$0.229124^{\{9\}}$	$0.191328^{\{1\}}$	$0.227549^{\{8\}}$	$0.266782^{\{10\}}$	$0.21646^{\{6\}}$	$0.202964^{\{2\}}$	$0.226916^{\{7\}}$	$0.208951^{\{4\}}$
	$\sum Ranks$	$10^{\{3\}}$	$15^{\{5\}}$	$26^{\{9\}}$	$3^{\{1\}}$	$23^{\{7.5\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$23^{\{7.5\}}$	$11^{\{4\}}$
35	BIAS	$0.310225^{\{2\}}$	$0.336527^{\{4\}}$	$0.35256^{\{7\}}$	$0.305786^{\{1\}}$	$0.353501^{\{8\}}$	$0.392075^{\{10\}}$	$0.346403^{\{5\}}$	$0.316516^{\{3\}}$	$0.385454^{\{9\}}$	$0.349171^{\{6\}}$
	MSE	$0.162031^{\{2\}}$	$0.189306^{\{4\}}$	$0.213022^{\{7\}}$	$0.146212^{\{1\}}$	$0.216481^{\{8\}}$	$0.270749^{\{10\}}$	$0.205619^{\{6\}}$	$0.167467^{\{3\}}$	$0.255861^{\{9\}}$	$0.194947^{\{5\}}$
	MRE	$0.12409^{\{2\}}$	$0.134611^{\{4\}}$	$0.141024^{\{7\}}$	$0.122314^{\{1\}}$	$0.141401^{\{8\}}$	$0.15683^{\{10\}}$	$0.138561^{\{5\}}$	$0.126606^{\{3\}}$	$0.154182^{\{9\}}$	$0.139668^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$24^{\{8\}}$	$30^{\{10\}}$	$16^{\{5\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$17^{\{6\}}$
80	BIAS	$0.208083^{\{2\}}$	$0.218222^{\{4\}}$	$0.229612^{\{7\}}$	$0.199422^{\{1\}}$	$0.2256^{\{6\}}$	$0.256566^{\{9\}}$	$0.218298^{\{5\}}$	$0.208806^{\{3\}}$	$0.258095^{\{10\}}$	$0.235311^{\{8\}}$
	MSE	$0.070904^{\{3\}}$	$0.076897^{\{4\}}$	$0.085731^{\{7\}}$	$0.062938^{\{1\}}$	$0.084107^{\{6\}}$	$0.111063^{\{10\}}$	$0.078212^{\{5\}}$	$0.070724^{\{2\}}$	$0.109313^{\{9\}}$	$0.085962^{\{8\}}$
	MRE	$0.083233^{\{2\}}$	$0.087289^{\{4\}}$	$0.091845^{\{7\}}$	$0.079769^{\{1\}}$	$0.09024^{\{6\}}$	$0.102626^{\{9\}}$	$0.087319^{\{5\}}$	$0.083523^{\{3\}}$	$0.103238^{\{10\}}$	$0.094124^{\{8\}}$
	$\sum Ranks$	$7^{\{2\}}$	$12^{\{4\}}$	$21^{\{7\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$28^{\{9\}}$	$15^{\{5\}}$	$8^{\{3\}}$	$29^{\{10\}}$	$24^{\{8\}}$
120	BIAS	$0.165564^{\{2\}}$	$0.179962^{\{5\}}$	$0.191356^{\{7\}}$	$0.164768^{\{1\}}$	$0.188021^{\{6\}}$	$0.206747^{\{9\}}$	$0.178297^{\{4\}}$	$0.168783^{\{3\}}$	$0.208252^{\{10\}}$	$0.192033^{\{8\}}$
	MSE	$0.044233^{\{2\}}$	$0.05139^{\{5\}}$	$0.059069^{\{8\}}$	$0.042927^{\{1\}}$	$0.056551^{\{6\}}$	$0.07033^{\{10\}}$	$0.051082^{\{4\}}$	$0.045143^{\{3\}}$	$0.069937^{\{9\}}$	$0.057437^{\{7\}}$
	MRE	$0.066226^{\{2\}}$	$0.071985^{\{5\}}$	$0.076542^{\{7\}}$	$0.065907^{\{1\}}$	$0.075208^{\{6\}}$	$0.082699^{\{9\}}$	$0.071319^{\{4\}}$	$0.067513^{\{3\}}$	$0.083301^{\{10\}}$	$0.076813^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$15^{\{5\}}$	$22^{\{7\}}$	$3^{\{1\}}$	$18^{\{6\}}$	$28^{\{9\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$29^{\{10\}}$	$23^{\{8\}}$
180	BIAS	$0.134608^{\{1\}}$	$0.14442^{\{4\}}$	$0.153182^{\{6\}}$	$0.135821^{\{2\}}$	$0.153406^{\{7\}}$	$0.166779^{\{9\}}$	$0.146094^{\{5\}}$	$0.139999^{\{3\}}$	$0.170301^{\{10\}}$	$0.156523^{\{8\}}$
	MSE	$0.02868^{\{1\}}$	$0.033322^{\{4\}}$	$0.037211^{\{6\}}$	$0.028881^{\{2\}}$	$0.038118^{\{7\}}$	$0.045103^{\{9\}}$	$0.033849^{\{5\}}$	$0.031686^{\{3\}}$	$0.046179^{\{10\}}$	$0.038438^{\{8\}}$
	MRE	$0.053843^{\{1\}}$	$0.057768^{\{4\}}$	$0.061273^{\{6\}}$	$0.054328^{\{2\}}$	$0.061362^{\{7\}}$	$0.066712^{\{9\}}$	$0.058438^{\{5\}}$	$0.055999^{\{3\}}$	$0.06812^{\{10\}}$	$0.062609^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$
250	BIAS	$0.114682^{\{2\}}$	$0.122735^{\{4\}}$	$0.129587^{\{6\}}$	$0.110574^{\{1\}}$	$0.132528^{\{7\}}$	$0.141999^{\{9\}}$	$0.124578^{\{5\}}$	$0.11625^{\{3\}}$	$0.147439^{\{10\}}$	$0.13394^{\{8\}}$
	MSE	$0.020761^{\{2\}}$	$0.023659^{\{4\}}$	$0.026451^{\{6\}}$	$0.019421^{\{1\}}$	$0.027773^{\{7\}}$	$0.032593^{\{9\}}$	$0.024667^{\{5\}}$	$0.021345^{\{3\}}$	$0.034133^{\{10\}}$	$0.028098^{\{8\}}$
	MRE	$0.045873^{\{2\}}$	$0.049094^{\{4\}}$	$0.051835^{\{6\}}$	$0.04423^{\{1\}}$	$0.053011^{\{7\}}$	$0.056799^{\{9\}}$	$0.049831^{\{5\}}$	$0.0465^{\{3\}}$	$0.058976^{\{10\}}$	$0.053576^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$18^{\{6\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$
500	BIAS	$0.07906^{\{2\}}$	$0.087704^{\{5\}}$	$0.089675^{\{6\}}$	$0.078979^{\{1\}}$	$0.090961^{\{7\}}$	$0.099542^{\{9\}}$	$0.08615^{\{4\}}$	$0.082347^{\{3\}}$	$0.100966^{\{10\}}$	$0.095352^{\{8\}}$
	MSE	$0.009995^{\{1\}}$	$0.012162^{\{5\}}$	$0.0128^{\{6\}}$	$0.010137^{\{2\}}$	$0.013077^{\{7\}}$	$0.015728^{\{9\}}$	$0.011674^{\{4\}}$	$0.010582^{\{3\}}$	$0.016268^{\{10\}}$	$0.014279^{\{8\}}$
	MRE	$0.031624^{\{2\}}$	$0.035082^{\{5\}}$	$0.03587^{\{6\}}$	$0.031592^{\{1\}}$	$0.036384^{\{7\}}$	$0.039817^{\{9\}}$	$0.03446^{\{4\}}$	$0.032939^{\{3\}}$	$0.040386^{\{10\}}$	$0.038141^{\{8\}}$
	$\sum Ranks$	$5^{\{2\}}$	$15^{\{5\}}$	$18^{\{6\}}$	$4^{\{1\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$

TABLE 6. Simulation values of BIAS, MSE and MRE of IXLD for ($\theta = 4$).

n	Est.	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
15	BIAS	$0.846319^{\{3\}}$	$0.867557^{\{4\}}$	$0.967914^{\{8\}}$	$0.795308^{\{1\}}$	$0.948681^{\{7\}}$	$1.140412^{\{10\}}$	$0.919615^{\{6\}}$	$0.83177^{\{2\}}$	$0.968646^{\{9\}}$	$0.872303^{\{5\}}$
	MSE	$1.345118^{\{4\}}$	$1.357288^{\{5\}}$	$1.824064^{\{9\}}$	$1.046334^{\{1\}}$	$1.699417^{\{7\}}$	$2.904661^{\{10\}}$	$1.580291^{\{6\}}$	$1.230171^{\{2\}}$	$1.819101^{\{8\}}$	$1.287198^{\{3\}}$
	MRE	$0.21158^{\{3\}}$	$0.216889^{\{4\}}$	$0.241979^{\{8\}}$	$0.198827^{\{1\}}$	$0.23717^{\{7\}}$	$0.285103^{\{10\}}$	$0.229904^{\{6\}}$	$0.207943^{\{2\}}$	$0.242161^{\{9\}}$	$0.218076^{\{5\}}$
	$\sum Ranks$	$10^{\{3\}}$	$13^{\{4.5\}}$	$25^{\{8\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$18^{\{6\}}$	$6^{\{2\}}$	$26^{\{9\}}$	$13^{\{4.5\}}$
35	BIAS	$0.527838^{\{2\}}$	$0.544991^{\{4\}}$	$0.611545^{\{8\}}$	$0.508347^{\{1\}}$	$0.59428^{\{7\}}$	$0.665411^{\{10\}}$	$0.569593^{\{5\}}$	$0.534908^{\{3\}}$	$0.634747^{\{9\}}$	$0.587853^{\{6\}}$
	MSE	$0.472314^{\{2\}}$	$0.499771^{\{4\}}$	$0.648194^{\{8\}}$	$0.409448^{\{1\}}$	$0.595716^{\{7\}}$	$0.771601^{\{10\}}$	$0.563354^{\{6\}}$	$0.478991^{\{3\}}$	$0.686715^{\{9\}}$	$0.558028^{\{5\}}$
	MRE	$0.131959^{\{2\}}$	$0.136248^{\{4\}}$	$0.152886^{\{8\}}$	$0.127087^{\{1\}}$	$0.14857^{\{7\}}$	$0.166353^{\{10\}}$	$0.142398^{\{5\}}$	$0.133727^{\{3\}}$	$0.158687^{\{9\}}$	$0.146963^{\{6\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$24^{\{8\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$30^{\{10\}}$	$16^{\{5\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$17^{\{6\}}$
80	BIAS	$0.336702^{\{1\}}$	$0.364301^{\{4\}}$	$0.39081^{\{8\}}$	$0.339879^{\{2\}}$	$0.387841^{\{6\}}$	$0.431374^{\{10\}}$	$0.367714^{\{5\}}$	$0.344281^{\{3\}}$	$0.42444^{\{9\}}$	$0.389661^{\{7\}}$
	MSE	$0.184216^{\{2\}}$	$0.215182^{\{4\}}$	$0.248803^{\{8\}}$	$0.181573^{\{1\}}$	$0.246621^{\{7\}}$	$0.310827^{\{10\}}$	$0.215756^{\{5\}}$	$0.190739^{\{3\}}$	$0.292803^{\{9\}}$	$0.235014^{\{6\}}$
	MRE	$0.084175^{\{1\}}$	$0.091075^{\{4\}}$	$0.097702^{\{8\}}$	$0.08497^{\{2\}}$	$0.09696^{\{6\}}$	$0.107843^{\{10\}}$	$0.091929^{\{5\}}$	$0.08607^{\{3\}}$	$0.10611^{\{9\}}$	$0.097415^{\{7\}}$
	$\sum Ranks$	$4^{\{1\}}$	$12^{\{4\}}$	$24^{\{8\}}$	$5^{\{2\}}$	$19^{\{6\}}$	$30^{\{10\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$20^{\{7\}}$
120	BIAS	$0.271679^{\{1\}}$	$0.292434^{\{4\}}$	$0.319886^{\{7\}}$	$0.278625^{\{2\}}$	$0.318298^{\{6\}}$	$0.350675^{\{9\}}$	$0.30163^{\{5\}}$	$0.285786^{\{3\}}$	$0.35745^{\{10\}}$	$0.320724^{\{8\}}$
	MSE	$0.11862^{\{1\}}$	$0.136399^{\{4\}}$	$0.164118^{\{8\}}$	$0.12485^{\{2\}}$	$0.163922^{\{7\}}$	$0.198703^{\{9\}}$	$0.144515^{\{5\}}$	$0.130742^{\{3\}}$	$0.2042^{\{10\}}$	$0.16111^{\{6\}}$
	MRE	$0.06792^{\{1\}}$	$0.073109^{\{4\}}$	$0.079972^{\{7\}}$	$0.069656^{\{2\}}$	$0.079575^{\{6\}}$	$0.087669^{\{9\}}$	$0.075407^{\{5\}}$	$0.071446^{\{3\}}$	$0.089363^{\{10\}}$	$0.080181^{\{8\}}$
	$\sum Ranks$	$3^{\{1\}}$	$12^{\{4\}}$	$22^{\{7.5\}}$	$6^{\{2\}}$	$19^{\{6\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$22^{\{7.5\}}$
180	BIAS	$0.222081^{\{2\}}$	$0.24723^{\{5\}}$	$0.258318^{\{6\}}$	$0.220523^{\{1\}}$	$0.259158^{\{7\}}$	$0.293942^{\{10\}}$	$0.246371^{\{4\}}$	$0.234493^{\{3\}}$	$0.288311^{\{9\}}$	$0.261901^{\{8\}}$
	MSE	$0.079601^{\{2\}}$	$0.096683^{\{5\}}$	$0.106855^{\{7\}}$	$0.078856^{\{1\}}$	$0.106253^{\{6\}}$	$0.136874^{\{10\}}$	$0.09656^{\{4\}}$	$0.088157^{\{3\}}$	$0.13325^{\{9\}}$	$0.109069^{\{8\}}$
	MRE	$0.05552^{\{2\}}$	$0.061807^{\{5\}}$	$0.064579^{\{6\}}$	$0.055131^{\{1\}}$	$0.064789^{\{7\}}$	$0.073486^{\{10\}}$	$0.061593^{\{4\}}$	$0.058623^{\{3\}}$	$0.072078^{\{9\}}$	$0.065475^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$15^{\{5\}}$	$19^{\{6\}}$	$3^{\{1\}}$	$20^{\{7\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$24^{\{8\}}$
250	BIAS	$0.18938^{\{2\}}$	$0.203017^{\{4\}}$	$0.215812^{\{6\}}$	$0.184372^{\{1\}}$	$0.21939^{\{7\}}$	$0.239758^{\{9\}}$	$0.207722^{\{5\}}$	$0.195755^{\{3\}}$	$0.243253^{\{10\}}$	$0.222716^{\{8\}}$
	MSE	$0.056853^{\{2\}}$	$0.065547^{\{4\}}$	$0.073078^{\{6\}}$	$0.054681^{\{1\}}$	$0.075911^{\{7\}}$	$0.093271^{\{9\}}$	$0.068403^{\{5\}}$	$0.061126^{\{3\}}$	$0.094006^{\{10\}}$	$0.077766^{\{8\}}$
	MRE	$0.047345^{\{2\}}$	$0.050754^{\{4\}}$	$0.053953^{\{6\}}$	$0.046093^{\{1\}}$	$0.054848^{\{7\}}$	$0.05994^{\{9\}}$	$0.051931^{\{5\}}$	$0.048939^{\{3\}}$	$0.060813^{\{10\}}$	$0.055679^{\{8\}}$
	$\sum Ranks$	$6^{\{2\}}$	$12^{\{4\}}$	$18^{\{6\}}$	$3^{\{1\}}$	$21^{\{7\}}$	$27^{\{9\}}$	$15^{\{5\}}$	$9^{\{3\}}$	$30^{\{10\}}$	$24^{\{8\}}$
500	BIAS	$0.132678^{\{2\}}$	$0.146507^{\{5\}}$	$0.154622^{\{6\}}$	$0.128449^{\{1\}}$	$0.154893^{\{7\}}$	$0.170336^{\{10\}}$	$0.143698^{\{4\}}$	$0.137441^{\{3\}}$	$0.166852^{\{9\}}$	$0.16051^{\{8\}}$
	MSE	$0.027774^{\{1\}}$	$0.034342^{\{5\}}$	$0.037583^{\{7\}}$	$0.027821^{\{2\}}$	$0.037555^{\{6\}}$	$0.046354^{\{10\}}$	$0.03272^{\{4\}}$	$0.029844^{\{3\}}$	$0.044299^{\{9\}}$	$0.039993^{\{8\}}$
	MRE	$0.033169^{\{2\}}$	$0.036627^{\{5\}}$	$0.038655^{\{6\}}$	$0.032112^{\{1\}}$	$0.038723^{\{7\}}$	$0.042584^{\{10\}}$	$0.035924^{\{4\}}$	$0.03436^{\{3\}}$	$0.041713^{\{9\}}$	$0.040128^{\{8\}}$
	$\sum Ranks$	$5^{\{2\}}$	$15^{\{5\}}$	$19^{\{6\}}$	$4^{\{1\}}$	$20^{\{7\}}$	$30^{\{10\}}$	$12^{\{4\}}$	$9^{\{3\}}$	$27^{\{9\}}$	$24^{\{8\}}$

lengths of time it takes for 20 different components to fail. It was given by Murthy et al. [20], and its values are as follows: 70.175, 8.851, 2.968, 9.763, 57.637, 57.337, 9.773, 48.442, 6.662, 37.386, 79.333, 85.283, 8.608, 6.56, 54.145, 4.229, 7.11, 10.578, 30.112, 19.136. The second real world

TABLE 7. Partial and overall ranks of all the methods of estimation of IXLD by various values of model parameters.

Parameter	n	MLE	ADE	CVME	MPSE	LSE	RTADE	WLSE	LTADE	MSADE	MSALDE
$\theta = 0.25$	15	2.0	4.0	8.0	1.0	7.0	10.0	5.0	3.0	9.0	6.0
	35	2.0	4.0	7.0	1.0	6.0	9.0	5.0	3.0	10.0	8.0
	80	1.0	4.0	6.0	3.0	7.0	9.0	5.0	2.0	10.0	8.0
	120	4.0	3.0	7.0	1.0	6.0	9.0	5.0	2.0	10.0	8.0
	180	1.0	5.0	7.0	2.0	6.0	9.0	4.0	3.0	10.0	8.0
	250	1.0	5.0	6.0	3.0	7.0	9.0	4.0	2.0	10.0	8.0
	500	2.0	4.0	8.0	1.0	6.0	10.0	5.0	3.0	7.0	9.0
$\theta = 0.75$	15	4.0	3.0	8.0	1.0	7.0	10.0	5.0	2.0	9.0	6.0
	35	5.0	1.0	7.0	2.5	6.0	8.5	4.0	2.5	10.0	8.5
	80	5.0	1.0	6.0	4.0	3.0	8.5	7.0	2.0	10.0	8.5
	120	4.0	1.0	6.0	2.0	5.0	7.0	9.0	3.0	10.0	8.0
	180	1.5	1.5	6.0	4.0	5.0	8.0	9.0	3.0	10.0	7.0
	250	3.0	4.0	7.0	1.0	6.0	8.0	9.0	2.0	10.0	5.0
	500	1.0	2.0	6.0	3.0	7.0	8.0	10.0	4.0	9.0	5.0
$\theta = 1.0$	15	3.5	3.5	8.0	1.0	7.0	10.0	6.0	2.0	9.0	5.0
	35	3.0	5.0	8.0	1.0	6.5	10.0	4.0	2.0	9.0	6.5
	80	3.0	2.0	7.0	5.5	5.5	8.0	4.0	1.0	10.0	9.0
	120	4.0	3.0	5.0	6.0	7.0	8.0	2.0	1.0	10.0	9.0
	180	5.0	2.0	4.0	7.5	6.0	7.5	3.0	1.0	10.0	9.0
	250	5.0	2.0	4.0	7.0	6.0	8.0	3.0	1.0	10.0	9.0
	500	5.0	2.0	3.0	4.0	6.0	9.0	7.0	1.0	10.0	8.0
$\theta = 2.5$	15	3.0	5.0	9.0	1.0	7.5	10.0	6.0	2.0	7.5	4.0
	35	2.0	4.0	7.0	1.0	8.0	10.0	5.0	3.0	9.0	6.0
	80	2.0	4.0	7.0	1.0	6.0	9.0	5.0	3.0	10.0	8.0
	120	2.0	5.0	7.0	1.0	6.0	9.0	4.0	3.0	10.0	8.0
	180	1.0	4.0	6.0	2.0	7.0	9.0	5.0	3.0	10.0	8.0
	250	2.0	4.0	6.0	1.0	7.0	9.0	5.0	3.0	10.0	8.0
	500	2.0	5.0	6.0	1.0	7.0	9.0	4.0	3.0	10.0	8.0
$\theta = 4.0$	15	3.0	4.5	8.0	1.0	7.0	10.0	6.0	2.0	9.0	4.5
	35	2.0	4.0	8.0	1.0	7.0	10.0	5.0	3.0	9.0	6.0
	80	1.0	4.0	8.0	2.0	6.0	10.0	5.0	3.0	9.0	7.0
	120	1.0	4.0	7.5	2.0	6.0	9.0	5.0	3.0	10.0	7.5
	180	2.0	5.0	6.0	1.0	7.0	10.0	4.0	3.0	9.0	8.0
	250	2.0	4.0	6.0	1.0	7.0	9.0	5.0	3.0	10.0	8.0
	500	2.0	5.0	6.0	1.0	7.0	10.0	4.0	3.0	9.0	8.0
$\sum$ Ranks		92.0	123.	5 231.5	78.5	223.5	316.5	183.0	85.5	333.5	257.5
Overall Rank		3	4	7	1	6	9	5	2	10	8

TABLE 8. Numerical values for analyzing the failure real data set.

Model	AC	CAC	BC	HC	F_1	F_2	F_3	$F_3(p)$	Est. parameters (SEs)
IXLD	179.174	179.396	180.169	179.368	0.644124	0.106358	0.156424	0.711985	$\hat{\theta} = 11.375$ (2.51345)
XL	182.005	182.227	183.001	182.2	2.28279	0.372274	0.323007	0.0308015	$\hat{\theta} = 0.061498$ (0.00973999)
PXL	181.365	182.071	183.357	181.754	0.925262	0.164139	0.231357	0.234683	$\hat{\alpha} = 0.806945$ (0.11294) $\hat{\theta} = 0.120863$ (0.0498503)
LD	182.624	182.846	183.619	182.818	2.61818	0.411485	0.334687	0.0226526	$\hat{\alpha} = 0.0632012$ (0.0100018)
WL	182.809	184.309	185.796	183.392	0.917479	0.164326	0.234539	0.221231	$\hat{\alpha} = 1.06888$ (0.216503) $\hat{\lambda} = 0.00395764$ (0.0340549) $\hat{\beta} = 0.0313653$ (0.00825867)
TPL	180.901	181.606	182.892	181.289	0.828547	0.144275	0.218699	0.294275	$\hat{\alpha} = 0.0122613$ (0.0398799) $\hat{\theta} = 0.040183$ (0.0182127)
QL	180.901	181.606	182.892	181.289	0.828547	0.144275	0.218699	0.294275	$\hat{\alpha} = 3.27724$ (9.32825) $\hat{\theta} = 0.040183$ (0.0182127)
GL	188.923	189.629	190.915	189.312	1.20231	0.181282	0.210009	0.340942	$\hat{\alpha} = 0.0284764$ (0.00972155) $\hat{a} = 0.428945$ (0.118199)
EXL	201.032	201.254	202.028	201.227	6.9511	1.3774	0.475084	0.000239961	$\hat{\lambda} = 0.691053$ (0.268356) $\hat{\alpha} = 0.955171$ (0.264794) $\hat{\beta} = 0.471578$ (0.0356972)
W	180.815	181.52	182.806	181.203	0.920548	0.164954	0.234796	0.220169	$\hat{a} = 1.0754$ (0.191493) $\hat{b} = 31.6069$ (6.95357)
ILL	223.703	223.925	224.699	223.898	12.8012	2.72979	0.63583	< 0.00001	$\hat{\alpha} = 0.512287$ (0.0902661)
IPLE	183.019	184.519	186.006	183.602	0.685175	0.112581	0.159365	0.689997	$\hat{\alpha} = 0.862769$ (0.511305) $\hat{\beta} = 1.19757$ (0.615642) $\hat{\lambda} = 17.948$ (26.7311)
IW	181.073	181.779	183.064	181.462	0.707628	0.119133	0.161147	0.676607	$\hat{\alpha} = 1.05766$ (0.184225) $\hat{\lambda} = 11.0416$ (2.4702)
IWL	180.987	181.693	182.979	181.376	0.748945	0.127082	0.168565	0.6207	$\hat{\alpha} = 1.12874$ (0.31877) $\hat{\lambda} = 13.621$ (4.51066)
INM	182.229	182.935	184.221	182.618	0.751121	0.118901	0.176226	0.563607	$\hat{\alpha} = 0.399727$ (0.102249) $\hat{\lambda} = 0.0148783$ (0.00526206)

TABLE 9. Numerical values for analyzing the repair time real data set.

Model	AC	CAC	BC	HC	F_1	F_2	F_3	$F_3(p)$	Est. parameters (SEs)
IXLD	182.773	182.878	184.462	183.383	0.478557	0.0571118	0.0904688	0.898846	$\hat{\theta} = 1.74964$ (0.23767)
XL	197.026	197.131	198.715	197.637	1.7312	0.284439	0.183717	0.134348	$\hat{\theta} = 0.381191$ (0.044247)
PXL	196.242	196.566	199.619	197.463	1.07368	0.139945	0.138667	0.425246	$\hat{\alpha} = 0.854674$ (0.086109) $\hat{\theta} = 0.476243$ (0.0819999)
LD	199.583	199.688	201.272	200.193	2.43938	0.412189	0.215695	0.0483734	$\hat{\alpha} = 0.42421$ (0.0485164)
WL	197.023	197.689	202.089	198.855	1.02116	0.135904	0.129041	0.518136	$\hat{\alpha} = 0.960359$ (0.108864) $\hat{\lambda} = 1.1811 \times 10^{-9}$ (0.0541138) $\hat{\beta} = 0.254644$ (0.0445577)
TPL	195.153	195.477	198.531	196.374	1.09572	0.157425	0.138093	0.430522	$\hat{\alpha} = 0.000$ (0.0502191) $\hat{\theta} = 0.249221$ (0.0638337)
QL	195.153	195.477	198.531	196.374	1.09572	0.157425	0.138093	0.430522	$\hat{\alpha} = 2.86815 \times 10^{6}$ (9.1869×10^{11}) $\hat{\theta} = 0.249221$ (0.0482434)
GL	215.012	215.336	218.39	216.233	3.36143	0.557059	0.197084	0.0894283	$\hat{\alpha} = 0.21701$ (0.0462074) $\hat{a} = 0.494563$ (0.0924458)
EXL	200.123	200.228	201.812	200.734	1.8998	0.223112	0.194746	0.0962283	$\hat{\lambda} = 0.627259$ (0.181589) $\hat{\alpha} = 0.918778$ (0.194457) $\hat{\beta} = 0.841764$ (0.0230003)
W	195.023	195.347	198.4	196.244	1.02116	0.135904	0.129041	0.518136	$\hat{a} = 0.960359$ (0.108864) $\hat{b} = 3.92706$ (0.687158)
ILL	205.638	205.743	207.326	206.248	9.20838	1.76469	0.309551	0.0009372	$\hat{\alpha} = 1.3462$ (0.177977)
IPLE	183.875	184.541	188.941	185.707	0.485365	0.0889935	0.102409	0.79558	$\hat{\alpha} = 0.616222$ (0.258219) $\hat{\beta} = 1.82068$ (0.657385) $\hat{\lambda} = 1.93911$ (0.462155)
IW	182.898	183.222	186.276	184.119	0.530118	0.0704862	0.0952589	0.860995	$\hat{\alpha} = 1.20778$ (0.151779) $\hat{\lambda} = 1.45177$ (0.200576)
IWL	182.921	183.245	186.299	184.142	0.529969	0.0701772	0.0944512	0.867757	$\hat{\alpha} = 1.26956$ (0.268988) $\hat{\lambda} = 2.48746$ (0.518827)
INM	183.247	183.572	186.625	184.469	0.515138	0.0892427	0.103308	0.786761	$\hat{\alpha} = 0.479887$ (0.088247) $\hat{\lambda} = 0.693045$ (0.158184)

data set describes repair times for an airborne communication transceiver. It was given by Lemonte et al. [16], and its values are as follows: 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

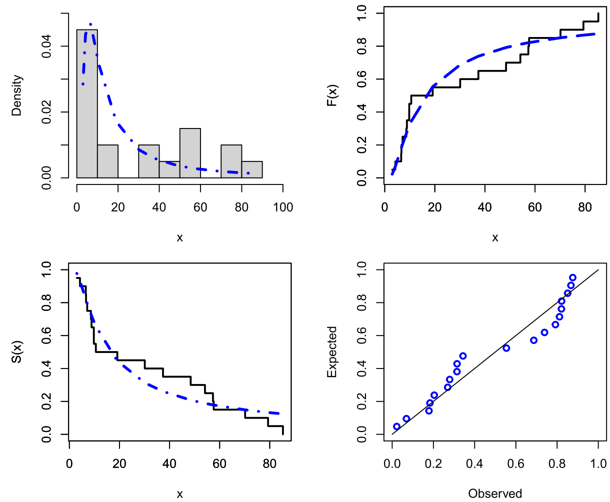

FIGURE 3. Histogram of the failure real data set with the fitted PDF, CDF, SF and P-P plots.

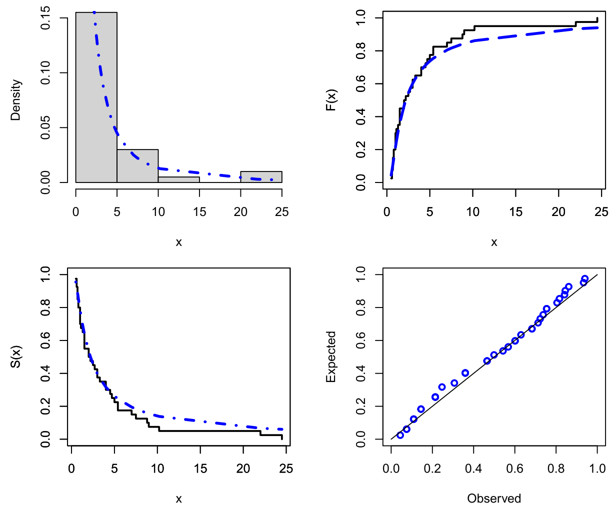

FIGURE 4. Histogram of the repair time real data set with the fitted PDF, CDF, SF and P-P plots.

In order to demonstrate how adaptable IXLD is, we are going to test it in contrast to a variety of well recognised models, including as XL, power XLindley (PXL) [19], Lindley (L), Weibull Lindley (WL) [6], two parameter Lindley (TPL) [27], Quasi Lindley (QL) [24], gamma Lindely (GL), extend Lindley (EXL) [7], Weibull (W), inverse log-logistic (ILL) [23], inverse-power logistic-exponential (IPLE) [3], inverse Weibull (IW), inverse weighted Lindley (IWL) [25], inverse Nakagami-M (INM) [17] distributions.

We make use of a variety of analytical criteria in order to identify which of the models available to us is the most relevant one to apply with the failure data set. These analytical criteria are Akaike information criterion (AC), the correct Akaike information criterion (CAC), Bayesian information criterion (BC), Hannan information criterion (HC). In addition to this, we base our decision on a range of other variables on the overall goodness-of-fit of the model,

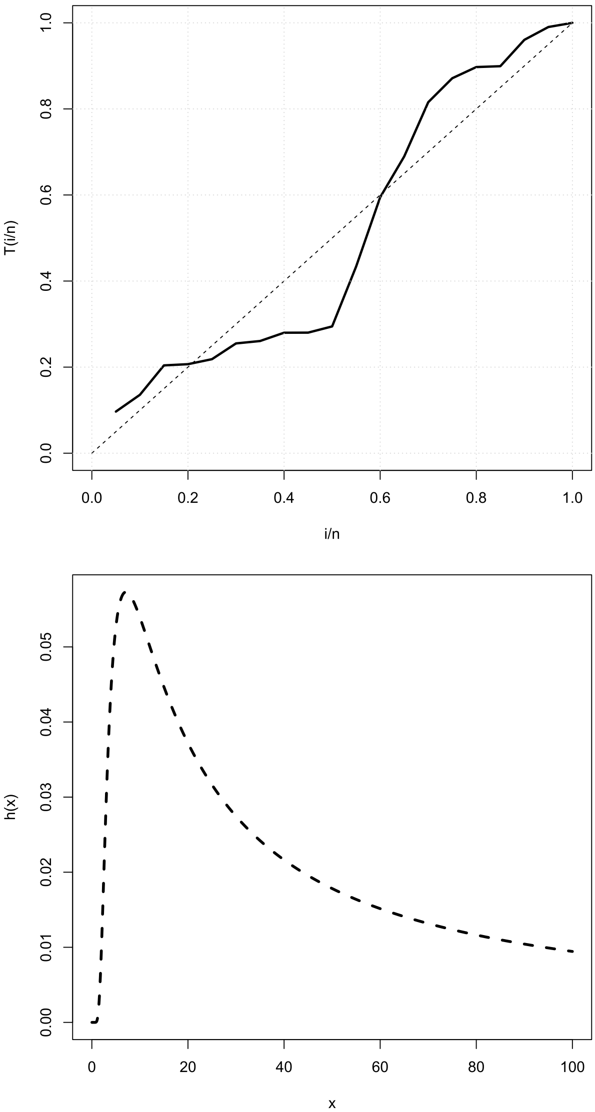

FIGURE 5. TTT plot and fitted HRF of the IXL model for the failure real data set.

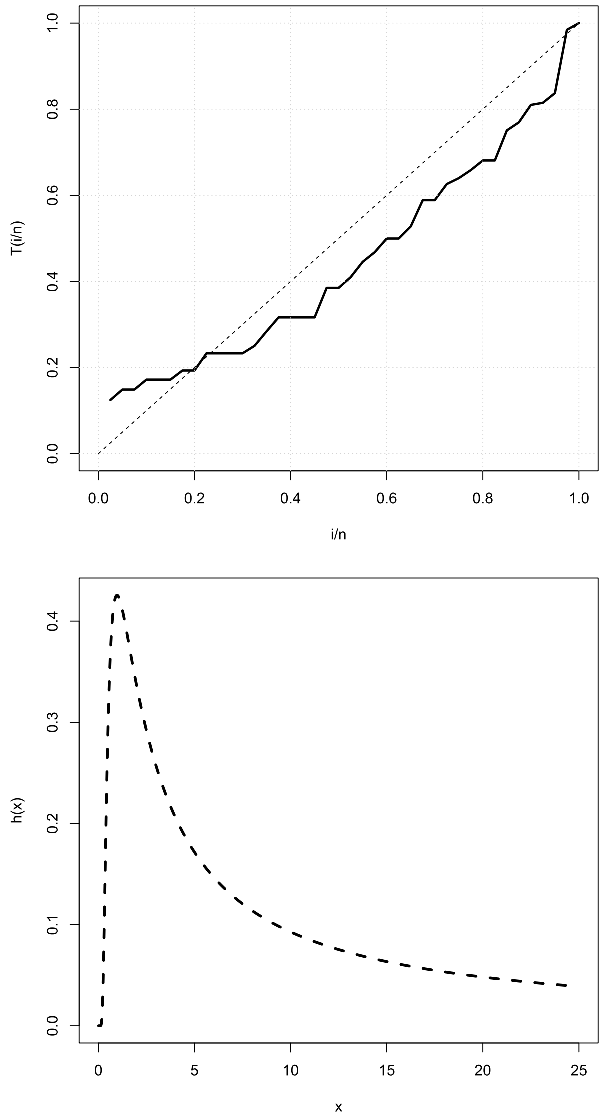

FIGURE 6. TTT plot and fitted HRF of the IXL model for the repair time real data set.

such as Anderson Darling (F_1), Cramér–von Mises (F_2) and Kolmogorov–Smirnov (F_3) with its p-value ($F_3(p)$). The model that has the smallest values for all of these metrics, with the exception of $F_3(p)$, is the best one that will match the failure real data set.

For the two real data sets that were being taken into consideration for assessment, analytical measurements are provided together with the estimations made by MLE and the related standard errors (SE). Tables 8 and 9 display a report of these numerical values. This leads us to conclude that the IXLD outperforms other compared models. The P-P plot and the fitted PDF, CDF, and SF plots are used to fit IXLD to the two real data sets which shown in Figures 3 and 4. The IXLD was shown to be a good match using the two actual data sets. For the two real data sets, the TTT and estimated HRF of the IXLD plots are shown in Figures 5 and 6, respectively. Figures 7 and 8 illustrate the behaviour of the unimodal log-likelihood function with estimated parameter for the two real data set, respectively.

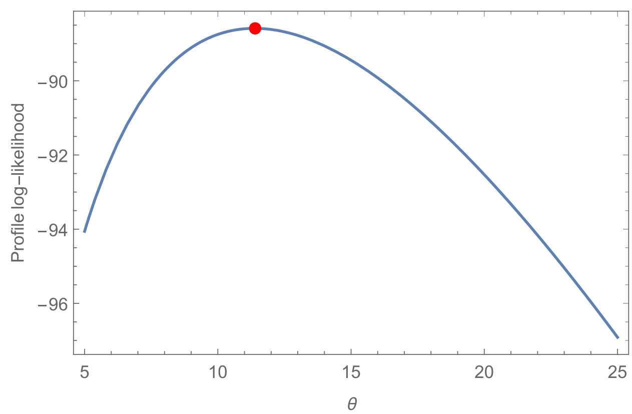

FIGURE 7. The profile of the log-likelihood function for θ parameter of the failure real data set.

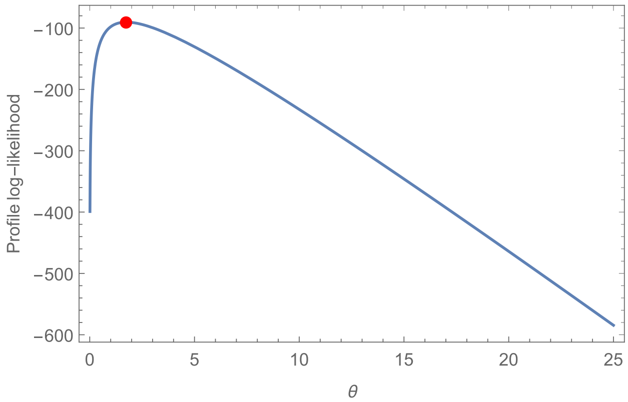

FIGURE 8. The profile of the log-likelihood function for θ parameter of the repair time real data set.

VII. CONCLUSION

In this study, we presented the IXLD, a relatively new extension of the Lindley distribution. The inverse transformation along with the Lindley distribution is used to derive it. A variety of the recommended model's statistical characteristics were demonstrated. Traditional estimation techniques, such as maximum likelihood estimation and nine other methods, were used to estimate the indicated model parameters. The behaviour of the IXLD parameters was examined using estimation approaches and data sets created at random. The IXLD's usefulness and superiority over competing models were shown using a real-world data sets. We came to the conclusion that it was the greatest option out of all its rivals since it had the lowest values of the determined measures and the highest P-value. Additionally, we graphed the profile-likelihood function of the IXLD with its parameter for the real data sets to confirm that the roots of the proposed distribution's MLE give a maximum value. This figure show that the profile-likelihood function of the estimated parameter is unimodal.

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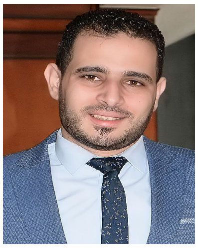

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