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RESEARCH ARTICLE

Asymptotical Stability and Exponential Stability in Mean Square of Impulsive Stochastic Time-Varying Neural Network

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ABSTRACT The effect of impulse on stability of neural network is evident not only in performance, that is, impulsive control and impulsive interference. The amount of impulse has a certain impact on stability of neural network. Unlike traditional average impulsive interval, a new strategy is applied in this paper, namely, impulsive density. Based on this strategy, by constructing Lyapunov function, we establish sufficient conditions for mean square asymptotical stability of impulsive stochastic time-varying neural network without time delay. As well as, under this strategy and uniformly asymptotically stable function, by combining trajectory based approach and improved Razumikhin method, mean square exponential stability criterion of impulsive stochastic time-varying neural network with time delay is established. Finally, to demonstrate the viability of our theoretical findings, some instances are provided.

INDEX TERMS Impulsive stochastic neural network, impulsive density, asymptotical stability, exponential stability.

I. INTRODUCTION

Neural network (NN) is an algorithmic mathematical model that simulates animal brain network function and implements distribute parallel information processing. This kind of network achieves the purpose of processing information by adjusting interconnecting relationship among numerous internal nodes. In recent decades, NN has been widely used in the fields of economy [1], information [2], and pattern recognition [3], etc. In order to improve their design and application, it is essential to consider the dynamic behaviors of NN, such as, stability [4], synchronization [5], etc.

As we all know, random noise is one of important factors which maybe influence the stability of NN. Thus, numerous control approaches for stochastic neural networks have been established, for example, event-trigger control [6], output feedback control [7], impulsive control [8], [9], [10], etc.

For the purpose of lowering communication costs, impulsive control has received a lot of attention in control theory. Guo developed a novel simple sufficient condition for the globally robustly asymptotic stability of stochastic time-varying delayed NN based on impulsive control in [8]. Using pinning impulsive control approach, Fu et al. discussed projective synchronization of memristive neural network and exponential synchronization of fuzzy memristive neural network in [9] and [10]. The uncertainty and complexity of stochastic perturbation may have a significant impact on performance of neural network. Furthermore, research on dynamic behavior of impulsive stochastic neural network (ISNN) has a long way to go.

Time delay is unavoidable in addition to stochastic impact. Time delays maybe occur due to processing time and signal propagation in actuators and sensors, especially when some devices differ significantly. The presence of time delay in neural network might alter related dynamic behavior and lead to instability. Therefore, it is of practical significance to

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consider ISNN with time delay. Currently, there are numerous theoretical findings concerning ISNN with delay [10], [11], [12], [13], [14]. In [10], Yu et al. analyzed stability of delayed ISNN driven by Lé vy noise using average impulsive interval (AII) and Lyapunov-Krasovskii function. In [11], employing Razumikhin approach and stochastic analysis method, Li et al. investigated robust exponential stability of uncertain ISNN with delayed impulse. In [13], Hu and Zhu used the Razumikhin technology and Lyapunov technique to probe mean square exponential stability of stochastic neural network with delay impulse, and so on.

In addition, it is noted that the number of impulses is one of critical factors that govern behavior of state of NN. However, the existing literatures analyze stability of ISNN from impact of impulses, namely, impulsive control and impulsive interference. Few scholars have paid attention to impact of impulsive amount on stability. In fact, if quantity of activated impulses is too little, the ability to stabilize system is compromised. If impulses is redundant, communication cost will rise and be squandered. For characterizing the amount of activated impulses, one of the most used ways is AII [15], [16], [17], [18]. AII approach was initially presented in 2010 for stability of impulse systems [16], which has the obvious advantage of being able to analyze impulsive signals with a larger variety of impulsive intervals. But in the existing documents, it can be noted that both amount of triggered impulses and impulsive strength are considered to remain constant at each same time interval. However, because of complexity of the real condition, quantity and strength of activated impulses are not always constant at a given period, and they will vary as impulsive time changes. Therefore, the shortcoming of AII is naturally exposed: quantity of impulses specified by AII is inadequate or exaggerated to stabilize the neural network. Moreover, the quantity of impulses triggered between two impulsive instants does not always have a linear connection with impulsive time interval. So then, conventional AII technique for analyzing stability of impulsive stochastic time-varying neural network (ISTVNN) has some limitations due to time-varying impulsive intensity and the amount of impulses. Thus, how to overcome this restraint has always been a mystery. This method was not improved until 2022 by [19], where the authors established the concept of impulsive density based on definition of probability density. Impulsive density can more accurately and deeply reflect quantity of activated impulses. Therefore, it is of great significance to explore the stability of ISTVNN under impulse density. Besides, trajectory based approach in [20] is more beneficial tool for handling the stability issues of time-varying systems. In addition, as far as we know, there is little research work on the analysis of delayed ISTVNN based on impulsive density by combining improved Razumikhin method, uniformly asymptotically stable function (UASF) and trajectory based approach, which urges us to improve current related works.

TABLE 1. Symbols and their meanings.

Symbol	Meaning
$\mathbb{N}^+$	$\{1, 2, \cdots, n\}$
$\mathbb{N}$	$0 \cup \mathbb{N}^+$
$\mathbb{R}^+$	$[0, +\infty)$
$\mathbb{R}$	$(-\infty, +\infty)$
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t_0 \le t}, P)$	complete probability space
$\{\mathcal{F}_t\}_{t_0 \le t}$	a filtration and satisfying standard requirement
$V(t, x(t))$	$[t_0 - \tau, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$
A^T	transpose of A
$\omega(t)$	Brownian motion
$\|\| \cdot \|\|$	Euclidean norm
$\mathbb{E}$	mathematical expectation operator
$\kappa : [-\tau, 0] \to \mathbb{R}$	piecewise continuous function on $[-\tau, 0]$
$\mathcal{PC}([-\tau, 0]; \mathbb{R})$	set formed by κ with $\|\|\kappa\|\| = \sup_{\theta \in [-\tau,0] \| \kappa(\theta) \|}$
$\hbar(\theta)$,$\theta \in [-\tau, 0]$	$\mathcal{F}_t$-measurable and $\|\|\hbar(\theta)\|\|_{L^2} = \sup_{\theta \in [-\tau,0]} \mathbb{E}\|\hbar(\theta)\|^2 < \infty$.
$\mathcal{PL}^2_{\mathcal{F}_t}$	the set of $\mathcal{PC}([-\tau, 0]; \mathbb{R})$-valued random variables $\hbar$
Θ	$\Theta = \{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_s\}$, $\mathcal{S}_i \in \mathbb{R}^+$, which is impulsive strength

Inspired by the above discussions, stability of ISTVNN is taken into account. The primary contributions of this paper are that: (1) Based on impulsive density, mean square asymptotical stability (MSAS) of ISTVNN without time delay is analyzed by constructing Lyapunov function. In addition, under impulsive density and notion of UASF, mean square exponential stability (MSES) of ISTVNN with time delay is analyzed by using improved Razumikhin method and trajectory based method. (2) Impulsive density is introduced into the neural network to depict the number of impulsive occurrence. Under impulsive density, amount of impulses is restricted an integrable function. It establishes relationship between time interval and number of impulsive occurrence, which is linear or nonlinear. To some extent, this lowers conservatism. The quantity of impulses is only limited by linear relation in previous literature. (3) The approach discussed in this paper has a broader variety of applications. Specifically, some constraints of [13] and [14] are eased in this paper. The impulsive intensity varies over time, and it can range from 0 to $+\infty$. In [10], [13], and [14], impulsive intensity is constant over time and is required to be from 0 to 1 or from 1 to ∞. AII in [15], [16], [17], and [18] can be used as a special case of impulsive density. (4) The theoretical results are applied to the Chua's circuit, which supports validity of the derived conclusions.

The following sections make up the framework of this paper. In Section II, model description and some prerequisites are given. In Section III, asymptotic stability is investigated for ISTVNN without time delay, and MSES of ISTVNN with time delay is studied. In Section IV, to show the feasibility and usefulness of proposed conclusions, some simulated examples are presented.

In this paper, meanings of symbols and abbreviations are listed in Tables 1 and 2, respectively.

TABLE 2. Abbreviations and their meanings.

Abbreviation	Meaning
NN	neural network
ISNN	impulsive stochastic neural network
ISTVNN	impulsive stochastic time-varying neural network
AII	average impulsive interval
UASF	uniformly asymptotically stable function
MSAS	mean square asymptotical stability
MSES	mean square exponential stability

II. MODEL DESCRIPTION AND PRELIMINARIES

In this part, we consider the following ISTVNN,

$$\begin{cases} dx_h(t) = [-a_h x_h(t) + \sum_{r=1}^{m} b_{hr} f_r(x_r(t+\theta))]dt \\ \qquad + \sum_{r=1}^{m} e_{hr} g_r(x_r(t+\theta)) d\omega_r(t), & t \neq t_k, \\ x_h(t_k) = \mathcal{I}(t_k^{-}, x_h(t_k^{-})), & k \in \mathbb{N}, \\ x_h(\theta) = \eta_h(\theta), & -\tau \le \theta \le 0, \end{cases} \tag{1}$$

where $1 \le h \le m$, m represents number of neurons in ISTVNN (1), $x_h(t)$ serve as state of the hth neuron, for impulsive instant t_k, $x_h(t_k^+) = x_h(t_k)$, and $x_h(t_k^-) = \lim_{s \to t_k^-} x_h(s)$. f_r and g_r are the activation functions of rth neuron, respectively. $\theta \in [-\tau, 0]$ is time delay. $a_h > 0$, $b_{hr} \in \mathbb{R}$ and $e_{hr} \in \mathbb{R}$, which mean intensity of the rth neuron on the hth neuron at current time, respectively. $\omega_r(t)$, which defined in complete probability space, is a Brownian movement. $\mathcal{I} : [t_0, +\infty) \times \mathbb{R} \to \mathbb{R}$. Any initial function $\eta_h(\theta) \in \mathcal{PL}^2_{\mathcal{F}_{t_0}}$.

Definition 1: $N(t,s)$ represents quantity of activated impulses on $(s,t]$. $\phi(u)$ is said to be impulsive density, if there are an integrable function $\phi(u) > 0$ and a constant $N_0 > 0$ such that

$$N(t,s) \le N_0 + \int_s^t \phi(u)du, \quad t \ge s \ge t_0$$

holds for any instant $u \in (s,t]$.

Remark 1: Nothing that when $\phi(u) \equiv \frac{1}{\mathcal{T}_a}$, it can be yielded that $N(t,s) \le N_0 + \frac{t-s}{\mathcal{T}_a}$, where $\mathcal{T}_a$ is called as AII. It means that average impulsive interval is regarded as a special case of impulsive density.

Assumption 1: Functions $f_h(\cdot)$, $g_h(\cdot)$ satisfy $f_h(0) = 0$, $g_h(0) = 0$.

Assumption 2: For $\forall h \in \{1, 2, \cdots, m\}$, there exist two Lipschitz constants $\varsigma_h > 0$, $\varrho_h > 0$ such that

$$|f_h(\varpi_1) - f_h(\varpi_2)| \le \varsigma_h |\varpi_1 - \varpi_2|,$$
$$|g_h(\varpi_1) - g_h(\varpi_2)| \le \varrho_h |\varpi_1 - \varpi_2|.$$

Definition 2: For impulsive strength $\mathcal{S}_k \in \Theta$, $1 \le k \le s$, $\phi_k(u)$ is said to be kth impulsive density function, if for any instant $u \in [s,t]$, there are an integrable function $\phi_k(u) > 0$ and a constant $N_{0k} > 0$ such that

$$\begin{cases} N_k(t,s) \le N_{0k} + \int_s^t \phi_k(u)du, & \mathcal{S}_k \ge 1, \ t \ge s \ge t_0, \\ N_k(t,s) \ge -N_{0k} + \int_s^t \phi_k(u)du, & \mathcal{S}_k < 1, \ t \ge s \ge t_0, \end{cases} \tag{2}$$

where $N_k(t,s)$ is quantity of activated kth impulsive strength on interval $(s,t]$.

Let

$$\Xi = \max_{1 \le h \le m} (-a_h + \varsigma_h \sum_{r=1}^{m} |b_{rh}|),$$

$$\Gamma = \max_{1 \le h \le m} (\varrho_h^2 \sum_{r=1}^{m} |e_{rh}|^2).$$

Now, we define a differential generator $\mathcal{L}$ for ISTVNN (1),

$$\begin{aligned} \mathcal{L}V(t, x_h(t)) &= V_t(t, x_h(t)) \\ &\quad + V_{x_h}(t, x_h(t))\Big[-a_h x_h(t) + \sum_{r=1}^{m} b_{hr} f_r(x_r(t))\Big] \\ &\quad + \frac{1}{2}\text{trace}\Big[(\sum_{r=1}^{m} e_{hr} g_r(x_r(t)))^T V_{x_h x_r}(t, x_h(t)) \\ &\quad \times \sum_{r=1}^{m} e_{hr} g_r(x_r(t))\Big], \end{aligned}$$

besides,

$$\begin{aligned} \mathcal{L}V(t, x_h(t+\theta)) &= V_t(t, x_h(t)) \\ &\quad + V_{x_h}(t, x_h(t))\Big[-a_h x_h(t) + \sum_{r=1}^{m} b_{hr} f_r(x_r(t+\theta))\Big] \\ &\quad + \frac{1}{2}\text{trace}\Big[(\sum_{r=1}^{m} e_{hr} g_r(x_r(t+\theta)))^T V_{x_h x_r}(t, x_h(t)) \\ &\quad \times \sum_{r=1}^{m} e_{hr} g_r(x_r(t+\theta))\Big]. \end{aligned}$$

Definition 3: The state of ISTVNN (1) is called mean square asymptotically stable, if for $\forall x(t_0) \in \mathcal{PL}^2_{\mathcal{F}_{t_0}}$ and $\forall \varepsilon > 0$, there exists a constant $\bar{\delta} = \bar{\delta}(\varepsilon) > 0$, when $\mathbb{E}||x(t_0)||^2 \le \bar{\delta}$,

$$\mathbb{E}||x(t)||^2 \le \varepsilon, \quad t_0 \le t$$

holds. Namely,

$$\lim_{t \to \infty} \mathbb{E}||x(t)||^2 = 0$$

holds for $\forall x(t_0) \in \mathcal{PL}^2_{\mathcal{F}_{t_0}}$

Definition 4: The state of ISTVNN (1) *is called mean square exponentially stable, if*

$$\lim_{t \to \infty} \sup \frac{1}{t} \ln \mathbb{E}||x(t)||^2 < 0, \quad \forall x(t_0) \in \mathcal{PL}^2_{\mathcal{F}_{t_0}}$$

holds.

III. MAIN RESULTS

This section consists of two subsections. We will analyze asymptotical stability and exponential stability of ISTVNN, respectively.

A. ASYMPTOTICAL STABILITY OF ISTVNN

In this subsection, we assume that time delay $\tau \equiv 0$, MSAS of ISTVNN (1) will be considered.

Theorem 1: Let Assumption 1-Assumption 2 *hold, if there exist constants* $\mathcal{S}(t_k) \in \Theta$, $\mathcal{S}_i \in \mathbb{R}^+$ *and integrable function* $\phi_i(t)$ *such that*

$$\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 \leq \mathcal{S}(t_k)\frac{t_k^-}{2}\sum_{h=1}^{m}|x_h(t_k^-)|^2, \tag{3}$$

$$\int_{t_0}^{\infty}\phi_i(u)\ln\mathcal{S}_i + \frac{1}{u} + 2\Xi + n\Gamma du = -\infty, \tag{4}$$

then ISTVNN (1) *is asymptotically stable in mean square.*

Proof: We take $V(t,x(t)) = \frac{t}{2}||x(t)||^2$. According to definition of generator $\mathcal{L}$, we have

$$\begin{aligned}
&\mathcal{L}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\
&\quad = \frac{1}{2}\sum_{h=1}^{m}|x_h(t)|^2 + t\sum_{h=1}^{m}sgn(x_h(t))x_h(t) \\
&\qquad \times\left[-a_hx_h(t) + \sum_{r=1}^{m}b_{hr}f_r(x_r(t))\right] \\
&\qquad + \frac{1}{2}\sum_{h=1}^{m}(tsgn(x_h(t)))(\sum_{r=1}^{m}e_{hr}g_r(x_r(t)))^2.
\end{aligned} \tag{5}$$

According to Cauchy inequality and Assumption 1-Assumption 2, we can derive that

$$\begin{aligned}
&\mathcal{L}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\
&\leq \frac{1}{2}\sum_{h=1}^{m}|x_h(t)|^2 + t\sum_{h=1}^{m}(-a_h|x_h(t)|^2 + \sum_{r=1}^{m}\varsigma_r|b_{hr}| \\
&\quad \times|x_r(t)|^2) + \frac{mt}{2}\sum_{h=1}^{m}\sum_{r=1}^{m}|e_{hr}|^2\varrho_r^2|x_r(t)|^2 \\
&\leq \frac{1}{2}\sum_{h=1}^{m}|x_h(t)|^2 + t\max_{1\leq h\leq m}(-a_h + \varsigma_h\sum_{r=1}^{m}|b_{rh}|)\sum_{h=1}^{m}|x_h(t)|^2 \\
&\quad + \frac{mt}{2}\max_{1\leq h\leq m}(\varrho_h^2\sum_{r=1}^{m}|e_{rh}|^2)\sum_{h=1}^{m}|x_h(t)|^2 \\
&= \frac{1}{2}\sum_{h=1}^{m}|x_h(t)|^2 + t\Xi\sum_{h=1}^{m}|x_h(t)|^2 + \frac{mt}{2}\Gamma\sum_{h=1}^{m}|x_h(t)|^2 \\
&= (\frac{1}{t} + 2\Xi + m\Gamma)(\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2).
\end{aligned} \tag{6}$$

We can confirm that for $t \in [t_k, t_{k+1})$ by Ito formula,

$$\begin{aligned}
&\mathbb{E}e^{-\int_{t_k}^{t}\frac{1}{u}+2\Xi+m\Gamma du}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 - \mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 \\
&\quad = \mathbb{E}\int_{t_k}^{t}e^{-(\int_{t_k}^{u}\frac{1}{s}+2\Xi+m\Gamma ds)} \\
&\qquad \times\left[-(\frac{1}{u} + 2\Xi + m\Gamma)\frac{u}{2}\sum_{h=1}^{m}|x_h(u)|^2\right. \\
&\qquad \left.+ \mathcal{L}\frac{u}{2}\sum_{h=1}^{m}|x_h(u)|^2\right]du.
\end{aligned} \tag{7}$$

Combining (6) with (7), we can check that

$$\mathbb{E}e^{-\int_{t_k}^{t}\frac{1}{u}+2\Xi+m\Gamma du}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 - \mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 \leq 0. \tag{8}$$

From (8), we can get that

$$\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \leq \mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2e^{\int_{t_k}^{t}\frac{1}{u}+2\Xi+m\Gamma du}. \tag{9}$$

According to (3), for $t \in [t_k, t_{k+1})$ one can judge that

$$\begin{aligned}
&\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\
&\quad \leq \mathbb{E}\mathcal{S}(t_k)\frac{t_k^-}{2}\sum_{h=1}^{m}|x_h(t_k^-)|^2e^{\int_{t_k}^{t}\frac{1}{u}+2\Xi+m\Gamma du}.
\end{aligned} \tag{10}$$

The following inequality follows via mathematical induction,

$$\begin{aligned}
&\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\
&\quad \leq \mathbb{E}\prod_{l=1}^{k}\mathcal{S}(t_l)\frac{t_0}{2}\sum_{h=1}^{m}|x_h(t_0)|^2e^{\int_{t_0}^{t}\frac{1}{u}+2\Xi+m\Gamma du}.
\end{aligned} \tag{11}$$

We notice that

$$\prod_{l=1}^{N(t,t_0)}\mathcal{S}(t_l) = \prod_{i=1}^{s}\mathcal{S}_i^{N_i(t,t_0)}. \tag{12}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\
&\quad \leq \mathbb{E}\prod_{i=1}^{s}\mathcal{S}_i^{N_i(t,t_0)}\frac{t_0}{2}\sum_{h=1}^{m}|x_h(t_0)|^2e^{\int_{t_0}^{t}\frac{1}{u}+2\Xi+m\Gamma du} \\
&\quad \leq \mathbb{E}\frac{t_0}{2}||x(t_0)||^2e^{\sum_{i=1}^{s}N_i(t,t_0)\ln\mathcal{S}_i}e^{\int_{t_0}^{t}\frac{1}{u}+2\Xi+m\Gamma du}.
\end{aligned} \tag{13}$$

For $t \geq t_0$, we claim that

$$N_i(t,t_0)\ln\mathcal{S}_i \leq N_{0i}|\ln\mathcal{S}_i| + \int_{t_0}^{t}\phi_i(u)\ln\mathcal{S}_idu \tag{14}$$

holds. Firstly, it can be deduced that from Definition 2, if $\mathcal{S}_i > 1$,

$$N_i(t,t_0)\ln\mathcal{S}_i \le N_{0i}\ln\mathcal{S}_i + \int_{t_0}^{t}\phi_i(u)\ln\mathcal{S}_i du.$$

Secondly, if $\mathcal{S}_i < 1$, we get that

$$N_i(t,t_0)\ln\mathcal{S}_i \le -N_{0i}\ln\mathcal{S}_i + \int_{t_0}^{t}\phi_i(u)\ln\mathcal{S}_i du.$$

Thus, (14) holds.

Replacing (14) with (13), we have

$$\begin{aligned}&\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2\\&\quad\le \mathbb{E}\frac{t_0}{2}||x(t_0)||^2\\&\quad\times e^{\sum_{i=1}^{s}N_{0i}|\ln\mathcal{S}_i|}e^{\int_{t_0}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+2\Xi+m\Gamma du}.\end{aligned}\tag{15}$$

Obviously, for $t \ge t_0$,

$$\begin{aligned}&\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2\\&\quad\le \mathbb{E}\frac{t_0}{t}||x(t_0)||^2\\&\qquad\times e^{\sum_{i=1}^{s}N_{0i}|\ln\mathcal{S}_i|}e^{\int_{t_0}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+2\Xi+m\Gamma du}\\&\quad\le \mathbb{E}||x(t_0)||^2e^{\sum_{i=1}^{s}N_{0i}|\ln\mathcal{S}_i|}\\&\qquad\times e^{\int_{t_0}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+2\Xi+m\Gamma du}.\end{aligned}\tag{16}$$

It can be got from (4) that

$$e^{\sum_{i=1}^{s}N_{0i}|\ln\mathcal{S}_i|}e^{\int_{t_0}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+2\Xi+m\Gamma du}\le \mathbf{C},$$

where $\mathbf{C}$ is a positive constant. It can be judged easily that there is arbitrary small constants $\bar{\delta} \le \frac{\varepsilon}{\mathbf{C}}$, when $\mathbb{E}||x(t_0)||^2 \le \bar{\delta}$, and $x(t_0)\in\mathcal{PL}^2_{\mathcal{F}_{t_0}}$, we have

$$\mathbb{E}||x(t)||^2 \le \bar{\delta}\mathbf{C} \le \varepsilon.\tag{17}$$

Inequality (17) is equivalent to

$$\lim_{t\to\infty}\mathbb{E}||x(t)||^2 = 0.\tag{18}$$

Thus, ISTVNN (1) is mean square asymptotically stable. □

Remark 2: In Theorem 1, number of impulsive occurrence, the strengths of impulses and the dynamics behavior of impulsive neural network are regarded as time varying.

Remark 3: When $\mathcal{S}_i > 1$, impulses is unstable. When $0 < \mathcal{S}_i < 1$, impulses is stable. The theoretical result of Theorem 1 allows stable and unstable pulses to occur simultaneously in NN.

Remark 4: Theorem 1 provides certain stability criteria for ISTVNN (1) without delay. It is worth emphasizing that our contribution is to illustrate efficacy of deploying impulsive density technique to study stability of ISTVNN (1) without delay based on Lyapunov function and Ito formula. Condition (4) is a limit on impulsive density, where $\frac{1}{t}+2\Xi+n\Gamma$ controls the number of impulses.

B. EXPONENTIAL STABILITY OF ISTVNN

Now, combining the trajectory based method and improved Razumikhin method, it will be investigated that MSES of ISTVNN (1) with time delay.

Before presenting our conclusion, it will be required that following important lemmas and definition. And we take $V(t,x_h(t)) = \frac{t}{2}\sum_{h=1}^{m}x_h^2(t)$. Set $\aleph = \max_{1\le h\le m}\Big[-2a_h + \sum_{r=1}^{m}|b_{hr}|\varsigma_r\Big]$ and $\psi = \max_{1\le h\le m}\Big[\sum_{r=1}^{m}(e_{rh}^2\varrho_h^2+|b_{rh}|\varsigma_h)\Big]$.

Lemma 1: Let Assumption 1-Assumption 2 *hold, for any $t\in\mathbb{R}^+$, $\mu > 1$, $\zeta > 1$, the following inequality*

$$\mathbb{E}\mathcal{L}V(t,x_h(t+\theta)) \le (\frac{1}{t}+\aleph+m\mu\psi)\mathbb{E}V(t,x_h(t))\tag{19}$$

holds, if $\mathbb{E}\sum_{h=1}^{m}x_h^2(t+\theta) \le \mu\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2$ for $-\tau \le \theta \le 0$.

Proof: For $\theta\in[-\tau,0]$, $t\in\mathbb{R}^+$, it can be verified from definition of $\mathcal{L}$ that

$$\begin{aligned}&\mathcal{L}V(t,x_h(t+\theta))\\&\quad=\frac{1}{2}\sum_{h=1}^{m}x_h^2(t)+t\sum_{h=1}^{m}sgn(x_h(t))x_h(t)\Big[-a_hx_h(t)\\&\qquad+\sum_{r=1}^{m}b_{hr}f_r(x_r(t+\theta))\Big]+\frac{t}{2}\Big[\sum_{h=1}^{m}sgn(x_h(t))\\&\qquad\times(\sum_{r=1}^{n}e_{hr}g_r(x_r(t+\theta)))^2\Big].\end{aligned}\tag{20}$$

According to Assumption 1 and Assumption 2, we can test that

$$\begin{aligned}&\mathbb{E}\mathcal{L}V(t,x_h(t+\theta))\\&\le \mathbb{E}\frac{1}{2}\sum_{h=1}^{m}x_h^2(t)\\&\quad+\mathbb{E}t\sum_{h=1}^{m}\Big[-a_hx_h^2(t)+\sum_{r=1}^{m}|b_{hr}|\varsigma_r|x_r(t+\theta)||x_h(t)|\Big]\\&\quad+\mathbb{E}\frac{mt}{2}\sum_{h=1}^{m}\sum_{r=1}^{m}e_{hr}^2\varrho_r^2x_r^2(t+\theta)\\&\le(-a_ht+\frac{1}{2})\mathbb{E}\sum_{h=1}^{m}x_h^2(t)+\frac{mt}{2}\mathbb{E}\sum_{h=1}^{m}\sum_{r=1}^{m}e_{rh}^2\varrho_h^2x_h^2(t+\theta)\\&\quad+\frac{t}{2}\mathbb{E}\sum_{h=1}^{m}\sum_{r=1}^{m}|b_{hr}|\varsigma_r(x_h^2(t)+x_r^2(t+\theta))\\&\le\frac{1}{2}\sum_{h=1}^{m}x_h^2(t)+\max_{1\le h\le m}\Big[-2a_h+\sum_{r=1}^{m}|b_{hr}|\varsigma_r\Big]\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}x_h^2(t)\\&\quad+m\max_{1\le h\le m}\Big[\sum_{r=1}^{m}(e_{rh}^2\varrho_h^2+|b_{rh}|\varsigma_h)\Big]\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}x_h^2(t+\theta)\\&=(\frac{1}{t}+\aleph+m\mu\psi)\mathbb{E}V(t,x(t)).\end{aligned}\tag{21}$$

Thus, (19) holds. □

Remark 5: In Lemma 1, we obtain a specific improved Razumikhin condition by taking a time-dependent Lyapunov function, which is an important method for dealing with time delay. The classic Razumikhin approach described in [13] and [18] is not appropriate for this paper and has several limitations, namely, $\mathbb{E}\mathcal{L}V(t, x(t+\theta)) \leq c\mathbb{E}V(t, x(t))$, if $\mathbb{E}V(t, x(t+\theta)) \leq qV(t, x(t))$. On the one hand, c does not exist due to time-varying impulsive strength and impulsive number. Additionally, if c is overly big, this condition becomes highly conservative. In this paper, the number of impulses needed to stabilize ISTNVNN (1) with time delay is determined by $\frac{1}{t} + \aleph + m\mu\psi$.

A trajectory based approach for validating certain inequalities is described below.

Lemma 2 [20]: If there exist a constant $0 < \rho < 1$, a piecewise continuous function $\kappa : [-T-\tau, \infty) \to [0, \infty)$ and $T + \tau > 0$ such that

$$\kappa(t) \leq \rho \sup_{s\in[t-T-\tau,t]} \kappa(s), t \geq t_0,$$

then,

$$\kappa(t) \leq \sup_{s\in[t_0-T-\tau,t_0]} \kappa(s)e^{\frac{(t-t_0)\ln\rho}{T+\tau}}, t \geq t_0.$$

For time-varying function, a definition is required before proceeding to main findings.

Definition 5: Let $\mathfrak{Y}$ be a set of functions. Any $\chi \in \mathfrak{Y}$ is called UASF, *if there exist finite positive constant $\bar{\mathfrak{C}}$ and $\breve{\delta} > 1$ such that*

$$\int_t^{t+s} \chi(u)du < \ln\bar{\mathfrak{C}}, t \geq t_0$$

holds when $s \geq T > 0$; and

$$\int_t^{t+s} \chi(u)du < \ln\breve{\delta}, t \geq t_0$$

holds when $0 \leq s \leq T$.

The MSES criteria for delayed ISTVNN (1) are provided as follows. The impulse here is a hybrid impulse, which can be stable or unstable at various moments.

Theorem 2: Let Assumption 1-Assumption 2 *hold, if there exist constants $\rho > 0$, $\breve{\delta}e^{\sum_{i=1}^{s} N_{0i}|\ln\mathcal{S}_i|} > 0$ satisfy*

$$\mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 \leq \mathcal{S}(t_k)\mathbb{E}\frac{t_k^-}{2}\sum_{h=1}^{m}|x_h(t_k^-)|^2, \tag{22}$$

$$\breve{\delta}e^{\sum_{i=1}^{s} N_{0i}|\ln\mathcal{S}_i|} < \mu, \tag{23}$$

$$\rho \in (\nu, 1), \quad \nu = \max\{\frac{\breve{\delta}}{\mu}e^{\sum_{i=1}^{s} N_{0i}|\ln\mathcal{S}_i|}, \frac{1}{\mu}\} < 1, \tag{24}$$

then ISTVNN (1) *is mean square exponentially stable, where $\mathcal{S}(t_k) \in \Theta$.*

Proof: Function $\sum_{i=1}^{s}\phi_i(t)\ln\mathcal{S}_i + \frac{1}{t} + \aleph + m\mu\psi \in \mathfrak{Y}$, therefore, there is a $T > 0$, for $T < t - t_0$,

$$e^{\int_{t-T}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i + \frac{1}{u} + \aleph + m\mu\psi du} \leq \bar{\mathfrak{C}} \tag{25}$$

holds, where $\bar{\mathfrak{C}} = \frac{\rho}{1+e^{\sum_{i=1}^{s} N_{0i}(t,t-T)|\ln\mathcal{S}_i|}}$. Therefore, for $t > t_0 + T$, one firstly prove that

$$\mathbb{E}\sum_{h=1}^{m}x_h^2(t) \leq \rho \sup_{u\in[t-T-\tau,t]} \mathbb{E}\sum_{h=1}^{m}x_h^2(u). \tag{26}$$

Let $t_0 + T \leq t_k \leq t < t_{k+1}$, $k \in \mathbb{N}$, we assume that interval $[t-T, t)$ contains β impulsive strength, equivalent to $t - T \leq t_{k-\beta+1} \leq \cdots \leq t_k \leq t$.

For $t \geq t_0$ and $-\tau \leq \theta \leq 0$, let

$$\Delta = \{t : \mathbb{E}\sum_{h=1}^{m}|x_h(t+\theta)|^2 \leq \mu\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2\}.$$

Two situations are considered, namely, $[t-T, t] \nsubseteq \Delta$ and $[t-T, t] \subseteq \Delta$.

Case I: If $[t-T, t] \subseteq \Delta$, then $\mathbb{E}\sum_{h=1}^{m}|x_h(u+\theta)|^2 \leq \mu\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2$ for any $u \in [t_k, t]$. It will be obtained that by using (9) and (22)

$$\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \leq \mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 e^{\int_{t_k}^{t}\frac{1}{u}+\aleph+m\mu\psi du}, \tag{27}$$

and

$$\mathbb{E}\frac{t_k}{2}\sum_{h=1}^{m}|x_h(t_k)|^2 \leq \mathcal{S}(t_k)\mathbb{E}\frac{t_k^-}{2}\sum_{h=1}^{m}|x_h(t_k^-)|. \tag{28}$$

By combining (27) and (28), we have

$$\begin{aligned}\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 \\ \leq \mathcal{S}(t_k)\mathbb{E}\frac{t_k^-}{2}\sum_{h=1}^{m}|x_h(t_k^-)|^2 e^{\int_{t_k}^{t}\frac{1}{u}+\aleph+m\mu\psi du}.\end{aligned}$$

Similar to Theorem 1, one can infer that

$$\begin{aligned}
&\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}x_h^2(t) \\
&\leq \mathbb{E}\frac{t-T}{2}\sum_{h=1}^{m}|x_h(t-T)|^2\prod_{j=1}^{\beta}\mathcal{S}(t_{k-j+1}) \\
&\quad\times e^{\int_{t-T}^{t}\frac{1}{u}+\aleph+m\mu\psi du} \\
&= \mathbb{E}\frac{t-T}{2}\sum_{h=1}^{m}|x_h(t-T)|^2\prod_{i=1}^{s}\mathcal{S}_i^{N_i(t,t-T)} \\
&\quad\times e^{\int_{t-T}^{t}\frac{1}{u}+\aleph+m\mu\psi du} \\
&\leq \mathbb{E}\frac{t-T}{2}\sum_{h=1}^{m}|x_h(t-T)|^2 e^{\sum_{i=1}^{s}N_i(t,t-T)\ln\mathcal{S}_i} \\
&\quad\times e^{\int_{t-T}^{t}\frac{1}{u}+\aleph+m\mu\psi du} \\
&\leq \mathbb{E}\frac{t-T}{2}\sum_{h=1}^{m}|x_h(t-T)|^2 e^{\sum_{i=1}^{s}N_{0i}(t,t-T)|\ln\mathcal{S}_i|} \\
&\quad\times e^{\int_{t-T}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+\aleph+m\mu\psi du}.
\end{aligned} \tag{29}$$

Replacing (25) into (29) generates that

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 &\le \rho\mathbb{E}\frac{t-T}{t}\sum_{h=1}^{m}|x_h(t-T)|^2\\ &\le \rho\mathbb{E}\sum_{h=1}^{m}|x_h(t-T)|^2\\ &\le \rho\sup_{u\in[t-T-\tau,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2.\end{aligned}$$

Thus, (26) holds when $[t-T,t]\subseteq\Delta$.

Case II: If $[t-T,t]\nsubseteq\Delta$, for some instants $t-T\le u\le t$, we can derive

$$\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2<\mu^{-1}\mathbb{E}\sum_{h=1}^{m}|x_h(u+\theta)|^2.$$

Let

$$\check{t}=\sup\Big\{t-T\le u\le t:\ \mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2<\mu^{-1}\mathbb{E}\sum_{h=1}^{m}|x_h(u+\theta)|^2\Big\}.$$

Then, it is true that

$$\mathbb{E}\sum_{h=1}^{m}|x_h(\check{t})|^2\le\mu^{-1}\sup_{u\in[t-T-\tau,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2. \tag{30}$$

Actually, it is confirmed that providing $\check{t}$ is not an impulsive instant

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(\check{t})|^2 &= \mu^{-1}\mathbb{E}\sum_{h=1}^{m}|x_h(\check{t}+\theta)|^2\\ &\le \mu^{-1}\sup_{u\in[t-T-\tau,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2.\end{aligned}$$

Otherwise, it is also validated that providing $\check{t}$ is an impulsive instant

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(\check{t})|^2 &< \mu^{-1}\mathbb{E}\sum_{h=1}^{m}|x_h((\check{t}+\theta)^-)|^2\\ &\le \mu^{-1}\sup_{u\in[t-T-\tau,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2.\end{aligned}$$

Consequently, (30) holds. Next, it is illustrated that (26) also holds for $[t-T,t]\nsubseteq\Delta$.

From (24) and (30), when $t=\check{t}$, we can get that

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 &\le \mu^{-1}\mathbb{E}\sum_{h=1}^{m}|x_h(t+\theta)|^2\\ &\le \rho\sup_{u\in[t-T,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u+\theta)|^2.\end{aligned}$$

Therefore, (26) holds.

Besides, when $t>\check{t}$,

$$\mathbb{E}\sum_{h=1}^{m}|x_h(u+\theta)|^2\le\mu\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2$$

holds. And $\sum_{i=1}^{s}\phi_i(t)\ln\mathcal{S}_i+\frac{1}{u}+\aleph+m\mu\psi\in\mathfrak{Y}$, we have

$$e^{\int_{\check{t}}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+\aleph+m\mu\psi du}\le\check{\delta},\ \check{\delta}>1. \tag{31}$$

Similar to *Case I*, we can derive

$$\begin{aligned}\mathbb{E}\frac{t}{2}\sum_{h=1}^{m}|x_h(t)|^2 &\le \mathbb{E}\frac{\check{t}}{2}\sum_{h=1}^{m}|x_h(\check{t})|^2\prod_{j=1}^{\beta}\mathcal{S}_{k-j+1}e^{\int_{\check{t}}^{t}\frac{1}{u}+\aleph+m\mu\psi du}\\ &= \mathbb{E}\frac{\check{t}}{2}\sum_{h=1}^{m}|x_h(\check{t})|^2\prod_{i=1}^{s}\mathcal{S}_i^{N_i(t,\check{t})}e^{\int_{\check{t}}^{t}\frac{1}{u}+\aleph+m\mu\psi du}\\ &\le \mathbb{E}\frac{\check{t}}{2}\sum_{h=1}^{n}|x_h(\check{t})|^2e^{\sum_{i=1}^{s}N_i(t,\check{t})\ln\mathcal{S}_i}\\ &\quad\times e^{\int_{\check{t}}^{t}\frac{1}{u}+\aleph+m\mu\psi du}\\ &\le \mathbb{E}\frac{\check{t}}{2}\sum_{h=1}^{m}|x_h(\check{t})|^2e^{\sum_{i=1}^{s}N_{0i}(t,\check{t})|\ln\mathcal{S}_i|}\\ &\quad\times e^{\int_{\check{t}}^{t}\sum_{i=1}^{s}\phi_i(u)\ln\mathcal{S}_i+\frac{1}{u}+\aleph+m\mu\psi du}.\end{aligned} \tag{32}$$

It can be inferred that by substituting (31) into (32)

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 &\le \check{\delta}e^{\sum_{i=1}^{s}N_{0i}(t,\check{t})|\ln\mathcal{S}_i|}\mathbb{E}\frac{\check{t}}{t}\sum_{h=1}^{m}|x_h(\check{t})|^2\\ &\le \check{\delta}e^{\sum_{i=1}^{s}N_{0i}(t,\check{t})|\ln\mathcal{S}_i|}\mathbb{E}\sum_{h=1}^{m}|x_h(\check{t})|^2.\end{aligned} \tag{33}$$

Replacing (30) into (33) infers that

$$\begin{aligned}\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 &\le \frac{\check{\delta}}{\mu}e^{\sum_{i=1}^{s}N_{0i}(t,\check{t})|\ln\mathcal{S}_i|}\sup_{u\in[t-T-\tau,t]}\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2.\end{aligned} \tag{34}$$

Noticing that $\frac{\breve{\delta}}{\mu}e^{\sum_{i=1}^{s}N_{0i}(t,\breve{t})|\ln\mathcal{S}_i|} < \rho$, thus,

$$\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 \leq \rho \sup_{u\in[t-T-\tau,t]} \mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2.$$

Therefore, when $[t-T,t] \nsubseteq \Delta$, (26) also holds. According to Lemma 2, for $t \geq t_0 + T$, we can get from (26)

$$\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 \leq \sup_{u\in[t_0-\tau,t_0+T]} \mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2 e^{\frac{(t-T-t_0)\ln\rho}{T+\tau}}, \tag{35}$$

where $\frac{\ln\rho}{T+\tau} < 0$. Except for some finite impulsive instants, for $u \in [t_0, t_0+T]$, $x_h(u)$ is continuous, so $\mathbb{E}\sum_{h=1}^{m}|x_h(u)|^2 = \mathbb{E}||x(t)||^2 < +\infty$. Thus, there is a constant Π to satisfy

$$\sup_{u\in[t_0-\tau,t_0+T]} \mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 \leq \Pi. \tag{36}$$

Substituting (36) with (35), then,

$$\mathbb{E}\sum_{h=1}^{m}|x_h(t)|^2 \leq \Pi e^{\beta(t-T-t_0)}, \tag{37}$$

where $\beta = \frac{\ln\rho}{T+\tau} < 0$. Furthermore,

$$\mathbb{E}||x(t)||^2 \leq \Pi e^{\beta(t-T-t_0)}. \tag{38}$$

It can be yield that

$$\lim_{t\to\infty}\sup\frac{1}{t}\ln\mathbb{E}||x(t)||^2 = \beta < 0.$$

Consequently, ISTVNN (1) with time delay is mean square exponentially stable. □

Remark 6: Due to occurrence of time delay, it is challenging to accurately calculate the number of impulsive occurrence during the operation of ISTVNN, so state trajectory of ISTVNN with time delay is relatively tough to investigate. Lemma 1 is crucial in addressing the influence of time delay on ISTVNN.

Remark 7: In Theorem 2, based on condition of Lemma 1, we mainly prove that (26) holds for any t, and then obtain (35) by using Lemma 2. Through further analysis, we verify that stability criterion in Theorem 2 is valid.

Remark 8: In Theorem 1 and Theorem 2, $\mathcal{S}(t_k)$ represents an impulsive strength, and $\mathcal{S}(t_k) \in \Theta$, which means strength of impulses is time varying.

Remark 9: In [15], $\mu(t) + \frac{\ln\beta}{T_a}$ is required to be an exponential stable function, where β is the constant impulsive intensity and T_a is AII. This condition has strict restrictions on the function, so this paper relaxs this restriction. $\sum_{i=1}^{s}\phi_i(t)\ln\mathcal{S}_i + \frac{1}{t} + \aleph + m\mu\psi$ only needs to satisfy UASF condition, where $\phi_i(t)$ is impulsive density function and $\mathcal{S}_i$ represents time-varying impulses intensity. $\frac{1}{t} + \aleph + m\mu\psi$ only needs to be a piecewise continuous.

Remark 10: In [16], [17], and [18], authors consider impulsive number through AII, which is only limited by a linear relation. But in our Theorem 1 and Theorem 2, the quantity of impulses is depicted by impulsive density, which is limited by an integrable function. This relationship can be both linear and nonlinear. Since AII is constant, AII can be considered as a special case of impulsive density. Thus, when $p = 2$ in the above literatures, theoretical results are included in this paper.

IV. NUMERICAL EXAMPLES

The following some instances are given to show feasibility of our theoretical findings.

Example 1: We consider the following one-dimensional ISTVNN without time delay:

$$\begin{cases} dx(t) = [-Ax(t) + Bf(x(t))]dt + Eg(x(t))d\omega(t), \\ x(t_k) = \mathcal{U}_k x(t_k^-), \quad k \in \mathbb{N}^+. \end{cases} \tag{39}$$

Taking that $A = 1/2$, $B = 2$, $f(x(t)) = \frac{x(t)}{2}$, $E = 1$, $g(x(t)) = x(t)$, $\mathcal{U}_k \equiv \mathcal{U} = e^{-2}$, Lyapunov function $V(t,x(t)) = \frac{t}{2}x^2(t)$. Then, for $t \geq t_0 = 1$, we can calculate that

$$\mathcal{L}\frac{t}{2}x^2(t) = (\frac{1}{t} + 2)\frac{t}{2}x^2(t).$$

Let impulsive strength $\mathcal{S}_k \equiv \mathcal{S} = e^{-1}$, according to Remark 1 and (10), then,

$$\begin{aligned}
&\mathbb{E}\frac{t}{2}x^2(t)\\
&\quad = \mathbb{E}\frac{t_0}{2}x^2(t_0)\mathcal{S}^{N(t,t_0)}e^{\int_{t_0}^{t}1/u+2du}\\
&\quad \geq \mathbb{E}\frac{t_0}{2}x^2(t_0)e^{N_0\ln\mathcal{S}}e^{\int_{t_0}^{t}1/u+2+\frac{\ln\mathcal{S}}{\mathcal{T}_a}du}.
\end{aligned}$$

We can easily judge for any $\mathcal{T}_a > 0$ that

$$\lim_{t\to\infty}\int_{t_0}^{t}1/u + 2 + \frac{\ln\mathcal{S}}{\mathcal{T}_a}du = +\infty.$$

It implies that, when quantity of activated impulses is given by AII, it is inadequate to stabilize ISTVNN (39). Therefore, more impulses are needed to stabilize ISTVNN (39).

However, compared with AII, impulsive number, which is depicted by impulsive density, can stabilize ISTVNN (39). Set impulsive density $\phi(t) = t + \frac{1}{t} + 2$, it can be deduced that

$$\int_{t_0}^{t}(u + \frac{1}{u} + 2)\ln\mathcal{S} + \frac{1}{u} + 2du = -\frac{1}{2}(t^2 - t_0^2).$$

Thus, according to (3) and (4), ISTVNN (39) with $\phi(t) \geq t + \frac{1}{t} + 2$ is mean square asymptotically stable.

Remark 11: Fig. 1 not only describes the state trajectory of ISTVNN (39) without time delay, which number of impulsive occurrence is characterized by AII, but also indicates that ISTVNN (39) with impulsive density $\phi(t) = t + \frac{1}{t} + 2$ is mean square asymptotically stable under conditions of $t_k = \inf\{t > t_0 : \int_{t_0}^{t} u + \frac{1}{u} + 2du = k\}$ and $\mathcal{S}_k \equiv e^{-1}$. The results show that the amount of impulses triggered by AII is not enough to stabilize the system, and the amount of impulses represented by impulsive density is effective for stabilizing the system.

Example 2: Based on the ISTVNN (39) in Example 1, take $t_0 = 1$, $A = \frac{1}{2}$, $B = 1$, $f(x(t)) = (\frac{1}{2} + \frac{1}{t})x(t)$, $E = 1$,

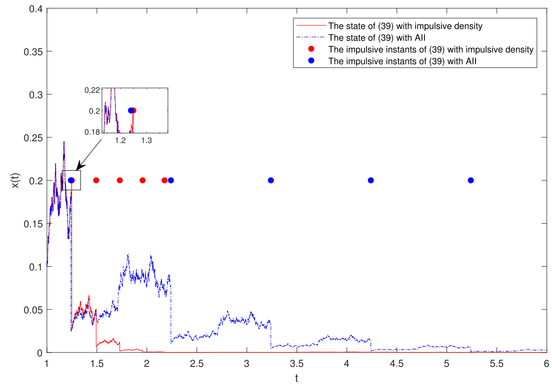

FIGURE 1. The comparison of states of (39) with impulsive density $\phi(t) = 1/t + t + 2$ and AII $\mathcal{T}_a = 0.24$.

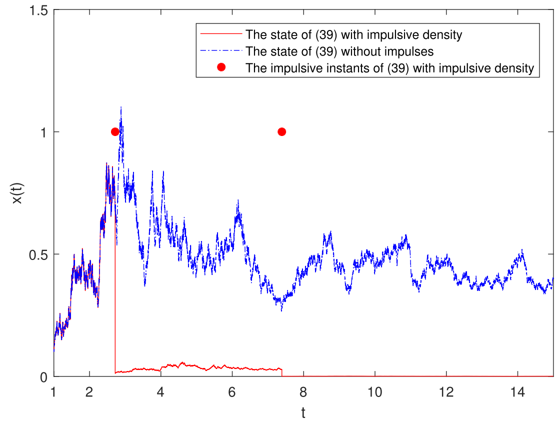

FIGURE 2. The comparison of (39) with $\phi(t) = 1/t$ and without impulses.

$\mathcal{U}_k \equiv \mathcal{U} = e^{-4}$. Set $V(t, x(t)) = tx^2(t)$, for $t \geq t_0$, $g(x(t)) = \frac{x(t)}{\sqrt{t}}$, we can get that

$$\mathcal{L}tx^2(t) = 4x^2(t).$$

From (10), it can be obtained that

$$\mathbb{E}tx^2(t) = e^{\int_{t_0}^{t} 4/u du}\mathbb{E}t_0x^2(t_0)$$

and

$$\lim_{t\to\infty} \mathbb{E}tx^2(t) = +\infty.$$

So, ISTVNN (39) without impulses is unstable. By impulses, ISTVNN (39) is asymptotical stable. we assume that impulsive strength $\mathcal{S}_k \equiv \mathcal{S} = e^{-5}$, impulsive density $\phi(t) = \frac{1}{t}$, yield that

$$\int_{t_0}^{t} \frac{1}{u}\ln \mathcal{S} + \frac{4}{u}du \leq \ln t_0 - \ln t,\ \lim_{t\to\infty}(\ln t_0 - \ln t) \to -\infty.$$

From Theorem 1, ISTVNN (39) is mean square asymptotically stable, if impulsive density $\phi(t) \geq \frac{1}{t}$.

By utilizing Definition 1, one can deduce for any $\mathcal{T}_a > 0$ that

$$N(t, t_0) \leq N_0 + \ln t - \ln t_0 \leq N_0 + \frac{t - t_0}{\mathcal{T}_a}.$$

It fully demonstrates that number of impulsive occurrence depicted by impulsive density is less than impulses described by AII. In other word, number of impulsive occurrence described by AII is excessive to stabilize ISTVNN (39). Number of impulsive occurrence depicted by impulsive density is more reasonable.

Remark 12: Fig. 2 illustrates ISTVNN (39) is mean square asymptotically stable with $\mathcal{S}_k \equiv e^{-5}$ and $t_k = \{t \geq t_0 : \int_{t_0}^{t} \frac{1}{s}ds = k\}$. According to notion of impulsive density, we have $N(t, t_0) \leq N_0 + \ln t - \ln t_0 \leq N_0 + \frac{t-t_0}{\mathcal{T}_a}$. This demonstrates that impulsive amount defined by AII is exaggerated for system stabilization.

Example 3: We will consider Chua's circuit with time delay in [20]:

$$\begin{aligned}
\dot{x}_1(t) &= 9(x_2(t) - \frac{2}{7}x_1(t) + g(x_1(t))) - 0.1x_1(t + \theta),\\
\dot{x}_2(t) &= x_1(t) - x_2(t) + x_3(t) - 0.1x_1(t + \theta),\\
\dot{x}_3(t) &= -14.28x_2(t) + 0.1(2x_1(t + \theta) - x_3(t + \theta)),
\end{aligned}$$

where $g(x_1(t)) = \frac{3}{14}(|x_1(t) + 1| - |x_1(t) - 1|)$. The delayed Chua's circuit can be recast the following form,

$$\dot{x}(t) = Ax(t) + Bx(t + \theta) + Df(x(t)), \tag{40}$$

where

$$A = \begin{bmatrix} -\frac{18}{7} & 9 & 0 \\ 1 & -1 & 1 \\ 0 & -14.28 & 0 \end{bmatrix},\quad B = \begin{bmatrix} -0.1 & 0 & 0 \\ -0.1 & 0 & 0 \\ 0.2 & 0 & -0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} \frac{27}{7} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $f = (\frac{1}{2}(|x_1(t) + 1| - |x_1(t) - 1|), 0, 0)^T$. The slave system is constructed using the same manner as master system, but with random disturbance and a controller in mind,

$$\begin{aligned}
d\Gamma(t) = {}& [A\Gamma(t) + B\Gamma(t + \theta) + Df(\Gamma(t)) + u(t)]dt\\
&+ g(t, \Gamma(t), \Gamma(t + \theta))d\omega(t),
\end{aligned}$$

where $u(t) = \sum_{k=1}^{\infty} \alpha(y(t) - x(t))\delta(t - t_k)$ is impulsive controller, $\delta(t)$ is Dirac function, $\alpha = e^{-3} - 1$. The synchronization error is specified as

$$\begin{cases} de(t) = [Ae(t) + Be(t + \theta) + D\bar{f}(e(t))]dt \\ \qquad + g(t, e(t), e(t + \theta))d\omega(t), \\ e(t_k) = (1 + \alpha)e(t_k^-), k \in \mathbb{N}^+, \end{cases} \tag{41}$$

where $-\tau < \theta < 0, \bar{f}(e(t)) = f(\Gamma(t) - f(x(t)))$.

We take $V(t, e(t)) = t||e(t)||^2$. And let

$$\mathbb{E}e^2(t + \theta) \leq q\mathbb{E}e^2(t),$$

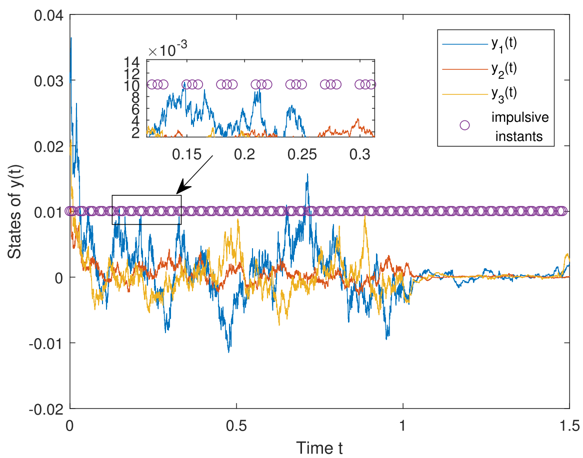

FIGURE 3. State of (41) with AII=0.01.

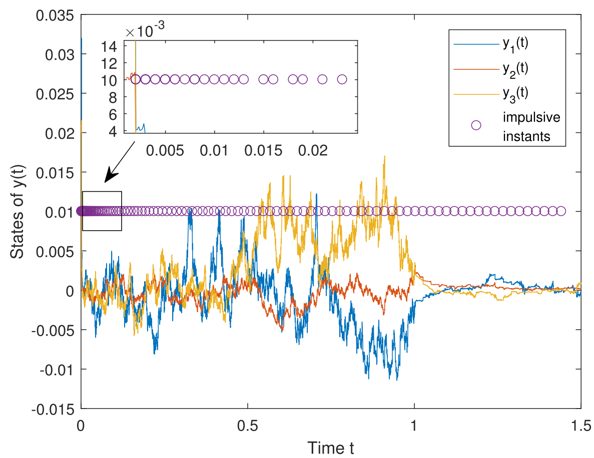

FIGURE 5. State of (41) with $\phi(t) = \frac{1}{t} + \frac{3q}{t} + 32.8481$.

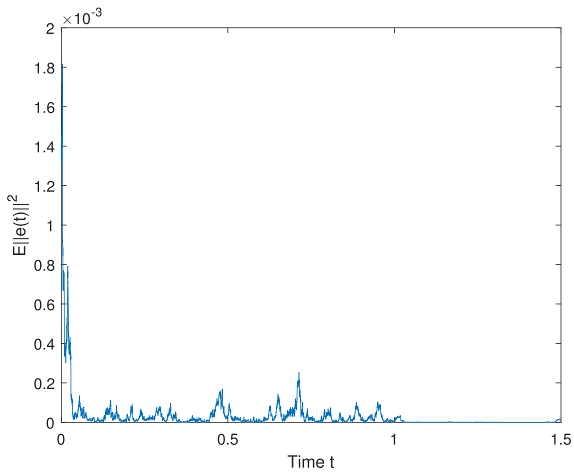

FIGURE 4. Mean square trajectory of (41) of under AII=0.01.

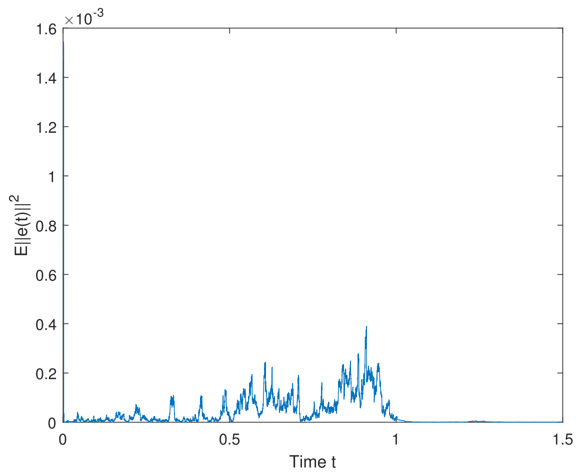

FIGURE 6. Mean square trajectory of (41) under $\phi(t) = \frac{1}{t} + \frac{3q}{t} + 32.8481$.

where $q > 1$. Setting that $\alpha = -0.3$, $g(t, e(t), e(t+\theta)) = e(t)+e(t+\theta)$, $\mathcal{S}_k \equiv \mathcal{S} = e^{-3}$. Through calculation, we know that

$$\mathbb{E}\mathcal{L}V(t, e(t+\theta)) \leq (\frac{1+3q}{t} + 32.808)\mathbb{E}V(t, e(t)).$$

Take impulsive density $\phi(t) = \frac{1}{t} + \frac{3q}{t} + 32.8481$, $N_0 = 1$, $\mathcal{S}_k \equiv \mathcal{S} = e^{-1}$, $\breve{\delta} = 1.01$, $T = 1$, $q = 3$, then, we have

$$e^{\int_t^{t+1} \phi(u) \ln \mathcal{S} + \frac{1+3q}{t} + 32.808 du} = 0.9608 < \breve{\delta} = 1.01.$$

Further, we can calculate that

$$\breve{\delta} e^{N_0 |\ln \mathcal{S}|} = 2.7455 < q = 3.$$

Besides, we can find a positive $\rho \in (0.9152, 1)$, thus, according to Theorem 2, we can deduce that error system (41) is mean square exponentially stable when impulsive density $\phi(t) = \frac{1}{t} + \frac{3q}{t} + 32.8481$. Furthermore, master system and slave system achieve synchronization.

Remark 13: Example 3 uses the same parameters as [17] to illustrate effectiveness and efficiency of impulsive density. Fig. 3 and Fig. 5 depict state trajectory of (41) based on AII and impulsive density, respectively. Fig. 4 and Fig. 6 describe mean square trajectory of (41) based on AII and impulsive density, respectively. The results show that, when AII = 0.01, synchronization error system (41) reaches MSES. The frequency of impulses triggered, however, is high in this situation. In reality, amount of impulses shown by impulsive density is less than that shown by AII, but overall impact is the same, namely, synchronization error system (41) can attain MSES under $\phi(t) = \frac{1}{t} + \frac{3q}{t} + 32.8481$, and the quantity of impulses is less than that shown by AII. From these comparisons, we can see that impulsive density is more effective than AII to characterize amount of impulses to stabilize system (41).

V. CONCLUSION

The stability of ISTVNN is investigated in this paper by utilizing impulsive density. Impulsive density technique efficiently eliminates issue of excessive or inadequate amount of impulse as characterized by AII. Stability conditions for ISTVNN with and without delay are developed by using impulsive density, respectively. Finally, we use numerical examples to show the effectiveness of theoretical results. Further work will focus on applying the analytical framework here to hybrid impulsive systems on time scales.

Data Availability: No data were used to support this study.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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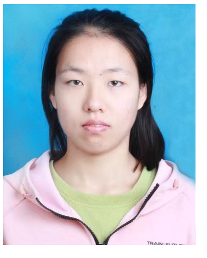

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