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RESEARCH ARTICLE

Fermatean Hesitant Fuzzy Choquet Integral Aggregation Operators

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ABSTRACT Fermatean fuzzy hesitant sets provide a flexible and powerful tool for decision making, allowing decision makers to incorporate uncertainty and hesitation into their decision making process, and enabling them to make more informed and effective decisions in complex and uncertain environments. Considering that many factors are interdependent in reality while existing methods cannot solve this problem, we propose Fermatean hesitant fuzzy Choquet integral ordered aggregation operators based on the Fermatean hesitant fuzzy set and Choquet integral, and establish their related properties. These operators including averaging and geometric operators not only handle situations where decision criteria or preferences are interdependent, but also provides the decision under ideal and non-ideal situation. Additionally, we present a multi-attribute decision-making method for Fermatean hesitant fuzzy information using these operators. We have validated the proposed method, and its practicality and effectiveness are demonstrated through numerical examples from previous research regarding Fermatean hesitant fuzzy set, including sensitivity analysis and comparison.

INDEX TERMS Fermatean hesitant fuzzy set, Choquet integral, Choquet integral aggregation operators, multi-attribute decision making.

I. INTRODUCTION

A. BACKGROUND

Fuzzy set theory, introduced by Zadeh [1], is a valuable tool for addressing uncertainty problems and has been successfully applied in various fields [2], [3], [4], [5], [6]. However, when dealing with ambiguous information sources, fuzzy sets have limitations. To overcome these limitations, researchers have proposed several extended models of fuzzy sets [7], [8], [9]. One such extension is the intuitionistic fuzzy set (IFS) [10], which takes the degree of membership, non-membership, and hesitation into account. However, the sum of membership and non-membership degrees must be less than or equal to 1, restricting the range of applications for fuzzy sets. To address this issue, Yager proposed two new types of fuzzy sets: the Pythagorean fuzzy set (PFS) and the Fermatean fuzzy set (FFS) [11], [12], [13]. PFS and FFS only require the sum of squares and sum of cubes, respectively, of the membership and non-membership degrees to be less than or equal to 1, expanding the range of applications for fuzzy sets.

In the decision-making process, every decision maker needs to judge alternatives based on various attributes. However, due to subjective consciousness, decision results are often indecisive. To address this issue, Torra proposed the hesitant fuzzy set (HFS) [14], which has been effective in dealing with ambiguity. Based on the advantages of HFS and PFS, Khan et al. proposed the Pythagorean hesitant fuzzy set (PHFS) [15], which expands the fuzzy set model. Furthermore, the Fermatean hesitant fuzzy set (FHFS) was proposed by Kirişci in order to further handle ambiguities [16]. However, obtaining a good decision result is not solely based on the decision information provided by a single expert or one type of decision attribute. Therefore, it is crucial to study the aggregation of multiple decision information. Grabisch introduced the fuzzy integral and its applications

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in decision-making problems [17]. Xia and Xu proposed a set of aggregation operators for hesitant fuzzy information and discussed their correlations [18]. In 2014, Wang proposed the extended hesitant fuzzy linguistic term sets and their aggregation operators, which can be applied to group decision making [19]. In 2019, Senapati et al [20] proposed Fermatean fuzzy weighted averaging and geometric operators(FFWA,FFWG) which can be used in multi-attribute decision-making(MADM) under Fermatean fuzzy condition. In 2021, Shahzadi et al. [21] presented group decision-making for the selection of an antivirus mask under fermatean fuzzy soft information. In 2022, Akram [22] proposed complex fermatean fuzzy N-soft sets which can process two-dimensional information related to levels of satisfaction and dissatisfaction implicit in the nature of human decision-making. At the same time, Mishra et al. [23] proposed the COPRAS method based on interval-valued hesitant Fermatean fuzzy sets and its application in selecting desalination technology.

It is significant to note that the aggregation operators mentioned above assume that decision conditions are independent, which is not always the case. In many decision-making problems, the interdependence of decision conditions is often overlooked.

B. LITERATURE REVIEW

In 1974, Sugeno [24], [25] first proposed the concept of a fuzzy measure, which is a regular, monotone, continuous, non-negative set function. Fuzzy measure is a continuation and extension of classical additive measure. The Choquet integral for fuzzy measures was originally proposed by Murofushi and Sugeno [26], [27], [28]. In 1989, they first combined the fuzzy measure with Choquet integral in [29]. Using these ideas, the problem of mutual dependence of decision criteria can be solved well. In 2000, Marichal [30] proposed an axiomatic method of the discrete Choquet integral as a tool to aggregate interacting criteria. In 2011, Tan and Chen [32] proposed the induced intuitionistic fuzzy Choquet integral operator and gave a corresponding method to solve the multiple criteria decision-making problem. In [33], Tan extended the multi-criteria interval-valued intuitionistic fuzzy decision-making technology to group decision-making environment. Considering the interdependence between criteria and decision makers' preferences, he proposed a TOPSIS-based Choquet integral for multi-criteria interval-valued intuitionistic fuzzy group decision-making. In 2012, Wei et al. [31] proposed the hesitant fuzzy Choquet integral aggregation operator and studied the MADM problem of attribute values in the form of hesitant fuzzy information. In 2016, Peng and Yang [34] proposed the Pythagorean Fuzzy Choquet Integral Operator, which not only considers the importance of its elements or their ordered positions, but also reflects the correlation between its elements or their ordered positions. In 2018, Khan et al. [35], [36] proposed the information aggregation and hybrid aggregation operator based on Pythagorean hesitant fuzzy sets, respectively, and studied their applications in group decision making.

In recent years, there have been increasing studies on the MADM problem. In 2020, Akram et al. [37] extended Dombi aggregation operators to handle uncertainty in m-polar fuzzy information and investigates their properties. In 2023, Akram et al. [38] proposed a new hybrid model called Pythagorean fuzzy n-soft expert sets, which combines Pythagorean fuzzy sets and n-soft expert sets, to handle uncertain parameterized information in multi-attribute group decision-making problems. Meanwhile, Sarwar et al. [39] presents a novel linguistic assessment model,which integrates rough fuzzy numbers with cloud model theory to handle uncertainties.

C. MOTIVATION AND INNOVATION

In this paper, we present an extended approach to the Choquet integral operator based on the Fermatean hesitant fuzzy set, which offers several significant improvements. Our motivation and innovations can be summarized as follows:

(1) Although numerous studies have been conducted to solve Multiple Attribute Decision Making MADM) problems [38], [39], [40], there is a lack of research on the Fermatean hesitant fuzzy set. In Fermatean fuzzy environments, where high ambiguity and uncertainty make it difficult for experts to make decisions, it is common to describe a problem using two sets of possible values denoting membership and non-membership, respectively. Therefore, we propose a method that is adapted to the Fermatean hesitant fuzzy condition.

(2) One of the main drawbacks of the recent existing methods for solving MADM problems is the failure to consider the correlation and interdependence between different attributes. The latest research on the Choquet integral has been effective in addressing this issue, such as the Pythagorean hesitant fuzzy Choquet integral aggregation operators proposed by Khan et al. [41]. However, research on the Choquet integral remains limited. Therefore, we provide a systematic system of Fermatean hesitant fuzzy Choquet integral aggregation operators to address this gap in the literature.

D. STRUCTURE

The paper is organized as follows. In Section II, we introduce the basic concepts of the Fermatean hesitant fuzzy set accompanied with fuzzy measure and Choquet integral. In Section III, we define the Fermatean hesitant fuzzy Choquet integral aggregation operators, and prove their related properties. In Section IV, we develop MADM based on the Fermatean hesitant fuzzy Choquet integral aggregation operators. In Section V, we give a numerical example with sensitivity analysis and comparison to verify the validity and practicality of the aggregation operators proposed in this paper. In Section VI, we present some concluding remarks and future research.

II. PRELIMINARY

In this section, we introduce some basic concepts of Fermatean hesitant fuzzy set and Choquet integral operator.

A. FERMATEAN HESITANT FUZZY SET

Definition 1 [15]: Let X be a fixed set. A Fermatean hesitant fuzzy set abbreviated as FHFS F_H in X is an object with the following notion:

$$F_H = \{\langle x, \Lambda_{F_H}(x), \Gamma_{F_H}(x)\rangle | x \in X\}, \quad (1)$$

where $\Lambda_{F_H}(x)$ and $\Gamma_{F_H}(x)$ are mappings from X to $[0, 1]$, denoting a possible degree of membership and non-membership degree of element $x \in X$ in F_H respectively, and for each element $x \in X$, $\forall h_{F_H}(x) \in \Lambda_{F_H}(x)$, $\exists h'_{F_H}(x) \in \Gamma_{F_H}(x)$, such that, and $\forall h'_{F_H}(x) \in \Gamma_{F_H}(x)$, $\exists h_{F_H}(x) \in \Lambda_{F_H}(x)$, such that $0 \leq h^3_{F_H}(x) + h'^3_{F_H}(x) \leq 1$. For any FHFS $F_H = \{\langle x, \Lambda_{F_H}(x), \Gamma_{F_H}(x)\rangle | x \in X\}$ and for all $x \in X$, $\Pi_{F_H}(x) = \cup_{h_{F_H}(x)\in\Lambda_{F_H}(x), h'_{F_H}(x)\in\Gamma_{F_H}(x)} \sqrt[3]{1 - h^3_{F_H} - h'^3_{F_H}}$ is said to be the degree of indeterminacy of x to F_H, where $1 - h^3_{F_H} - h'^3_{F_H} > 0$.

Moreover, $FHFS(X)$ denotes the set of all elements of FHFSs. If X has only one element $\langle x, \Lambda_{F_H}(x), \Gamma_{F_H}(x)\rangle$ is said to be Fermatean hesitant fuzzy number and is denoted by $\hat{h} = \langle \Lambda_{F_H}(x), \Gamma_{F_H}(x)\rangle$ for convenience. We denoted the set of all FHFNs by FHFNS.

Remark for all $x \in X$ if $\Lambda_{F_H}(x)$ and $\Gamma_{F_H}(x)$ have only one element, then the FHFS become a PFS. If the non-membership degree is $\{0\}$, then the FHFS become a HFS.

Definition 2 [15]: Let $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$, $\hat{h_1} = \langle \Lambda_{\hat{h}_1}, \Gamma_{\hat{h}_1}\rangle$, $\hat{h_2} = \langle \Lambda_{\hat{h}_2}, \Gamma_{\hat{h}_2}\rangle$ are three FHFNs and $\lambda > 0$, then the following operational laws are valid.

(1) $\hat{h}_1 \cup \hat{h}_2 = \langle \max\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\}, \min\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\}\rangle$,

(2) $\hat{h}_1 \cap \hat{h}_2 = \langle \min\{\Lambda_{\hat{h}_1}, \Lambda_{\hat{h}_2}\}, \max\{\Gamma_{\hat{h}_1}, \Gamma_{\hat{h}_2}\}\rangle$,

(3) $\hat{h} = \langle \Gamma_{\hat{h}}, \Lambda_{\hat{h}}\rangle$,

(4) $\hat{h}_1 \oplus \hat{h}_2 = \langle \cup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2}} \{\sqrt[3]{h^3_{\hat{h}_1} + h^3_{\hat{h}_2} - h^3_{\hat{h}_1} h^3_{\hat{h}_2}}\}$, $\cup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2}} \{h'_{\hat{h}_1} h'_{\hat{h}_2}\}\rangle$,

(5) $\hat{h}_1 \otimes \hat{h}_2 = \langle \cup_{h_{\hat{h}_1}\in\Lambda_{\hat{h}_1}, h_{\hat{h}_2}\in\Lambda_{\hat{h}_2}} \{h_{\hat{h}_1} h_{\hat{h}_2}\}$, $\cup_{h'_{\hat{h}_1}\in\Gamma_{\hat{h}_1}, h'_{\hat{h}_2}\in\Gamma_{\hat{h}_2}} \{\sqrt[3]{h'^3_{\hat{h}_1} + h'^3_{\hat{h}_2} - h'^3_{\hat{h}_1} h'^3_{\hat{h}_2}}\}\rangle$,

(6) $\lambda\hat{h} = \langle \cup_{h_{\hat{h}}\in\Lambda_{\hat{h}}} \{\sqrt[3]{1 - (1 - (h_{\hat{h}})^3)^\lambda}\}, \cup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} \{(h'_{\hat{h}})^\lambda\}\rangle$, $\lambda > 0$,

(7) $\hat{h}^\lambda = \langle \cup_{h_{\hat{h}}\in\Lambda_{\hat{h}}} \{h^\lambda_{\hat{h}}\}, \cup_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} \{\sqrt[3]{1 - (1 - (h'_{\hat{h}})^3)^\lambda}\}\rangle$, $\lambda > 0$,

To compare two FHFNs, in following [15], the score function, accuracy function, and some basic laws on the basis of the score function are defined.

Definition 3 [15]: Let $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ be a FHFN. Then, we defined the score function of $\hat{h}$ as follows:

$$S(\hat{h}) = (\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}} \sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}})^3 - (\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}} \sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}})^3 \quad (2)$$

where $S(\hat{h}) \in [-1, 1]$. $l_{h_{\hat{h}}}$ denotes the number of elements in $\Lambda_{\hat{h}}$ and $l_{h'_{\hat{h}}}$ denotes the number of elements in $\Gamma_{\hat{h}}$.

Definition 4 [15]: Let $\hat{h} = \langle \Lambda_{\hat{h}}, \Gamma_{\hat{h}}\rangle$ be a FHFN. Then, we defined the accuracy function of $\hat{h}$ as follows:

$$\bar{\sigma}(\hat{h}) = (\frac{1}{l_{h_{\hat{h}}\in\Lambda_{\hat{h}}}} \sum_{h_{\hat{h}}\in\Lambda_{\hat{h}}} h_{\hat{h}} - S(\hat{h}))^3 + (\frac{1}{l_{h'_{\hat{h}}\in\Gamma_{\hat{h}}}} \sum_{h'_{\hat{h}}\in\Gamma_{\hat{h}}} h'_{\hat{h}} - S(\hat{h}))^3 \quad (3)$$

Here, we can see that $S(\hat{h})$ is just the mean value in statistics, and $\bar{\sigma}(\hat{h})$ is just the standard variance, which reflects the accuracy degree between all values in the FHFN $\hat{h}$ and their mean value. Let $\hat{h}_1$ and $\hat{h}_2$ be two FHFNs, $S(\hat{h}_1)$ be the score of $\hat{h}_1$, $S(\hat{h}_2)$ be the score of $\hat{h}_2$, and $\bar{\sigma}(\hat{h}_1)$ be the deviation degree of $\hat{h}_1$, $\bar{\sigma}(\hat{h}_2)$ be the deviation degree of $\hat{h}_2$. Then

(1) If $S(\hat{h}_1) < S(\hat{h}_2)$, then $\hat{h}_1 < \hat{h}_2$.

(2) If $S(\hat{h}_1) > S(\hat{h}_2)$, then $\hat{h}_1 > \hat{h}_2$.

(3) If $S(\hat{h}_1) = S(\hat{h}_2)$, then $\hat{h}_1 \sim \hat{h}_2$.

a) If $\bar{\sigma}(\hat{h}_1) < \bar{\sigma}(\hat{h}_2)$, then $\hat{h}_1 < \hat{h}_2$.

b) If $\bar{\sigma}(\hat{h}_1) > \bar{\sigma}(\hat{h}_2)$, then $\hat{h}_1 > \hat{h}_2$.

c) If $\bar{\sigma}(\hat{h}_1) = \bar{\sigma}(\hat{h}_2)$, then $\hat{h}_1 \sim \hat{h}_2$.

Definition 5 [35]: Let $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs, $\hat{h}_{\sigma(i)}$ be the largest in them, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0(i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\Sigma^n_{i=1} w_i = 1$. Then Fermatean hesitant fuzzy ordered weighted averaging (FHFOWA) operator is a mapping $FHFOWA : FHFN^n \rightarrow FHFN$ can be defined by

$$\begin{aligned}
&FHFOWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \\
&= w_1\hat{h}_{\sigma(1)} \oplus w_2\hat{h}_{\sigma(2)}\oplus, \ldots, \oplus w_n\hat{h}_{\sigma(n)} \\
&= \langle \cup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}}\in\Lambda_{\hat{h}_{\sigma(n)}}} \\
&\times\{\sqrt[3]{1 - \prod_{i=1}^{n}(1 - h^3_{\hat{h}_{\sigma(i)}})^{w_i}}\}, \\
&\cup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}}\in\Gamma_{\hat{h}_{\sigma(n)}}} \\
&\times \times \{\prod_{i=1}^{n}(h_{\hat{h}_{\sigma(i)}})^{w_i}\}\rangle.
\end{aligned}$$

Definition 6 [35]: Let $\hat{h}_i = \langle \Lambda_{\hat{h}_i}, \Gamma_{\hat{h}_i}\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs, $\hat{h}_{\sigma(i)}$ be the largest in them, and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of $\hat{h}_i(i = 1, 2, 3, \ldots, n)$ with $w_i \geq 0(i = 1, 2, 3, \ldots, n)$ such that $w_i \in [0, 1]$ and $\Sigma^n_{i=1} w_i = 1$. Then Fermatean hesitant fuzzy ordered weighted geometric (FHFOWG) operator is a mapping $FHFOWG : FHFN^n \rightarrow FHFN$ can be defined by

$$\begin{aligned}
&FHFOWA(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_n) \\
&= \hat{h}^{w_1}_{\sigma(1)} \otimes \hat{h}^{w_2}_{\sigma(2)}\otimes, \ldots, \otimes\hat{h}^{w_n}_{\sigma(n)} \\
&= \langle \cup_{h_{\hat{h}_{\sigma(1)}}\in\Lambda_{\hat{h}_{\sigma(1)}}, h_{\hat{h}_{\sigma(2)}}\in\Lambda_{\hat{h}_{\sigma(2)}}, \ldots, h_{\hat{h}_{\sigma(n)}}\in\Lambda_{\hat{h}_{\sigma(n)}}} \\
&\times\{\prod_{i=1}^{n}(h_{\hat{h}_{\sigma(i)}})^{w_i}\}, \\
&\cup_{h'_{\hat{h}_{\sigma(1)}}\in\Gamma_{\hat{h}_{\sigma(1)}}, h'_{\hat{h}_{\sigma(2)}}\in\Gamma_{\hat{h}_{\sigma(2)}}, \ldots, h'_{\hat{h}_{\sigma(n)}}\in\Gamma_{\hat{h}_{\sigma(n)}}}
\end{aligned}$$

$$\{\sqrt[3]{1-\prod_{i=1}^{n}(1-h'^{3}_{\hat{h}_{\sigma(i)}})^{w_i}}\}\rangle.$$

B. FUZZY MEASURE AND CHOQUET INTEGRAL OPERATOR

Fuzzy measure, a non-additive measure originally proposed by Sugeno [24], has become an essential tool for solving Multiple Attribute Decision Making (MADM) problems. However, since decision criteria or a decision maker's preferences are often interdependent, traditional weighted arithmetic average operators are often inadequate for addressing this issue. The Choquet integral model [26], [27] has been shown to be effective in addressing this problem by allowing for interdependence among multiple criteria. In this subsection, we present the definitions of fuzzy measure, λ-fuzzy measure, and discrete Choquet integral as follows:

Definition 7 [24]: A fuzzy measure on X is a set function $\mu : P(X) \to [0, 1]$, satisfying the following conditions:

(1) $\mu(\emptyset) = 0, \mu(X) = 1$ (boundary conditions);
(2) If $A, B \in X$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ (monotonicity).

While adding the axiom of continuity is necessary when X is infinite, for practical purposes, it is sufficient to consider a finite universal set. In this context, $\mu(x_1, x_2, \ldots, x_n)$ can be interpreted as the degree of subjective importance of the decision criteria set $x_1, x_2, \ldots, x_n$. Furthermore, by assigning separate weights to each criterion, it is possible to define the weights of any combination of criteria. It is worth noting that some remarks pertain to any pair of criteria sets $A, B \in X$ with the condition $A \cap B \in \emptyset$:

(i) A and B are independent (without interaction) if $\mu(A \cup B) = \mu(A) + \mu(B)$. It is called an additive measure.
(ii) Positive interaction between A and B is exhibited if $\mu(A \cup B) > \mu(A) + \mu(B)$. It is called a super-additive measure.
(iii) Negative interaction between A and B is exhibited if $\mu(A \cup B) < \mu(A) + \mu(B)$. It is called a sub-additive measure.

Given the difficulty in determining the fuzzy measure as per Definition (2), Sugeno [24] proposed the following λ-fuzzy measure to establish a fuzzy measure in MAGDM problems:

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B), \quad \lambda \in [-1, \infty), A \cap B = \emptyset \tag{4}$$

λ plays a crucial role in determining the interaction between the criteria. Specifically, in Equation 6, when $\lambda = 0$, the λ-fuzzy measure is reduced to a simple additive measure. However, when λ is negative or positive, the λ-fuzzy measure is reduced to a sub-additive or super-additive measure, respectively. Furthermore, in cases where all elements in X are independent, and we have:

$$\mu(A) = \sum_{x_i \in A} \mu(\{x_i\}). \tag{5}$$

Definition 8 [24]: Let X be a finite set which satisfies $\bigcup_{i=1}^{n}\{x_i\} = X$ and $x_i \cap x_j = \emptyset$ for all $i, j = 1, 2, \ldots, n$ while $i \neq j$. If X_S is a subset of X, then

$$\mu(X_S) = \begin{cases} \frac{1}{\lambda}(\prod_{i=1}^{n}[1+\lambda\mu(x_i)] - 1) \text{ if } \lambda \neq 0 \\ \sum_{i=1}^{n}\mu(x_i) \text{ if } \lambda = 0 \end{cases} \tag{6}$$

especially when X_S contains only two elements, the equation is equal to (4).

According to the (6), if we let $X_S = X$, as $\mu(X) = 1$, we obtain

$$1 = \frac{1}{\lambda}(\prod_{i=1}^{n}[1+\lambda\mu(x_i)] - 1)$$
$$\Leftrightarrow \lambda + 1 = \prod_{i=1}^{n}[1+\lambda\mu(x_i)] \tag{7}$$

formula (7) shows that λ is uniquely determined by set X

Definition 9 [25]: Let f be a positive real-valued function on X and μ be a fuzzy measure on X. The discrete Choquet integral of f with respect to μ is defined by

$$C_\mu(f) = \sum_{i=1}^{n} f_{\sigma(i)[\mu(A_{\sigma(i)})-\mu(A_{\sigma(i-1)})]}, \tag{8}$$

where $\sigma(i)$ indicates a permutation on X such that $f_{\sigma(1)} \geq f_{\sigma(2)} \geq \cdots \geq f_{\sigma(n)}, A_{\sigma(i)} = \{1, 2, \ldots, i\}, A_{\sigma(i)} = \emptyset$.

III. FERMATEAN HESITANT FUZZY CHOQUET ORDERED AVERAGING/GEOMETRIC OPERATOR

In this section, we present Fermatean hesitant fuzzy Choquet ordered operators based on Fermatean hesitant fuzzy set and Choquet integral. Then their idempotency, boundedness and monotonicity are proved respectively.

Definition 10: Let $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then, the Fermatean hesitant fuzzy Choquet ordered averaging (FHFCOA) operator is a mapping $FHFCOA : FHFN^n \to FHFN$ can be defined by

$$FHFCOA(p_1, p_2, \ldots, p_n) = \bigoplus_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))p_{\sigma(i)} \tag{9}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $p_{\sigma(i-1)} \geq p_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \emptyset$.

Theorem 11: Let $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then, the aggregation result using FHFCOA operator is also a FHFN and

$$FHFCOA(p_1, p_2, \ldots, p_n)$$
$$= \bigoplus_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))p_{\sigma(i)}$$
$$= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}, \gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}, \ldots, \gamma_{\sigma(n)} \in \Gamma_{p_{\sigma(n)}}}$$

$$\{\sqrt[3]{1-\prod_{i=1}^{n}(1-\gamma_{\sigma(i)}^{3})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(n)}\in\Psi_{p_{\sigma(n)}}}$$
$$\{\prod_{i=1}^{n}(\tau_{\sigma(i)})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}\}\rangle \tag{10}$$

Proof: By mathematical induction we prove that Equation (10) holds for all n. For this first we show that Equation (10) holds for $n = 2$.

Since,

$$(\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)}))p_{\sigma(1)}$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}}}\{\sqrt[3]{1-(1-\gamma_{\sigma(1)}^{3})^{\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}}}\{(\tau_{\sigma(1)})^{\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)})}\}\rangle,$$

and

$$(\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)}))p_{\sigma(2)}$$
$$=\langle\cup_{\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}}}\{\sqrt[3]{1-(1-\gamma_{\sigma(2)}^{3})^{\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)})}}\},$$
$$\cup_{\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}}}\{(\tau_{\sigma(2)})^{\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)})}\}\rangle.$$

So,

$$FHFCOA(p_1,p_2)$$
$$=(\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)}))p_{\sigma(1)}$$
$$\oplus(\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)}))p_{\sigma(2)}$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}}}\{\sqrt[3]{1-(1-\gamma_{\sigma(1)}^{3})^{\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}}}\{(\tau_{\sigma(1)})^{\varphi(A_{\sigma(1)})-\varphi(A_{\sigma(0)})}\}\rangle$$
$$\oplus\langle\cup_{\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}}}\{\sqrt[3]{1-(1-\gamma_{\sigma(2)}^{3})^{\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)})}}\},$$
$$\cup_{\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}}}\{(\tau_{\sigma(2)})^{\varphi(A_{\sigma(2)})-\varphi(A_{\sigma(1)})}\}\rangle$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{3}(1-\gamma_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}}}$$
$$\{\prod_{i=1}^{3}(\tau_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}\rangle$$

Thus, the Equation (8) is hold for $n = 2$. Suppose the equation is hold for $n = k$, i.e.,

$$FHFCOA(p_1,p_2,\ldots,p_k)$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\ldots,\gamma_{\sigma(k)}\in\Gamma_{p_{\sigma(k)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{k}(1-\gamma_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k)}\in\Psi_{p_{\sigma(k)}}}$$
$$\{\prod_{i=1}^{k}(\tau_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}\rangle$$

We show that the equation is hold for $n = k + 1$, i.e.,

$$FHFCOA(p_1,p_2,\ldots,p_k,p_{k+1})$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\ldots,\gamma_{\sigma(k)}\in\Gamma_{p_{\sigma(k)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{k}(1-\gamma_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k)}\in\Psi_{p_{\sigma(k)}}}$$
$$\{\prod_{i=1}^{k}(\tau_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}\rangle$$
$$\oplus\langle\cup_{\gamma_{\sigma(k+1)}\in\Gamma_{p_{\sigma(k+1)}}}$$
$$\{\sqrt[3]{1-(1-\gamma_{\sigma(k+1)}^{3})^{\varphi(A_{\sigma(k+1)})-\varphi(A_{\sigma(k)})}}\},$$
$$\cup_{\tau_{\sigma(k+1)}\in\Psi_{p_{\sigma(k+1)}}}\{(\tau_{\sigma(k+1)})^{\varphi(A_{\sigma(k+1)})-\varphi(A_{\sigma(k)})}\}\rangle$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\ldots,\gamma_{\sigma(k+1)}\in\Gamma_{p_{\sigma(k+1)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{k+1}(1-\gamma_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k+1)}\in\Psi_{p_{\sigma(k+1)}}}$$
$$\{\prod_{i=1}^{k+1}(\tau_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}\rangle$$

Hence the equation is hold for $n = k + 1$. Therefore, the equation is hold for all n.

Example 12: Let φ be a fuzzy measure on $X, X = \{x_1, x_2, x_3\}$ in which

$\varphi(\{x_1\}) = 0.5$, $\varphi(\{x_2\}) = 0.2$, $\varphi(\{x_3\}) = 0.3$.

Parameter $\lambda = 0.5$ is obtained using Equation (4), and the following are obtained. $\varphi(\{x_1, x_2\}) = 0.75$, $\varphi(\{x_1, x_3\}) = 0.875$, $\varphi(\{x_2, x_3\}) = 0.53$, $\varphi(\{x_1, x_2, x_3\}) = 1$.

Suppose there are three experts who invited to evaluate some decision alternatives. The evaluation of the expert is denoted by FHFNs. $p_1 = \langle\{0.5, 0.6, 0.9\}, \{0.3, 0.6\}\rangle$, $p_2 = \langle\{0.3, 0.4, 0.7, 0.9\}, \{0.2, 0.8, 0.9\}\rangle$, $p_3 = \langle\{0.3, 0.6, 0.8\}, \{0.2, 0.7, 0.9\}\rangle$, To calculate the comprehensive evaluation of the three experts on the decision alternation through using the FHFCOA operator, we have

$$FHFCOA(p_1,p_2,p_3)$$
$$=\langle\cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\gamma_{\sigma(3)}\in\Gamma_{p_{\sigma(3)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{3}(1-\gamma_{\sigma(i)}^{3})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\tau_{\sigma(3)}\in\Psi_{p_{\sigma(3)}}}$$
$$\{\prod_{i=1}^{3}(\tau_{\sigma(i)})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}\}\rangle$$

First, we calculate the score functions of p_1, p_2 and p_3 by equation (2). For this, we have

$$S(p_1) = 0.205,\ S(p_2) = -0.064,\ S(p_3) = -0.034.$$

Thus, $S(p_1) > S(p_3) > S(p_2)$. Then we rearrange the three FHFNs in descending order as follows,

$$p_{\sigma(1)} = p_1 = \langle\{0.5, 0.6, 0.9\}, \{0.3, 0.6\}\rangle,$$
$$p_{\sigma(2)} = p_3 = \langle\{0.3, 0.4, 0.7, 0.9\}, \{0.2, 0.8, 0.9\}\rangle,$$
$$p_{\sigma(3)} = p_2 = \langle\{0.3, 0.6, 0.8\}, \{0.2, 0.7, 0.9\}\rangle.$$

Now,

$$\begin{aligned} &FHFCOA(p_1, p_2, p_3) \\ &= \langle\{0.426, 0.467, 0.5355, 0.4492, 0.4861, 0.5493, \\ &\quad 0.5883, 0.6081, 0.6459, 0.754, 0.7628, 0.7807, 0.5021, \\ &\quad 0.5312, 0.5836, 0.5183, 0.5453, 0.5947, 0.6267, 0.6433, \\ &\quad 0.6757, 0.7715, 0.7795, 0.7958, 0.7865, 0.7938, 0.8087, \\ &\quad 0.7905, 0.7976, 0.8122, 0.8227, 0.8285, 0.8403, 0.8802, \\ &\quad 0.8839, 0.8914\}, \{0.2449, 0.2865, 0.2956, 0.412, 0.4818, \\ &\quad 0.4972, 0.4306, 0.5035, 0.5196, 0.3464, 0.4051, 0.4181, \\ &\quad 0.5826, 0.6814, 0.7031, 0.6089, 0.7121, 0.7348\}\rangle \end{aligned}$$

Theorem 13: Let $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then

(1) (Idempotency) If all $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle (i = 1, 2, 3, \ldots, n)$ are equal, i.e. $p_i (i = 1, 2, 3, \ldots, n) = p$, then

$$FHFCOA(p_1, p_2, \ldots, p_n) = p. \tag{11}$$

(2) (Boundedness)

$$p^- \leq FHFCOA(p_1, p_2, \ldots, p_n) \leq p^+, \tag{12}$$

where $p^- = \langle\cup_{\gamma_i \in \Gamma_{p_i}} \min_i\{\gamma_i\}, \cup_{\tau_i \in \Psi_{p_i}} \max_i\{\tau_i\}\rangle$, $p^+ = \langle\cup_{\gamma_i \in \Gamma_{p_i}} \max_i\{\gamma_i\}, \cup_{\tau_i \in \Psi_{p_i}} \min_i\{\tau_i\}\rangle$.

(3) (Monotonicity) If $p_i > p_i^*$, then

$$\begin{aligned} &FHFCOA(p_1, p_2, \ldots, p_n,) \\ &\quad \leq FHFCOA(p_1^*, p_2^*, \ldots, p_n^*,). \end{aligned} \tag{13}$$

Proof: (1) By Theorem (11), we have

$$\begin{aligned} &FHFCOA(p_1, p_2, \ldots, p_n) \\ &= \langle\cup_{\gamma_{p_i} \in \Gamma_{p_i}}\{\sqrt[3]{1 - \prod_{i=1}^{n}(1 - \gamma_{p_i}^3)^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}}\}, \\ &\quad \cup_{\tau_{p_i} \in \Psi_{p_i}}\{\prod_{i=1}^{n}(\tau_{p_i})^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}\}\rangle \\ &= \langle\cup_{\gamma_p \in \Gamma_p}\{\sqrt[3]{1 - \prod_{i=1}^{n}(1 - \gamma_p^3)^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}}\}, \\ &\quad \cup_{\tau_p \in \Psi_p}\{\prod_{i=1}^{n}(\tau_p)^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}\}\rangle \\ &= \langle\cup_{\gamma_p \in \Gamma_p}\{\sqrt[3]{1 - (1 - \gamma_p^3)^{\Sigma_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}}\}, \\ &\quad \cup_{\tau_p \in \Psi_p}\{(\tau_p)^{\Sigma_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}\}\rangle \end{aligned}$$

Because of

$$\begin{aligned} &\Sigma_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})) \\ &= \varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)}) + \varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)}) \\ &\quad + \varphi(A_{\sigma(3)}) - \varphi(A_{\sigma(2)}) + \ldots + \varphi(A_{\sigma(n-1)}) \\ &\quad - \varphi(A_{\sigma(n-2)}) + \varphi(A_{\sigma(n)}) - \varphi(A_{\sigma(n-1)}) \\ &= \varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i)}) = 1 - 0 = 1 \end{aligned}$$

So

$$\begin{aligned} &FHFCOA(p_1, p_2, \ldots, p_n) \\ &\quad = \langle\cup_{\gamma_p \in \Gamma_p}\{\sqrt[3]{1 - (1 - \gamma_p^3)}\}, \cup_{\tau_p \in \Psi_p}\{\tau_p\}\rangle \\ &\quad = \langle\cup_{\gamma_p \in \Gamma_p}\{\gamma_p\}, \cup_{\tau_p \in \Psi_p}\{\tau_p\}\rangle = p \end{aligned}$$

(2) As

$$\cup_{\gamma_i \in \Gamma_{p_i}} \min_i\{\gamma_i\} \leq \cup_{\gamma_i \in \Gamma_{p_i}}\{\gamma_i\} \leq \cup_{\gamma_i \in \Gamma_{p_i}} \max_i\{\gamma_i\}, \tag{14}$$

and

$$\cup_{\tau_i \in \Psi_{p_i}} \min_i\{\tau_i\} \leq \cup_{\tau_i \in \Psi_{p_i}}\{\tau_i\} \leq \cup_{\tau_i \in \Psi_{p_i}} \max_i\{\tau_i\}, \tag{15}$$

Thus, from Equation (14), we have

$$\begin{aligned} &\cup_{\gamma_i \in \Gamma_{p_i}} \min_i\{\gamma_i\} \leq \cup_{\gamma_i \in \Gamma_{p_i}}\{\gamma_i\} \leq \cup_{\gamma_i \in \Gamma_{p_i}} \max_i\{\gamma_i\} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\min_i\{(\gamma_i)^3\}} \leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\{(\gamma_i)^3\}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\max_i\{(\gamma_i)^3\}} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1 - \max_i\{(\gamma_i)^3\}} \leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1 - \{(\gamma_i)^3\}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1 - \min_i\{(\gamma_i)^3\}} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \max_i\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \min_i\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1 - \max_i\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1 - \{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1 - \min_i\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \max_i\{(\gamma_i)^3\})^{\Sigma_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1 - \{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}} \\ &\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \min_i\{(\gamma_i)^3\})^{\Sigma_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}} \\ &\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1 - \max_i\{(\gamma_i)^3\})} \end{aligned}$$

$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1-\min_i\{(\gamma_i)^3\})}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(-1+\min_i\{(\gamma_i)^3\})}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(-1+\max_i\{(\gamma_i)^3\})}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1-1+\min_i\{(\gamma_i)^3\})}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1-\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1-1+\max_i\{(\gamma_i)^3\})}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\min_i\{(\gamma_i)^3\}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1-\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\max_i\{(\gamma_i)^3\}}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \min_i\{\gamma_i\}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1-\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i \in \Gamma_{p_i}} \max_i\{\gamma_i\}$$

Then, from Equation (15), we have

$$\cup_{\tau_i \in \Psi_{p_i}} \min_i\{\tau_i\} \leq \cup_{\tau_i \in \Psi_{p_i}}\{\tau_i\} \leq \cup_{\tau_i \in \Psi_{p_i}} \max_i\{\tau_i\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}} \min_i\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}}\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}} \max_i\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}} \prod_{i=1}^{n} \min_i\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}} \prod_{i=1}^{n}\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}} \prod_{i=1}^{n} \max_i\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}} \min_i\{(\tau_i)^{\sum_{i=1}^{n}(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}} \prod_{i=1}^{n}\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\leq \cup_{\tau_i \in \Psi_{p_i}} \max_i\{(\tau_i)^{\sum_{i=1}^{n}(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}} \min_i\{\tau_i\} \leq \cup_{\tau_i \in \Psi_{p_i}} \prod_{i=1}^{n} \{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\} \leq \cup_{\tau_i \in \Psi_{p_i}} \max_i\{\tau_i\}$$

According to the score function, we have

$$FHFCOA(p_1, p_2, \ldots, p_n) \geq p^-$$

if and only if $p^- = FHFCOA(p)$.

Similarly, $FHFCOA(p_1, p_2, \ldots, p_n) \leq p^+$ is with equality if and only if $FHFCOA(p) = p^+$. Hence, $p^- \leq FHFCOA(p_1, p_2, \ldots, p_n) \leq p^+$.

(3) If $p_i > p_i^*$, we have $\Gamma_{p_i} \leq \Gamma_{p_i^*}$ and $\Psi_{p_i} \geq \Psi_{p_i^*}$.

If $\Gamma_{p_i} \leq \Gamma_{p_i^*}$, then

$$\cup_{\gamma_i \in \Gamma_{p_i}}\{\gamma_i\} \leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}}\{\gamma_i^*\}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}}\{(\gamma_i)^3\} \leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}}\{(\gamma_i^*)^3\}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\{(\gamma_i)^3\}} \leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}} \sqrt[3]{\{(\gamma_i^*)^3\}}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1-\{(\gamma_i)^3\}} \leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}} \sqrt[3]{1-\{(\gamma_i^*)^3\}}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}} \sqrt[3]{(1-\{(\gamma_i^*)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}} \sqrt[3]{\prod_{i=1}^{n}(1-\{(\gamma_i^*)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\Leftrightarrow \cup_{\gamma_i \in \Gamma_{p_i}} \sqrt[3]{1-\prod_{i=1}^{n}(1-\{(\gamma_i)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}$$
$$\leq \cup_{\gamma_i^* \in \Gamma_{p_i^*}} \sqrt[3]{1-\prod_{i=1}^{n}(1-\{(\gamma_i^*)^3\})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}} \quad (16)$$

Now, if $\Psi_{p_i} \geq \Psi_{p_i^*}$, then

$$\cup_{\tau_i \in \Psi_{p_i}}\{\tau_i\} \geq \cup_{\tau_i^* \in \Psi_{p_i^*}}\{\tau_i^*\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}}\{(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\geq \cup_{\tau_i^* \in \Psi_{p_i^*}}\{(\tau_i^*)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\Leftrightarrow \cup_{\tau_i \in \Psi_{p_i}}\{\prod_{i=1}^{n}(\tau_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\}$$
$$\geq \cup_{\tau_i^* \in \Psi_{p_i^*}}\{\prod_{i=1}^{n}(\tau_i^*)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\} \quad (17)$$

Let $p = FHFCOA(p_1, p_2, \ldots, p_n)$ and $p^* = FHFCOA(p_1^*, p_2^*, \ldots, p_n^*)$. Then, from Equation (14) and (15), we have $S(p) \leq S(p^*)$.

If $S(p) < S(p^*)$, then $FHFCOA(p_1, p_2, \ldots, p_n < FHFCOA(p_1^*, p_2^*, \ldots, p_n^*)$. If $S(p) = S(p^*)$, then

$$(\frac{1}{l_{\gamma_p \in \Gamma_p}} \sum_{\gamma_p \in \Gamma_p} \gamma_p)^3 - (\frac{1}{l_{\tau_p \in \Psi_p}} \sum_{\tau_p \in \Psi_p} \tau_p)^3$$
$$= (\frac{1}{l_{\gamma_{p^*}^* \in \Gamma_{p^*}}} \sum_{\gamma_{p^*}^* \in \Gamma_{p^*}} \gamma_{p^*}^*)^3 - (\frac{1}{l_{\tau_{p^*}^* \in \Psi_{p^*}}} \sum_{\tau_{p^*}^* \in \Psi_{p^*}} \tau_{p^*}^*)^3$$
$$\Leftrightarrow (\frac{1}{l_{\gamma_p \in \Gamma_p}} \sum_{\gamma_p \in \Gamma_p} \gamma_p)^3 = (\frac{1}{l_{\gamma_{p^*}^* \in \Gamma_{p^*}}} \sum_{\gamma_{p^*}^* \in \Gamma_{p^*}} \gamma_{p^*}^*)^3 \quad \text{and}$$
$$(\frac{1}{l_{\tau_p \in \Psi_p}} \sum_{\tau_p \in \Psi_p} \tau_p)^3 = (\frac{1}{l_{\tau_{p^*}^* \in \Psi_{p^*}}} \sum_{\tau_{p^*}^* \in \Psi_{p^*}} \tau_{p^*}^*)^3$$
$$\Leftrightarrow \frac{1}{l_{\gamma_p \in \Gamma_p}} \sum_{\gamma_p \in \Gamma_p} \gamma_p = \frac{1}{l_{\gamma_{p^*}^* \in \Gamma_{p^*}}} \sum_{\gamma_{p^*}^* \in \Gamma_{p^*}} \gamma_{p^*}^* \quad \text{and}$$
$$\frac{1}{l_{\tau_p \in \Psi_p}} \sum_{\tau_p \in \Psi_p} \tau_p = \frac{1}{l_{\tau_{p^*}^* \in \Psi_{p^*}}} \sum_{\tau_{p^*}^* \in \Psi_{p^*}} \tau_{p^*}^*$$

As

$$\bar{\sigma}(p)$$
$$= (\frac{1}{l_{\gamma_p \in \Gamma_p}} \sum_{\gamma_p \in \Gamma_p} \gamma_p - S(p))^3 + (\frac{1}{l_{\tau_p \in \Psi_p}} \sum_{\tau_p \in \Psi_p} \tau_p - S(p))^3$$
$$= (\frac{1}{l_{\gamma_{p^*}^* \in \Gamma_{p^*}}} \sum_{\gamma_{p^*}^* \in \Gamma_{p^*}} \gamma_{p^*}^* - S(p^*))^3$$
$$-(\frac{1}{l_{\tau_{p^*}^* \in \Psi_{p^*}}} \sum_{\tau_{p^*}^* \in \Psi_{p^*}} \tau_{p^*}^* - S(p^*))^3$$
$$= \bar{\sigma}(p^*),$$

therefor,

$$FHFCOA(p_1, p_2, \ldots, p_n) = FHFCOA(p_1^*, p_2^*, \ldots, p_n^*).$$

So, if $p_i > p_i^*$, then

$$FHFCOA(p_1, p_2, \ldots, p_n,) \leq FHFCOA(p_1^*, p_2^*, \ldots, p_n^*,).$$

Definition 14: Let $p_i = \langle \Gamma_{p_i}, \Psi_{p_i} \rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then, the Fermatean hesitant fuzzy choquet ordered geometric (FHFCOG) operator is a mapping $FHFCOG : FHFN^n \rightarrow FHFN$ can be defined by

$$FHFCOG(p_1, p_2, \ldots, p_n) = \bigotimes_{i=1}^{n} (p_{\sigma(i)})^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))} \tag{18}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ is a permutation of $(1, 2, \ldots, n)$, such that $p_{\sigma(i-1)} \geq p_{\sigma(i)}$ for all $i = 1, 2, \ldots, n$, $A_{\sigma(k)} = \{x_{\sigma(i)} | i \leq k\}$ for $k \geq 1$ and $A_{\sigma(0)} = \emptyset$.

Theorem 15: Let $p_i = \langle \Gamma_{p_i}, \Psi_{p_i} \rangle (i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then, the aggregation result using FHFCOG operator is also a FHFN and

$$\begin{aligned} &FHFCOG(p_1, p_2, \ldots, p_n) \\ &= \bigotimes_{i=1}^{n} (p_{\sigma(i)})^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))} \\ &= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}, \gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}, \ldots, \gamma_{\sigma(n)} \in \Gamma_{p_{\sigma(n)}}} \\ &\{\prod_{i=1}^{n} (\gamma_{\sigma(i)})^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}\}, \\ &\cup_{\tau_{\sigma(1)} \in \Psi_{p_{\sigma(1)}}, \tau_{\sigma(2)} \in \Psi_{p_{\sigma(2)}}, \ldots, \tau_{\sigma(n)} \in \Psi_{p_{\sigma(n)}}} \\ &\{\sqrt[3]{1 - \prod_{i=1}^{n} (1 - \tau_{\sigma(i)}^3)^{(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))}}\}\rangle \end{aligned} \tag{19}$$

Proof: By mathematical induction we prove that Equation (19) holds for all n. For this first we show that Equation (19) holds for $n = 2$.

Since,

$$\begin{aligned} &(p_{\sigma(1)})^{(\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)}))} \\ &= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}} \{(\gamma_{\sigma(1)})^{\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)})}\}, \\ &\cup_{\tau_{\sigma(1)} \in \Psi_{p_{\sigma(1)}}} \{\sqrt[3]{1 - (1 - \tau_{\sigma(1)}^3)^{\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)})}}\}\rangle, \end{aligned}$$

and

$$\begin{aligned} &(p_{\sigma(2)})^{(\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)}))} \\ &= \langle \cup_{\gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}} \{(\gamma_{\sigma(2)})^{\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)})}\}, \\ &\cup_{\tau_{\sigma(2)} \in \Psi_{p_{\sigma(2)}}} \{\sqrt[3]{1 - (1 - \tau_{\sigma(2)}^3)^{\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)})}}\}\rangle, \end{aligned}$$

So,

$$\begin{aligned} &FHFCOG(p_1, p_2) \\ &= (p_{\sigma(1)})^{(\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)}))} \otimes (p_{\sigma(2)})^{(\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)}))} \\ &= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}} \{(\gamma_{\sigma(1)})^{\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)})}\}, \\ &\cup_{\tau_{\sigma(1)} \in \Psi_{p_{\sigma(1)}}} \{\sqrt[3]{1 - (1 - \tau_{\sigma(1)}^3)^{\varphi(A_{\sigma(1)}) - \varphi(A_{\sigma(0)})}}\}\rangle \\ &\otimes \langle \cup_{\gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}} \{(\gamma_{\sigma(2)})^{\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)})}\}, \\ &\cup_{\tau_{\sigma(2)} \in \Psi_{p_{\sigma(2)}}} \{\sqrt[3]{1 - (1 - \tau_{\sigma(2)}^3)^{\varphi(A_{\sigma(2)}) - \varphi(A_{\sigma(1)})}}\}\rangle \\ &= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}, \gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}} \\ &\{\prod_{i=1}^{3} (\gamma_{\sigma(i)})^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}\}, \\ &\cup_{\tau_{\sigma(1)} \in \Psi_{p_{\sigma(1)}}, \tau_{\sigma(2)} \in \Psi_{p_{\sigma(2)}}} \\ &\{\sqrt[3]{1 - \prod_{i=1}^{3} (1 - \tau_{\sigma(i)}^3)^{\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})}}\}\rangle \end{aligned}$$

Thus, the Equation (19) is hold for $n = 2$. Suppose the equation is hold for $n = k$, i.e.,

$$\begin{aligned} &FHFCOG(p_1, p_2, \ldots, p_k) \\ &= \langle \cup_{\gamma_{\sigma(1)} \in \Gamma_{p_{\sigma(1)}}, \gamma_{\sigma(2)} \in \Gamma_{p_{\sigma(2)}}, \ldots, \gamma_{\sigma(k)} \in \Gamma_{p_{\sigma(k)}}} \end{aligned}$$

$$\{\prod_{i=1}^{k}(\gamma_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\},$$
$$\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k)}\in\Psi_{p_{\sigma(k)}}}$$
$$\{\sqrt[3]{1-\prod_{i=1}^{k}(1-\tau_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\}\rangle$$

We show that the equation is hold for $n = k + 1$, i.e.,

$$\begin{aligned}
&FHFCOG(p_1, p_2, \ldots, p_k, p_{k+1})\\
&= \langle \cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\ldots,\gamma_{\sigma(k)}\in\Gamma_{p_{\sigma(k)}}}\\
&\{\prod_{i=1}^{k}(\gamma_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\},\\
&\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k)}\in\Psi_{p_{\sigma(k)}}}\\
&\{\sqrt[3]{1-\prod_{i=1}^{k}(1-\tau_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\}\rangle\\
&\otimes\langle\cup_{\gamma_{\sigma(k+1)}\in\Gamma_{p_{\sigma(k+1)}}}\{(\gamma_{\sigma(k+1)})^{\varphi(A_{\sigma(k+1)})-\varphi(A_{\sigma(k)})}\},\\
&\cup_{\tau_{\sigma(k+1)}\in\Psi_{p_{\sigma(k+1)}}}\\
&\{\sqrt[3]{1-(1-\tau_{\sigma(k+1)}^{3})^{\varphi(A_{\sigma(k+1)})-\varphi(A_{\sigma(k)})}}\}\rangle\\
&= \langle \cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\ldots,\gamma_{\sigma(k+1)}\in\Gamma_{p_{\sigma(k+1)}}}\\
&\{\prod_{i=1}^{k+1}(\gamma_{\sigma(i)})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}\},\\
&\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\ldots,\tau_{\sigma(k+1)}\in\Psi_{p_{\sigma(k+1)}}}\\
&\{\sqrt[3]{1-\prod_{i=1}^{k+1}(1-\tau_{\sigma(i)}^{3})^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})}}\}\rangle
\end{aligned}$$

Hence the equation is hold for $n = k + 1$. Therefore, the equation is hold for all n.

Example 16: [Continued Example (12)] To calculate the comprehensive evaluation of the three experts on the decision alternation through using the FHFCOG operator, we have

$$\begin{aligned}
&FHFCOG(p_1, p_2, p_3)\\
&= \langle \cup_{\gamma_{\sigma(1)}\in\Gamma_{p_{\sigma(1)}},\gamma_{\sigma(2)}\in\Gamma_{p_{\sigma(2)}},\gamma_{\sigma(3)}\in\Gamma_{p_{\sigma(3)}}}\\
&\{\prod_{i=1}^{3}(\gamma_{\sigma(i)})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}\},\\
&\cup_{\tau_{\sigma(1)}\in\Psi_{p_{\sigma(1)}},\tau_{\sigma(2)}\in\Psi_{p_{\sigma(2)}},\tau_{\sigma(3)}\in\Psi_{p_{\sigma(3)}}}\\
&\{\sqrt[3]{1-\prod_{i=1}^{3}(1-\tau_{\sigma(i)}^{3})^{(\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)}))}}\}\rangle
\end{aligned}$$

By Example (12), we know $S(p_1) > S(p_3) > S(p_2)$, and

$$p_{\sigma(1)} = p_1, p_{\sigma(2)} = p_3, p_{\sigma(3)} = p_2.$$

Thus, we have

$$\begin{aligned}
&FHFCOG(p_1, p_2, p_3)\\
&= \langle\{0.3873, 0.4224, 0.4378, 0.4314, 0.4705,\\
&0.4877, 0.5322, 0.5803, 0.6016, 0.5847, 0.6377, 0.661,\\
&0.4243, 0.4627, 0.4796, 0.4726, 0.5154, 0.5342, 0.5829,\\
&0.6357, 0.659, 0.6406, 0.6985, 0.7241, 0.5196, 0.5666,\\
&0.5874, 0.5788, 0.6312, 0.6543, 0.714, 0.7786, 0.8071,\\
&0.7845, 0.8555, 0.8868\}, \{0.2599, 0.4059, 0.5481,\\
&0.6275, 0.658, 0.7112, 0.7344, 0.7527, 0.7865, 0.4906,\\
&0.5456, 0.6301, 0.6869, 0.7101, 0.752, 0.7708, 0.7857,\\
&0.8139, 0.8070\}\rangle.
\end{aligned}$$

Theorem 17: Let $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle(i = 1, 2, 3, \ldots, n)$ be a collection of all FHFNs on X, and φ is a fuzzy measure on X. Then

(1) (Idempotency) If all $p_i = \langle\Gamma_{p_i}, \Psi_{p_i}\rangle(i = 1, 2, 3, \ldots, n)$ are equal, i.e. $p_i(i = 1, 2, 3, \ldots, n) = p$, then

$$FHFCOG(p_1, p_2, \ldots, p_n) = p. \tag{20}$$

(2) (Boundedness)

$$p^- \leq FHFCOG(p_1, p_2, \ldots, p_n) \leq p^+, \tag{21}$$

where $p^- = \langle\cup_{\gamma_i\in\Gamma_{p_i}} \min_i\{\gamma_i\}, \cup_{\tau_i\in\Psi_{p_i}} \max_i\{\tau_i\}\rangle, p^+ = \langle\cup_{\gamma_i\in\Gamma_{p_i}} \max_i\{\gamma_i\}, \cup_{\tau_i\in\Psi_{p_i}} \min_i\{\tau_i\}\rangle$.

(3) (Monotonicity) If $p_i > p_i^*$, then

$$\begin{aligned}&FHFCOG(p_1, p_2, \ldots, p_n,)\\ &\quad\leq FHFCOG(p_1^*, p_2^*, \ldots, p_n^*,).\end{aligned} \tag{22}$$

Proof: Proof of this theorem is similarly as the proof of Theorem (13).

Lemma 18: [34] Let φ be a fuzzy measure, $A \in X$, and $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ is a permutation of $\{1, 2, \ldots, n\}$, then

$$\sum_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})) = 1.$$

Lemma 19: Let $p_i > 0$, $\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}) > 0(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})) = 1$. Then

$$\prod_{i=1}^{n}(p_i)^{\varphi(A_{\sigma(i)})-\varphi(A_{\sigma(i-1)})} \leq \sum_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}))p_i,$$

where the equality holds if and only if $p_1 = p_2 = p_3 = \ldots = p_n$.

Theorem 20: Let $p_i =< \Gamma_{p_i}, \Psi_{p_i} > (i = 1, 2, \ldots, n)$ be a collection of FHFNs. Then

$$FHFCOG(p_1, p_1, \ldots, p_n) \leq FHFCOA(p_1, p_2, \ldots, p_n),$$

where $\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)}) > 0(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n}(\varphi(A_{\sigma(i)}) - \varphi(A_{\sigma(i-1)})) = 1$.

Proof: We can easily prove this Theorem based on Lemma (19).

IV. MULTI-ATTRIBUTE DECISION MAKING WITH FERMATEAN HESITANT FUZZY INFORMATION

In multi-attribute decision making (MADM) problems, multiple decision experts and multiple decision criteria are often needed to obtain optimal decision results. However, due to the subjectivity of decision experts and the interdependence of decision criteria, the decision results will be affected. The Fermatean hesitant fuzzy Choquet integral aggregation operators proposed in this paper can solve this problem well.

Consider a MADM with anonymity where there is a discrete set of m alternatives $X = \{x_1, x_2, \ldots, x_m\}$. Suppose $D = \{d_1, d_2, \ldots, d_l\}$ be a set of l decision makers (DMs) that have the important degree of $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_l\}$, and $\lambda_k, (k = 1, 2, \ldots, l)$ is a fuzzy number which is not required that $\Sigma_{k=1}^{n} \lambda_k = 1$. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a collection of n attributes. To evaluate the performance of the alternative x_i under the attributes A_j, the decision maker is required to provide not only the information that the alternative x_i satisfies the attributes A_j, but also the information that the alternative x_i does not satisfy the attributes A_j. This two-part information can be expressed by Γ_{ij} and Ψ_{ij}, which denote the degrees that the alternative x_i satisfy the criterion A_j and does not satisfy the criterion A_j, then the performance of the alternative x_i under the criteria A_j can be expressed by a FHFN $p_{ij} = \langle \Gamma_{ij}, \Psi_{ij} \rangle$ with the condition that for all $\gamma_{ij} \in \Gamma_{ij}, \exists \tau_{ij} \in \Psi_{ij}$ such that $0 \leq (\gamma_{ij})^3 + (\tau_{ij})^3 \leq 1$, and for all $\tau_{ij} \in \Psi_{ij}, \exists \gamma_{ij} \in \Gamma_{ij}$ such that $0 \leq (\gamma_{ij})^3 + (\tau_{ij})^3 \leq 1, (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$. To obtain the ranking of the alternatives, We give the steps as follows.

Step 1. In this step, we construct the Fermatean hesitant fuzzy decision matrices $C = (p_{ij})_{m\times n}$ for the decision where $p_{ij} = \langle \Gamma_{ij}, \Psi_{ij} \rangle (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$.

If the attribute has two types, such as cost and benefit attributes, then the Fermatean hesitant decision matrix can be converted into the normalized Fermatean hesitant fuzzy decision matrix

$$D_N = (\kappa_{ij})_{m\times n},$$
$$\text{where} \kappa_{ij} = \begin{cases} p_{ij} & \text{if the attribute is of benefit type} \\ p_{ij}^c & \text{if the attribute is of cost type} \end{cases}$$

where $p_{ij}^c = \langle \Psi_{ij}, \Gamma_{ij} \rangle (i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$. If all the attributes have the same type, then there is no need to normalize the decision matrix.

Step 2. Reorder the $p_{ij} (j = 1, 2, \ldots, n)$ for each alternative $x_i (i = 1, 2, \ldots, m)$ in a descending order by Equation (2) or (3).

Step 3. Confirm the fuzzy measure of attribute sets of A. We take Equation (4) for determining the fuzzy measure.

Step 4. Utilize the proposed aggregation operators (Equation (8) or (17)) to obtain the FHFNs $p_i (i = 1, 2, \ldots, m)$ for the alternatives x_i, that is the proposed operators to derive the collective overall preference values $p_i (i = 1, 2, \ldots, m)$ of the alternatives x_i.

Step 5. By using Equation (2), we calculate the scores $S(p_i)(i = 1, 2, \ldots, m)$ and the deviation degree $\bar{\sigma}(p_i)(i = 1, 2, \ldots, m)$ of all the overall values $p_i (i = 1, 2, \ldots, m)$.

Step 6. Rank all the alternatives $x_i (i = 1, 2, \ldots, m)$ and then select the most preferred alternative.

Step 7. End.

V. ILLUSTRATIVE EXAMPLE

In this section, we utilize the aggregation operators proposed in this paper to solve a practical problem. First, we employ data from [16] to address medical decision-making in the context of Fermatean hesitant fuzzy conditions. We conduct a sensitivity analysis on the parameter λ which determines whether the problem is super-additive or sub-additive. Additionally, we compare the effectiveness of our proposed method with existing methods in Fermatean fuzzy conditions.

A. PRACTICAL CALCULATION

Example 21: [16] Let the diseases be given by the set $D_i (i = 1, 2, 3, 4, 5) =$ {viral fever, malaria, typhoid, stomach problem, chest problem}, which are available for selection, and the symptoms by the set $S_i (i = 1, 2, 3, 4) =$ {temperature, headache, stomach pain, cough}.

The Fermatean hesitant fuzzy decision matrix C is given in Table (1).

Then, we utilize the developed method to get the most suitable disease.

Step 1. We know that, the Fermatean hesitant fuzzy decision matrix $C = (p_{ij})_{5\times 4}$ for the decision where $p_{ij} = \langle \Gamma_{ij}, \Psi_{ij} \rangle (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ in Table (1). No normalization is performed due to the identical measurement.

Step 2. By Equation (2), we have

$$S(p_{11}) = 0.2555, \quad S(p_{12}) = -0.4480,$$
$$S(p_{13}) = -0.1270, \quad S(p_{14}) = 0.5230.$$

Thus, $S(p_{14}) > S(p_{11}) > S(p_{13}) > S(p_{12})$. So, we can reorder $p_{11}, p_{12}, p_{13}, p_{14}$ as follows,

$$p_{1\sigma(1)} = p_{14}, \quad p_{1\sigma(2)} = p_{11}, \quad p_{1\sigma(3)} = p_{13}, \quad p_{1\sigma(4)} = p_{12}.$$

Similarly, we can reorder $p_{i1}, p_{i2}, p_{i3}, p_{i4} (i = 2, 3, 4, 5)$ as follows,

$$p_{2\sigma(1)} = p_{23}, p_{2\sigma(2)} = p_{24}, \quad p_{2\sigma(3)} = p_{22}, p_{2\sigma(4)} = p_{21}.$$
$$p_{3\sigma(1)} = p_{32}, p_{3\sigma(2)} = p_{33}, \quad p_{3\sigma(3)} = p_{31}, p_{3\sigma(4)} = p_{34}.$$
$$p_{4\sigma(1)} = p_{41}, p_{4\sigma(2)} = p_{42}, \quad p_{4\sigma(3)} = p_{43}, p_{4\sigma(4)} = p_{44}.$$
$$p_{5\sigma(1)} = p_{54}, p_{5\sigma(2)} = p_{52}, \quad p_{5\sigma(3)} = p_{53}, p_{5\sigma(4)} = p_{51}.$$

Step 3. Kirişci defined the weight vector $\omega = 0.15, 0.27, 0.33, 0.22$, whereas the interdependence between factors is not defined. We make a supplement using λ and observing the impact of different values of lambda on the decision result. Let φ be a fuzzy measure on $D_i, (i = 1, 2, 3, 4, 5)$ in which $\varphi(\{d_1\}) = 0.15$, $\varphi(\{d_2\}) = 0.27$, $\varphi(\{d_3\}) = 0.33$, $\varphi(\{d_4\}) = 0.22$.

We calculate by (7) and the parameter $\lambda = 0.09$ is obtained. The followings are obtained using Equation (4).

TABLE 1. Fermatean hesitant fuzzy decision matrix C.

	S_1	S_2	S_3	S_4
D_1	$\langle\{0.8,0.7\},\{0.6,0.5\}\rangle$	$\langle\{0.3,0.5\},\{0.7,0.9\}\rangle$	$\langle\{0.6\},\{0.6,0.7,0.8\}\rangle$	$\langle\{0.8,0.9\},\{0.4,0.5\}\rangle$
D_2	$\langle\{0.6,0.5,0.7\},\{0.8\}\rangle$	$\langle\{0.5,0.6\},\{0.6,0.7\}\rangle$	$\langle\{0.8,0.9\},\{0.4,0.5\}\rangle$	$\langle\{0.7,0.8\},\{0.4,0.6\}\rangle$
D_3	$\langle\{0.6,0.7\},\{0.6,0.8,0.9\}\rangle$	$\langle\{0.6,0.7,0.9\},\{0.3,0.4\}\rangle$	$\langle\{0.5,0.7\},\{0.5,0.6\}\rangle$	$\langle\{0.4,0.5\},\{0.7,0.9\}\rangle$
D_4	$\langle\{0.9\},\{0.3,0.4,0.6\}\rangle$	$\langle\{0.6,0.7\},\{0.6\}\rangle$	$\langle\{0.5,0.6,0.4\},\{0.7,0.8\}\rangle$	$\langle\{0.6,0.5\},\{0.8\}\rangle$
D_5	$\langle\{0.3,0.4\},\{0.8,0.9\}\rangle$	$\langle\{0.7,0.8\},\{0.5,0.7\}\rangle$	$\langle\{0.4,0.6\},\{0.6,0.7\}\rangle$	$\langle\{0.7,0.8,0.9\},\{0.5\}\rangle$

$\varphi(\{d_1,d_2\}) = 0.42$, $\varphi(\{d_1,d_3\}) = 0.48$, $\varphi(\{d_1,d_4\}) = 0.37$, $\varphi(\{d_2,d_3\}) = 0.61$, $\varphi(\{d_2,d_4\}) = 0.50$, $\varphi(\{d_3,d_4\}) = 0.56$, $\varphi(\{d_1,d_2,d_3\}) = 0.77$, $\varphi(\{d_1,d_2,d_4\}) = 0.65$, $\varphi(\{d_1,d_3,d_4\}) = 0.71$, $\varphi(\{d_2,d_3,d_4\} = 0.84$, $\varphi(\{d_1,d_2,d_3,d_4\} = 1$.

Step 4. We utilize the FHFCOA operator to obtain the overall preference values p_i of the disease $D_i (i = 1,2,3,4,5)$. We have

$$p_1 = \langle\{0.6811, 0.6562, 0.7484, 0.7311, 0.6976, 0.6748, 0.7601, 0.7440\}, \{0.5640, 0.5478, 0.5845, 0.5677, 0.6029, 0.5856, 0.6001, 0.5829, 0.6220, 0.6041, 0.6415, 0.6231, 0.6128, 0.5952, 0.6351, 0.6168, 0.6551, 0.6363, 0.6520, 0.6333, 0.6757, 0.6563, 0.6970, 0.6770\}\rangle.$$

$$p_2 = \langle\{0.6981, 0.7059, 0.6863, 0.6945, 0.7158, 0.7229, 0.7605, 0.7660, 0.7520, 0.7579, 0.7732, 0.7784, 0.7380, 0.7443, 0.7284, 0.7351, 0.7524, 0.7582, 0.7895, 0.7942, 0.7825, 0.7874, 0.8003, 0.8047\}, \{0.5013, 0.5138, 0.5334, 0.5467, 0.573, 0.5873, 0.6097, 0.6249\}\rangle.$$

$$p_3 = \langle\{0.547, 0.5568, 0.5822, 0.5905, 0.7023, 0.7070, 0.5885, 0.5966, 0.6178, 0.6249, 0.7229, 0.7271, 0.6207, 0.6277, 0.6461, 0.6523, 0.7401, 0.7439, 0.6508, 0.6568, 0.6729, 0.6784, 0.7570, 0.7606\}, \{0.4930, 0.5133, 0.5271, 0.5487, 0.5341, 0.5560, 0.5710, 0.5944, 0.5519, 0.5745, 0.5900, 0.6142, 0.5236, 0.5451, 0.5598, 0.5827, 0.5672, 0.5905, 0.6064, 0.6312, 0.5861, 0.6101, 0.6265, 0.6522\}\rangle.$$

$$p_4 = \langle\{0.7569, 0.7510, 0.7651, 0.7595, 0.7516, 0.7456, 0.7724, 0.7670, 0.7799, 0.7747, 0.7676, 0.7620\}, \{0.5180, 0.5343, 0.5696, 0.5875, 0.6511, 0.6716\}\rangle.$$

$$p_5 = \langle\{0.6219, 0.6259, 0.6472, 0.6508, 0.6700, 0.6732, 0.6903, 0.6932, 0.6779, 0.6810, 0.6974, 0.7002, 0.7154, 0.7179, 0.7317, 0.7340, 0.7565, 0.7586, 0.7696, 0.7715, 0.7819, 0.7836, 0.7932, 0.7948\}, \{0.5623, 0.5730, 0.5828, 0.5939, 0.6175, 0.6292, 0.6400, 0.6521\}\rangle.$$

Step 5. Calculate the scores $S(p_i)(i = 1,2,3,4,5)$ of the overall FHFNs $p_i (i = 1,2,3,4,5)$ as follows,

$$S(p_1) = 0.1226, S(p_2) = 0.2469, S(p_3) = 0.0957,$$
$$S(p_4) = 0.2397, S(p_5) = 0.1411.$$

Step 6. Rank all the alternatives $D_i (i = 1,2,3,4,5)$ in accordance with the scores $S(p_i)(i = 1,2,3,4,5)$ of the overall Fermatean hesitant fuzzy preference numbers. We have $S(p_2) > S(p_4) > S(p_5) > S(p_1) > S(p_3)$, which shows that $D_2 > D_4 > D_5 > D_1 > D_3$. That is, the most appropriate disease is D_2.

Step 7. End.

Similarly, we apply the FHFCOG operator to get the medical decision. The step 1 to step 3 is not change, we start from step 4.

Step 4. We utilize the FHFCOG operator to obtain the overall preference values p_i of the medical decision $D_i (i = 1,2,3,4,5)$. We have

$$p_1 = \langle\{0.5414, 0.5300, 0.5594, 0.5476, 0.6408, 0.6273, 0.6622, 0.6482\}, \{0.6065, 0.5937, 0.6335, 0.6224, 0.6724, 0.6632, 0.6193, 0.6073, 0.6448, 0.6343, 0.6818, 0.6730, 0.7489, 0.7427, 0.7625, 0.7568, 0.7832, 0.7781, 0.7552, 0.7493, 0.7683,$$

$$p_2 = \langle\{0.6642, 0.6839, 0.6367, 0.6556, 0.6884, 0.7088, 0.6863, 0.7066, 0.6579, 0.6774, 0.7113, 0.7324, 0.6941, 0.7147, 0.6654, 0.6851, 0.7194, 0.7407, 0.7173, 0.7385, 0.6875, 0.7079, 0.7434, 0.7654\}, \{0.6016, 0.6211, 0.6147, 0.6330, 0.6400, 0.6564, 0.6510, 0.6665\}\rangle.$$

$$p_3 = \langle\{0.5295, 0.5487, 0.5488, 0.5687, 0.5817, 0.6029, 0.5527, 0.5728, 0.5728, 0.5936, 0.6072, 0.6293, 0.5917, 0.6132, 0.6132, 0.6355, 0.6500, 0.6737, 0.6176, 0.6400, 0.6401, 0.6633, 0.6785, 0.7032\}, \{0.5530, 0.6533, 0.5610, 0.6583, 0.6477, 0.7165, 0.6528, 0.7202, 0.7253, 0.7738, 0.7288, 0.7764, 0.5835, 0.6728, 0.5905, 0.6773, 0.6677, 0.7309, 0.6723, 0.7343, 0.7390, 0.7843, 0.7423, 0.7868\}\rangle.$$

$$p_4 = \langle\{0.6575, 0.6386, 0.6859, 0.6662, 0.6243, 0.6064, 0.6863, 0.6666, 0.7159, 0.6954, 0.6516, 0.6329\}, \{0.6307, 0.6701, 0.6386, 0.6766, 0.6713, 0.7040\}\rangle.$$

$p_5 = \langle\{0.5368, 0.5621, 0.5898, 0.6176, 0.5571, 0.5834, 0.6121, 0.6409, 0.5610, 0.5874, 0.6164, 0.6454, 0.5822, 0.6097, 0.6397, 0.6698, 0.5833, 0.6107, 0.6408, 0.6710, 0.6053, 0.6338, 0.6650, 0.6964\}, \{0.6065, 0.6641, 0.6335, 0.6849, 0.6562, 0.7027, 0.6778, 0.7200\}\rangle.$

Step 5. Calculate the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall FHFNs $p_i(i = 1, 2, 3, 4, 5)$ as follows,

$S(p_1) = -0.1347, \quad S(p_2) = 0.0856, \quad S(p_3) = -0.1014,$
$S(p_4) = -0.0060, \quad S(p_5) = -0.0677.$

Step 6. Rank all the alternatives $D_i(i = 1, 2, 3, 4, 5)$ in accordance with the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall Fermatean hesitant fuzzy preference numbers. We have $S(p_2) > S(p_4) > S(p_5) > S(p_3) > S(p_1)$, which shows that $D_2 > D_4 > D_5 > D_3 > D_1$. That is, the most appropriate disease is D_2.

Step 7. End.

As we can see from the results, although both FHFCOA and FHFCOG produce the same final decision, there are significant differences in the score values. This is because FHFCOA tends to prioritize membership information, resulting in a forecast in better situation, while FHFCOG emphasizes non-membership information, leading to a forecast in worse situation.

B. SENSITIVITY ANALYSIS

The above process of decision is based on the fact that the problem is super-additive since the value of $\lambda > 0$.

We try another weight vector in which $\varphi(\{d_1\}) = 0.2$, $\varphi(\{d_2\}) = 0.35$, $\varphi(\{d_3\}) = 0.4$, $\varphi(\{d_4\}) = 0.3$. The ordering of weights is the same as in the first experiment, whereas we calculate by (7) and the parameter $\lambda = -0.48$ is obtained which demonstrate the problem is sub-additive. Similarly, the followings are obtained using Equation (4). $\varphi(\{d_1, d_2\}) = 0.52$, $\varphi(\{d_1, d_3\}) = 0.56$, $\varphi(\{d_1, d_4\}) = 0.47$, $\varphi(\{d_2, d_3\}) = 0.68$, $\varphi(\{d_2, d_4\}) = 0.60$, $\varphi(\{d_3, d_4\}) = 0.64$, $\varphi(\{d_1, d_2, d_3\}) = 0.82$, $\varphi(\{d_1, d_2, d_4\}) = 0.74$, $\varphi(\{d_1, d_3, d_4\}) = 0.78$, $\varphi(\{d_2, d_3, d_4\} = 0.88$, $\varphi(\{d_1, d_2, d_3, d_4\} = 1$.

Step 1. to Step 3. is identical.

Step 4. Use FHFCOA operator and the overall preference values p_i are as follows:

$p_1 = \langle\{0.6637, 0.6444, 0.7373, 0.7242, 0.6852, 0.6680, 0.7523, 0.7402\}, \{0.5690, 0.5572, 0.5870, 0.5748, 0.6030, 0.5904, 0.6061, 0.5935, 0.6252, 0.6122, 0.6423, 0.6289, 0.6292, 0.6161, 0.6491, 0.6355, 0.6668, 0.6529, 0.6702, 0.6562, 0.6913, 0.6769, 0.7102, 0.6954\}\rangle.$

$p_2 = \langle\{0.7067, 0.7121, 0.6969, 0.7026, 0.7215, 0.7266, 0.7675, 0.7714, 0.7605, 0.7646, 0.7782, 0.7819, 0.7522, 0.7564, 0.7445, 0.7490, 0.7638, 0.7678, 0.8008, 0.8040, 0.7951, 0.7984, 0.8096, 0.8126\}, \{0.4821, 0.4907, 0.5135, 0.5227, 0.5670, 0.5771, 0.6039, 0.6147\}\rangle.$

$p_3 = \langle\{0.5472, 0.5543, 0.5780, 0.5842, 0.6876, 0.6913, 0.5893, 0.5951, 0.6149, 0.6201, 0.7101, 0.7133, 0.6336, 0.6384, 0.6546, 0.6589, 0.7357, 0.7386, 0.6625, 0.6666, 0.6808, 0.6846, 0.7533, 0.7559\}, \{0.4937, 0.5083, 0.5232, 0.5387, 0.5356, 0.5514, 0.5676, 0.5843, 0.5537, 0.5700, 0.5868, 0.6041, 0.5311, 0.5467, 0.5628, 0.5794, 0.5761, 0.5931, 0.6105, 0.6285, 0.5956, 0.6132, 0.6312, 0.6498\}\rangle.$

$p_4 = \langle\{0.7808, 0.7771, 0.7870, 0.7834, 0.7768, 0.7730, 0.7945, 0.7911, 0.8002, 0.7969, 0.7909, 0.7874\}, \{0.4849, 0.4982, 0.5441, 0.5589, 0.6399, 0.6573\}\rangle.$

$p_5 = \langle\{0.6396, 0.6423, 0.6601, 0.6625, 0.6848, 0.6870, 0.7014, 0.7034, 0.7010, 0.7030, 0.7163, 0.7181, 0.7352, 0.7368, 0.7480, 0.7495, 0.7840, 0.7852, 0.7937, 0.7949, 0.8060, 0.8071, 0.8145, 0.8156\}, \{0.5477, 0.5552, 0.5650, 0.5727, 0.6023, 0.6106, 0.6214, 0.6299\}\rangle.$

Step 5. Calculate the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall FHFNs $p_i(i = 1, 2, 3, 4, 5)$ as follows,

$S(p_1) = 0.0948, S(p_2) = 0.2761, \quad S(p_3) = 0.0951,$
$S(p_4) = 0.3074, S(p_5) = 0.1903.$

Step 6. Rank all the alternatives $D_i(i = 1, 2, 3, 4, 5)$ in accordance with the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall Fermatean hesitant fuzzy preference numbers. We have $S(p_4) > S(p_2) > S(p_5) > S(p_3) > S(p_1)$, which shows that $D_4 > D_2 > D_5 > D_3 > D_1$. That is, the most appropriate disease is D_4.

Step 7. End. We utilize FHFCOG to obtain the preference values p_i which are as follows: Step 1. to Step 3. is identical

Step 4. Use FHFCOA operator and the overall preference values p_i are as follows:

$p_1 = \langle\{0.5099, 0.5021, 0.5272, 0.5191, 0.6255, 0.6159, 0.6467, 0.6368\}, \{0.6144, 0.6057, 0.6373, 0.6295, 0.6709, 0.6643, 0.6270, 0.6187, 0.6486, 0.6412, 0.6805, 0.6742, 0.7725, 0.7687, 0.7829, 0.7793, 0.7991, 0.7958, 0.7782, 0.7744, 0.7882, 0.7847, 0.8039, 0.8007\}\rangle.$

$p_2 = \langle\{0.6778, 0.6922, 0.6534, 0.6673, 0.6992, 0.7141, 0.7008, 0.7157, 0.6755, 0.6899, 0.7229, 0.7383, 0.7150, 0.7302, 0.6892, 0.7039, 0.7376,$

TABLE 2. Sensitivity analysis of λ.

ω	λ	problem	ranking by FHFCOA	ranking by FHFCOG
$\{0.15, 0.27, 0.33, 0.22\}$	0.09	super-additive	$D_2 > D_4 > D_5 > D_1 > D_3$	$D_2 > D_4 > D_5 > D_3 > D_1$
$\{0.20, 0.35, 0.40, 0.30\}$	-0.48	sub-additive	$D_4 > D_2 > D_5 > D_3 > D_1$	$D_2 > D_4 > D_5 > D_3 > D_1$
$\{0.35, 0.30, 0.20, 0.10\}$	0.15	super-additive	$D_2 > D_4 > D_5 > D_1 > D_3$	$D_2 > D_4 > D_5 > D_3 > D_1$
$\{0.60, 0.40, 0.30, 0.20\}$	-0.74	sub-additive	$D_4 > D_2 > D_5 > D_3 > D_1$	$D_4 > D_2 > D_5 > D_3 > D_1$
$\{0.12, 0.14, 0.20, 0.40\}$	0.52	super-additive	$D_4 > D_2 > D_5 > D_1 > D_3$	$D_2 > D_4 > D_5 > D_3 > D_1$
$\{0.18, 0.20, 0.24, 0.40\}$	-0.05	sub-additive	$D_4 > D_2 > D_5 > D_3 > D_1$	$D_2 > D_4 > D_5 > D_3 > D_1$

$0.7533, 0.7392, 0.7550, 0.7125, 0.7277, 0.7626, 0.7788\}, \{0.5807, 0.5964, 0.5954, 0.6100, 0.6307, 0.6431, 0.6423, 0.6541\}\rangle.$

$p_3 = \langle\{0.5323, 0.5462, 0.5491, 0.5634, 0.5776, 0.5927, 0.5560, 0.5705, 0.5735, 0.5885, 0.6034, 0.6191, 0.6090, 0.6249, 0.6282, 0.6446, 0.6608, 0.6781, 0.6361, 0.6527, 0.6562, 0.6733, 0.6903, 0.7083\}, \{0.5447, 0.6241, 0.5519, 0.6292, 0.6438, 0.6967, 0.6483, 0.7002, 0.7238, 0.7603, 0.7269, 0.7628, 0.5825, 0.6508, 0.5886, 0.6552, 0.6682, 0.7157, 0.6722, 0.7189, 0.7404, 0.7740, 0.7432, 0.7763\}\rangle.$

$p_4 = \langle\{0.6802, 0.6660, 0.7056, 0.6909, 0.6502, 0.6367, 0.7105, 0.6957, 0.7371, 0.7217, 0.6792, 0.6651\}, \{0.6029, 0.6420, 0.6137, 0.6510, 0.6571, 0.6879\}\rangle.$

$p_5 = \langle\{0.5670, 0.5861, 0.6153, 0.6361, 0.5888, 0.6087, 0.6390, 0.6606, 0.5981, 0.6183, 0.6491, 0.6710, 0.6211, 0.6421, 0.6740, 0.6968, 0.6269, 0.6481, 0.6804, 0.7034, 0.6511, 0.6731, 0.7066, 0.7304\}, \{0.5848, 0.6321, 0.6109, 0.6532, 0.6401, 0.6773, 0.6605, 0.6944\}\rangle.$

Step 5. Calculate the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall FHFNs $p_i(i = 1, 2, 3, 4, 5)$ as follows,

$$S(p_1) = -0.1763, \quad S(p_2) = 0.1277, \quad S(p_3) = -0.0818,$$
$$S(p_4) = 0.0585, \quad S(p_5) = -0.0017.$$

Step 6. Rank all the alternatives $X_i(i = 1, 2, 3, 4, 5)$ in accordance with the scores $S(p_i)(i = 1, 2, 3, 4, 5)$ of the overall Fermatean hesitant fuzzy preference numbers. We have $S(p_2) > S(p_4) > S(p_5) > S(p_3) > S(p_1)$, which shows that $X_2 > X_4 > X_5 > X_3 > X_1$. That is, the most appropriate disease is X_2.

Step 7. End.

We tried other weight vectors and the corresponding results are presented in Table (2).

The table reveals that the value of λ has a notable impact on the decision outcome. However, the robustness of the results is demonstrated by the consistency in the rankings of alternatives. Specifically, alternatives D_2 and D_4 are consistently ranked at the top, while alternatives D_1 and D_3 are consistently ranked at the bottom, regardless of the value of λ. Therefore, the result of the decision-making process is considered to be robust, despite the effect of the parameter λ on the outcome.

C. COMPARISON ANALYSIS

In this subsection, we compare our proposed approach with two existing methods which are specifically designed for solving decision-making problems in Fermatean Fuzzy environment: FHFWA and FHFWG by Kirişci [16] and FFWA, FFWG, FFWPA, and FFWPG by Senapati [20].

It is noted that the four operators proposed by Senapati [20] only consider that both the degree of membership and the degree of non-membership have only one value. Therefore, we computing the average of the Table (1) and the new table is depicted in Table (3). Moreover, it is required that the sum of weight vector must equal to 1 when using FFWA, FFWG, FFWPA and FFWPG, therefore, we must make propotional adjustment.

TABLE 3. Fermatean fuzzy decision matrix C.

	S_1	S_2	S_3	S_4
D_1	$\langle\{0.75, 0.55\}\rangle$	$\langle\{0.40, 0.80\}\rangle$	$\langle\{0.60, 0.70\}\rangle$	$\langle\{0.85, 0.45\}\rangle$
D_2	$\langle\{0.60, 0.80\}\rangle$	$\langle\{0.55, 0.65\}\rangle$	$\langle\{0.85, 0.45\}\rangle$	$\langle\{0.75, 0.50\}\rangle$
D_3	$\langle\{0.65, 0.77\}\rangle$	$\langle\{0.73, 0.35\}\rangle$	$\langle\{0.60, 0.55\}\rangle$	$\langle\{0.45, 0.80\}\rangle$
D_4	$\langle\{0.90, 0.43\}\rangle$	$\langle\{0.65, 0.60\}\rangle$	$\langle\{0.50, 0.75\}\rangle$	$\langle\{0.55, 0.80\}\rangle$
D_5	$\langle\{0.35, 0.85\}\rangle$	$\langle\{0.75, 0.60\}\rangle$	$\langle\{0.50, 0.65\}\rangle$	$\langle\{0.80, 0.50\}\rangle$

The outcome of the decision-making process is presented in Table(4). The table illustrates that while most of the methods regard D_2 as the most appropriate decision, our approach judges D_4 as the next best option, which happens to be the last one in the other methods. The reason for this is that our method takes into account the interdependence between each factor.

Furthermore, FHFWA and FHFWG do not allow the weight vectors to sum up to 1, which leads to the problem of super-additivity and sub-additivity. Therefore, it is recommended to use the Choquet Integral to solve the impact of super-additivity and sub-additivity. On the other hand, Fermatean fuzzy aggregation operators have several restrictions, such as requiring the weight vector sum to 1, and not being able to deal effectively with problems where experts have more than one value.

TABLE 4. Comparison between different method.

ω	method	ranking	ω	method	ranking
$\{0.15, 0.27, 0.33, 0.22\}$	FHFCOA	$D_2 > D_4 > D_5 > D_1 > D_3$	$\{0.20, 0.35, 0.40, 0.30\}$	FHFCOA	$D_4 > D_2 > D_5 > D_1 > D_3$
$\{0.15, 0.27, 0.33, 0.22\}$	FHFCOG	$D_2 > D_4 > D_5 > D_1 > D_3$	$\{0.20, 0.35, 0.40, 0.30\}$	FHFCOG	$D_2 > D_4 > D_5 > D_3 > D_1$
$\{0.15, 0.27, 0.33, 0.22\}$	*FHFWA*[16]	$D_2 > D_1 > D_5 > D_3 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FHFWA	$D_2 > D_1 > D_5 > D_3 > D_4$
$\{0.15, 0.27, 0.33, 0.22\}$	*FHFWG*[16]	$D_2 > D_1 > D_5 > D_3 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FHFWG	$D_2 > D_3 > D_5 > D_1 > D_4$
$\{0.15, 0.27, 0.33, 0.22\}$	*FFWA*[20]	$D_2 > D_3 > D_1 > D_5 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FFWA	$D_2 > D_3 > D_5 > D_1 > D_4$
$\{0.15, 0.27, 0.33, 0.22\}$	*FFWG*[20]	$D_2 > D_3 > D_1 > D_5 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FFWG	$D_2 > D_3 > D_1 > D_5 > D_4$
$\{0.15, 0.27, 0.33, 0.22\}$	*FFWPA*[20]	$D_2 > D_5 > D_1 > D_3 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FFWPA	$D_2 > D_5 > D_1 > D_3 > D_4$
$\{0.15, 0.27, 0.33, 0.22\}$	*FFWPG*[20]	$D_2 > D_5 > D_1 > D_3 > D_4$	$\{0.20, 0.35, 0.40, 0.30\}$	FFWPG	$D_2 > D_1 > D_5 > D_4 > D_3$

VI. CONCLUSION

In multi-attribute decision making problems, decision-makers tend to pay more attention to the optimal decision scheme, and thus ignore the alternatives.

In light of the fact that alternative evaluation often involves incomplete and uncertain information, this paper adopts the Fermatean hesitant fuzzy set to describe such information. Moreover, it is crucial to consider the interdependence of decision criteria and decision maker's preferences in multi-attribute decision making problems, as ignoring these factors may lead to decision errors.

To address these challenges, this paper proposes the Fermatean hesitant fuzzy Choquet ordered averaging (geometric) operators, which utilize fuzzy measures and Choquet integrals to aggregate hesitant fuzzy information. These operators not only effectively handle incomplete and uncertain evaluation information, but also account for the interdependence between decision criteria and decision maker's preferences while giving the decision in both good situation and bad situation at the same time.

To demonstrate the validity and practicality of the proposed approach, numerical examples and sensitivity analysis are presented. We also compare our proposed methods with existing methods, which indicates that our approach outperforms these methods in terms of rationality, practicality and robustness.

As we move forward with our research, we place a high priority on extending the FHFS methodology to address a wider range of constraints and practical examples that may arise in real-world applications. By doing so, we aim to enhance the versatility and effectiveness of the FHFS approach and establish its practical applicability in various categories.

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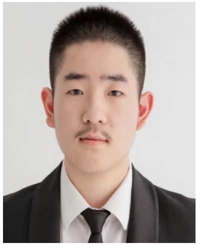

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