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RESEARCH ARTICLE

Bipartite Containment Control for Delayed Multiagent Systems With Markovian Switching Topologies Under Impulsive Attacks

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ABSTRACT This paper is concerned with the problem of bipartite containment control design for a class of nonlinear multiagent systems with time-delays in agents' states under impulsive false-data-injection attacks. The considered multiagent system is subject to Markovian variation in the signed communication topology. The graph among the followers is assumed to be structurally balanced for each Markovian switching mode. A memory distributed control protocol is proposed to achieve bipartite containment within the convex hull of leader agents as well as the symmetric convex hull. The bipartite containment control problem is solved by means of a Markovian switching Lyapunov function and the Razumikhin technique. In addition, the problem of bipartite leader-following consensus is also addressed for delayed nonlinear multiagent systems with one leader. Two examples are provided to show the effectiveness of the proposed control scheme, and comparison results with existing methods are given as well.

INDEX TERMS Bipartite containment control, multiagent systems, Markovian switching topologies, impulsive attacks, Razumikhin method.

I. INTRODUCTION

In recent years, the problem of distributed control for multiagent systems with cooperative and competitive interactions has attracted researchers' attention due to its wide applications including bipartite formation control [1], bipartite leader-following/containment motion of Lagrangian systems [2], social networks [3], etc. In particular, for multiagent systems with one leader and signed topology, the objective of designing distributed control protocols is to achieve bipartite tracking control [4]. Also, the cases of more than one leader often take place in many engineering applications such as localization and navigation, environment perception and human-robot interaction [5]. In those cases, containment control problems can easily occur. For multiagent systems with multiple leaders and signed topology, the objective of designing distributed control protocols is to achieve bipartite containment control, where the states of all followers in two competitive nodes' subsets can eventually converge to the convex hulls spanned by all the leaders' states or their symmetric states with the same modulus but different in sign [6].

In the context of multiagent systems, there are two categories of information graph: the fixed topology and the switching topologies. The latest works of bipartite containment control mostly focus on for multiagent systems with fixed topology [6], [7], whereas switching topologies involve more variable factors, and are more meaningful for application of realistic circumstances [8]. Up to now, few works focus on the problem of bipartite containment control for multiagent systems with switching topologies, such as [9] and [10]. In [9], bipartite containment control has been proposed for nonlinear fractional multiagent systems over signed networks with switching topologies. Based on the fractional Razumikhin technique and common Lyapunov function method, a delayed control protocol is proposed to ensure bipartite containment control. In [10], bipartite containment

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control has been proposed for discrete-time second-order linear multiagent systems over signed networks with switching topologies, where time-varying communication delays are taken into account. From the above analysis, it can be found that the problem of bipartite containment control for multiagent systems with Markovian switching topologies remains to be important and challenging, which is one of the reasons that motivate the current work.

Time delay is widely utilized in various practical systems, and it does cause many issues such as system instability, oscillation and performance degradation. Memory controllers can provide better performance than memoryless controllers for time-delay systems [11]. Particularly, the problem of bipartite leader-following synchronization have been investigated for multiagent systems with node state delays over fixed topology [12] and Markovian switching topologies [13]. In addition, in the above-mentioned works, memoryless control has been utilized to guarantee bipartite leader-following synchronization. In most cases the bipartite containment control deals with multiagent systems without node state delays [2], [9], [14]. If there are node state delays, however, it is essential to consider the effect of time delay on bipartite containment control. This motivates us to consider memory control protocols for delayed multiagent systems.

Security control of multiagent systems is of paramount importance due to cyber-physical attacks such as false-data-injection attacks and denial-of-service attacks [15]. In the context of secure bipartite tracking control under fixed topology, various dynamics of agents have been discussed, such as single integrator [16], linear systems [17], nonlinear systems [18]. In particular, the case of switching topology is also considered in [18]. By now, few works emphasize the problem of bipartite containment control for multiagent systems under cyber-attacks. In [19], bipartite containment control has been considered for networked agents with general linear dynamics and antagonistic interactions under denial-of-service attacks. Therefore, it is still a challenging work to study the problem of bipartite containment control for nonlinear multiagent systems with node state delays and switching topologies under cyber-attacks.

Inspired by the above analysis, this paper addresses the problem of bipartite containment control for nonlinear multiagent systems with node time-delays and Markovian switching topologies under impulsive false-data-injection attacks. The main contributions of this paper are outlined as follows:

1) Comparing with fixed topology considered in [4], [5], and [20], Markovian switching topologies are considered, which can be viewed as an extension of existing results for multiagent systems over unsigned and switching graphs [21].

2) Motivated by memory feedback that is capable of compensating for the effect of time-varying delays [22], a memory distributed control protocol is presented to solve the bipartite containment control problem.

3) Different from the impulsive effects on bipartite synchronization in [23], the impulses arising from false-data-injection attacks can destroy bipartite containment consensus.

4) By using a Markovian switching Lyapunov function and the Razumikhin technique, sufficient conditions are provided for achieving the bipartite containment consensus of the delayed multiagent systems considered in this paper.

Notations: I_n represents the $n \times n$ identity matrix. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and the maximum eigenvalue of the corresponding matrix, respectively. $\otimes$ stands for the Kronecker product for matrices. $\text{sign}(\cdot)$ represents the signum function. $\mathbb{E}\{x\}$ is the expectation of the stochastic variable x, and $\text{Prob}\{\cdot\}$ is the occurrence probability of an event.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider a bipartite containment control problem for a multiagent system with N followers and R leaders over a coopetition network. The dynamics of follower i is given by

$$\begin{aligned}\dot{x}_i(t) &= -Cx_i(t) + Af(x_i(t)) + Bf_d(x_i(t-d(t))) \\ &\quad + u_i(t), \quad i \in \underline{N} \triangleq \{1, \cdots, N\},\end{aligned} \tag{1}$$

and the dynamics of leader l is given by

$$\begin{aligned}\dot{x}_l(t) &= -Cx_l(t) + Af(x_l(t)) + Bf_d(x_l(t-d(t))), \\ &\quad l \in \underline{R} \triangleq \{N+1, \cdots, N+R\},\end{aligned} \tag{2}$$

where $x_i(t) \in \mathbb{R}^{n_x}$ is the state vector for the ith agent ($i \in \underline{N} \cup \underline{R}$), and $u_i(t) \in \mathbb{R}^{n_x}$ is the control input for the ith agent ($i \in \underline{N}$). The matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_x}$ and $C \in \mathbb{R}^{n_x \times n_x}$ are known constant matrices. For each $i \in \underline{N}$, $f(x_i(t)) \in \mathbb{R}^{n_x}$ and $f_d(x_i(t-d(t))) \in \mathbb{R}^{n_x}$ are nonlinear vector-valued functions without and with time delay, respectively. $d(t)$ denotes the time delay satisfying that $0 \leq d(t) \leq \bar{d}$, where $\bar{d}$ is a known constant.

Let $\theta(t), t \geq 0$ represent the Markovian switching process which takes values in a finite set $\underline{S} \triangleq \{1, \cdots, S\}$. The transition rates of the process $\theta(t)$ are defined as

$$\text{Prob}\{\theta(t+\Delta)=q|\theta(t)=p\}=\begin{cases}\mu_{pq}\Delta + o(\Delta), & p \neq q, \\ 1 + \mu_{pq}\Delta + o(\Delta), & p = q,\end{cases} \tag{3}$$

where $\Delta > 0$, $\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$. $\mu_{pq} \geq 0 (p, q \in \underline{S}, p \neq q)$ denotes the transition rate from p to q, and $\mu_{pp} = -\sum_{q=1, q\neq p}^{S} \mu_{pq}$. The transition rate matrix is given by $\Upsilon = [\mu_{pq}]_{S \times S}$.

For multiagent systems with R leaders and N followers over a Markovian switching coopetition network, a signed graph $\mathcal{G}^{\theta(t)} = \{\mathcal{V}, \mathcal{E}^{\theta(t)}, \mathcal{A}^{\theta(t)}\}$ is defined to a coopetition relationship among agents, where $\mathcal{V} = \underline{N} \cup \underline{R}$, $\mathcal{E}^{\theta(t)}$ denotes a set of edges, $\mathcal{A}^{\theta(t)} = [a_{ij}^{\theta(t)}] \in \mathbb{R}^{(N+R)\times(N+R)}$ represents a weighted adjacency matrix. If agent i can receive the information from agent j, then $a_{ij}^{\theta(t)} \neq 0$. The Laplacian matrix of $\mathcal{G}^{\theta(t)}$ is defined as $\mathcal{L}^{\theta(t)} = \mathcal{D}^{\theta(t)} - \mathcal{A}^{\theta(t)}$ with $\mathcal{D}^{\theta(t)} = \text{diag}\{d_1^{\theta(t)}, \cdots, d_{N+R}^{\theta(t)}\}$ and $d_i^{\theta(t)} = \sum_{j=1}^{N+R} |a_{ij}^{\theta(t)}|$. It is assumed that each leader has no neighbors, then the matrix $\mathcal{L}^{\theta(t)}$ is rewritten as

$$\mathcal{L}^{\theta(t)} = \begin{bmatrix} \mathcal{L}_1^{\theta(t)} & \mathcal{L}_2^{\theta(t)} \\ 0 & 0 \end{bmatrix}$$

where $\mathcal{L}_1^{\theta(t)} \in \mathbb{R}^{N\times N}$ and $\mathcal{L}_2^{\theta(t)} \in \mathbb{R}^{N\times R}$.

The communication between N followers is modeled as the signed subgraph $\hat{\mathcal{G}}^{\theta(t)} = \{\underline{N}, \hat{\mathcal{E}}^{\theta(t)}, \hat{\mathcal{A}}^{\theta(t)}\}$, where $\hat{\mathcal{E}}^{\theta(t)} \subseteq \underline{N} \times \underline{N}$, $\hat{\mathcal{A}}^{\theta(t)} = [a_{ij}^{\theta(t)}] \in \mathbb{R}^{N\times N}$. In particular, if $a_{ij}^{\theta(t)} > 0$, then the relationship between follower i and follower j is cooperative, else if $a_{ij}^{\theta(t)} < 0$, then the relationship between follower i and follower j is competitive.

For each $p \in \underline{S}$, the signed subgraph $\hat{\mathcal{G}}^p$ is said to be structurally balanced if the follower set $\underline{N}$ can be divided into two subsets $\underline{N}_1$ and $\underline{N}_2$, where $\underline{N}_1 \cap \underline{N}_2 = \emptyset$ and $\underline{N}_1 \cup \underline{N}_2 = \underline{N}$, such that $a_{ij}^p \geq 0$ for all $i, j \in \underline{N}_1$ or $i, j \in \underline{N}_2$, and $a_{ij}^p \leq 0$ for all $i \in \underline{N}_a$ and $j \in \underline{N}_b$, $a \neq b, a, b \in \{1, 2\}$. Define $\Theta = \mathrm{diag}\{\nu_1, \cdots, \nu_N\}$ with $\nu_i = 1$ if $i \in \underline{N}_1$ and $\nu_i = -1$ if $i \in \underline{N}_2$.

Assumption 1: The nonlinear function $f(\cdot)$ and $f_d(\cdot)$ are odd functions and for all $x, y_l \in \mathbb{R}^{n_x}$, $l = N+1, \cdots, N+R$,

$$\left\| f(x) - \sum_{l=N+1}^{N+R} \rho_l f(y_l) \right\| \leq \mu_1 \left\| x - \sum_{l=N+1}^{N+R} y_l \right\|, \tag{4}$$

and

$$\left\| f_d(x) - \sum_{l=N+1}^{N+R} \rho_l f_d(y_l) \right\| \leq \mu_2 \left\| x - \sum_{l=N+1}^{N+R} y_l \right\|, \tag{5}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are known constants, $\rho_l \geq 0$, $\sum\limits_{l=N+1}^{N+R} \rho_l = 1$.

Remark 1: Bipartite consensus in a structurally balanced signed graph means that agents are divided into two groups where agents in two groups reach consensus on state value but opposite in sign. Nonlinear functions are commonly required to be odd functions to achieve bipartite consensus for nonlinear multiagent systems over a signed network, see, e.g. [4], [9], and [12]. In the presence of multiple leaders in multiagent systems, the problem of containment control arises. In this case, the conditions (4) and (5) in Assumption 1 are often required for nonlinear trems in nonlinear multiagent systems [9].

Assumption 2: For each $p \in \underline{S}$, the signed subgraph $\hat{\mathcal{G}}^p$ is structurally balanced, and for all $p \in \underline{S}$, there exists the same bipartition $\{\underline{N}_1, \underline{N}_2\}$ of the follower set $\underline{N}$, and there is at least one leader with a directed path to each follower in the signed graph $\mathcal{G}^p$ ($p \in \underline{S}$).

Remark 2: Assumption 2 implies that for each switching mode p, the signed graph $\hat{\mathcal{G}}^p$ is structurally balanced, and the state information of at least one leader can be transmitted to and available over time to the unpinned agents as well. In addition, vertices of all possible topology graphs can be partitioned into subsets $\underline{N}_1$ and $\underline{N}_2$. Assumption 2 has also been made in multiagent networks with antagonistic interactions and switching topologies [9].

Lemma 1: [7] By Assumption 2, each eigenvalue exists positive real part for matrix $\Theta\mathcal{L}_1^p\Theta$, $\forall p \in \underline{S}$, where $\Theta = \mathrm{diag}\{\nu_1, \cdots, \nu_N\}$. Moreover, for each $p \in \underline{S}$, each element of $-\Theta(\mathcal{L}_1^p)^{-1}\Theta\mathcal{L}_2^p$ is nonnegative, and each row sum is 1.

Suppose that the containment controller $u_i(t)$ is subjected to impulsive false data injection (FDI) attacks. Then the memory and Markovian switching containment controller $u_i(t)$ under FDI attacks is designed as

$$u_i(t) = -\zeta_1(\theta(t))\psi_i(t) - \zeta_2(\theta(t))\psi_i(t-\bar{d}) + \sum_{k=1}^{\infty} q_i(t)\delta(t-t_k) \tag{6}$$

where

$$\psi_i(t) = \sum_{j\in\mathcal{N}_i} \left|a_{ij}^{\theta(t)}\right| \left(x_i(t) - \mathrm{sign}\left(a_{ij}^{\theta(t)}\right) x_j(t)\right) + \sum_{l=N+1}^{N+R} a_{il}^{\theta(t)} \left(x_i(t) - \nu_i x_l(t)\right), \tag{7}$$

where $\mathcal{N}_i = \{j \in \underline{N} \mid a_{ij}^{\theta(t)} \neq 0\}$. $\zeta_1(\theta(t)) > 0$ and $\zeta_2(\theta(t)) > 0$ are the controller gains. $q_i(t) : \mathbb{R}^{n_x} \to \mathbb{R}$ is an attack function, $\delta(\cdot)$ is the Dirac impulse, and $\{t_k\}_0^\infty$ is the impulsive time sequence to describe when an FDI attack occurs, where $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $\lim\limits_{k\to\infty} t_k = \infty$. It is assumed that $\inf_k\{t_k - t_{k-1}\} = h_1$, where $h_1 > 0$ is a known constant.

Therefore, the dynamics of follower i can be written as

$$\begin{cases} \dot{x}_i(t) = -Cx_i(t) + Af(x_i(t)) + Bf_d(x_i(t-d(t))) \\ \qquad - \zeta_1(\theta(t))\psi_i(t) - \zeta_2(\theta(t))\psi_i(t-\bar{d}), \quad t \neq t_k \\ x_i(t_k^+) = x_i(t_k^-) + q_i(t_k^-) \end{cases} \tag{8}$$

where $\psi_i(t)$ is defined by (7), $x_i(t_k^+) = \lim\limits_{\delta\to 0^+} x_i(t_k+\delta)$, $x_i(t_k) = x_i(t_k^-) = \lim\limits_{\delta\to 0^-} x_i(t_k+\delta)$.

It is assumed that the attack signal $q_i(\cdot)$ is bounded, $i \in \underline{N}$. That is, there exists a positive constant η such that $\|q(\cdot)\|^2 \leq \eta$, where $q(\cdot) = [q_1^T(\cdot) \cdots q_N^T(\cdot)]^T$.

Remark 3: In the literature, the interaction topologies among agents over signed networks can be roughly classified into the following two categories: the time-invariant topology and the time-varying ones that include Markovian switching topologies. In addition, different from the assumption on impulsive false-data-injection attacks in [24], here impulsive false-data-injection attacks are assumed to be bounded and have negative effects on bipartite containment consensus.

Remark 4: In this paper, it is assumed that leaders have no neighbors. In order to reach bipartite containment consensus, it is usually required that there is at least one leader that has a directed path to each follower in the signed communication graph [14]. In this case, leaders control followers in the architecture of decentralized control (see, e.g. [1], [9], [14]). When each leader can be autonomous or evolving dynamically via communicating with other leaders in its neighbourhood, leaders control followers in the architecture of distributed control (see, e.g. [20], [25]). It would be interesting to design a bipartite containment controller for nonlinear multiagent systems with multiple leaders where each leader has its neighboring leaders, which merits future research.

Let $x_F(t) = [x_1^T(t), \cdots, x_N^T(t)]^T$, $x_L(t) = [x_{N+1}^T(t), \cdots, x_{N+R}^T(t)]^T$, $\Psi(t) = [\psi_1^T(t), \cdots, \psi_N^T(t)]^T$, then

$$\Psi(t) = \left(\mathcal{L}_1^{\theta(t)} \otimes I_{n_x}\right) x_F(t) + \left(\Theta\mathcal{L}_2^{\theta(t)} \otimes I_{n_x}\right) x_L(t).$$

Furthermore, let $e(t) = \left(\Theta\left(\mathcal{L}_1^{\theta(t)}\right)^{-1} \otimes I_{n_x}\right)\Psi(t)$, which can be rewritten as

$$\begin{aligned} e(t) = &\left(\Theta \otimes I_{n_x}\right) x_F(t) \\ &+ \left(\Theta\left(\mathcal{L}_1^{\theta(t)}\right)^{-1} \Theta\mathcal{L}_2^{\theta(t)} \otimes I_{n_x}\right) x_L(t). \end{aligned}$$

Definition 1: Bipartite containment consensus with an error bound is said to be achieved if the bipartite containment error $e(t)$ converges into the set H in the mean-square sense for any initial conditions, where $H \triangleq \{e(t) \in \mathbb{R}^{n_xN} \mid E\{\|e(t)\|^2\} \leq c\}$ and c is a positive constant.

Definition 2: Let $m(t) : \mathbb{R} \to \mathbb{R}$ be a continuous function. Define the upper Dini-derivative of $m(t)$ by $D^+m(t)$, defined as

$$D^+m(t) = \limsup_{h\to 0^+} \frac{m(t+h) - m(t)}{h}.$$

Denote

$\bar{F}(x_F(t)) = [f^T(x_1(t)), \cdots, f^T(x_N(t))]^T$,
$\tilde{F}(x_L(t)) = [f^T(x_{N+1}(t)), \cdots, f^T(x_{N+R}(t))]^T$,
$\bar{F}_d(x_F(t-d(t))) = [f_d^T(x_1(t-d(t))), \cdots, f_d^T(x_N(t-d(t)))]^T$,
$\tilde{F}_d(x_L(t-d(t))) = [f_d^T(x_{N+1}(t-d(t))), \cdots, f_d^T(x_{N+R}(t-d(t)))]^T$,

$$\begin{aligned} &M(x_F(t), x_L(t), \theta(t)) = \left(\Theta \otimes I_{n_x}\right)\bar{F}(x_F(t)) \\ &+ \left(\Theta\left(\mathcal{L}_1^{\theta(t)}\right)^{-1}\Theta\mathcal{L}_2^{\theta(t)} \otimes I_{n_x}\right)\tilde{F}(x_L(t)), \\ &M_d(x_F(t-d(t)), x_L(t-d(t)), \theta(t)) \\ &= \left(\Theta \otimes I_{n_x}\right)\bar{F}_d(x_F(t-d(t))) \\ &+ \left(\Theta\left(\mathcal{L}_1^{\theta(t)}\right)^{-1}\Theta\mathcal{L}_2^{\theta(t)} \otimes I_{n_x}\right)\tilde{F}_d(x_L(t-d(t))). \end{aligned}$$

Then the dynamics of the bipartite containment error $e(t)$ is given by

$$\begin{aligned} \dot{e}(t) = &-(I_N \otimes C)e(t) \\ &+ (I_N \otimes A)M(x_F(t), x_L(t), \theta(t)) \\ &+ (I_N \otimes B)M_d(x_F(t-d(t)), x_L(t-d(t)), \theta(t)) \\ &- \zeta_1(\theta(t))\left(\mathcal{L}_1^{\theta(t)} \otimes I_{n_x}\right)e(t) \\ &- \zeta_2(\theta(t))\left(\mathcal{L}_1^{\theta(t)} \otimes I_{n_x}\right)e(t-\bar{d}), \quad t \neq t_k, \\ e(t_k^+) = &\, e(t_k^-) + \left(\Theta \otimes I_{n_x}\right)q(t_k^-). \end{aligned} \tag{9}$$

Suppose that the initial state of the error system (9) is given by $e(t) = \varphi(t), t \in [-\bar{d}, 0]$, where $\varphi : [-\bar{d}, 0] \to \mathbb{R}^{n_xN}$ is a continuous function.

Lemma 2: Let R_1, R_2, R_3 be any real matrices of appropriate dimensions with $R_3 > 0$. Then, for any vectors x and y with appropriate dimensions, the following inequality holds:

$$2x^TR_1^TR_2y \leq x^TR_1^TR_3R_1x + y^TR_2^TR_3^{-1}R_2y.$$

III. MAIN RESULTS

In this section, by using a Markovian switching Lyapunov function and the Razumikhin technique in [26], Razumikhin-type stability theorem is established for the bipartite containment error system (9).

Theorem 1: Suppose that for given positive scalars a_1, a_2, a_3, σ, there exist matrices $P_p > 0, p \in \underline{S}$, positive scalars ϵ_1, ϵ_2, such that

$$\begin{bmatrix} \Xi_p & 0 & P_pA & P_pB \\ 0 & -a_2P_p + \epsilon_2\mu_2^2I_{n_x} & 0 & 0 \\ A^TP_p & 0 & -\epsilon_1I_{n_x} & 0 \\ B^TP_p & 0 & 0 & -\epsilon_2I_{n_x} \end{bmatrix} < 0, \quad p \in \underline{S}, \tag{10}$$

$$-a_3(I_N \otimes P_p) + \zeta_2(p)((\mathcal{L}_1^p)^T\mathcal{L}_1^p \otimes P_p) < 0, \quad p \in \underline{S}, \tag{11}$$

and

$$a_1 - \frac{\ln(1+\sigma)}{h_1} > 0, \tag{12}$$

where $\Xi_p = -P_pC - C^TP_p + \sum_{q=1}^{S}\mu_{pq}P_q + \zeta_2(p)P_p + (-2\gamma + a_1 + (a_2 + a_3)(1+\sigma))P_p + \epsilon_1\mu_1^2I_{n_x}$, $0 < \gamma < \min_{p\in\underline{S}, i\in\underline{N}}\left(\zeta_1(p)\mathrm{Re}(r_i^p)\right)$, where $r_i^p, i \in \underline{N}$ are the eigenvalues of matrix $\mathcal{L}_1^p$, and $\mathrm{Re}(r_i^p)$ denotes the real part of the eigenvalue r_i^p. Then the error system (9) converges into the bounded set $H \triangleq \{e(t) \in \mathbb{R}^{n_xN} \mid E\{\|e(t)\|^2\} \leq c\}$ in the mean-square sense, where $c = \frac{(1+\sigma^{-1})\max_{p\in\underline{S}}(\lambda_{\max}(P_p))\eta}{\min_{p\in\underline{S}}(\lambda_{\min}(P_p))(e^{a_1h_1} - (1+\sigma))}$.

Proof. Construct the Markovian switching Lyapunov function as follows:

$$V(e(t), \theta(t)) = e^T(t)(I_N \otimes P_{\theta(t)})e(t) \tag{13}$$

where $P_p, \forall\theta(t) = p \in \underline{S}$, is a positive definite matrix of appropriate dimension to be determined.

Define the weak infinitesimal operator $\mathcal{A}$ of the random process $\{(e(t), \theta(t)), t \geq 0\}$ as follows

$$\begin{aligned} &\mathcal{A}V(e(t), \theta(t)) \\ &= \lim_{\delta\to 0^+}\frac{1}{\delta}[\mathbb{E}\{V(e(t+\delta), \theta(t+\delta))|e(t), \theta(t)\} - V(e(t), \theta(t))]. \end{aligned}$$

For each $\theta(t) = p \in \underline{S}$ and $t \in (t_k, t_{k+1}]$, $\mathcal{A}V(e(t), p)$ can be computed as follows [27]

$$\begin{aligned} \mathcal{A}V(e(t), p) = &\, 2e^T(t)(I_N \otimes P_p)\dot{e}(t) \\ &+ \sum_{q=1}^{S}\mu_{pq}e^T(t)(I_N \otimes P_q)e(t), \end{aligned} \tag{14}$$

and the weak infinitesimal operator along the system (9) leads to

$$\begin{aligned}\mathcal{A}V(e(t),p)\\ &= e^T(t)\left(I_N\otimes\left(-P_pC-C^TP_p+\sum_{q=1}^{S}\mu_{pq}P_q\right)\right)e(t)\\ &\quad +2e^T(t)(I_N\otimes P_pA)M(x_F(t),x_L(t),p)\\ &\quad +2e^T(t)(I_N\otimes P_pB)M(x_F(t-d(t)),x_L(t-d(t)),p)\\ &\quad -2\zeta_1(p)e^T(t)\left(\mathcal{L}_1^p\otimes P_p\right)e(t)\\ &\quad -2\zeta_2(p)e^T(t)\left(\mathcal{L}_1^p\otimes P_p\right)e(t-\bar{d}). \end{aligned}\tag{15}$$

By Lemma 2, it follows that

$$\begin{aligned}&-2\zeta_2(p)e^T(t)\left(\mathcal{L}_1^p\otimes P_p\right)e(t-\bar{d})\\ &\quad\le \zeta_2(p))e^T(t)(I_N\otimes P_p)e(t)\\ &\qquad +\zeta_2(p)e^T(t-\bar{d})((\mathcal{L}_1^p)^T\mathcal{L}_1^p\otimes P_p)e(t-\bar{d}).\end{aligned}\tag{16}$$

By Assumption 1, it follows that $\forall\epsilon_1>0$,

$$\epsilon_1[\mu_1^2e^T(t)e(t)-M^T(x_F(t),x_L(t),p)M(x_F(t),x_L(t),p)]\ge 0,\tag{17}$$

and $\forall\epsilon_2>0$,

$$\begin{aligned}&\epsilon_2[\mu_2^2e^T(t-d(t))e(t-d(t))\\ &\quad -M_d^T(x_F(t-d(t)),x_L(t-d(t)),p)\\ &\quad M_d(x_F(t-d(t)),x_L(t-d(t)),p)]\ge 0.\end{aligned}\tag{18}$$

By (10) and (12), there exists a sufficiently small positive scalar ε such that $r\triangleq 1+\sigma+\varepsilon<e^{a_1h_1}$ and

$$\Omega_p\triangleq\begin{bmatrix}\Gamma_p & 0 & P_pA & P_pB\\ 0 & -a_2P_p+\epsilon_2\mu_2^2I_{n_x} & 0 & 0\\ A^TP_p & 0 & -\epsilon_1I_{n_x} & 0\\ B^TP_p & 0 & 0 & -\epsilon_2I_{n_x}\end{bmatrix}<0,\quad p\in\underline{S},\tag{19}$$

where $\Gamma_p=-P_pC-C^TP_p+\sum_{q=1}^{S}\mu_{pq}P_q+\zeta_2(p)P_p+(-2\gamma+a_1+(a_2+a_3)(1+\sigma+\varepsilon))P_p+\epsilon_1\mu_1^2I_{n_x}$.

Suppose that

$$\begin{aligned}rV(e(t),p)-V(e(t-d(t)),p)&\ge 0,\\ rV(e(t),p)-V(e(t-\bar{d}),p)&\ge 0.\end{aligned}\tag{20}$$

Then, by (15), (16), (17) and (18), it follows that for any $a_1>0$, $a_2>0$, $a_3>0$, $\epsilon_1>0$, $\epsilon_2>0$,

$$\begin{aligned}&\mathcal{A}V(e(t),p)+a_1V(e(t),p)\\ &\le \mathcal{A}V(e(t),p)+a_1V(e(t),p)\\ &\quad +a_2(rV(e(t),p)-V(e(t-d(t)),p))\\ &\quad +a_3(rV(e(t),p)-V(e(t-\bar{d}),p))\\ &\quad +\epsilon_1[\mu_1^2e^T(t)e(t)\\ &\quad -M^T(x_F(t),x_L(t),p)M(x_F(t),x_L(t),p)]\\ &\quad +\epsilon_2[\mu_2^2e^T(t-d(t))e(t-d(t))\\ &\quad -M_d^T(x_F(t-d(t)),x_L(t-d(t)),p)\\ &\quad M_d(x_F(t-d(t)),x_L(t-d(t)),p)]\\ &=\xi^T(t)(I_N\otimes\Omega_p)\xi(t)\\ &\quad +e^T(t)\big(2\gamma(I_N\otimes P_p)-\zeta_1(p)(\mathcal{L}_1^p\otimes P_p)\\ &\quad -\zeta_1(p)((\mathcal{L}_1^p)^T\otimes P_p)\big)e(t)\end{aligned}\tag{21}$$

where $\xi(t)=[e^T(t)\quad e^T(t-d(t))\quad M^T(x_F(t),x_L(t),p)\quad M_d^T(x_F(t-d(t)),x_L(t-d(t)),p)]^T$.

Since $0<\gamma<\min_{p\in\underline{S},i\in\underline{N}}\left(\zeta_1(p)\mathrm{Re}(r_i^p)\right)$, where r_i^p, $i\in\underline{N}$ are the eigenvalues of matrix $\mathcal{L}_1^p$, $e^T(t)(2\gamma(I_N\otimes P_p)-\zeta_1(p)(\mathcal{L}_1^p\otimes P_p)-\zeta_1(p)((\mathcal{L}_1^p)^T\otimes P_p))e(t)\le 0$.

Therefore, for $t\in(t_k,t_{k+1}]$,

$$\mathcal{A}V(e(t),p)+a_1V(e(t),p)\le\xi^T(t)(I_N\otimes\Omega_p)\xi(t),\tag{22}$$

where Ω_p is defined in (19).

By (19) and (22), it follows that for $t\in(t_k,t_{k+1}]$,

$$\mathcal{A}V(e(t),p)\le -a_1V(e(t),p).\tag{23}$$

It follows from (23) that for $t\in(t_k,t_{k+1}]$,

$$\mathbb{E}\{\mathcal{A}V(e(t),\theta(t))\}\le -a_1\mathbb{E}\{V(e(t),\theta(t))\},\tag{24}$$

Moreover, $\forall\sigma>0$,

$$\begin{aligned}&V(e(t_k^+),\theta(t_k^+))\\ &=V(e(t_k^+),\theta(t_k^-))\\ &=e^T(t_k^-)(I_N\otimes P_{\theta(t_k^-)})e(t_k^-)\\ &\quad +2e^T(t_k^-)(I_N\otimes P_{\theta(t_k^-)})\left(\Theta\otimes I_{n_x}\right)q(t_k^-)\\ &\quad +q^T(t_k^-)(I_N\otimes P_{\theta(t_k^-)})q(t_k^-)\\ &\le (1+\sigma)e^T(t_k^-)(I_N\otimes P_{\theta(t_k^-)})e(t_k^-)\\ &\quad +(1+\sigma^{-1})q^T(t_k^-)(I_N\otimes P_{\theta(t_k^-)})q(t_k^-)\\ &\le (1+\sigma)V(e(t_k^-),\theta(t_k^-))\\ &\quad +(1+\sigma^{-1})\max_{p\in\underline{S}}(\lambda_{\max}(P_p))\|q(t_k^-)\|^2\\ &\le (1+\sigma)V(e(t_k^-),\theta(t_k^-))+b\end{aligned}\tag{25}$$

where $b=(1+\sigma^{-1})\max_{p\in\underline{S}}(\lambda_{\max}(P_p))\eta$.

In the sequel, it will be proved by the method of mathematical induction that for $t\in(t_k,t_{k+1}]$,

$$\begin{aligned}\mathbb{E}\{V(e(t),\theta(t))\}&\le(1+\sigma)^ke^{-a_1t}\mathbb{E}\{V(e(0),\theta(0))\}\\ &\quad +(1+\sigma)^{k-1}be^{-a_1(t-t_1)}\\ &\quad +(1+\sigma)^{k-2}be^{-a_1(t-t_2)}\\ &\quad +\cdots+be^{-a_1(t-t_k)}\triangleq\varphi_k(t).\end{aligned}\tag{26}$$

Firstly, when $t\in(t_0,t_1]$, the following inequality (27) will be shown.

$$\mathbb{E}\{V(e(t),\theta(t))\}\le e^{-a_1t}\mathbb{E}\{V(e(0),\theta(0))\}=\varphi_0(t).\tag{27}$$

If the inequality (27) does not hold, then there exists some $t\in(t_0,t_1]$ such that

$$\mathbb{E}\{V(e(t),\theta(t))\}>\varphi_0(t).\tag{28}$$

Let $\hat{t}_0 = \inf\{t \in (t_0, t_1] | \mathbb{E}\{V(e(t), \theta(t))\} > \varphi_0(t)\}$, then

$$\mathbb{E}\{V(e(\hat{t}_0), \theta(\hat{t}_0))\} > \varphi_0(\hat{t}_0). \quad (29)$$

and

$$\mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_0(t), \forall t \in (t_0, \hat{t}_0]. \quad (30)$$

Let $\check{t}_0 = \sup\{t \in (t_0, \hat{t}_0] | \mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_0(t)\}$, then

$$\mathbb{E}\{V(e(t), \theta(t))\} > \varphi_0(t), \forall t \in (\check{t}_0, \hat{t}_0]. \quad (31)$$

By Definition 2, one has

$$D^+\mathbb{E}\{V(e(t), \theta(t))\} = \mathbb{E}\{\mathcal{A}V(e(t), \theta(t))\}. \quad (32)$$

By using Dynkin's formula, it follows that

$$\mathbb{E}\{V(e(\hat{t}_0), \theta(\hat{t}_0))\} \leq e^{-a_1(\hat{t}_0 - \check{t}_0)}\mathbb{E}\{V(e(\check{t}_0), \theta(\check{t}_0))\}. \quad (33)$$

Note that $\mathbb{E}\{V(e(\check{t}_0), \theta(\check{t}_0))\} \leq \varphi_0(\check{t}_0)$, it follows that

$$\mathbb{E}\{V(e(\hat{t}_0), \theta(\hat{t}_0))\} \leq e^{-a_1\hat{t}_0}\mathbb{E}\{V(e(0), \theta(0))\} = \varphi_0(\hat{t}_0)), \quad (34)$$

which contradicts with (29).

Secondly, suppose that (26) holds for $t \in (t_k, t_{k+1}]$, $k = 1, 2, \cdots, l-1 (l > 1)$, then

$$\mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_{l-1}(t), \forall t \in (t_{l-1}, t_l] \quad (35)$$

and

$$\mathbb{E}\{V(e(t_l^-), \theta(t_l^-))\} \leq \varphi_{l-1}(t_l^-). \quad (36)$$

It will be proved that (26) holds for $k = l$, i.e.,

$$\mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_l(t), \forall t \in (t_l, t_{l+1}]. \quad (37)$$

If the inequality (37) does not hold, then there exists some $t \in (t_l, t_{l+1}]$ such that

$$\mathbb{E}\{V(e(t), \theta(t))\} > \varphi_l(t). \quad (38)$$

Let $\hat{t}_l = \inf\{t \in (t_l, t_{l+1}] | \mathbb{E}\{V(e(t), \theta(t))\} > \varphi_l(t)\}$, then

$$\mathbb{E}\{V(e(\hat{t}_l), \theta(\hat{t}_l))\} > \varphi_l(\hat{t}_l). \quad (39)$$

and

$$\mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_l(t), \quad \forall t \in (t_l, \hat{t}_l]. \quad (40)$$

Let $\check{t}_l = \sup\{t \in (t_l, \hat{t}_l] | \mathbb{E}\{V(e(t), \theta(t))\} \leq \varphi_l(t)\}$, then

$$\mathbb{E}\{V(e(t), \theta(t))\} > \varphi_l(t), \quad \forall t \in (\check{t}_l, \hat{t}_l]. \quad (41)$$

By using Dynkin's formula, it follows that

$$\begin{aligned}\mathbb{E}\{V(e(\hat{t}_l), \theta(\hat{t}_l))\} &\leq e^{-a_1(\hat{t}_l - \check{t}_l)}\mathbb{E}\{V(e(\check{t}_l), \theta(\check{t}_l))\} \\ &\leq e^{-a_1(\hat{t}_l - t_l)}\mathbb{E}\{V(e(t_l^+), \theta(t_l^+))\}. \quad (42)\end{aligned}$$

In addtion,

$$\mathbb{E}\{V(e(t_l^+), \theta(t_l^+))\} \leq (1+\sigma)\mathbb{E}\{V(e(t_l^-), \theta(t_l^-))\} + b \quad (43)$$

From (42), (43) and (36), it follows that

$$\mathbb{E}\{V(e(\hat{t}_l), \theta(\hat{t}_l))\} \leq \varphi_l(\hat{t}_l), \quad (44)$$

which contradicts with (39).

By mathematical induction it can be concluded that (26) holds.

By (26) and the assumption that $\inf_k\{t_k - t_{k-1}\} = h_1$, it follows that

$$\begin{aligned}\mathbb{E}\{V(e(t), \theta(t))\} &\leq (1+\sigma)^k e^{-a_1 t}\mathbb{E}\{V(e(0), \theta(0))\} \\ &\quad + \frac{be^{-a_1h_1}\left(1 - (1+\sigma)^k e^{-a_1h_1k}\right)}{1 - (1+\sigma)e^{-a_1h_1}}. \quad (45)\end{aligned}$$

Denote $\alpha = a_1 - \frac{\ln(1+\sigma)}{h_1}$ and note that $\frac{t-t_0}{h_1} - 1 \geq k$, then

$$\begin{aligned}\mathbb{E}\{V(e(t), \theta(t))\} \leq\ & \frac{1}{1+\sigma}e^{-\alpha t}\mathbb{E}\{V(e(0), \theta(0))\} \\ & - \frac{be^{a_1h_1}e^{-\alpha t}}{(1+\sigma)(1-(1+\sigma)e^{-a_1h_1})} \\ & + \frac{b}{e^{a_1h_1} - (1+\sigma)}. \quad (46)\end{aligned}$$

Moreover,

$$\mathbb{E}\{V(e(t), \theta(t))\} \geq \min_{p\in\underline{S}} \lambda_{\min}(P_p)\mathbb{E}\{\|e(t)\|^2\}. \quad (47)$$

(46) and (47) imply that

$$\lim_{t\to\infty} \mathbb{E}\{\|e(t)\|^2\} \leq c \quad (48)$$

where $c = \frac{(1+\sigma^{-1})\max_{p\in\underline{S}}(\lambda_{\max}(P_p))\eta}{\min_{p\in\underline{S}}(\lambda_{\min}(P_p))(e^{a_1h_1} - (1+\sigma))}$.

This completes the proof.

Remak 5: Based on a Razumikhin-type technique, Theorem 1 is presented on the stability of the bipartite containment error system, which is modeled as a Markovian switching system with time delay and impulsive effects. Recently, bipartite quasi-synchronization was investigated in [28] for delayed neural networks under time-invariant signed graphs and single leader via a distributed impulsive control strategy. All the results and analysis methods in [28] cannot be directly extended to the case of delayed neural networks under time-varying signed graphs and multiple leaders. Hence, results presented in this paper are new and more general than those in [28].

Remak 6: From the perspective of practical applications, Lyapunov functions contain more information about the considered multiagent network, which could reduce the conservatism of the conditions on the stability and performane of the bipartite containment error system. In light of the consideration of less conservatism, the impulse-time-dependent Lyapunov function method has been proposed for time-delay systems with impulsive effects [29], but the computational complexity could be raised.

If $\theta(t) \equiv 1$ in (6)-(7), that is, the signed graph is fixed, then the bipartite containment error system becomes

$$\begin{aligned}\dot{e}(t) =\ & -(I_N \otimes C)e(t) \\ & + (I_N \otimes A)M(x_F(t), x_L(t), \theta(t)) \\ & + (I_N \otimes B)M_d(x_F(t - d(t)), x_L(t - d(t)), \theta(t)) \\ & - \zeta_1\left(\mathcal{L}_1 \otimes I_{n_x}\right)e(t)\end{aligned}$$

$$-\zeta_2\left(\mathcal{L}_1 \otimes I_{n_x}\right) e(t-\bar{d}), \quad t \neq t_k,$$
$$e(t_k^+) = e(t_k^-) + \left(\Theta \otimes I_{n_x}\right) q(t_k^-), \tag{49}$$

and the following Corollary 1 holds.

Corollary 1: Suppose that for given positive scalars a_1, a_2, a_3, σ, there exist matrices $P > 0$, positive scalars ϵ_1, ϵ_2, such that

$$\begin{bmatrix} \Xi & 0 & PA & PB \\ 0 & -a_2P+\epsilon_2\mu_2^2 I_{n_x} & 0 & 0 \\ A^TP & 0 & -\epsilon_1 I_{n_x} & 0 \\ B^TP & 0 & 0 & -\epsilon_2 I_{n_x} \end{bmatrix} < 0, \tag{50}$$
$$-a_3(I_N \otimes P) + \zeta_2(\mathcal{L}_1^T\mathcal{L}_1 \otimes P) < 0, \tag{51}$$

and

$$a_1 - \frac{\ln(1+\sigma)}{h_1} > 0, \tag{52}$$

where $\Xi = -PC - C^TP + \zeta_2 P + (-2\gamma + a_1 + (a_2+a_3)(1+\sigma))P + \epsilon_1\mu_1^2 I_{n_x}$, $0 < \gamma < \min_{i\in \underline{N}} (\zeta_1 \mathrm{Re}(r_i))$, where $r_i, i \in \underline{N}$ are the eigenvalues of matrix $\mathcal{L}_1$, and $\mathrm{Re}(r_i)$ denotes the real part of the eigenvalue r_i. Then the bipartite containment error system (49) converges into the bounded set $H \triangleq \{e(t) \in \mathbb{R}^{n_xN} \mid E\{\|e(t)\|^2\} \leq c\}$ in the mean-square sense, where $c = \frac{(1+\sigma^{-1})\lambda_{\max}(P)\eta}{\lambda_{\min}(P)(e^{a_1h_1}-(1+\sigma))}$.

If there is just one leader ($R = 1$), then the bipartite consensus problem for delayed multiagent systems (1)-(2) is a bipartite tracking control problem. Assumption 1 is replaced by the following Assumption 3.

Assumption 3: The nonlinear function $f(\cdot)$ and $f_d(\cdot)$ are odd functions and for all $x, y \in \mathbb{R}^{n_x}$,

$$\|f(x) - f(y)\| \leq \mu_1 \|x-y\|, \tag{53}$$

and

$$\|f_d(x) - f_d(y)\| \leq \mu_2 \|x-y\|, \tag{54}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are known constants.

Let $\hat{\mathcal{L}}^{\theta(t)} = [\hat{l}_{ij}^{\theta(t)}]_{N\times N}$ be the Laplacian matrix of the signed subgraph $\hat{\mathcal{G}}^{\theta(t)}$,
$\mathcal{A}_{N+1}^{\theta(t)} = \mathrm{diag}\{a_{1(N+1)}^{\theta(t)}, \cdots, a_{N(N+1)}^{\theta(t)}\}$,
$\mathcal{J}^{\theta(t)} \triangleq \hat{\mathcal{L}}^{\theta(t)} + \mathcal{A}_{N+1}^{\theta(t)}$,
$\varphi_i(t) = f(x_i(t)) - \nu_i f(x_{N+1}(t))$,
$\Phi(t) = [\varphi_1^T(t), \cdots, \varphi_N^T(t)]^T$,
$\varphi_{di}(t-d(t)) = f_d(x_i(t-d(t))) - \nu_i f_d(x_{N+1}(t-d(t)))$,
$\Phi_d(t-d(t)) = [\varphi_{d1}^T(t-d(t)), \cdots, \varphi_{dN}^T(t-d(t))]^T$,
$e_i(t) = x_i(t) - \nu_i x_{N+1}(t)$, $e(t) = [e_1^T(t), \cdots, e_N^T(t)]^T$.
Then, the bipartite tracking error system is given by

$$\begin{aligned} \dot{e}(t) &= -(I_N \otimes C)e(t) + (I_N \otimes A)\Phi(t) \\ &\quad + (I_N \otimes B)\Phi_d(t-d(t)) \\ &\quad - \zeta_1(\theta(t))\left(\mathcal{J}^{\theta(t)} \otimes I_{n_x}\right)e(t) \\ &\quad - \zeta_2(\theta(t))\left(\mathcal{J}^{\theta(t)} \otimes I_{n_x}\right)e(t-\bar{d}), \quad t \neq t_k, \\ e(t_k^+) &= e(t_k^-) + \left(\Theta \otimes I_{n_x}\right) q(t_k^-), \end{aligned} \tag{55}$$

and the following Corollary 2 holds.

Corollary 2: Suppose that for given positive scalars a_1, a_2, a_3, σ, there exist matrices $P_p > 0, p \in \underline{S}$, positive scalars ϵ_1, ϵ_2, such that

$$\begin{bmatrix} \Xi_p & 0 & P_pA & P_pB \\ 0 & -a_2P_p+\epsilon_2\mu_2^2 I_{n_x} & 0 & 0 \\ A^TP_p & 0 & -\epsilon_1 I_{n_x} & 0 \\ B^TP_p & 0 & 0 & -\epsilon_2 I_{n_x} \end{bmatrix} < 0, \quad p \in \underline{S}, \tag{56}$$
$$-a_3(I_N \otimes P_p) + \zeta_2(p)((\mathcal{J}^p)^T\mathcal{J}^p \otimes P_p) < 0, \quad p \in \underline{S}, \tag{57}$$

and

$$a_1 - \frac{\ln(1+\sigma)}{h_1} > 0, \tag{58}$$

where $\Xi_p = -P_pC - C^TP_p + \sum_{q=1}^{S}\mu_{pq}P_q + \zeta_2(p)P_p + (-2\gamma + a_1 + (a_2+a_3)(1+\sigma))P_p + \epsilon_1\mu_1^2 I_{n_x}$, $0 < \gamma < \min_{p\in\underline{S}, i\in\underline{N}} \left(\zeta_1(p)\mathrm{Re}(r_i^p)\right)$, where $r_i^p, i \in \underline{N}$ are the eigenvalues of matrix $\mathcal{J}^p$, and $\mathrm{Re}(r_i^p)$ denotes the real part of the eigenvalue r_i^p. Then the bipartite tracking error system (55) converges into the bounded set $H \triangleq \{e(t) \in \mathbb{R}^{n_xN} \mid E\{\|e(t)\|^2\} \leq c\}$ in the mean-square sense, where $c = \frac{(1+\sigma^{-1})\max_{p\in\underline{S}}(\lambda_{\max}(P_p))\eta}{\min_{p\in\underline{S}}(\lambda_{\min}(P_p))(e^{a_1h_1}-(1+\sigma))}$.

Remak 7: Many practical systems can be modeled by multiagent systems with switching agent dynamics. Under the framework of multiagent systems with switching agent dynamics over unsigned graphs, the problems of consensus tracking and containment control have been studied in [30] and [31], respectively. From a practical perspective, it is interesting to study the problem of bipartite containment control for multiagent systems with switching agent dynamics over signed graphs, which would be one of our future research directions.

IV. TWO NUMERICAL EXAMPLES

In this section, two numerical examples are presented respectively to illustrate the effectiveness of the proposed bipartite containment and leader-following consensus control methods.

Example 1 (Bipartite containment control). Consider the 3-D delayed neural network [32] as follows:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf_d(z(t-d(t))) \tag{59}$$

where $z = [z_1, z_2, z_3]^T \in \mathbb{R}^3$, $C = 2I_3$, and

$$A = \begin{bmatrix} 1.25 & 3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1.0 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & -2 & 1 \\ 2 & -0.5 & 1 \\ 1 & -1 & -2 \end{bmatrix}.$$

$f(z(t)) = f_d(z(t)) = [f(z_1(t)), f(z_2(t)), f(z_3(t))]^T$ with $f(z_m(t)) = 0.5(|z_m(t)+1| - |z_m(t)-1|)(m = 1, 2, 3)$, and time delay $d(t) = \frac{1}{1+e^{-t}}$.

Let us consider a multiagent system which consists of seven followers and three leaders, and the Markovian switching topologies are given by signed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ shown

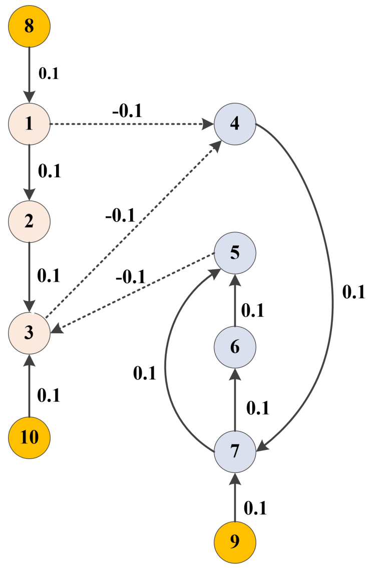

FIGURE 1. Signed graph $\mathcal{G}_1$.

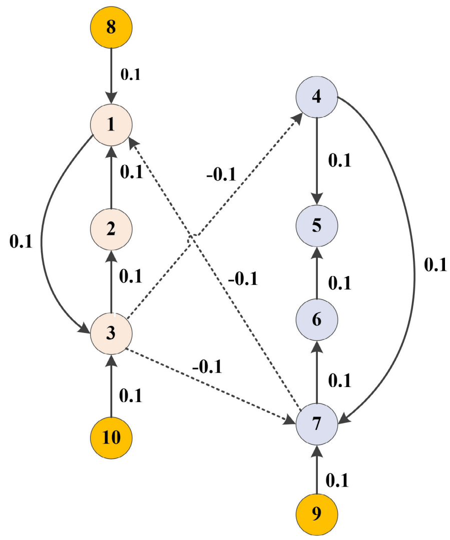

FIGURE 2. Signed graph $\mathcal{G}_2$.

in Figures 1 and 2, where three leaders are labeled by 8,9 and 10, and seven followers are labeled by 1-7.

The coopetition relationship between 7 followers is modeled by signed subgraphs $\hat{\mathcal{G}}_1$ and $\hat{\mathcal{G}}_2$, which are structurally balanced. Let $\underline{N}_1 = \{1, 2, 3\}$, $\underline{N}_2 = \{4, 5, 6, 7\}$ and $\Theta = \text{diag}\{1, 1, 1, -1, -1, -1, -1\}$. The dynamics of three leaders satisfies (59), and the transition rate matrix is chosen as

$$\Upsilon = \begin{bmatrix} -0.35 & 0.35 \\ 0.65 & -0.65 \end{bmatrix}.$$

Let $a_1 = 0.5, a_2 = 0.9, a_3 = 1, h_1 = 2, \sigma = 0.8, \zeta_2(1) = 5, \zeta_2(2) = 4.5$. By solving the linear matrix inequalities (10)-(11) in Theorem 1 by Matlab LMI toolbox, it is obtained that

$$P_1 = \begin{bmatrix} 0.4383 & -0.0094 & -0.0021 \\ -0.0094 & 0.4336 & -0.0046 \\ -0.0021 & -0.0046 & 0.4309 \end{bmatrix},$$

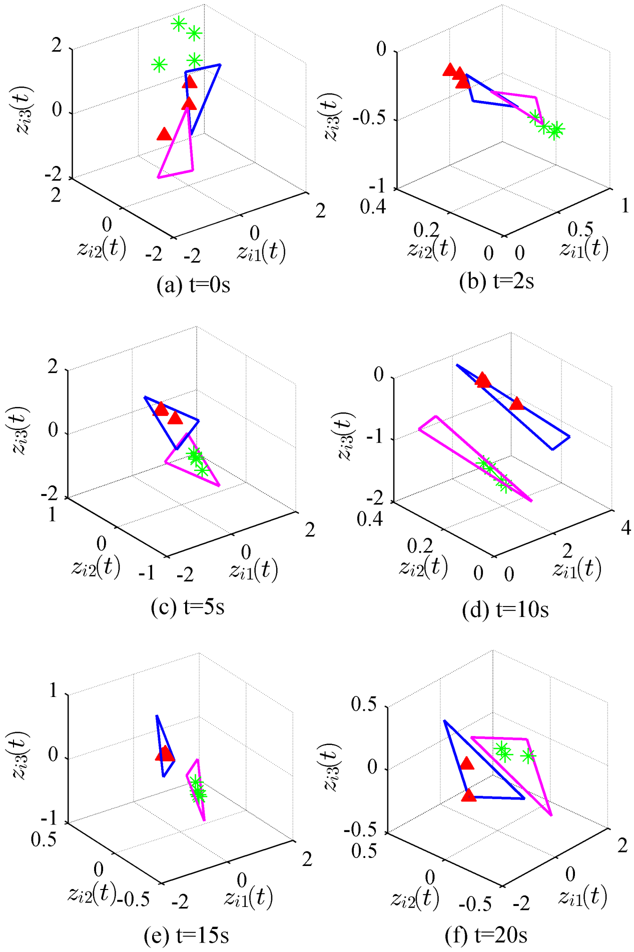

FIGURE 3. The positions of all agents at different instants, where the vertex of the blue triangle is the leader $\{8, 9, 10\}$, while the green $*$ and red Δ are the follower $\{4, 5, 6, 7\}$ and $\{1, 2, 3\}$ respectively.

$$P_2 = \begin{bmatrix} 0.4272 & -0.0084 & -0.0020 \\ -0.0084 & 0.4229 & -0.0042 \\ -0.0020 & -0.0042 & 0.4206 \end{bmatrix}.$$

Moreover, $\zeta_1(1)$ and $\zeta_1(2)$ can be chosen as $\zeta_1(1) = 258$ and $\zeta_1(2) = 328$ respectively.

Assume that $q_i(t) = [0.085, -0.1, 0.04]^T$. Then, $\eta = 0.0987$, and based on Theorem 1, it can be obtained that the containment error bound $c = 0.2606$. Figure 3 depicts the positions of all agents at different instants. Figure 4 displays the time evolution of $\|e(t)\|^2$, which shows that the containment errors have satisfactory upper bounds.

If we consider the memoryless control protocol, that is, $\zeta_2(\theta(t)) = 0$ in (6), then the maximum of $\|e(t)\|^2$ is $c = 0.4131$. It can be found that with the memory control protocol, the maximum of $\|e(t)\|^2$ is reduced.

Example 2 (Bipartite leader-following control). In this example, the bipartite tracking control issue is taken into consideration. We remove the leaders 9 and 10 from signed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ in Example 1. The transition rate matrix and the attack functions $q_i(t)(i = 1, \cdots, 7)$ are same as in

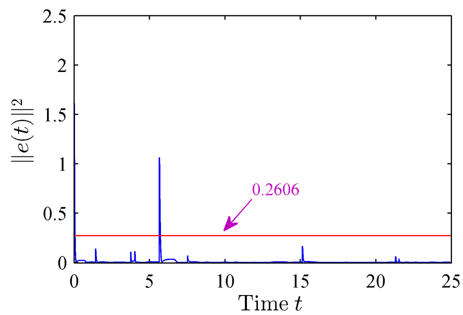

FIGURE 4. $\|e(t)\|^2$ in Example 1.

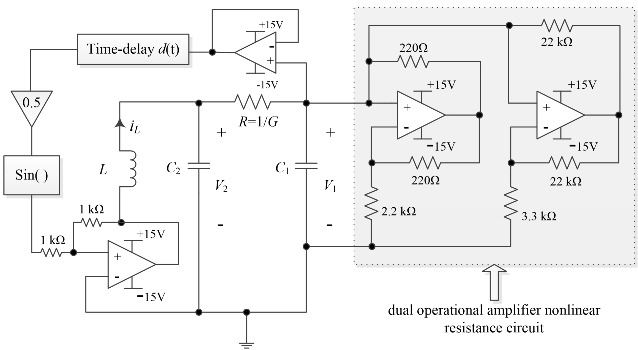

FIGURE 5. Experimental diagram of the delayed Chua's circuit.

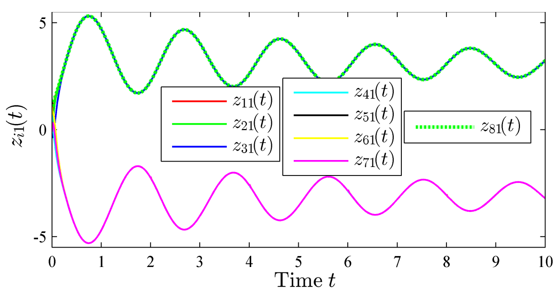

FIGURE 6. Trajectories of states $z_{i1}(i = 1, 2, \cdots, 8)$.

Example 1. The dynamic of the leader is the delayed Chua's circuit [33], where the system parameters are taken as

$$C = \begin{bmatrix} 1.269 & -10 & 0 \\ -1 & 1 & -1 \\ 0 & 19.53 & 0.1636 \end{bmatrix},$$

$$A = \begin{bmatrix} 5.594 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -3.906 & 0 & 0 \end{bmatrix},$$

and the nonlinear functions $f(z_8(t)) = [f(z_{81}(t)), 0, 0]^T$ with $f(z_{81}(t)) = 0.5(|z_{81}(t) + 1| - |z_{81}(t) - 1|)$, and $f_d(z_8(t - d(t))) = [f_d(z_{81}(t - d(t))), 0, 0]^T$ with $f_d(z_{81}(t - d(t))) = \sin(0.5z_{81}(t - d(t)))$. $d(t) = 0.02$.

Let $a_1 = 0.5, a_2 = 0.9, a_3 = 0.5, h_1 = 2, \sigma = 0.8, \zeta_2(1) = 3, \zeta_2(2) = 2.8$. By solving the linear matrix inequalities (56)-(57) in Corollary 2 by Matlab LMI toolbox, it is obtained that

$$P_1 = \begin{bmatrix} 4.1293 & 0.0269 & 0.1856 \\ 0.0269 & 4.6159 & -0.0380 \\ 0.1856 & -0.0380 & 3.7830 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 4.1321 & 0.0229 & 0.1792 \\ 0.0229 & 4.5922 & -0.0254 \\ 0.1792 & -0.0254 & 3.7969 \end{bmatrix}.$$

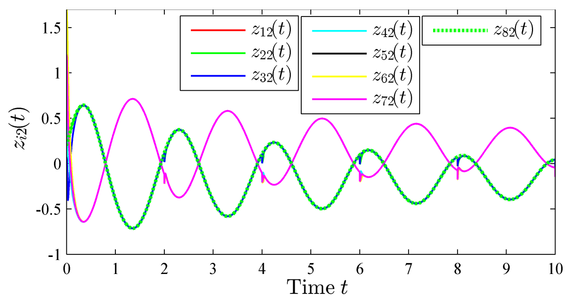

FIGURE 7. Trajectories of states $z_{i2}(i = 1, 2, \cdots, 8)$.

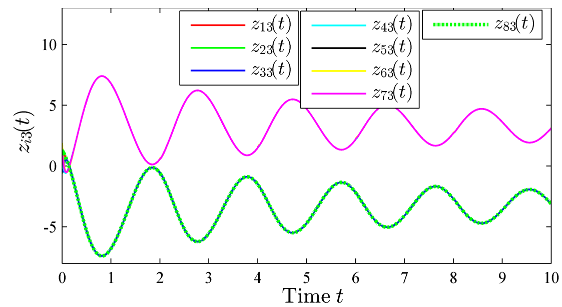

FIGURE 8. Trajectories of states $z_{i3}(i = 1, 2, \cdots, 8)$.

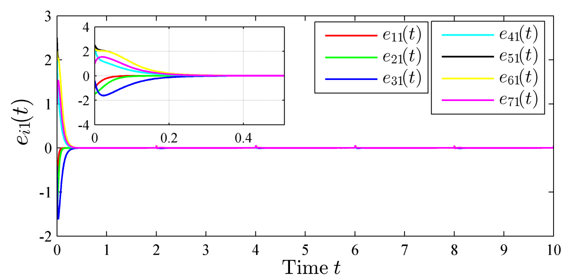

FIGURE 9. Trajectories of bipartite tracking errors $e_{i1}(i = 1, 2, \cdots, 7)$.

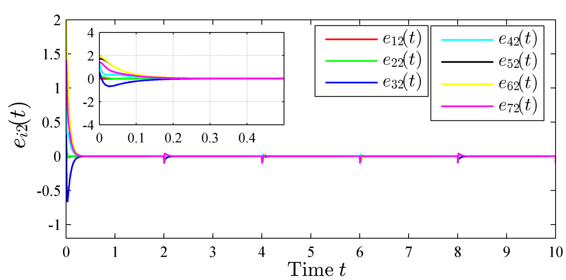

FIGURE 10. Trajectories of bipartite tracking errors $e_{i2}(i = 1, 2, \cdots, 7)$.

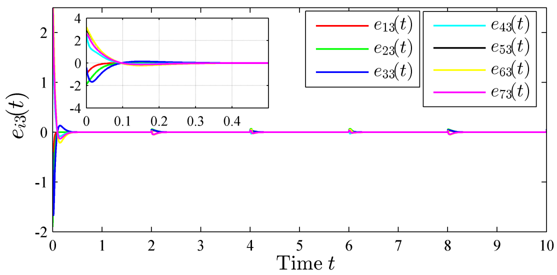

FIGURE 11. Trajectories of bipartite tracking errors $e_{i3}(i = 1, 2, \cdots, 7)$.

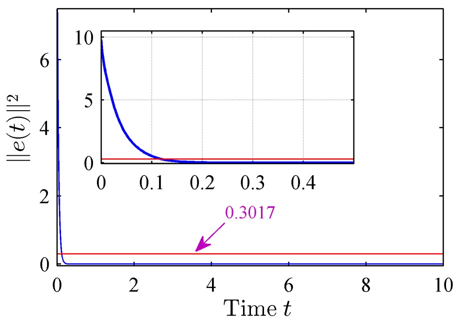

FIGURE 12. $\|e(t)\|^2$ in Example 2.

Moreover, $\zeta_1(1)$ and $\zeta_1(2)$ can be chosen as $\zeta_1(1) = 478$ and $\zeta_1(2) = 1120$, respectively. based on Corollary 2, one gets that the containment error bound $c = 0.3017$. The experimental circuit diagram of the delayed Chua's circuit is shown in Figure 5. Figures 6-8 show time evolutions of state variables of seven followers and one leader. Figures 9-11 display time evolutions of the containment error signals. Figure 12 depicts the time evolution of $\|e(t)\|^2$, which demonstrates that the experimental upper bound of $\|e(t)\|^2$ is less than the theoretical upper bound.

Remak 8: Example 2 cannot be studied by applying the results in [12], because the nonlinear function without delay is different from the one with time-varying delay in this paper, and a switching topology and impulsive attacks were not taken in account in [12].

V. CONCLUSION

The problem of bipartite containment control design has been investigated for a class of nonlinear multiagent systems with node-delay under signed switching topologies and impulsive false-data-injection attacks. A memory control protocol has been designed to guarantee that the dynamics of the bipartite containment error system is ultimately bounded in mean square. The simulation examples have been presented to demonstrate the effectiveness of the proposed design scheme of bipartite containment/leader-tracking controllers. Further research topics include the extension of the main results to the bipartite containment/leader-tracking problem under event-triggered mechanism [34], [35].

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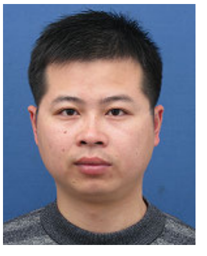

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