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RESEARCH ARTICLE

State Estimation for a Class of Discrete-Time BAM Neural Networks With Multiple Time-Varying Delays

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ABSTRACT For a class of discrete-time bidirectional associative memory neural networks (DTBAMNNs) with multiple time-varying delays, the issue of state estimation is studied. By propose a mathematical induction method, we first investigate novel delay-dependent and -independent global exponential stability (GES) criteria of the error system. The obtained GES criteria are described by linear scalar inequalities. Then, a state observer is derived via the theory of generalized matrix inverses. These exponential stability conditions are very simple, which is convenient to verify based on the standard software tools (for example, YALMIP). Finally, we present two illustrative examples to present the effectiveness of the theoretical results.

INDEX TERMS State observer, discrete-time BAM neural network, multiple time-varying delays, global exponential stability, linear scalar inequalities.

I. INTRODUCTION

In recent years, many excellent results on neural networks (NNs) have been addressed extensively, since they were applicable to many fields such as pattern recognition, artificial intelligence, optimization, etc. [1], [2], [3], [4]. Generally, many NNs including biological NNs are composed of many interconnected man-made or/and natural dynamical units. As one of the interconnected NNs, BAMNNs [5], [6] is composed of two-layer heteroassociative circuits, which generalizes the single-layer NNs and possesses the functions on memory and association of information. Therefore, it has large theoretical and practical significance to research stability of DTBAMNNs.

As everyone knows, time delays can not omitted in the realization of NNs because of the communication time among neurons, and further, their existence will result in performance degradation of NNs, even instability. Motivated this idea, the problem of testing stability of delayed NNs has received more attention, and stability conditions for BAMNNs with various delays were developed to assure the asymptotic or exponential stability [7], [8], [9], [10], [11], [12], [13].

It is worth noting that the neuronal states in large-scale NNs are usually not fully measurable. Thus, in many practice applications, estimating the states of neurons through available measurements is important. The state observer of NNs is designed by means of the measurement output, so as to realize the state estimation of the original NNs. Recently, more and more learners were interested in the problem to estimate the states of NNs [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]. The state estimation for delayed NNs was introduced in [16], [17], [18], [19], [21], and [29]. Through the available outputs and feasible solutions of some linear matrix inequalities (LMIs), general full-order state observers are designed, which guaranteed GES or global asymptotic stability. While, the relevant research on discrete-time complex-valued NNs is mentioned in the literature [30], [31]. To handle possible fluctuations in the state observer gains during implementation of the state observer, a resilient H_∞ state observer in the light of discrete-time delayed NNs is adopted in [28] and [32].

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For memristor-based stochastic DTBAMNNs with mixed delays and additive delays, the problem of state observer design is investigated in [33] and [34], respectively. For the delayed switched DTBAMNNs of stochastic perturbations and parameter uncertainties, Arunkumar et al. [23] proved sufficient conditions guaranteeing the existence of state observer by making use of the so-called average dwell time method and constructing a piecewise LKF. Then, the problem of robust state estimation for delayed uncertain DTBAMNNs is derived in [24]. Firstly, based on appropriate LKF and LMI methods, they obtain a asymptotic stability condition of the error system. Secondly, this result spread to the problem of designing robust state observer for time-varying delayed uncertain DTBAMNNs. In [25], under a weaker hypothesis on the neuron activation functions for time-varying delayed DTBAMNNs, by establishing a new LKF, conditions in form of LMIs are got for GES of the error system, and a state observer is designed. For time-varying delayed Markovian jump DTBAMNNs, Ali et al. [35] designed finite-time H_∞ filtering based on the suitable LKF and Jensen inequality. By solving LMI with a fixed parameter, the filter gains are got.

However, as everyone knows, the problem of designing state observer for BAMNNs with time-varying delays is rarely studied, which retains big space to develop further state estimation methods of BAMNNs with multiple time-varying delays. This paper continues this work. On this basis, we aim at establishing some novel sufficient criteria for the existence of state observers, and GES of the error system is guaranteed. The contributions of this article are:

(1) Delay-dependent and -independent GES criteria, described by linear scalar inequalities, are developed by using the mathematical induction method;
(2) Via generalized inverse theory of matrices, a state observer is given for DTBAMNNs with multiple time-varying delays;
(3) These obtained exponential stability criteria are simple, that can easily be solved by the software YALMIP.

Structure of the rest is below. The problem formulation and preliminaries are proceeded in the next section. The main results of the study, novel criteria of GES are given in Section III, then new state estimator is designed. Section IV gives illustrative examples to demonstrate the effectiveness of derived results. Lastly, in Section V, we give a conclusion.

Notation. Let $\mathbb{R}$ be the real number field. The symbol $\mathbb{R}^{s\times t}$ refers to the set of $s \times t$ matrices. Let $\mathbb{R}^{s\times t}_{\succeq}$ and $\mathbb{R}^{s\times t}_{\succ}$ are the subsets of $\mathbb{R}^{s\times t}$ containing all nonnegative and positive matrices, respectively. Similarly, we also use $\mathbb{R}_{\succeq}$, $\mathbb{R}_{\succ}$, etc. Let $\mathbb{Z}$ be the integer set. For $p, q \in \mathbb{Z}$ with $p \le q$, let $[p, q]_{\mathbb{Z}}$ denote the set which consists of all integers between p and q. When $hq \to \infty$, the limit case of $[p, q]_{\mathbb{Z}}$ is written by $[p, \infty)_{\mathbb{Z}}$. For $U = [u_{ij}] \in \mathbb{R}^{p\times q}$ and $V = [v_{ij}] \in \mathbb{R}^{p\times q}$, the matrix $[u_{ij}v_{ij}]$, denoted by $U \circ V$, refers to the Hadamard product of U and V, and the notation $U \succeq V$ (or $V \preceq U$) denotes $u_{ij} \ge v_{ij}$, $\forall i \in [1, p]_{\mathbb{Z}}, j \in [1, q]_{\mathbb{Z}}$. If $u_{ij} > v_{ij}$, $\forall i \in [1, p]_{\mathbb{Z}}, j \in [1, q]_{\mathbb{Z}}$, we say $U \succ V$ (or $V \prec U$). If all off-diagonal entries of a square matrix are nonnegative, then it is called a Metzler matrix. Let $|U| = [|u_{ij}|]$. Then $|MN| \preceq |M||N|$ for all $M \in \mathbb{R}^{p\times q}$ and $N \in \mathbb{R}^{q\times r}$. The identity matrix in $\mathbb{R}^{n\times n}$ is defined by I_n. For $\Omega \in \mathbb{R}^{n\times n}$, set $\lambda(\Omega) = \{z \in \mathbb{C} : \det(zI_n - \Omega) = 0\}$, $\rho(\Omega) = \max\{|\lambda| : \lambda \in \lambda(\Omega)\}$ and $s(\Omega) = \max\{\mathrm{Re}\lambda : \lambda \in \lambda(\Omega)\}$.

II. PROBLEM DESCRIPTION AND PRELIMINARY RESULTS

A class of DTBAMNN with multiple time-varying delays is described as [36]:

$$x_i(t+1) = a_i x_i(t) + \sum_{j=1}^{n} \Big[c_{ij} f_j(y_j(t)) + e_{ij} h_j(y_j(t - \delta_{ij}(t))) \Big], \tag{1a}$$

$$y_j(t+1) = b_j y_j(t) + \sum_{i=1}^{n} \Big[d_{ji} \tilde{f}_i(x_i(t)) + w_{ji} \tilde{h}_i(x_i(t - \sigma_{ji}(t))) \Big], \tag{1b}$$

where $i, j \in [1, n]_{\mathbb{Z}}, t \in [0, \infty)_{\mathbb{Z}}$, $x_i(t)$ is the ith neuronal state of layer-X, $y_j(t)$ is the jth neuronal state of layer-Y; $a_i, b_j : [0, \infty)_{\mathbb{Z}} \to (-1, 1)$ describe the state feedback coefficient, respectively; $f_j : \mathbb{R} \to [-s_j^{(1)}, s_j^{(1)}]$, $h_j : \mathbb{R} \to [-s_j^{(2)}, s_j^{(2)}]$, $\tilde{f}_i : \mathbb{R} \to [-\tilde{s}_i^{(1)}, \tilde{s}_i^{(1)}]$ and $\tilde{h}_i : \mathbb{R} \to [-\tilde{s}_i^{(2)}, \tilde{s}_i^{(2)}]$ are the neuronal activation functions; $s_j^{(1)}, s_j^{(2)}, \tilde{s}_i^{(1)}$ and $\tilde{s}_i^{(2)}$ are known positive constants; Constants c_{ij}, d_{ji}, e_{ij} and w_{ji} represent the connection weights; $\delta_{ij} : [0, \infty)_{\mathbb{Z}} \to [0, \bar{\delta}_{ij}]$ and $\sigma_{ji} : [0, \infty)_{\mathbb{Z}} \to [0, \bar{\sigma}_{ji}]$ denote the multiple time-varying delays, $\bar{\sigma}_{ji} > 0$ and $\bar{\delta}_{ij} > 0$ are known integers.

Remark 1: The BAMNN is the minimal two-layer nonlinear feedback network. Bidirectionality, forward and backward information flows are introduced in neural nets to produce two-way associative search for stored associations. BAMNN generalizes the single layer network model of Hopfield and some unidirectional network models of Cohen and Grossberg. It has been shown that BAMNN is capable on storing paired patterns or memories and the search mode for stored patterns can be accomplished via both directions, i.e. forward and backward directions.

We require this assumption:

Assumption 1: There are $\beta_j^{(1)}, \tilde{\beta}_i^{(1)}, \beta_j^{(2)}, \tilde{\beta}_i^{(2)} \in \mathbb{R}_{\succ}$ such that

$$f_j(0) = \tilde{f}_i(0) = h_j(0) = \tilde{h}_i(0) = 0,$$
$$0 \le \frac{f_j(\alpha_1) - f_j(\alpha_2)}{\alpha_1 - \alpha_2} \le \beta_j^{(1)}, \; 0 \le \frac{\tilde{f}_i(\alpha_1) - \tilde{f}_i(\alpha_2)}{\alpha_1 - \alpha_2} \le \tilde{\beta}_i^{(1)},$$
$$0 \le \frac{h_j(\alpha_1) - h_j(\alpha_2)}{\alpha_1 - \alpha_2} \le \beta_j^{(2)}, \; 0 \le \frac{\tilde{h}_i(\alpha_1) - \tilde{h}_i(\alpha_2)}{\alpha_1 - \alpha_2} \le \tilde{\beta}_i^{(2)},$$

for any $i, j \in [1, n]_{\mathbb{Z}}, \alpha_1, \alpha_2 \in \mathbb{R}$ subject to $\alpha_1 \ne \alpha_2$.

Let the network measurements of DTBAMNN (1) be given by:

$$Z(t) = Mx(t), \tilde{Z}(t) = \tilde{M}y(t), \tag{2}$$

in which $Z(t)$ and $\tilde{Z}(t)$ are the measurement outputs, $M \in \mathbb{R}^{m_1\times n}$ and $\tilde{M} \in \mathbb{R}^{m_2\times n}$ denote the known full-row-rank matrices of appropriate dimensions.

For some large scale NNs, obtaining all the information of neuronal state is difficult. So people often need to employ the neuronal estimations to realize specific design goals. Thus, this paper aims at designing the state observer for DTBAMNNs (1):

$$\hat{x}_i(t+1) = a_i\hat{x}_i(t) + \sum_{j=1}^{n}\Big[c_{ij}f_j(\hat{y}_j(t)) + e_{ij}h_j(\hat{y}_j(t-\delta_{ij}(t)))\Big] + \sum_{j=1}^{m_1} r_{ij}\left[Z_j(t) - \sum_{l=1}^{n} m_{jl}\hat{x}_l(t)\right], \tag{3a}$$

$$\hat{y}_j(t+1) = b_j\hat{y}_j(t) + \sum_{i=1}^{n}\Big[d_{ji}\tilde{f}_i(\hat{x}_i(t)) + w_{ji}\tilde{h}_i(\hat{x}_i(t-\sigma_{ji}(t)))\Big] + \sum_{i=1}^{m_2} \tilde{r}_{ji}\left[\tilde{Z}_i(t) - \sum_{l=1}^{n} \tilde{m}_{il}\hat{y}_l(t)\right], \tag{3b}$$

where $t \in [0,\infty)_{\mathbb{Z}}$, $i,j \in [1,n]_{\mathbb{Z}}$, r_{ij} and $\tilde{r}_{ij}$ are the observer gains that will be determined later, $\hat{x}_i(t)$ and $\hat{y}_j(t)$ are respectively the estimations of $x_i(t)$ and $y_j(t)$, m_{ij} and $\tilde{m}_{ij}$ are respectively the (i,j)th entries of M and $\tilde{M}$, $Z_i(t)$ and $\tilde{Z}_i(t)$ are the ith components of $Z(t)$ and $\tilde{Z}(t)$, respectively.

Define the corresponding error variables

$$\kappa_i(t) = x_i(t) - \hat{x}_i(t),\ \tilde{\kappa}_j(t) = y_j(t) - \hat{y}_j(t),\quad i,j \in [1,n]_{\mathbb{Z}},\ t \in [0,\infty)_{\mathbb{Z}}.$$

Then, one can be easily obtain from (1)–(3) that the error system:

$$\kappa_i(t+1) = a_i\kappa_i(t) + \sum_{j=1}^{n}\Big[c_{ij}f_j^*(\tilde{\kappa}_j(t)) + e_{ij}h_j^*(\tilde{\kappa}_j(t-\delta_{ij}(t)))\Big] + \sum_{j=1}^{m_1}\sum_{l=1}^{n} r_{ij}m_{jl}\kappa_l(t), \tag{4a}$$

$$\tilde{\kappa}_j(t+1) = b_j\tilde{\kappa}_j(t) + \sum_{i=1}^{n}\Big[d_{ji}\tilde{f}_i^*(\kappa_i(t)) + w_{ji}\tilde{h}_i^*(\kappa_i(t-\sigma_{ji}(t)))\Big] + \sum_{i=1}^{m_2}\sum_{l=1}^{n} \tilde{r}_{ji}\tilde{m}_{il}\tilde{\kappa}_l(t), \tag{4b}$$

where $t \in [0,\infty)_{\mathbb{Z}}$, $i,j \in [1,n]_{\mathbb{Z}}$ and

$f_j^*(\tilde{\kappa}_j(\cdot)) = f_j(\tilde{\kappa}_j(\cdot) + \hat{y}_j(\cdot)) - f_j(\hat{y}_j(\cdot))$,
$h_j^*(\tilde{\kappa}_j(\cdot)) = h_j(\tilde{\kappa}_j(\cdot) + \hat{y}_j(\cdot)) - h_j(\hat{y}_j(\cdot))$,
$\tilde{f}_i^*(\kappa_i(\cdot)) = \tilde{f}_i(\kappa_i(\cdot) + \hat{x}_i(\cdot)) - \tilde{f}_i(\hat{x}_i(\cdot))$,
$\tilde{h}_i^*(\kappa_i(\cdot)) = \tilde{h}_i(\kappa_i(\cdot) + \hat{x}_i(\cdot)) - \tilde{h}_i(\hat{x}_i(\cdot))$.

Due to Assumption 1, we derive that

$$|f_j^*(u)| \le \beta_j^{(1)}|u|,\ |h_j^*(u)| \le \beta_j^{(2)}|u|,\ |\tilde{f}_i^*(u)| \le \tilde{\beta}_i^{(1)}|u|,\ |\tilde{h}_i^*(u)| \le \tilde{\beta}_i^{(2)}|u|,\ u\in\mathbb{R},\ i,j\in[1,n]_{\mathbb{Z}}. \tag{5}$$

Let

$$\vartheta = \max_{1\le i,j\le n} \max\left\{\delta_{ij}, \sigma_{ji}\right\}.$$

The symbol $C([-\vartheta,0]_{\mathbb{Z}},\mathbb{R}^n)$ refers to the set containing all functions $\varphi : [-\vartheta,0]_{\mathbb{Z}} \to \mathbb{R}^n$. The symbol $\|\cdot\|_2$ refers to the Euclidean norm of vectors. Let the norm $\|\cdot\|$ on $\mathbb{R}^n\times\mathbb{R}^n$ is defined by $\|(c,d)\| = (\|c\|_2^2 + \|d\|_2^2)^{1/2}$, $c,d \in \mathbb{R}^n$, and $\|(\cdot,\cdot)\|_\vartheta$ on $C([-\vartheta,0]_{\mathbb{Z}},\mathbb{R}^n)\times C([-\vartheta,0]_{\mathbb{Z}},\mathbb{R}^n)$ via

$$\|(\omega,\varpi)\|_\vartheta = \sup_{s\in[-\vartheta,0]_{\mathbb{Z}}} \max\left\{\|\omega(s)\|_2, \|\varpi(s)\|_2\right\}.$$

Definition 1: If there are $\lambda,\beta \in \mathbb{R}_{>}$ such that every solution $(\kappa(t),\tilde{\kappa}(t))$ of (4), corresponding to the initial functions $(\omega,\varpi) \in C([-\vartheta,0]_{\mathbb{Z}},\mathbb{R}^n)\times C([-\vartheta,0]_{\mathbb{Z}},\mathbb{R}^n)$, satisfies

$$\|(\kappa(t),\tilde{\kappa}(t))\| \le \beta e^{-\gamma t}\|(\omega,\varpi)\|_\vartheta,\ \forall t\in[0,\infty)_{\mathbb{Z}},$$

where
$\kappa(t) = [\kappa_1(t)\ \ldots\ \kappa_n(t)]^T$,
$\tilde{\kappa}(t) = [\tilde{\kappa}_1(t)\ \ldots\ \tilde{\kappa}_n(t)]^T$,
then the error system (4) subject to (5) is globally exponentially stable.

The purpose of this paper is to design a state observer (3) for DTBAMNN (1) via the measurements (2), that is, determine observer gains $R := [r_{ij}] \in \mathbb{R}^{n\times m_1}$ and $\tilde{R} := [\tilde{r}_{ij}] \in \mathbb{R}^{n\times m_2}$ guaranteeing GES of the error system (4).

Lemma 1: [38] Let $A \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{s\times n}$, and A^+ is the Moore-Penrose generalized inverse of A. Then CA^+ is a solution of $XA = C$ when

$$rank\begin{bmatrix}A\\C\end{bmatrix} = rankA.$$

III. MAIN RESULTS

In this part, using the generalized matrix inverses and the definition of GES, we study directly the GES criteria for the error system (4), and give a novel approach of designing a state observer (3) for DTBAMNN (1).

Set

$$A = \mathrm{diag}(a_1,\ldots,a_n),\ B = \mathrm{diag}(b_1,\ldots,b_n),$$
$$C = [c_{ij}],\ E = [e_{ij}],\ D = [d_{ij}],\ W = [w_{ij}],$$
$$B_\beta = e^{\beta\bar{\delta}} \circ |E|\Gamma_2 + |C|\Gamma_1,\ e^{\beta\bar{\delta}} = [e^{\beta\bar{\delta}_{ij}}],$$
$$C_\beta = e^{\beta\bar{\sigma}} \circ |W|\tilde{\Gamma}_2 + |D|\tilde{\Gamma}_1,\ e^{\beta\bar{\sigma}} = [e^{\beta\bar{\sigma}_{ji}}],$$
$$A_\beta = |A| - e^{-\beta}I_n,\ D_\beta = |B| - e^{-\beta}I_n,$$
$$\Gamma_1 = \mathrm{diag}(\beta_1^{(1)},\ldots,\beta_n^{(1)}),\ \tilde{\Gamma}_1 = \mathrm{diag}(\tilde{\beta}_1^{(1)},\ldots,\tilde{\beta}_n^{(1)}),$$
$$\Gamma_2 = \mathrm{diag}(\beta_1^{(2)},\ldots,\beta_n^{(2)}),\ \tilde{\Gamma}_2 = \mathrm{diag}(\tilde{\beta}_1^{(2)},\ldots,\tilde{\beta}_n^{(2)}).$$

Theorem 1: If there are $\beta \in \mathbb{R}_{\succ}$, $\tilde{u}, \tilde{v} \in \mathbb{R}^n_{\succ}$ and $\zeta, \eta \in \mathbb{R}^n$ such that

$$A_\beta \tilde{u} + B_\beta \tilde{v} + \zeta \preceq 0, \quad (6)$$
$$C_\beta \tilde{u} + D_\beta \tilde{v} + \eta \preceq 0, \quad (7)$$

then the error system (4) subject to (5) is globally exponentially stable. Furthermore, the desired state observer is given by (3) with the observer gains $R = \zeta(|M|\tilde{u})^+$ and $\tilde{R} = \eta(|\tilde{M}|\tilde{v})^+$.

Proof: For any fixed $\omega, \varpi \in C([-\vartheta, 0]_{\mathbb{Z}}, \mathbb{R}^n)$, let $(\kappa(t), \tilde{\kappa}(t))$ be the solution of (4) with the initial functions (ω, ϖ). We can a choose $\lambda > 0$ such that

$$\lambda\tilde{u} \succ [1 \ \cdots \ 1]^T, \ \lambda\tilde{v} \succ [1 \ \cdots \ 1]^T.$$

Define

$$\hat{u}(t) = \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta t}\tilde{u}, \ t \in [-\vartheta, \infty)_{\mathbb{Z}}, \quad (8)$$
$$\hat{v}(t) = \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta t}\tilde{v}, \ t \in [-\vartheta, \infty)_{\mathbb{Z}}, \quad (9)$$

where
$\tilde{u} = [\tilde{u}_1 \ \cdots \ \tilde{u}_n]^T, \tilde{v} = [\tilde{v}_1 \ \cdots \ \tilde{v}_n]^T,$
$\hat{u}(t) = [\hat{u}_1(t) \ \cdots \ \hat{u}_n(t)]^T, \hat{v}(t) = [\hat{v}_1(t) \ \cdots \ \hat{v}_n(t)]^T.$

Now the mathematical induction method will be employed to investigate the following conclusions

$$|\kappa(t)| \preceq \hat{u}(t), \ |\tilde{\kappa}(t)| \preceq \hat{v}(t), \ t \in [-\vartheta, \infty)_{\mathbb{Z}}. \quad (10)$$

Clearly, combined with the choice of λ and the definition of $\|\cdot\|_\vartheta$, one can obtain

$$|\kappa(k)| \preceq \hat{u}(k), \ |\tilde{\kappa}(k)| \preceq \hat{v}(k), \ \forall k \in [-\vartheta, 0]_{\mathbb{Z}}.$$

Assume that inequality (10) holds when $t \leq k$ for arbitrary but fixed $k \geq 0$. When $t = k + 1$, for any $i \in [1, n]_{\mathbb{Z}}$, using (4a) and (5), we get

$$\begin{aligned}|\kappa_i(k+1)| \leq\ & |a_i||\kappa_i(k)| + \sum_{j=1}^{n}\Big[|c_{ij}||f_j^*(\tilde{\kappa}_j(k))| \\ & + |e_{ij}||h_j^*(\tilde{\kappa}_j(k - \delta_{ij}(k)))|\Big] \\ & + um_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}||\kappa_l(k)| \\ \leq\ & |a_i||\kappa_i(k)| + \sum_{j=1}^{n}\Big[|c_{ij}|\beta_j^{(1)}|\tilde{\kappa}_j(k)| \\ & + |e_{ij}|\beta_j^{(2)}|\tilde{\kappa}_j(k - \delta_{ij}(k))|\Big] \\ & + \sum_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}||\kappa_l(k)|.\end{aligned}$$

By using the inductive hypothesis, we can obtain

$$\begin{aligned}|\kappa_i(k+1)| \leq\ & |a_i|\hat{u}_i(k) + \sum_{j=1}^{n}\Big[|c_{ij}|\beta_j^{(1)}\hat{v}_j(k) \\ & + |e_{ij}|\beta_j^{(2)}\hat{v}_j(k - \delta_{ij}(k))\Big] \\ & + \sum_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}||\hat{u}_l(k)|, \quad (11)\end{aligned}$$

Substituting (8) and (9) into (11), we get

$$\begin{aligned}|\kappa_i(k+1)| \leq\ & |a_i|\lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta k}\tilde{u}_i \\ & + \sum_{j=1}^{n}|c_{ij}|\beta_j^{(1)}\lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta k}\tilde{v}_j \\ & + \sum_{j=1}^{n}|e_{ij}|\beta_j^{(2)}\lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta(k-\delta_{ij}(k))}\tilde{v}_j \\ & + \sum_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}||\lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta k}\tilde{u}_l \\ \leq\ & \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta k} \\ & \times \Bigg[|a_i|\tilde{u}_i + \sum_{j=1}^{n}(|c_{ij}|\beta_j^{(1)} \\ & + |e_{ij}|\beta_j^{(2)}\mathrm{e}^{\beta\bar{\delta}_{ij}})\tilde{v}_j \\ & + \sum_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}|\tilde{u}_l\Bigg], \quad (12)\end{aligned}$$

Due to rank$M = m_1$ and $\tilde{u} \in \mathbb{R}^n_{\succ}$, we have

$$\mathrm{rank}\begin{bmatrix}|M|\tilde{u} \\ \zeta\end{bmatrix} = \mathrm{rank}(|M|\tilde{u}) = 1.$$

Applying Lemma 1, one can obtain that $\zeta(|M|\tilde{u})^+$ is a solution of $X|M|\tilde{u} = \zeta$. Set $R = \zeta(|M|\tilde{u})^+$. Then

$$\sum_{j=1}^{m_1}\sum_{l=1}^{n}|r_{ij}||m_{jl}|\tilde{u}_l = \zeta_i,$$

where ζ_i is the ith component of ζ. According to the arbitrariness of $i \in [1, n]_{\mathbb{Z}}$, we get that (12) is equivalent to

$$|\kappa(k+1)| \preceq \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta k}(|A|\tilde{u} + B_\beta\tilde{v} + \zeta).$$

By using (6) and (8), we have

$$|\kappa(k+1)| \preceq \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta(k+1)}\tilde{u} = \hat{u}(k+1). \quad (13)$$

Similarly, through a process similar to derivation (13), we obtain

$$|\tilde{\kappa}(k+1)| \preceq \lambda\|(\omega, \varpi)\|_\vartheta \mathrm{e}^{-\beta(k+1)}\tilde{v} = \hat{v}(k+1).$$

Therefore, (10) is true.

Then, together with (8)–(10), we have

$$\begin{aligned}\|(\kappa(t), \tilde{\kappa}(t))\| &= (\|\kappa(t)\|_2^2 + \|\tilde{\kappa}(t)\|_2^2)^{\frac{1}{2}} \\ &\leq (\|\hat{u}(t)\|_2^2 + \|\hat{v}(t)\|_2^2)^{\frac{1}{2}} \\ &= \lambda\mathrm{e}^{-\beta t}\|(\omega, \varpi)\|_\vartheta(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2)^{\frac{1}{2}}, \\ &\quad \forall t \in [0, \infty)_{\mathbb{Z}}.\end{aligned}$$

Let $H = \lambda(\|\tilde{u}\|_2^2 + \|\tilde{v}\|_2^2)^{\frac{1}{2}}$, Then

$$\|(\kappa(t), \tilde{\kappa}(t))\| \leq H\mathrm{e}^{-\beta t}\ \|(\omega, \varpi)\|_\vartheta\ , \ \forall t \in [0, \infty)_{\mathbb{Z}}.$$

The arbitrariness of $\varpi, \omega \in C([-\vartheta, 0]_{\mathbb{Z}}, \mathbb{R}^n)$ guarantees GES of the error system (4). In addition, one can easily design the state observer (3) with the gains $R = \zeta(|M|\tilde{u})^{+}$ and $\tilde{R} = \eta(|\tilde{M}|\tilde{v})^{+}$. ■

Remark 2: In the study, the involved main difficulty is how to determine the observer gain matrices. In the existing results (see [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32] and the references therein), they mostly are derived by using the LKF method that is difficult to be extended to the multiple time-varying delays; even it is done, there will greatly increase the computational complexity. While this paper overcomes the shortcoming of LKF method by proposing a mathematical induction method associated with the generalized matrix inverses, that decreases a lot of decision variables in the obtained stability conditions, so the computation amount is reduced. In addition, the observer gain of the designed state observer can be easily obtained by the generalized inverse theory of the matrices.

Remark 3: In this paper, a neuron state estimator is designed based on available output measurements. Theorem 1 shows a delay-dependent GES condition of (4), the method is applicable to the case that the numbers of neurons in the two neural domains are different. In addition, the obtained stability criterion can used to give the relation between upper-bounds of delays and decay rate.

Set

$$A_0 = |A| - I_n,\ B_0 = |E|\Gamma_2 + |C|\Gamma_1,$$
$$C_0 = |D|\tilde{\Gamma}_1 + |W|\tilde{\Gamma}_2,\ D_0 = |B| - I_n.$$

Combining [39, Theorems 1 and 3], one can easily get the following conclusion.

Theorem 2: If there are $\tilde{u}, \tilde{v} \in \mathbb{R}^n_{\succ}$ and $\zeta, \eta \in \mathbb{R}^n$ such that

$$A_0\tilde{u} + B_0\tilde{v} + \zeta \preceq 0, \tag{14}$$
$$C_0\tilde{u} + D_0\tilde{v} + \eta \preceq 0. \tag{15}$$

then (4) subject to (5) is globally exponentially stable. Furthermore, the desired state observer is given by (3) with the observer gains $R = \zeta(|M|\tilde{u})^{+}$ and $\tilde{R} = \eta(|\tilde{M}|\tilde{v})^{+}$.

Remark 4: In [23, Theorem 3.1], [24, Theorem 3.1], and [25, Theorem 1], by constructing LKFs the state observers similar to (3) are designed. The proposed method here is directly based on the generalized matrix inverses and the definition of GES. We overcome the difficulties of constructing appropriate LKFs and offer simple sufficient conditions which is convenient to use.

We conclude this section by demonstrating the following result.

Lemma 2: [37] For a Metzler matrix $A_0 \in \mathbb{R}^{n\times n}$ and matrices $B_0, C_0, D_0 \in \mathbb{R}^{n\times n}_{\succeq}$, the items (a)–(c) are equivalent:

(a) There are $\phi, \varphi \in \mathbb{R}^n_{\succ}$ such that $A_0\phi + B_0\varphi \prec 0$ and $C_0\phi + D_0\varphi \prec \varphi$.
(b) $\rho(D_0) < 1, s(A_0 + B_0(I_n - D_0)^{-1}C_0) < 0$.
(c) $s(A_0) < 0$,d $\rho(C_0(-A_0)^{-1}B_0 + D_0) < 1$.

Theorem 3: Assume that a number $\beta > 0$ and diagonal matrices $\wedge_1$ and $\wedge_2$ satisfy $\hat{A}_\beta = |A| - e^{-\beta}I_n + \wedge_1 \geq 0$ and $\hat{D}_\beta := |B| - e^{-\beta}I_n + \wedge_2 \geq 0$. The error system (4) subject to (5) is globally exponentially stable, when

(a) $\rho(\hat{D}_\beta + I_n) < 1, s(\hat{A}_\beta - B_\beta\hat{D}_\beta^{-1}C_\beta) < 0$; or
(b) $\rho(\hat{D}_\beta + I_n - C_\beta\hat{A}_\beta^{-1}B_\beta) < 1, s(\hat{A}_\beta) < 0$.

Furthermore, the desired state observer is given by (3) with $\zeta = \wedge_1 x^*$ and $\eta = \wedge_2 y^*$, where $x^*, y^* \in \mathbb{R}^n_{\succ}$.

Proof: Observe, that $\hat{A}_\beta$ is a Metzler matrix, and B_β, C_β and $\hat{D}_\beta + I_n$ are non-negative matrices. If one of (a) and (b) in Theorem 3 is true, using Lemma 2, there exist $x^*, y^* \in \mathbb{R}^n_{\succ}$ satisfy

$$\hat{A}_\beta x^* + B_\beta y^* \prec 0,\ C_\beta x^* + \hat{D}_\beta y^* \prec 0.$$

Let $\zeta = \wedge_1 x^*$, $\eta = \wedge_2 y^*$, by Theorem 1, which ensures GES of the error system (4) subject to (5). ■

IV. ILLUSTRATIVE EXAMPLES

Two illustrative examples will be offered to explain the merits of the proposed approach.

Example 1: A DTBAMNN in the form of (1) is involved, where

$$A = \begin{bmatrix} 0.15 & 0 \\ 0 & 0.25 \end{bmatrix},\ B = \begin{bmatrix} 0.14 & 0 \\ 0 & 0.23 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.03 & -0.01 \\ 0.02 & 0.06 \end{bmatrix},\ E = \begin{bmatrix} 0.05 & -0.04 \\ 0.2 & 0.01 \end{bmatrix},$$
$$D = \begin{bmatrix} 0.04 & 0.03 \\ 0.02 & 0.01 \end{bmatrix},\ W = \begin{bmatrix} 0.03 & 0.05 \\ -0.2 & 0.01 \end{bmatrix},$$
$$f_1(s) = 0.2\tanh(s),\ h_1(s) = -0.2\tanh(s),$$
$$\tilde{f}_1(s) = 0.2\tanh(s),\ \tilde{h}_1(s) = 0.1\tanh(s),$$
$$f_2(s) = 0.8\tanh(s),\ h_2(s) = 0.9\tanh(s),$$
$$\tilde{f}_2(s) = 0.3\tanh(s),\ \tilde{h}_2(s) = 0.4\tanh(s),\ s \in \mathbb{R},$$
$$\delta_{ij}(t) = r_{ij} + s_{ij}\sin(t\pi/2),$$
$$\sigma_{ji}(t) = p_{ji} + q_{ji}\cos(t\pi),\ t \in [0,\infty)_{\mathbb{Z}}, i, j \in [1,2]_{\mathbb{Z}},$$
$$r_{11} = 6, r_{12} = 5, r_{21} = 8, r_{22} = 7,$$
$$s_{11} = s_{12} = s_{21} = s_{22} = 1,$$
$$p_{11} = 6, p_{12} = 5, p_{21} = p_{22} = 7,$$
$$q_{11} = q_{21} = q_{12} = q_{22} = 1.$$

Clearly, $\bar{\delta}_{11} = \bar{\sigma}_{11} = 7$, $\bar{\delta}_{12} = \bar{\sigma}_{12} = 6$, $\bar{\delta}_{21} = 9$ and $\bar{\delta}_{22} = \bar{\sigma}_{21} = \bar{\sigma}_{22} = 8$. Furthermore, when $\beta_1^{(1)} = \tilde{\beta}_1^{(1)} = \beta_1^{(2)} = 0.2$, $\tilde{\beta}_1^{(2)} = 0.1$, $\beta_2^{(2)} = 0.9$, $\tilde{\beta}_2^{(1)} = 0.3$ and $\tilde{\beta}_2^{(2)} = 0.4$, Assumption 1 is satisfied.

Assume that the measurements of DTBAMNN under consideration are given by (2) with

$$M = \begin{bmatrix} 0.7 & 0.66 \end{bmatrix},\ \tilde{M} = \begin{bmatrix} 0.4 & 0.7 \end{bmatrix}.$$

By solving the inequalities (14)–(15) in Theorem 2, and feasible solutions are obtained below:

$$\tilde{u} = [0.25 \times 10^3, 0.75 \times 10^3]^T,$$
$$\tilde{v} = [0.15 \times 10^3, 2.48 \times 10^3]^T,$$
$$\zeta = [-0.5956 \times 10^2, 0.2741 \times 10^3]^T,$$
$$\eta = [-0.1591 \times 10^3, -0.8807 \times 10^2]^T.$$

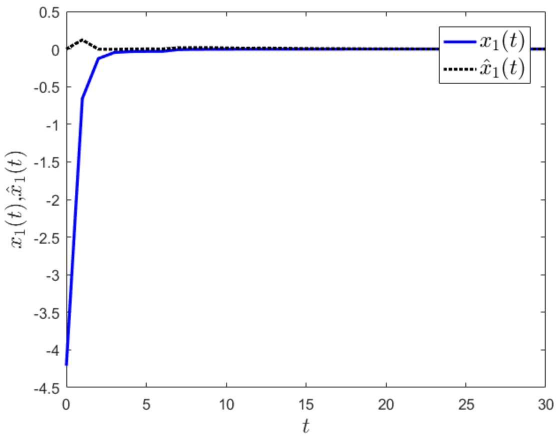

FIGURE 1. $x_1(t)$ and its estimations $\hat{x}_1(t)$ (Example 1).

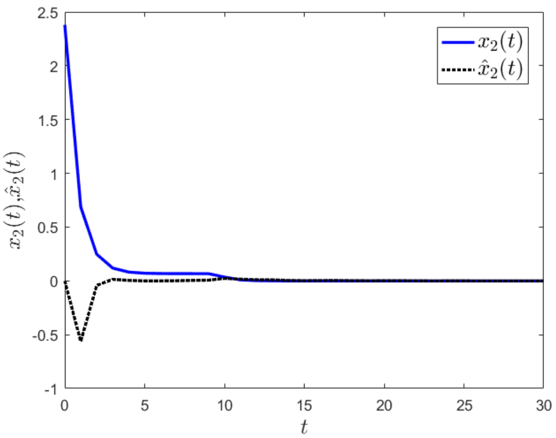

FIGURE 2. $x_2(t)$ and its estimations $\hat{x}_2(t)$ (Example 1).

Accordingly,the gain matrices of the desired state observer are given follows:

$$R = \zeta(|M|\tilde{u})^+ = \begin{bmatrix} -0.0899 \\ 0.4091 \end{bmatrix},$$

$$\tilde{R} = \eta(|\tilde{M}|\tilde{v})^+ = \begin{bmatrix} -0.6810 \\ -0.3769 \end{bmatrix}.$$

Moreover, when

$x(s) = [-4.21 \;\; 2.38]^T$, $y(s) = [-3.23 \;\; 2.90]^T$,

$\hat{x}(s) = [0 \;\; 0]^T$, $\hat{y}(s) = [0 \;\; 0]^T$, $s \in [-9, 0]_{\mathbb{Z}}$,

the trajectories of the considered DTBAMNN, designed observer and corresponding error system are given in Figures 1–6, respectively. Further, we can find the observer trajectories are convergent to state trajectories little by little, and the error system trajectories approach zero. So, the obtained observer is applicable, which explains the theoretical results presented in Theorem 2.

Example 2: Consider a delayed DTBAMNN (1) with

$$A = \begin{bmatrix} 0.08 & 0 \\ 0 & 0.09 \end{bmatrix}, \; B = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.06 \end{bmatrix},$$

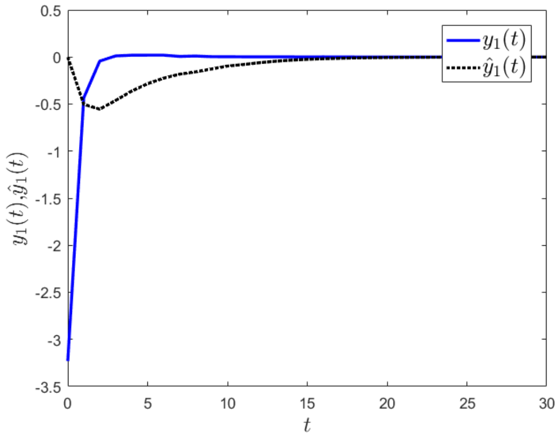

FIGURE 3. $y_1(t)$ and its estimations $\hat{y}_1(t)$ (Example 1).

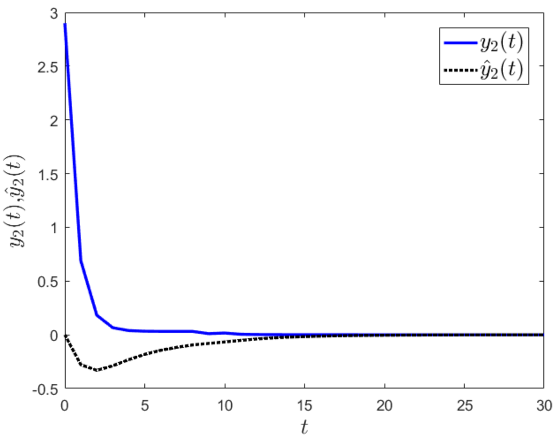

FIGURE 4. $y_2(t)$ and its estimations $\hat{y}_2(t)$ (Example 1).

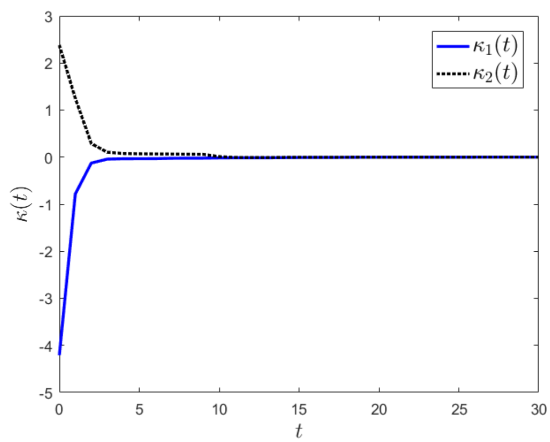

FIGURE 5. The error system. (Example 1).

$$C = \begin{bmatrix} -0.03 & -0.01 \\ 0.02 & 0.02 \end{bmatrix}, \; E = \begin{bmatrix} 0.05 & -0.02 \\ 0.02 & 0.01 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.04 & 0.03 \\ 0.03 & 0.01 \end{bmatrix}, \; W = \begin{bmatrix} 0.03 & 0.06 \\ -0.02 & 0.01 \end{bmatrix},$$

$$f_j(\theta) = \tilde{f}_i(\theta) = h_j(\theta) = \tilde{h}_i(\theta) = \tanh(\theta), \;\; \theta \in \mathbb{R},$$

$$\delta_{ij}(t) \equiv \sigma_{ji}(t) \equiv 7, \;\; t \in [0, \infty)_{\mathbb{Z}}, \;\; i, j \in [1, 2]_{\mathbb{Z}}.$$

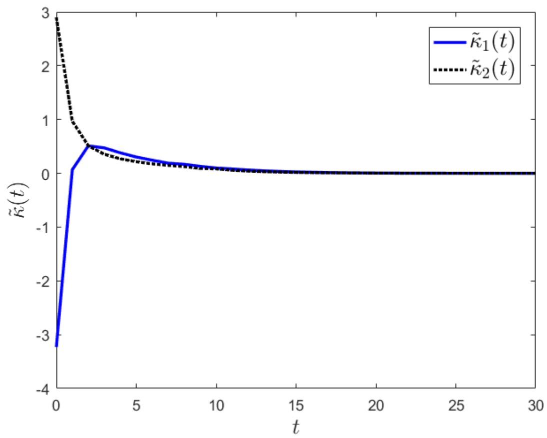

FIGURE 6. The error system. (Example 1).

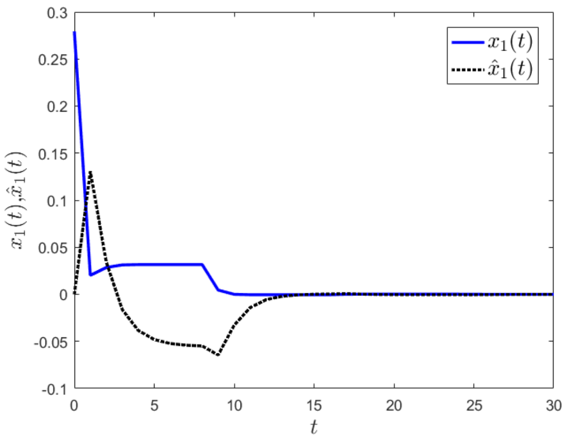

FIGURE 7. $x_1(t)$ and its estimations $\hat{x}_1(t)$ (Example 2).

If we choose $\beta_1^{(1)} = \beta_1^{(2)} = \tilde{\beta}_1^{(2)} = \tilde{\beta}_1^{(1)} = \beta_2^{(1)} = \beta_2^{(2)} = \tilde{\beta}_2^{(2)} = \tilde{\beta}_2^{(1)} = 1$ (i.e., $\Gamma_1 = \Gamma_2 = \tilde{\Gamma}_1 = \tilde{\Gamma}_2 = I_2$), then Assumption 1 is satisfied.

Assume that the measurements of DTBAMNN (1) are given by (2), where

$$M = \begin{bmatrix}0.50 \; 0.90\end{bmatrix}, \; \tilde{M} = \begin{bmatrix}0.10 \; 0.60\end{bmatrix}.$$

Choose $\beta = 0.02$ and solve the inequalities (6) and (7) in Theorem 1 by the software YALMIP, we can get

$$\tilde{u} = [50, 550]^T, \; \tilde{v} = [418.9, 600]^T,$$
$$\zeta = [-430.3, 33.78]^T, \; \eta = [-87.42, 118.73]^T.$$

Accordingly, the gain matrices of the desired state observer are given follows:

$$R = \zeta(|M|\tilde{u})^+ = \begin{bmatrix}-0.8726 \\ 0.0650\end{bmatrix},$$
$$\tilde{R} = \eta(|\tilde{M}|\tilde{v})^+ = \begin{bmatrix}-0.2175 \\ 0.2955\end{bmatrix}.$$

Moreover, when

$$x(s) = [0.2794 \;\; -0.3307]^T, y(s) = [1.1674 \;\; 0.7279]^T,$$
$$\hat{x}(s) = [0 \;\; 0]^T \text{ and } \hat{y}(s) = [0 \;\; 0]^T, s \in [-7, 0]_{\mathbb{Z}},$$

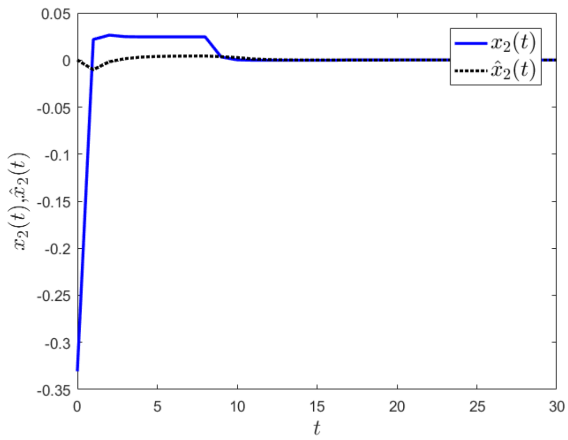

FIGURE 8. $x_2(t)$ and its estimations $\hat{x}_2(t)$ (Example 2).

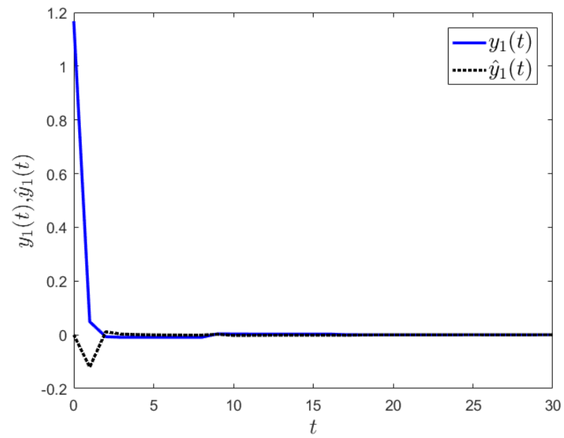

FIGURE 9. $y_1(t)$ and its estimations $\hat{y}_1(t)$ (Example 2).

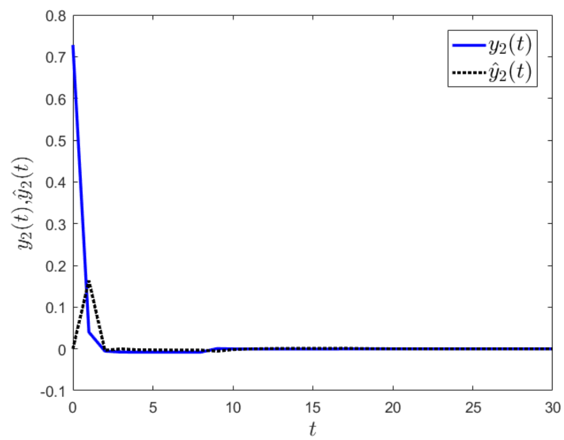

FIGURE 10. $y_2(t)$ and its estimations $\hat{y}_2(t)$ (Example 2).

Figures 7–12 describes the trajectories of DTBAMNN, the trajectories of state observer and the trajectories of error system, respectively. Further, we can find that the observer trajectories are convergent to state trajectories little by little, and the trajectories of error system approach zero. This also

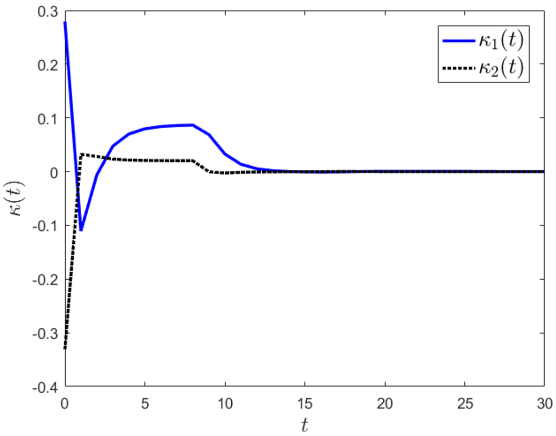

FIGURE 11. **The error system. (Example 2).**

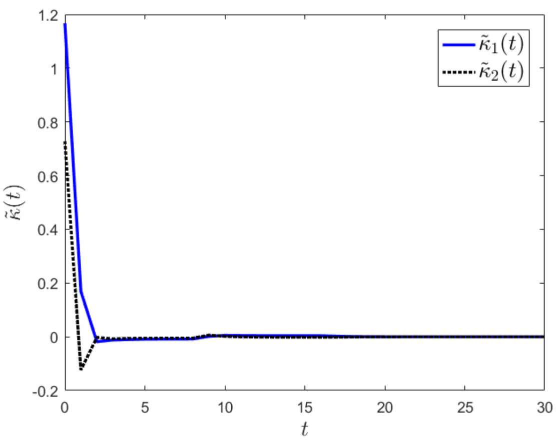

FIGURE 12. **The error system. (Example 2).**

proves the effectiveness of the state observer designed by Theorem 1.

V. CONCLUSION

This paper involves the problem designing state observer for DTBAMNNs with multiple time-varying delays. By using the definition of GES and generalize inverse theory of matrices, we first derive delay-dependent and -independent GES criteria for the error system. Then, by using Moore-Penrose inverses of matrices to represent observer gains, a state observer is given. Finally, we offer two illustrative examples to illustrate the applicability of conclusions. Compared with the previous conclusions, the proposed method has three merits:

(1) The method directly uses the generalized matrix inverses and the definitions of GES, and it avoids the construction of any LKF;
(2) The obtained sufficient conditions are composed of linear scalars inequalities that is easy to solve;
(3) It is suitable for the more general neural network models after a small modification. For example, memristor-based NNs [40], inertial neural works [41] and high-order NNs [4], [42], [43].

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