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RESEARCH ARTICLE

State-Space Region-Invariance of Discrete-Time Control Systems

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ABSTRACT In the paper, the so-called region-invariance properties of discrete-time control systems are considered. An already known notion of positivity of control systems has been generalized (using a kind of geometric insight instead of a pure algebraic approach) to the general nonlinear discrete-time control systems, general regions in state-space, and controls from polyhedral cones in input-space. The class of such control systems has been characterized. Numerous numerical examples illustrating individual cases under consideration are presented in the paper.

INDEX TERMS Linear and nonlinear discrete-time control systems, state-space invariance, positive systems.

I. INTRODUCTION

It is generally known that modeling real phenomena and objects that we want to control plays a huge role in controlling. When determining the mathematical model of a phenomenon, we try to make sure that it reflects best, from the interest point of view, the phenomenon being described. One thing is the need to know as much as possible about a real object in order to determine its model, and another very important issue is the knowledge and a good understanding of the mathematical model itself, i.e., the properties that characterize it. Commonly desired information about a real control object that we infer and derive from a mathematical model are, e.g., stability, reachability, controllability, observability. When designing object control systems, we very often limit ourselves only to this very information, without knowing anything else. Is such knowledge really sufficient and we do not need additional knowledge? Such statement may sometimes be true. But as we know, there is never enough knowledge. In recognition of the need for additional knowledge, this article has been written to provide results in a form that gives the opportunity to verify some additional properties about an object model.

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This paper focuses on certain properties that describe the behavior of mathematical models of discrete-time control systems with respect to certain regions of its state-space. For example, it may be desirable to know whether the trajectories of a system are confined to a limited region or may never reach it. In this article, systems whose trajectories evolve only in constrained regions are defined as region-invariant (or just invariant) with respect to these regions. The possibility and ability to check such properties carries a lot of information in itself, but can also provide very important additional complementary knowledge regarding, for example, the issue of testing reachability or controllability. For example, if we know about a system that it is invariant with respect to a certain region in the state-space, then we automatically learn more about possible reachable sets (narrowing the search area), or we can prejudge the controllability of the system—if the target state of the system is outside the invariant region, it will be known that it will never be reached (from a starting point belonging to the invariant region) using admissible controls.

The theory of invariant systems has been developed for years. First significant concepts and results of the invariance theory, concerning continuous dynamics, were established by [1], [2], and [3]. The invariance and viability theory for differential inclusions has been investigated and developed by [4], [5], [6], and [7]. Invariance theory of continuous-time linear dynamical and control systems were investigated in [8], [9], [10], [11], [12], [13], and [14]. Invariance of nonlinear continuous-time dynamical and control systems were studied in [15], [16], and [17]. The theory of linear systems dealing with the invariance of linear control systems in the areas that are vector subspaces of the state-space—the issue of the so-called geometric approach for linear systems—is well known [11], [18]. This theory allows, for example, to define the so-called controllability subspaces of linear systems. Also the theory of invariant systems in areas that are not linear subspaces is being developed. The flagship type of such systems are positive systems, in which all state variables take only nonnegative values with nonnegative initial state and non-negative input values [19], [20]. The invariant region, therefore, for such systems is the non-negative orthant. The class of such systems is very much reflected in many real-life phenomena or processes, e.g. in engineering, biology, economics, chemistry, social science, etc. Well-known examples of invariant systems are numerous phenomena occurring in ecosystems, the behavior of which (from the point of view of the population dynamics of different individuals) can be described, for example, using Lotka-Volterra equations [21].

In this work, we focus on discrete-time control systems, where we deal with the generalization of non-negative orthants (as in positive systems) into more general areas, namely, we focus on: nonlinear regions that are the intersections of n inequalities defined by n independent functions in n-dimensional state-space, and polyhedral cones in the m-dimensional input-space. The great value of such generalized regions is their non-singular transformability into corresponding regions in the form of non-negative orthants, which yields the possibility of formulating quite simple and practically verifiable conditions. A generalization of this kind has been made before, but for continuous-time systems [21], [22], and linear discrete-time systems (with polyhedral cones in state- and input-space) [23]. The concept of invariance of discrete-time control systems has been already made in [10] with a very general result obtained for a specific class of regions that do not have the property of transformability onto non-negative orthants, so that the presented there invariance condition—in a very general form—may not always be practically useful. Therefore, the motivation and purpose of this work, is to contribute to existing knowledge by providing a complete characterization of nonlinear discrete-time control systems (both in general and control-affine form) on invariant nonlinear regions (with a corner), together with the formulation of relatively simple and practically verifiable conditions, which is made possible thanks to the chosen specific (although, as mentioned, quite general) form, and, especially—their mapping onto non-negative orthants, of both the regions in the state- and in input-space. Such a result would constitute an innovation with respect to existing knowledge.

In this work, we consider the following to be an original new contribution to existing knowledge: generalization of nonlinear positive discrete-time control systems into nonlinear systems invariant on more general regions in state-space—their characterization; providing verifiable conditions for region-invariance of general and control-affine form of nonlinear discrete-time systems; obtaining results on invariance of discrete-time systems with respect to specific regions in state- and input-space by approaching the problem with a kind of geometrical insight that allows a better view and understanding of the issue, in contrast to the purely algebraic approach, known from the literature so far, to such a class of systems. What distinguishes, among other things, this contribution from that used in the literature, for example, of positive systems, is that derivation and proving these conditions is obtained without the need to know the solution to the dynamics of the system.

Thanks to the fact that the region-invariant nature of dynamical control systems is an intrinsic property of those systems, i.e. independent of the choice of the coordinate system in which the system is expressed, a geometric approach in this work will be used.

The paper is organised as follows. Section II presents the characterization of nonlinear discrete-time invariant control systems on nonlinear regions in the state-space with controls belonging to a region that is a polyhedral cone in the input-space. In addition, cases resulting from the general characterization applied to special classes of discrete-time systems such as control-affine systems and linear systems, and the special form of invariant regions in the state-space, are given, thus obtaining some of the well-known results from the literature achieved there by using an algebraic approach. Section 4 summarizes the considerations presented in the paper. The presented theory is illustrated with numerous computational examples.

A. NOTATION AND PRELIMINARIES

Throughout the paper the following notation will be used. The sets of natural numbers, and naturals with zero will be denoted by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The set of all real numbers is denoted by $\mathbb{R}$. The notation $\mathbb{R}^n$ refers to n-dimensional vector space over the field of the real numbers $\mathbb{R}$. Non-negative and non-positive real numbers (both including zero) are denoted $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. By $\mathbb{R}^n_+$ ($\mathbb{R}^n_-$) we mean the Cartesian product of n copies of $\mathbb{R}_+$ ($\mathbb{R}_-$) and call it a non-negative (non-positive) orthant. By $\mathbb{R}^{n\times m}$ we denote the set of $n\times m$ matrices with entries from the field $\mathbb{R}$. The identity matrix of dimension $n \times n$ is denoted as $I_{n\times n}$. Let P denotes a matrix, vector or vector-valued function. The notation $P > 0$ ($P < 0$) means that all elements of P are positive (negative). The notation $P \geq 0$ ($P \leq 0$) means that all elements of P are non-negative (non-positive). By $P \ngeq 0$ we mean that at least one element of P is negative. If P is a matrix, its ith column vector is denoted P_i. For a set $S \subset V$, by S^c we mean the complement of S, i.e the set of elements of V that are not in S. Operation "$\circ$" denotes function composition.

II. MAIN RESULTS

Let us consider a discrete-time control system

$$\Pi : \quad x_{k+1} = f(x_k, u_k), \tag{1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are the values of state and input vectors at time index k, respectively, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a system map. Superscripts will be used to denote the components of a vector or vector-valued function (which directly depend on time), e.g., x_k^i denotes the value at time instant k of the ith component of the state vector x_k. By $\bar{x}_k = \bar{x}_k(x_0, \bar{u}_k)$ we mean a trajectory of Π, i.e. the sequence $(x_0, \ldots, x_k)$ of states $x_0, \ldots, x_k$, issued from x_0 and satisfying $x_{j+1} = f(x_j, u_j)$, $0 \le j \le k-1$, for a sequence $\bar{u}_k = (u_0, \ldots, u_k)$ of controls $u_0, \ldots, u_k$. Sometimes, for the reason of presentation transparency, we will use the simplified notation by omitting the time index k, where its appearance is irrelevant (or not essential) from the point of view of an issue under consideration, e.g., $f(x_k, u_k)$ may be written as $f(x, u)$, where x and u obviously depend on k.

We define the following regions in both state- and input-space. The so called nonlinear corner region in state-space $\mathbb{R}^n$ is the following

$$\begin{aligned}\mathscr{K} &= \{x \in \mathbb{R}^n : \ \varphi_i(x) \ge 0, \ 1 \le i \le n\} \\ &= \bigcap_{i=1}^{n} \{\varphi_i \ge 0\},\end{aligned} \tag{2}$$

where $\Phi = (\varphi_1, \ldots, \varphi_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism. On the other hand, in the input-space we define a polyhedral cone in the form

$$\begin{aligned}\mathcal{W} &= \{u \in \mathbb{R}^m : \ w^i u \ge 0, \ 1 \le i \le m\} \\ &= \bigcap_{i=1}^{m} \{w^i u \ge 0\},\end{aligned} \tag{3}$$

where w^i for $i = 1, \ldots, m$ are rows of a non-singular matrix $W \in \mathbb{R}^{m \times m}$.

In order to present the crucial, albeit rather obvious, properties of the region $\mathscr{K}$, let us recall the notation in which $\mathscr{K}^{\mathrm{c}}$ and $\left(\mathbb{R}_+^n\right)^{\mathrm{c}}$ denote the complements in $\mathbb{R}^n$ of the sets $\mathscr{K}$ and $\mathbb{R}_+^n$, respectively.

Lemma 1: The map $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ defining $\mathscr{K}$ via (2) is given in x-coordinates of the source $\mathbb{R}^n$ and $\tilde{x}$-coordinates of the target $\mathbb{R}^n$ as $\tilde{x} = \Phi(x)$ and satisfies

(i) $\Phi(\mathscr{K}) = \mathbb{R}_+^n = \{\tilde{x}_i \ge 0\}$ and $\Phi^{-1}(\mathbb{R}_+^n) = \mathscr{K}$;
(ii) $\Phi(\mathscr{K}^{\mathrm{c}}) = \left(\mathbb{R}_+^n\right)^{\mathrm{c}}$ and $\Phi^{-1}\left(\left(\mathbb{R}_+^n\right)^{\mathrm{c}}\right) = \mathscr{K}^{\mathrm{c}}$.

These quite evident properties follow naturally from the definition of $\mathscr{K}$ and the bijectivity of the diffeomorphism Φ. Still, for a reason of clarity and in order to better understand the essence of the future considerations, we will provide, in a specially intended form, the proof of Lemma 1.

Proof of Lemma 1: (i) For any $x \in \mathscr{K}$, from definition of $\mathscr{K}$, we have $\Phi(x) = \tilde{x} \ge 0$, i.e., $\tilde{x} \in \mathbb{R}_+^n$. Now, let us assume that there exists some $\tilde{x} \in \mathbb{R}_+^n$ such that $\Phi^{-1}(\tilde{x}) = x \notin \mathscr{K}$. This in turn means that $\Phi(x) = \tilde{x} \ngeq 0$ which leads to a contradiction.

(ii) Let $x \in \mathscr{K}^{\mathrm{c}}$, which means that $x \notin \mathscr{K}$, then $\tilde{x} = \Phi(x) \ngeq 0$, i.e., $\tilde{x} \notin \mathbb{R}_+^n$, so $\tilde{x} \in \left(\mathbb{R}_+^n\right)^{\mathrm{c}}$. Now, let us assume that there exists some $\tilde{x} \in \left(\mathbb{R}_+^n\right)^{\mathrm{c}}$ such that $\Phi^{-1}(\tilde{x}) = x \notin \mathscr{K}^{\mathrm{c}}$, which corresponds to $x \in \mathscr{K}$. This in turn means that $\Phi(x) = \tilde{x} \ge 0$, i.e., $\tilde{x} \in \mathbb{R}_+^n$, which leads to a contradiction.

Remark 1: Analogous results can be obtained for the linear map $u \mapsto Wu$, being a linear isomorphism from $\mathbb{R}^m$ to $\mathbb{R}^m$, defining a polyhedral cone $\mathcal{W}$ via (3).

Definition 1: Let $\mathscr{K}$ be a nonlinear corner region in $\mathbb{R}^n$ and $\mathcal{W}$ a linear corner region in $\mathbb{R}^m$. A nonlinear system Π of the form (1), is said to be $(\mathscr{K}, \mathcal{W})$-invariant if $\bar{x}_k(x_0, \bar{u}_k) \in \mathscr{K}$ for each $x_0 \in \mathscr{K}$, each $\bar{u}_k \in \mathcal{W}$, and all $k \in \mathbb{N}$.

Below, we provide a general characterization of invariant nonlinear discrete-time control systems.

Theorem 1: The following conditions are equivalent for a nonlinear control system Π:

(i) Π is $(\mathscr{K}, \mathcal{W})$-invariant;
(ii) $(\varphi_i \circ f)(x_k, u_k) \ge 0$ for all $1 \le i \le n$, for each $x_k \in \mathscr{K}$, $u_k \in \mathcal{W}$ and any $k \in \mathbb{N}_0$;
(iii) $(\Phi \circ f)\left(\Phi^{-1}(\tilde{x}_k), W^{-1}\tilde{u}_k\right) \ge 0$ for each $\tilde{x}_k \in \mathbb{R}_+^n$, $\tilde{u}_k \in \mathbb{R}_+^m$ and any $k \in \mathbb{N}_0$.

Before proceeding to the proof of Theorem 1, we will briefly explain the meaning of its conditions. Item (ii) requires us to check the behavior of the vector-valued function $f(x_k, u_k)$ with respect to each constraint function φ_i, $1 \le i \le n$, at each point $x_k \in \mathscr{K}$ and for all inputs $u_k \in \mathcal{W}$. The nonnegativity of this condition guarantees that the trajectory $\bar{x}_k$ will remain within $\mathscr{K}$. Condition (iii), alternative to condition (ii), results from the reformulation of condition (ii) by checking the behavior of the modified function f for all $\tilde{x}_k \in \mathbb{R}_+^n$ and all $\tilde{u}_k \in \mathbb{R}_+^m$, which in many practical cases may simplify the verification of system's invariance.

The modification of the function f mentioned above is just a transformation of the system Π, both in the state-space—by means of a nonlinear transformation $\tilde{x} = \Phi(x)$, as well as in the input-space—by means of the linear transformation $\tilde{u} = Wu$. Indeed, in order to get the transformed system dynamics $\tilde{\Pi}$, we express $\tilde{x} = \Phi(x)$ at time instant $k+1$ obtaining

$$\begin{aligned}\tilde{\Pi} : \quad \tilde{x}_{k+1} &= \Phi(x_{k+1}) \\ &= (\Phi \circ f)(x_k, u_k)\big|_{(\Phi^{-1}(\tilde{x}), W^{-1}\tilde{u})} \\ &= \left(\Phi \circ f \circ \Phi^{-1}\right)(\tilde{x}_k, W^{-1}\tilde{u}_k) \\ &= \tilde{f}(\tilde{x}_k, \tilde{u}_k).\end{aligned}$$

Obviously (we already know it from the definition of $\mathscr{K}$ or from Lemma 1), the nonlinear corner region $\mathscr{K}$ is simultaneously (together with the transformation of Π into $\tilde{\Pi}$) mapped, by means of $\tilde{x} = \Phi(x)$, to the region $\tilde{\mathscr{K}} = \Phi(\mathscr{K}) = \mathbb{R}_+^n$ being a non-negative orthant.

Proof of Theorem 1: (i) $\Rightarrow$ (ii): Since Π is $(\mathscr{K}, \mathcal{W})$-invariant, for any $x_0 \in \mathscr{K}$, state $x_k = f(x_{k-1}, u_{k-1}) \in \mathscr{K}$ for all $k = 1, 2, \ldots$. It follows that $x_{k+1} = f(x_k, u_k) \in \mathscr{K}$.

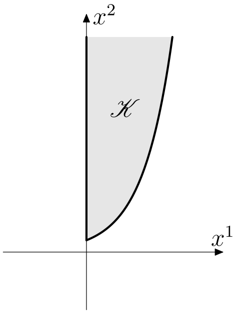

FIGURE 1. Nonlinear corner region $\mathscr{K}$ in the state-space $\mathbb{R}^2$ from Ex. 1.

Therefore, from definition of $\mathscr{K}$, we have $\varphi_i\left(f(x_k,u_k)\right) \geq 0$, i.e., $(\varphi_i \circ f)(x_k,u_k) \geq 0$ for all $1 \leq i \leq n$, for each $x_k \in \mathscr{K}$, $u_k \in \mathcal{W}$ and $k \in \mathbb{N}_0$.

(ii) $\Rightarrow$ (iii): Since the nonlinear corner region $\mathscr{K}$ is transformed onto $\mathbb{R}^n_+$ with the help of the diffeomorphism Φ (by definition of $\mathscr{K}$), and the cone $\mathcal{W}$ is transformed onto $\mathbb{R}^m_+$ by means of the transformation matrix W (by definition of $\mathcal{W}$), putting $\tilde{x} = \Phi(x)$ and $\tilde{u} = Wu$, there exist $x = \Phi^{-1}(\tilde{x})$ and $u = W^{-1}\tilde{u}$ such that for any $\tilde{x} \in \mathbb{R}^n_+$ and $\tilde{u} \in \mathbb{R}^m_+$, we have $x \in \mathscr{K}$ and $u \in \mathcal{W}$, respectively. Therefore, the condition $(\varphi_i \circ f)(x_k,u_k) \geq 0$ for all $1 \leq i \leq n$, all $x_k \in \mathscr{K}$ and all $u_k \in \mathcal{W}$ is equivalent to $(\varphi_i \circ f)(\Phi^{-1}(\tilde{x}_k), W^{-1}\tilde{u}_k) \geq 0$ for all $\tilde{x}_k \in \mathbb{R}^n_+$ and all $\tilde{u}_k \in \mathbb{R}^m_+$. So, taking into account all φ_i, $1 \leq i \leq n$, we get $(\Phi \circ f)\left(\Phi^{-1}(\tilde{x}), W^{-1}\tilde{u}\right) \geq 0$ for each $\tilde{x} \in \mathbb{R}^n_+$ and $\tilde{u} \in \mathbb{R}^m_+$.

(iii) $\Rightarrow$ (i): Since the dynamics f, being transformed by means of $\tilde{x}_k = \Phi(x_k)$, and being a composition $\tilde{f}(\tilde{x},\tilde{u}) = \left(\Phi \circ f \circ \Phi^{-1}\right)(\tilde{x}, W^{-1}\tilde{u})$ is always non-negative at any $\tilde{x}_k \in \mathbb{R}^n_+$ for $\tilde{u}_k \in \mathbb{R}^m_+$, and $k \in \mathbb{N}_0$, it means, thanks to Lemma 1, that $x_k = \Phi^{-1}(\tilde{x}_k)$ always remains within $\mathscr{K}$, because Φ^{-1} maps $\mathbb{R}^n_+$ onto $\mathscr{K}$.

Remark 2: It is worth to notice that the above invariance conditions, due to discrete-time nature of dynamics, have to be verified at all points x of the region $\mathscr{K}$ (interior and boundary of $\mathscr{K}$), and not only at the boundary of $\mathscr{K}$ as it is the case of continuous-time systems.

Example 1: Let us consider the nonlinear control system Π, where (we dropped for transparency the time index k)

$$f(x,u) = \begin{pmatrix} x^1u^2 \\ e^{x^1u^2} + \frac{x^1}{x^2+u^1-u^2} \end{pmatrix}, \tag{4}$$

where

$$x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in \mathbb{R}^2,\ u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \mathbb{R}^2,$$

and the following regions $\mathscr{K} \subset \mathbb{R}^2$ and $\mathcal{W} \subset \mathbb{R}^2$ defined (see Figs. 1 and 2, respectively) by

$$\Phi(x) = \begin{pmatrix} x^1 \\ -e^{x^1} + x^2 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

respectively.

It is worth noticing that (4) is well defined and C^∞-smooth at any $x \in \mathscr{K}$ and $u \in \mathcal{W}$; indeed $x^2 \geq 1$ and $u^1 - u^2 \geq 0$, thus excluding the denominator to vanish.

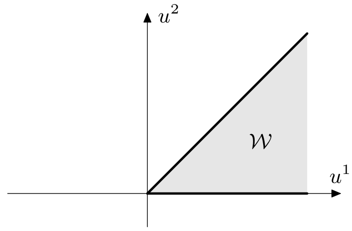

FIGURE 2. Cone $\mathcal{W}$ in the input-space $\mathbb{R}^2$ from Ex. 1.

We will verify, using conditions of Theorem 1, whether system (4) is $(\mathscr{K},\mathcal{W})$-invariant. Using (ii) we obtain

$$(\varphi_1 \circ f)(x,u) = x^1u^2 \geq 0$$
$$(\varphi_2 \circ f)(x,u) = \frac{x^1}{x^2+u^1-u^2} \geq 0$$

for all $x \in \mathscr{K}$ and $u \in \mathcal{W}$. From condition (iii), we get

$$\begin{aligned}(\Phi \circ f)\left(\Phi^{-1}(\tilde{x}), W^{-1}\tilde{u}\right) &= \begin{pmatrix} x^1u^2 \\ \frac{x^1}{x^2+u^1-u^2} \end{pmatrix}\left(\Phi^{-1}(\tilde{x}), W^{-1}\tilde{u}\right) \\ &= \begin{pmatrix} \tilde{x}^1\tilde{u}^2 \\ \frac{\tilde{x}^1}{e^{\tilde{x}^1}+\tilde{x}^2+\tilde{u}^1} \end{pmatrix} \geq 0\end{aligned}$$

for all $\tilde{x} = \left(\tilde{x}^1, \tilde{x}^2\right)^T \in \mathbb{R}^2_+$ and all $\tilde{u} \in \mathbb{R}^2_+$, where

$$\Phi^{-1}(\tilde{x}) = \begin{pmatrix} \tilde{x}^1 \\ e^{\tilde{x}^1} + \tilde{x}^2 \end{pmatrix} \quad \text{and} \quad W^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Finally, both conditions (ii) and (iii) show that system (4) is $(\mathscr{K},\mathcal{W})$-invariant.

Based on Theorem 1 we can establish the following particular cases of region-invariance for different types of control systems and regions in state-space.

Let us consider the identity map $\Phi = \mathrm{id}_{\mathbb{R}^n}$ defining a corner region $\mathscr{K}$, for which we have

$$\mathscr{K} = \left\{x = \left(x^1,\ldots,x^n\right)^T \in \mathbb{R}^n:\ x^i \geq 0,\ 1 \leq i \leq n\right\},$$

therefore $\mathscr{K} = \mathbb{R}^n_+$. Consider also a polyhedral cone $\mathcal{W}$ in the input-space with $W = I_{m\times m}$, for which

$$\mathcal{W} = \left\{u = \left(u^1,\ldots,u^m\right)^T \in \mathbb{R}^m:\ u^i \geq 0,\ 1 \leq i \leq m\right\},$$

therefore $\mathcal{W} = \mathbb{R}^m_+$. For such regions, we can present the following, already known (see, e.g., [24]), result.

Corollary 1: The nonlinear system Π is $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-invariant if and only if

$$f_i(x_k,u_k) \geq 0, \quad 1 \leq i \leq n \tag{5}$$

for all $x_k \in \mathbb{R}^n_+$ and all $u_k \in \mathbb{R}^m_+$.

Proof: Having known that the non-negative orthant $\mathbb{R}^n_+$ is described by Φ being the identity map, the property (ii) of Theorem 1 implies $(\varphi_i \circ f)(x_k,u_k) \equiv f_i(x_k,u_k) \geq 0$ for all $1 \leq i \leq n$ and each $x_k \in \mathbb{R}^n_+$ and $u_k \in \mathbb{R}^m_+$. □

Remark 3: A system Π that is $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-invariant is usually called a positive system, see e.g. [20] and [19] for an extensive literature on positive systems. Corollary 1 is then a well-known characterization of positive systems, see e.g. [24].

Example 2: Let us consider a system Π, where

$$f(x, u) = \begin{pmatrix} (x^1)^2 + x^2\sqrt{u} \\ (x^2)^2 \end{pmatrix},$$

with the regions $\mathcal{K} = \mathbb{R}^2_+$ and $\mathcal{W} = \mathbb{R}_+$. Checking condition (5) of Corollary 1

$$f(x, u) = \begin{pmatrix} (x^1)^2 + x^2\sqrt{u} \\ (x^2)^2 \end{pmatrix} \geq 0$$

for all $x \in \mathbb{R}^2_+$ and all $u \in \mathbb{R}_+$, and finding that it is fulfilled, yields the $(\mathbb{R}^2_+, \mathbb{R}_+)$-invariance of the system.

Quick analysis of the above system dynamics shows that the values of x that constitute the trajectories emanating from any given point $x_0 \in \mathbb{R}^2_+$ can only increase in value with respect to both components, or remain the same, which yields them being always fully within the desired region, implying the $(\mathbb{R}^2_+, \mathbb{R}_+)$-invariance of the system.

Let us consider a control-affine system of the form

$$\Sigma: \quad x_{k+1} = f(x_k) + G(x_k)u_k, \tag{6}$$

where state vector $x_k \in \mathbb{R}^n$, input vector $u_k \in \mathbb{R}^m$, matrix $G(x) = (g_1(x), \ldots, g_m(x))$, with vector-valued functions $g_j(x) = \left(g_j^1(x), \ldots, g_j^n(x)\right)^T$, $1 \leq j \leq m$, and obviously

$$G(x_k)u_k = \sum_{j=1}^{m} g_j(x_k)u_k^j.$$

For such a system, we can present the following corollaries.

Corollary 2: The control-affine system Σ is $(\mathscr{K}, \mathcal{W})$-invariant if and only if

$$\varphi_i \circ (f(x) + G(x)u) \geq 0, \quad 1 \leq i \leq n$$

for all $x \in \mathscr{K}$ and all $u \in \mathcal{W}$ or, equivalently,

$$\Phi \circ \left(f\left(\Phi^{-1}(\tilde{x})\right) + G\left(\Phi^{-1}(\tilde{x})\right) W^{-1}\tilde{u}\right) \geq 0$$

for all $\tilde{x} \in \mathbb{R}^n_+$ and all $\tilde{u} \in \mathbb{R}^m_+$.

Proof: It follows directly from Theorem 1, where the right-hand side of the system Π is just replaced by the right-hand side of the system Σ. □

Remark 4: Although Corollary 2 does not add anything particularly new, it is presented here to signal that for a control-affine system Σ and a nonlinear corner region $\mathscr{K}$, the conditions are not simplified, unlike for the continuous counterpart [21]. This is due to the lack, in general, of the linearity property of function composition operation, i.e. with Φ being nonlinear, as opposed to the continuous-time case, where the conditions are expressed using directional derivatives, which are linear.

The only additional information arising from the control-affine form of the system is the following necessary condition that gives a preliminary opportunity to verify the invariance of Σ on the nonlinear corner region $\mathscr{K}$.

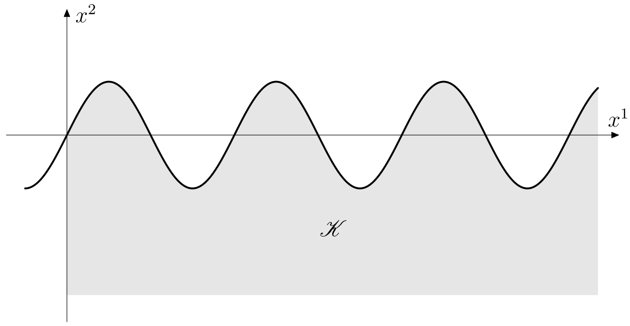

FIGURE 3. **The nonlinear corner region $\mathscr{K}$ from Ex. 1.**

Corollary 3: A necessary condition for the $(\mathscr{K}, \mathcal{W})$-invariance of the control-affine system Σ is

$$(\varphi_i \circ f)(x) \geq 0, \quad 1 \leq i \leq n$$

for all $x \in \mathscr{K}$.

Proof: This condition follows from the 1st condition of Corollary 2 expressed for $u = 0$, because it holds for all $u \in \mathcal{W}$, in particular for $u = 0$. □

Example 3: Let us consider the nonlinear control system Σ, where

$$f(x) = \begin{pmatrix} x^1 \\ \sin x^1 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ \sin x^1 - x^2 \end{pmatrix} \tag{7}$$

and the following regions $\mathscr{K} \subset \mathbb{R}^2$ and $\mathcal{W} \subset \mathbb{R}$ defined by (see Fig. 3)

$$\Phi(x) = \begin{pmatrix} \sin x^1 - x^2 \\ x^1 \end{pmatrix} \quad \text{and} \quad W = -1,$$

respectively. We will verify, using conditions of Corollary 2, whether system (7) is $(\mathscr{K}, \mathcal{W})$-invariant. We have

$$\begin{aligned} \varphi_1 \circ (f(x) + g(x)u) &= \sin x^1 - \sin x^1 - \left(\sin x^1 - x^2\right) u \\ &= -\left(\sin x^1 - x^2\right) u \geq 0 \\ \varphi_2 \circ (f(x) + g(x)u) &= x^1 \geq 0, \end{aligned}$$

for all $x \in \mathscr{K}$ and $u \in \mathcal{W}$. From the alternative condition, we get

$$\begin{aligned} &\Phi \circ \left(f\left(\Phi^{-1}(\tilde{x})\right) + G\left(\Phi^{-1}(\tilde{x})\right) W^{-1}\tilde{u}\right) \\ &= \begin{pmatrix} -\left(\sin x^1 - x^2\right) u \\ x^1 \end{pmatrix} \left(\Phi^{-1}(\tilde{x}), W^{-1}\tilde{u}\right) \\ &= \begin{pmatrix} \tilde{x}^1 \tilde{u} \\ \tilde{x}^2 \end{pmatrix} \geq 0, \end{aligned}$$

where

$$\Phi^{-1}(\tilde{x}) = \begin{pmatrix} \tilde{x}^2 \\ -\tilde{x}^1 + \sin \tilde{x}^2 \end{pmatrix} \quad \text{and} \quad W^{-1} = -1,$$

for all $\tilde{x} = \left(\tilde{x}^1, \tilde{x}^2\right)^T \in \mathbb{R}^2_+$ and $\tilde{u} \in \mathbb{R}_+$. Finally, from each of the two alternative conditions follows that system (7) is $(\mathscr{K}, \mathcal{W})$-invariant.

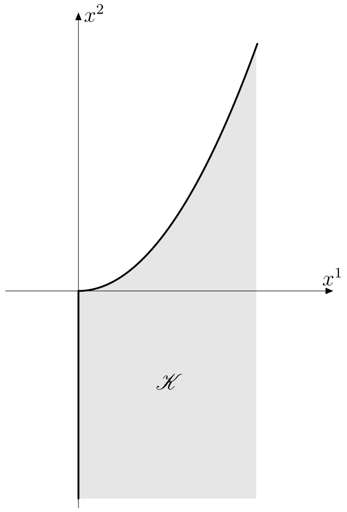

FIGURE 4. The nonlinear corner region $\mathscr{K}$ from Ex. 4.

The following example shows that it is incorrect to split the conditions of Corollary 2 into two conditions $\varphi \circ f$ and $\varphi \circ g$, in contrast to continuous-time nonlinear systems.

Example 4: Let us consider the following nonlinear control-affine system of the form (6), where

$$f(x) = \begin{pmatrix} x^1 \\ 0 \end{pmatrix}, \quad G(x) = \begin{pmatrix} e^{x^1} \\ e^{2x^1} \end{pmatrix} \tag{8}$$

and the following regions $\mathscr{K} \subset \mathbb{R}^2$ and $\mathcal{W} = \mathbb{R}_+$ defined by (see Fig. 4)

$$\Phi(x) = \begin{pmatrix} \left(x^1\right)^2 - x^2 \\ x^1 \end{pmatrix} \quad \text{and} \quad W = 1,$$

respectively. From Corollary 2 we get

$$\Phi \circ (f(x) + g(x)u) = \begin{pmatrix} \left(x^1 + e^{x^1}u\right)^2 - e^{2x^1}u \\ x^1 + e^{x^1}u \end{pmatrix},$$

which, e.g., for $x = 0 \in \mathscr{K}$, is

$$[\varphi \circ (f(x) + g(x)u)]_{x=0} = \begin{pmatrix} u(u-1) \\ u \end{pmatrix}$$

and thus takes negative values (actually, its first component does) for $u \in (0, 1) \in \mathcal{W}$. Therefore, system (8) is not $(\mathscr{K}, \mathcal{W})$-invariant.

However,

$$(\varphi \circ f)(x) = \begin{pmatrix} \left(x^1\right)^2 \\ x^1 \end{pmatrix} \geq 0$$

$$(\varphi \circ g)(x) = \begin{pmatrix} 0 \\ e^{x^1} \end{pmatrix} \geq 0$$

for each $x \in \mathscr{K}$ and all $u \in \mathcal{W}$.

Now, let us consider a corner region defined by a linear map $\Phi(x) = Kx$, where $K \in \mathbb{R}^{n \times n}$ is an invertible matrix, obtaining the following polyhedral cone

$$\begin{aligned} \mathcal{K} &= \{x \in \mathbb{R}^n \colon\ k^i x \geq 0,\ 1 \leq i \leq n\} \\ &= \bigcap_{i=1}^{n} \{k^i x \geq 0\}, \end{aligned} \tag{9}$$

where k^i, $1 \leq i \leq n$, are rows of K. Then, we get the following result.

Corollary 4: The control-affine system Σ is $(\mathcal{K}, \mathcal{W})$-invariant if and only if

$$k^i f(x) \geq 0 \quad \textit{and} \quad k^i G(x)u \geq 0, \quad 1 \leq i \leq n,$$

for all $x \in \mathcal{K}$ and all $u \in \mathcal{W}$ or, equivalently,

$$Kf(K^{-1}\tilde{x}) \geq 0 \quad \textit{and} \quad KG(K^{-1}\tilde{x})W^{-1} \geq 0$$

for all $\tilde{x} \in \mathbb{R}^n_+$.

Proof (Necessity): Since Σ is $(\mathcal{K}, \mathcal{W})$-invariant, first condition of Corollary 2 is satisfied, and thanks to the linearity of $\Phi = Kx$, takes the form

$$k^i\left(f(x) + G(x)u\right) = k^i f(x) + k^i G(x)u \geq 0, \quad 1 \leq i \leq n$$

for each $x \in \mathcal{K}$, $u \in \mathcal{W}$. Since this condition holds for $u = 0 \in \mathcal{W}$, we get $k^i f(x) \geq 0$, $1 \leq i \leq n$, for each $x \in \mathcal{K}$. Because $x = K^{-1}\tilde{x}$, we get $Kf\left(K^{-1}\tilde{x}\right) \geq 0$ for all $\tilde{x} \in \mathbb{R}^n_+$. Let us assume that $k^i G(x)u < 0$ for some $u \in \mathcal{W}$ and some $1 \leq i \leq n$. Since $u = W^{-1}\tilde{u}$, where $\tilde{u} \in \mathbb{R}^m_+$, we have $k^i G(x)W^{-1}\tilde{u} < 0$ for some $\tilde{u} \in \mathbb{R}^m_+$. Thus, we can find a large enough $\tilde{u}_{\text{big}} \in \mathbb{R}^m_+$ such that $k^i f(x) + k^i G(x)W^{-1}\tilde{u}_{\text{big}} < 0$, or thereby, $k^i f(x) + k^i G(x)u_{\text{big}} < 0$ for some $u_{\text{big}} = W^{-1}\tilde{u}_{\text{big}} \in \mathcal{W}$, which leads to a contradiction. Then, $k^i G(x)u \geq 0$ for each $u \in \mathcal{W}$ and also $k^i G(x)W^{-1}\tilde{u} \geq 0$ for each $\tilde{u} \in \mathbb{R}^m_+$, which obviously implies $k^i G(x)W^{-1} \geq 0$. Finally, since $x = K^{-1}\tilde{x}$, we get $KG(K^{-1}\tilde{x})W^{-1} \geq 0$ for each $\tilde{x} \in \mathbb{R}^n_+$.

(Sufficiency) Conditions $k^i f(x) \geq 0$ and $k^i G(x)u \geq 0$, $1 \leq i \leq n$, for all $x \in \mathcal{K}$ and all $u \in \mathcal{W}$, imply $k^i f(x) + k^i G(x)u = k^i\left(f(x) + G(x)u\right) \geq 0$, $1 \leq i \leq n$, for all $x \in \mathcal{K}$ and all $u \in \mathcal{W}$, which means that the first condition of Corollary 2 is satisfied, thus Σ is $(\mathcal{K}, \mathcal{W})$-invariant.

Likewise, conditions $Kf(K^{-1}\tilde{x}) \geq 0$ and $KG(K^{-1}\tilde{x})W^{-1} \geq 0$ for all $\tilde{x} \in \mathbb{R}^n_+$ imply $K\left(f(K^{-1}\tilde{x}) + G(K^{-1}\tilde{x})W^{-1}\tilde{u}\right) \geq 0$ for all $\tilde{x} \in \mathbb{R}^n_+$ and all $\tilde{u} \in \mathbb{R}^m_+$, which, thanks to the second condition of Corollary 2, means that Σ is $(\mathcal{K}, \mathcal{W})$-invariant.

For $\mathcal{K} \equiv \mathbb{R}^n_+$ and $\mathcal{W} \equiv \mathbb{R}^m_+$ we have a result, which immediately follows from Corollaries 1 and 4.

Corollary 5: The control-affine system Σ is $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-invariant if and only if

$$f(x) \geq 0 \quad \text{and} \quad g_j(x) \geq 0, \quad 1 \leq j \leq m$$

for all $x \in \mathbb{R}^n_+$.

Proof: It follows directly from Corollary 4, with $K = I_{n \times n}$ and $W = I_{m \times m}$. ☐

Example 5: Let us consider a control-affine system Σ described as follows

$$x_{k+1} = \begin{pmatrix} x^1_k + \left(x^2_k\right)^4 \\ \left(-x^1_k\right)^2 \end{pmatrix} + \begin{pmatrix} \left(-x^2_k\right)^2 \\ 0 \end{pmatrix} u_k\,,$$

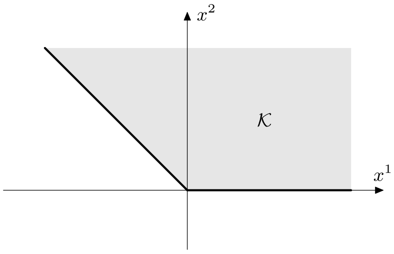

FIGURE 5. The cone $\mathcal{K}$ from Ex. 6.

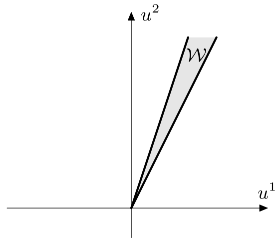

FIGURE 6. The cone $\mathcal{W}$ from Ex. 6.

with the regions given as $\mathcal{K} \equiv \mathbb{R}^2_+$ and $\mathcal{W} \equiv \mathbb{R}_+$. Checking the conditions of Corollary 5, we can trivially conclude that the system Σ is $(\mathbb{R}^2_+, \mathbb{R}_+)$-invariant.

Let us consider a linear discrete-time control system

$$\Xi: \quad x_{k+1} = Ax_k + Bu_k,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Consider also cones $\mathcal{K}$ and $\mathcal{W}$ defined by (9) and (3), respectively.

For such a system, based on Corollary 4, we can present the following (already known, see [23]) result.

Corollary 6: The linear system Ξ is $(\mathcal{K}, \mathcal{W})$-invariant if and only if

$$KAK^{-1} \geq 0 \quad and \quad KBW^{-1} \geq 0.$$

Proof: Making the substitution $f(K^{-1}\tilde{x}) = AK^{-1}\tilde{x}$ in the condition of Corollary 4, we get $KAK^{-1}\tilde{x} \geq 0$ for each $\tilde{x} \in \mathbb{R}^n_+$, which implies $KAK^{-1} \geq 0$. Similarly, substituting $G(K^{-1}\tilde{x}) = B$, we get $KBW^{-1} \geq 0$. □

Example 6: Let the linear system Ξ be described by the following matrices

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

with the cones $\mathcal{K}$ and $\mathcal{W}$ defined by (see Figs. 5 and 6, respectively)

$$K = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix},$$

respectively. Let us investigate, whether the given system is $(\mathcal{K}, \mathcal{W})$-invariant. From Corollary 6, we find that

$$KAK^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix} \geq 0$$

and

$$KBW^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 15 & 22 \\ 10 & 14 \end{pmatrix} \geq 0,$$

which means that the system is $(\mathcal{K}, \mathcal{W})$-invariant.

For $\mathcal{K} \equiv \mathbb{R}^n_+$ and $\mathcal{W} \equiv \mathbb{R}^m_+$ we have a result, which immediately follows from Corollary 6.

Corollary 7: A linear system Ξ is $(\mathbb{R}^n_+, \mathbb{R}^m_+)$-invariant (that is positive) if and only if

$$A \geq 0 \quad and \quad B \geq 0.$$

Proof: It follows directly from Corollary 6, with $K = I_{n\times n}$ and $W = I_{m\times m}$. □

Example 7: Consider the following system

$$x_{k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k .$$

Since $A \geq 0$ and $B \geq 0$ the system is $(\mathbb{R}^2_+, \mathbb{R}_+)$-invariant. Calculating the trajectories yields the same result, because $x^1_k = x^1_0 = \text{const}$ for each k and $x^2_k = u_{k-1}$ which can remain constant or grow depending on whether $u_k = 0$ or $u_k > 0$, respectively, therefore proving that any trajectory $\bar{x}_k = (x_0, \dots, x_k)$ satisfies $x_j \in \mathbb{R}^2_+$, $0 \leq j \leq k$.

Remark 5: Obviously, the system from Ex. 6 is both $(\mathcal{K}, \mathcal{W})$-invariant and $(\mathbb{R}^2_+, \mathbb{R}^2_+)$-invariant (that is positive) since $A > 0$ and $B > 0$, where $\mathbb{R}^2_+ \subsetneq \mathcal{K}$ and $\mathcal{W} \subsetneq \mathbb{R}^2_+$. It means that a trajectory $\bar{x}_k$ starting from $x_0 \in \mathbb{R}^2_+$ will remain within $\mathbb{R}^2_+$ for any $\bar{u}_k = (u_0, \dots, u_k)$, with $u_j \geq 0$, $0 \leq j \leq k$, which means that an invariant region can contain smaller invariant regions within it, however, in this case, with different cones in the input-space.

Below, some region-symmetry property of $(\mathcal{K}, \mathcal{W})$-invariant linear systems Ξ, is presented. In order to give it, let us define the cones that are reflections about their system coordinates origins of $\mathcal{K}$ and $\mathcal{W}$, respectively:

$$\bar{\mathcal{K}} = \{x \in \mathbb{R}^n : \; -Kx \geq 0\} = \{x \in \mathbb{R}^n : \; Kx \leq 0\}$$

and

$$\bar{\mathcal{W}} = \{u \in \mathbb{R}^m : \; -Wu \geq 0\} = \{u \in \mathbb{R}^m : \; Wu \leq 0\}.$$

That is, for any $x \in \mathcal{K}$ and any $u \in \mathcal{W}$, we have $-x \in \bar{\mathcal{K}}$ and $-u \in \bar{\mathcal{W}}$.

Corollary 8: If a system Ξ is $(\mathcal{K}, \mathcal{W})$-invariant, then it is also $(\bar{\mathcal{K}}, \bar{\mathcal{W}})$-invariant.

Proof: It follows directly from Corollary 6, that is

$$-KA(-K)^{-1} = KAK^{-1} \geq 0$$

and

$$-KB(-W)^{-1} = KBW^{-1} \geq 0$$

due to the $(\mathcal{K}, \mathcal{W})$-invariance of Ξ. □

III. CONCLUSION

This paper presents a characterization of nonlinear discrete-time control systems on invariant regions in the state-space with controls belonging to constrained sets in the form of polyhedral cones in the input-space. The obtained necessary and sufficient conditions for determining the invariance of discrete-time control systems are given in practically verifiable forms. They are distinguished from those for continuous-time systems primarily by the fact that there is a need to check the behavior of the dynamics at every point of the region in the state-space, as opposed to continuous-time systems, where it is sufficient to check the dynamics only at the edges of the region in the state-space, which is due to the infinitesimal nature of continuous-time systems. This difference is an illustration, for example for linear systems, of the fact that the state matrix of continuous-time systems is Metzler, and of discrete-time systems—positive. Another difference observed from the obtained results is the impossibility—for a nonlinear region in the state-space—of splitting the condition for discrete-time control-affine systems, as opposed to continuous-time systems. The relationships between the already known facts and the presented new approach, thus confirming its truthfulness and generality, have been presented.

The results obtained in this paper provide a set of tools which allow to gain additional insight into the nature and behavior of nonlinear discrete-time control systems in a relatively transparent way. These results can offer an important support, for example, in analysis of reachability or controllability of nonlinear discrete-time control systems.

Possible directions for further research on region-invariance may be to generalize the control cone region to a nonlinear region, or even depending on the state vector of the system. This could make it possible, for example, to study the equivalence of invariant nonlinear discrete-time control systems to invariant linear systems. In addition to the study of region-invariance, it may also be interesting to study, using the approach proposed here, other behaviors of system dynamics relative to fixed regions in the state-space, what will be addressed in a next paper.

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