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RESEARCH ARTICLE

The Exponentiated Cotangent Generalized Distributions: Characteristics and Applications Patients of Chemotherapy Treatments Data

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ABSTRACT Over the last several decades, many algebraic generalized families and classes of statistical distributions have been developed. This research aims to construct a new cotangent exponentiated generalized and generator of distributions with support on the real line. After that, two novel families of distributions incorporating the cotangent function are proposed: one called the cotangent exponentiated generalized (CE-G) family, and the other called the logistic cotangent exponentiated generalized (LCE-G) family. A comprehensive analysis of the mathematical and structural properties of the recently suggested G-family and a Burr-based novel model (LCEB) is presented here. The maximum likelihood method estimates model parameters and evaluates model performance in Monte Carlo simulation studies. These tasks are carried out using the maximum likelihood technique. The statistical analysis on the survival and waiting times data sets are carried out, and the outcomes confirm the competence, superiority, and utility of the suggested generator, G-family, and novel distribution compared to similar and competing Burr-based models already well-known.

INDEX TERMS Exponentiated generalized family, cotangent trigonometric function, reliability and probability functions, maximum likelihood methodology.

I. INTRODUCTION

Generalized distributions are important in statistics and data analysis. They are flexible and can model a wide range of data shapes and characteristics. They can be used to model data that does not fit well with traditional parametric distributions, such as the Normal or the Poisson distributions. Generalized distributions are often more robust than traditional distributions. They can handle outliers and heavy-tailed data better than some parametric distributions. They often have closed-form expressions for moments, quantiles, and other statistical properties, making them easier to work with and analyze. They often have clear interpretations for their parameters. For example, in the case of the Student-t distribution, the degrees of freedom parameter can be interpreted as the number of observations required to estimate the population mean with a given level of confidence. They have a wide range of applications in fields such as finance, engineering, biology, and social sciences. They are used to model phenomena such as financial returns, earthquake magnitudes, gene expression levels, and social network structures, among others. In summary, generalized distributions are a powerful tool for statisticians and data analysts, allowing them to model complex data shapes and characteristics and to make more accurate inferences and predictions.

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The flexibility of generalized distributions comes from their ability to model a wide range of data shapes and characteristics. Unlike traditional parametric distributions that assume a fixed shape, generalized distributions can accommodate a variety of shapes by introducing additional parameters that control the distribution's characteristics. For example, consider the Normal distribution, which assumes a bell-shaped curve. While the Normal distribution is appropriate for many types of data, it may not be suitable for data that is skewed or has heavier tails. In contrast, the Generalized Extreme Value (GEV) distribution can model a wider range of shapes, including both symmetric and asymmetric distributions with heavy or light tails. The GEV distribution includes the Normal, the Logistic, and the Gumbel distributions as special cases. Similarly, the Generalized Pareto distribution can model data with heavy tails, while the Beta distribution can model data that is bounded between 0 and 1. The Student-t distribution is another example of a generalized distribution that can accommodate a wider range of shapes than the Normal distribution. The Student-t distribution has heavier tails than the Normal distribution and can model data that may contain outliers. Overall, the flexibility of generalized distributions allows them to model a wide range of data shapes and characteristics, making them useful for modeling complex real-world phenomena. The choice of a specific generalized distribution depends on the specific characteristics of the data being modeled and the research question being addressed.

Statistical distributions play a crucial role in data analysis by providing a mathematical framework for describing the behavior of data. They allow us to summarize, model, and analyze data, making it easier to draw conclusions and make predictions. Statistical distributions are used to describe the probability of an event occurring, and the shape of the distribution can provide insights into the underlying causes of the data. For example, a Normal distribution may indicate that the data is symmetric around a mean value, while a skewed distribution may indicate that the data has a bias or preference towards one side. Different statistical distributions are used in different contexts depending on the type of data being analyzed. For example, the Poisson distribution is commonly used to model count data, while the Normal distribution is often used to model continuous data. Statistical distributions are also used in hypothesis testing, where the distribution of the test statistic is compared to a known distribution to assess the likelihood of the observed data under a specific hypothesis. This allows researchers to draw conclusions about the significance of their findings. The choice of a specific statistical distribution depends on the characteristics of the data being analyzed and the research question being addressed. It is important to choose a distribution that best represents the data in order to obtain accurate and meaningful results. In summary, statistical distributions are important tools for data analysis and provide a mathematical framework for summarizing, modeling, and analyzing data. They are used in a wide range of applications and help researchers draw meaningful conclusions from their data.

TABLE 1. Recent generalized families with aims/methodology.

Main Author	Suggested Generalized Family	Aim/Methodology
Hamdani (2021)	Type-I Quasi Lambert Family	Using Lambert Function.
Altun (2021)	Additive Odd-G Family	Additive Model Structure.
Aljarrah (2021)	Generators of Distributions	Disbursing Quantile Function.
Salahuddin (2021)	New Family of Distributions	Better Goodness of Fit and Flexibility.
Korkmaz (2021)	New Class of Distributions	Hjorth's IDB Generator of Distributions.
Cordeiro (2020)	XGamma Family of Distributions	Adopting Parameter Induction Technique.
Korkmaz (2019)	TL Gen.Odd Log-Logistic Family	Odd Generator of Distributions.
Alzaatreh (2019)	G-Classes of Distributions	T-X Methodology.
Bhatti (2019)	Burr III Family	Vide Hazard Rate Function.
Korkmaz (2018)	New G-Classes of Distributions	Extended Distribution.
ElSherpieny et al. (2022)	Exponentiated Generalized Alpha Power	T-X Methodology.
El-Sherpieny et al. (2022)	Bivariate Weibull-G Family	Copula.
Frederico Caeiro et al.(2023)	Generalized Probability-Weighted Moment Estimators for the Pareto Distribution	New class.
Ozkan Egemen et al.(2023)	Generalized Marshall-Olkin exponentiated exponential distribution	New G-class.
Alizadeh (2019)	Odd Log-Logistic Logarithmic-G	Odd Generator of Distributions.

Using the aid of differential aligns [55] carried out some of the initial studies on generalizing distributions. Several approaches, including transformation techniques, transmuted functions, compounding of discrete and continuous distributions, production of skewed models, injection of the new parameter(s), quantile technique, T-X generalizing system, extended T-X method, T-RY methodology, and others, are utilized in this context. etc.

The following Table 1 presents some recent important examples of generalized families of distributions, along with their aims or adopted methodologies:

The research that has been done in the field of generalized probability models has uncovered the following features, which will serve as the foundations of this article: (i) An overwhelming number of statistical families and models distribute algebraic generators (ignoring the non-algebraic especially trigonometric generators). (ii) Because researchers wanted to model directional-proportional data, they turned to trigonometric function-based models, which are better equipped to deal with such data sets. These models were developed as a result. (iii) The mixture and hybrid generalizers are neither investigated nor the algebraic nor trigonometric functions. (iv) By using the exponentiated cotangent generalizer that has been presented, it is possible to translate any traditional distribution into its corresponding trigonometric form easily.

Table 2 offers the previously conducted study on trigonometrically-based families and distributions, which served as the impetus for the inspiration behind the presentation of this paper.

The following is a list of the primary reasons why this study was carried out: being inspired by these concepts and being content with the results of the flexibility and the goodness-of-fit (gof), both of which are highly fulfilling.

- presenting an original distribution generator known as the CE-G that is based on the cotangent function.
- presenting a new distribution generator that is based on the cotangent exponentiated function and which at the same time incorporates the algebraic function, the algebraic generator, and the cotangent function.
- Introducing a new G-family termed the LCE-G (short for logistic cotangent exponentiated-G family of distributions) in the Lehmann Alternative-1 and trigonometric case;
- The suggested class has several benefits, such as simplicity and the absence of non-identifiability problems;

TABLE 2. Generalized Classes and Distributions Adopting Trigonometric Functions.

Proposed Trigonometric Model	Inventor(s)	Year
Cosine Distribution	Raab and Green	(1961)
Beta Trigonometric Distribution	Nadarajah and Kotz	(2006)
Sin-skew Logistic Distribution	Chakraborty	(2012)
New Trigonometric Classes of Probabilistic Distributions	Souza	(2015)
Hyperbolic Sine-Weibull Distribution	Kharazmi and Saadatinik	(2016)
Cosine Sine (CS) Distribution	Chesneau	(2019)
A New Sine-G Family of Distributions	Mahmood	(2019)
Sine Kumaraswamy-G Family of Distributions	Chesneau	(2020)
Sine Topp-Leone-G Family	Al-Babtain	(2020)
On Sine Power Lomax Model	Ali	(2021)
Estimation of Sine Inverse Exponential Model	Shrahili	(2021)
Sine Half-logistic Inverse Rayleigh Distribution	Shrahili	(2021)
Sine Entropy of Uncertain Random Variables	Gang Shi	(2021)
sine exponentiated Weibull-H	Alyami et al.	(2022)
An Extended Cosine-G Trig Family	Mahmood et al.	(2022)

- The new CDF has the capability of improving flexibility and goodness of fit owing to the inclusion of the cotangent function and one more shape parameter, which eventually leads in models that are one of a kind, more flexible, and more effective;
- Practically, for all base models, the new density also adopts bimodal shapes, and the hazard function adopts all monotone and non-monotone forms.

The primary motivations for introducing CE-G, LCE-G families, and LCEB model are:

- An exponentiated version of the cotangent-G family is developed by simultaneously disbursing the cotangent trigonometric function and the exponentiated families (EF) procedure.
- The proposed families (CE-G, LCE-G) utilize the exponentiated mixed algebraic and trigonometric generator of distributions, which has not been previously studied.
- Due to the addition of the cotangent function and one additional shape parameter, the new CDF improves the flexibility, accuracy, and gof, leading to the creation of new flexible and effective models.
- The numerical outcomes demonstrate the capability, superiority, and usefulness of the proposed generator, the G-family, and the novel distribution.
- Using the suggested extension allows for a straightforward rollback of any G-class or classical model.

A novel exponentiated trigonometric G-family of distributions, which the authors of this paper refer to as the new logistic cotangent exponentiated generalized family of distributions, is presented here. It has symmetrical, unimodal, and bimodal tails, in addition to a particular member that is predicated on the Burr distribution. Our goal is to provide proof that the LCEB model that has been proposed has excellent simulation qualities in addition to outstanding capacity to fit not only the survival data set but also the real data set.

The following is the order in which the paper should be folded. In Section II, the proposed exponentiated trigonometric family is described, whereas, in Section III, the family characteristics are discussed. Special models are presented with their graphs in Section IV. In Section V, the unique member based on Burr distribution is discussed in detail, while Section VI focuses on the estimate of parameters by employing the maximum likelihood method, conducting a simulation exercise, and applying the results to two different data sets. A few concluding feedback ends this study in Section VIII.

TABLE 3. The LCEG family members with essential description.

Name	New cdf $N(x)_{(Baseline)}$	Parameter(s)	Supporting Interval
LCEU	$N(x)_{(uniform)} = 1/\left(1+exp(\cot(\pi(\frac{x}{\theta})^{\alpha}))\right)$	(θ,α)	$(0,\theta)$
LCEE	$N(x)_{(exponential)} = 1/\left(1+exp(\cot(\pi((1-e^{-\lambda x}))^{\alpha}))\right)$	(λ,α)	$(0,+\infty)$

II. THE CE AND LCE GENERALIZED FAMILIES

A. DEVELOPMENT OF COTANGENT EXPONENTIATED-G (CE-G) FAMILY

Suppose $r(t)$ is the density of a r.v. $T \in [a, b]$ for $-\infty \leq a < b < \infty$. The suggested generator $W[L(X)]$ is a link function that satisfies the T-X family's required conditions. The main functions(cdf and pdf)of CE-G family, respectively, are:

$$N(x) = \int_{-\infty}^{(-\cot(\pi\, L^{\alpha}(x)))} r(t)\, dt. = R\left(-\cot\left(\pi\, L^{\alpha}(x)\right)\right) \quad (1)$$

$$n(x) = \pi\, \alpha\, L(X)\; [L(X)]^{(\alpha-1)}\; \csc^2(\pi\; (L(X))^{\alpha}) \times r\left[-\cot(\pi\; (L(X))^{\alpha})\right] \quad (2)$$

1) COTANGENT REPRESENTATION OF THE LCE-G

suppose that T is considered as a logistic r.v. and placing $W[L(X)] = -\cot[\pi(L(X))^{\alpha})]$ in logistic cdf $= \left(1 + e^{-t}\right)^{-1}$, then new cdf and pdf of LCE-G, respectively, are:

$$N(x) = \frac{1}{1 + exp(\cot(\pi(L(X))^{\alpha}))} \quad (3)$$

$$n(x) = \pi\, \alpha\, [L(X)]^{\alpha}\; \csc^2(\pi\; (L(X))^{\alpha})\, e^{-(-\cot(\pi\; (L(X))^{\alpha}))} \left[1 + e^{-(-\cot(\pi\; (L(X))^{\alpha}))}\right]^{-2} \quad (4)$$

The above central functions of LCE-G can be presented in tangent form simply using the trigonometric relationships.

2) MEMBERS OF LCEG FAMILY

Prominent members of the LCEG family are listed in Table 3. In Section V, the Burr $L(X)$ user will be the focus of all discussion.

III. CHARACTERISTICS OF LCEG FAMILY

Major characteristics of LCEG are presented in this section.

A. RELIABILITY AND NON-RELIABILITY FUNCTIONS

$$\text{Survival function} = S(x) = 1 - \left[\frac{1}{(1 + e^{-(-\cot(\pi\; (L(X))^{\alpha}))})}\right] = \frac{e^{-(-\cot(\pi\; (L(X))^{\alpha}))}}{1 + e^{-(-\cot(\pi(L(X))^{\alpha}))}}$$

$$\text{Hazard rate function} = h(x) = \pi\; \alpha\; \left[1 + e^{-(-\cot(\pi\; (L(X))^{\alpha}))}\right]^{-1} [L(X)]^{(\alpha)}\, \csc^2(\pi\; (L(X))^{\alpha}) \quad (5)$$

$$Cumulativehazardrate = H(x) = e^{-(-\cot(\pi\,(L(X))^{\alpha}))} + \ln\left[1 + e^{-(-\cot(\pi (L(X))^{\alpha}))}\right] \quad (6)$$

B. QUANTILE AND QUANTILE DENSITY FUNCTION

You may find the quantile function, also known as the qf, by immediately inverting the distribution. (3) as

$$Q(u) = Q_G\,(v) = G^{-1}(v), \quad (7)$$

such that $v = \left[\frac{1}{\pi}\cot^{-1}\left(-(-\ln\left[\frac{(1-u)}{u}\right])\right)\right]^{\frac{1}{\alpha}}$. If u follows a uniform $U(0, 1)$ distribution, then the density function in (4) is present in the solution of the nonlinear align $X = Q(u)$. The crucial uses of Eq. (7) are: (1). For example, substituting the Burr baseline model, we can use Eq. (7) to simulate 1000 LCEB (1.5,2.0) occurrences and onward, many useful graphical measures and respective graphs for these simulated data can be obtained. (2). Eq. (7) can be used to find median=Q(1/2), Bowley's skewness and Moors kurtosis. After differentiating $Q(u)$ with respect to u, one may derive the quantile density function (qdf) of X:

$$qdf = D\prime(t) = \frac{\frac{1}{\pi}\cot^{-1}\left(-\left(-\ln\frac{(1-t)}{t}\right)\right)^{(\frac{1-\alpha}{\alpha})}}{\pi\,\alpha\,t\,(1-t)\,\left(1 + (-(-\ln(\frac{(1-t)}{t})))\right)} \quad (8)$$

C. LINEAR REPRESENTATION OF NEW CDF

We can rewrite the cdf of LCEG (3) as

$$N(x) = \left[1 + e^{\cot(\pi\,(L(X))^{\alpha})}\right]^{-1}$$

Since $(1 + e^t)^{-1} = 1 - (1 + e^{-t})^{-1}$,then $N(x)$ is

$$N(x) = 1 - \left[1 + e^{-\cot(\pi\,(L(X))^{\alpha})}\right]^{-1}$$

by the following two expansions $(1 + v)^{-1} = \sum_{h=0}^{\infty}(-1)^h\,v^h$ for $|v| < 1$ and $e^x = \sum_{j=0}^{\infty}\frac{x^j}{j!}$, respectively, we have proposed model cdf as follows.

$$N(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+1}\,[i\cot\pi\,(L(X))^{\alpha}]^j}{j!}$$

We know $[\cot\pi\,(L(X))^{\alpha}]^j = \left[\frac{1}{\cot\pi\,(L(X))^{\alpha}}\right]^j$. Using the power series $[\tan(x)]^j = \sum_{k=0}^{\infty} a_k(j)(x)^{2k+j}$, where $a_0(j) = 1$, $a_1(j) = -j/3$, $a_2(j) = j(5j-7)/90$,etc.
Hence, $[\cot(\pi\,(L(X))^{\alpha})]^j = \sum_{k=0}^{\infty} b_k(j)\,(\pi)^{2k+j}\,(L(X))^{\alpha(2k+j)}$ where $b_k(j) = \frac{1}{a_k(j)}$. Finally,

$$N(x) = \sum_{j,k=0}^{\infty} W_{(i,j)}\,H_{k,j}(x)$$

such that $H_{k,j}(x))$ is the distribution function with power parameter that is exponentiated $\alpha(2k + j)$

$$W_{(i,j)} = \sum_{i=1}^{\infty}\frac{(i)^j(-1)^{(i+j+l)}}{j!}\,b_k(j)\,(\pi)^{2k+j}$$

$$H_{k,j}(x)) = (L(X))^{\alpha(2k+j)}.$$

whereas

$$n(x) = \sum_{j,k=0}^{\infty} v_{i,j}\,h_{(\alpha\,(2k+j))}(x) \quad (9)$$

here $h_{(\alpha(2k+j))}(x)$ a pdf of the exponentiated distribution with parameter $(\alpha(2k + j))$ in the power,

$$v_{i,j} = \sum_{i=1}^{\infty}\frac{(i)^j(-1)^{(i+j+l)}}{j!}\,b_k(j)\,(\pi)^{2k+j}$$
$$a_0(j) = 1,\, a_1(j) = -j/3,\, a_2(j) = j(5j-7)/90,\, etc.$$

and

$$h_{(\alpha\,(2k+j))}(x) = \alpha\,(2k+j)\,L(X)\,(L(X))^{(\alpha\,(2k+j))-1}$$

D. MATHEMATICAL DERIVATION OF MOMENTS, AND OTHER PROPERTIES

We can represent the nth moment of X by the aid of equation (9) as below:

$$\mathbb{E}(X^n) =, \sum_{j,k=0}^{\infty} v_{i,j}\,\mathbb{E}(Y^n_{\alpha(2k+j)}), \quad (10)$$

Formally speaking, the definition of the probability-weighted moment (PWM) of our proposed model is defined as follows:

$$\rho_{r,s} = \int_0^{\infty} x^r\,N(x)^s\,n(x)\,dx,\; r \geq 1, s \geq 0. \quad (11)$$

We consider

$$N(x)^s = [(1 + e^{\cot[\pi\,(L(X))^{\alpha}]})^{-1}]^s$$

Putting the pdf of the LCEG family(given below)

$$n(x) = \pi\alpha PL(X)\,(L(X))^{(\alpha-1)}\csc^2(\pi\,(L(X))^{\alpha})e^{-(-\cot(\pi(L(X))^{\alpha}))}\left[1 + e^{-(-\cot(\pi\,(L(X))^{\alpha}))}\right]^{-2}$$

in (11), then using the binomial expansion and exponential series, $PWM = \rho_{r,q}$ may also be interpreted as

$$\rho_{r,q} = \sum_{i,j=0}^{\infty}\pi\alpha\binom{-(s+2)}{i}\frac{(i+1)^j}{j!}\int_{-\infty}^{\infty} x^r\,L(X)\,(L(X))^{(\alpha-1)}\left(\csc(\pi\,(L(X))^{\alpha})\right)^2\left[\cot(\pi\,(L(X))^{\alpha})\right]^j\,dx. \quad (12)$$

For $[\cot(\pi(L(X))^{\alpha})]^j$, we used the following expansion $[\cot(x)]^s = \sum_{k=0}^{\infty} a_k(s)(x)^{2k-s}$, where $a_0(s) = 1$, $a_1(s) = -s/3$,$a_2(s) = s(5s-7)/90$,etc. and similarly for $[\csc(\pi\,(L(X))^{\alpha})]^2 = \sum_{l=0}^{\infty} c_l(2)(x)^{2l-2}$,we get

$$\rho_{r,q} = \sum_{i,j,k,l=0}^{\infty}\binom{-(s+2)}{i}\frac{(i+1)^j}{j!}\,\alpha\,\pi^{(2(k+l)-(j+2))-1}\,a_k(j)\,c_l(2)\int_{-\infty}^{\infty} x^r\,L(X)\,[L(X)]^{(\alpha\,(2k+2l+1)-(j+3))}\,dx$$

$$\rho_{r,q}=\sum_{i,j,k,l=0}^{\infty}U_{i,j,k,l}\int_{-\infty}^{\infty}x^{r}\left(\alpha(2k+2l+1)-(j+3)\right)L(X)\,[L(X)]^{(\alpha(2k+2l+1)-(j+3))}\,dx$$

where $U_{i,j,k,l}=\frac{\binom{-(s+2)}{i}\frac{(i+1)^j}{j!}\pi^{(2(k+l)-(j+2)-1)}a_k(j)\,c_l(2)}{(\alpha(2k+2l+1)-(j+3))}$. The corresponding moment-generating function is:

$$M_X(t)=E(e^{tx})=\int_{-\infty}^{\infty}(e^{tx})\,n(x)\,dx=\int_{-\infty}^{\infty}(e^{tx})\sum_{j,k=0}^{\infty}v_{i,j}\,(\alpha\,(2k+j))(L(X))^{(\alpha(2k+j)-1)}\,L(X)\,dx.$$

E. STOCHASTIC ORDERING

As the following theorem demonstrates, the LCEG family of distributions, denoted by the letter (α), is ranked according to the ''likelihood ratio'' ranking deemed the most authoritative. This demonstrates the adaptability of the parameter set used by the LCEG family of distributions.

Theorem. Let X follows LCEG(α_1) and Y follows LCEG(α_2).

If $\alpha_1 \geq \alpha_2$ *then*
$X \leqslant_{lr} (Y), X \leqslant_{hr} (Y), X \leqslant_{mrl} (Y) and X \leqslant_{st} (Y)$.

Proof. We can easily write the ratio of the likelihood as shown at the bottom of the page. Now if $\alpha_1 > \alpha_2$ then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0$, it suggests that $X \leqslant_{lr} (Y)$ and hence $X \leqslant_{lr} (Y), X \leqslant_{hr} (Y), X \leqslant_{mrl} (Y), X \leqslant_{st} (Y)$

IV. LCEG FAMILY's SPECIAL MODELS

Here, famous baseline models are utilized to develop the LCEG family's special models, which are under the:

A. THE LOGISTIC COT EXPONENTIATED EXPONENTIAL (LCEE) DISTRIBUTION

Using exponential as baseline with pdf $L(X) = \lambda\, e^{-\lambda x}$ and cdf $L(X) = 1 - e^{-\lambda x}$. Then, the cdf of the LCEE distribution is:

$$N(x) = (1 + e^{-(-\cot[\pi\Psi^{\alpha}])})^{-1},\ \ \Psi = 1 - \exp[-\lambda(x)]. \tag{13}$$

and its pdf is:

$$n(x) = \pi\,\lambda\alpha\,\exp[-\lambda(x)][1 - e^{-\lambda x}]^{(\alpha-1)}\,[\csc^2(\pi(1-\exp[-\lambda(x)]^{\alpha}))](e^{-(-\cot[\pi\Psi^{\alpha}])})\,(1 + e^{-(-\cot[\pi\Psi^{\alpha}])})^{-2}$$

and its hz is:

$$h(x) = \pi\,\lambda\alpha\,\exp[-\lambda(x)][1 - exp(-\lambda\,x)]^{(\alpha-1)}\,[\csc^2(\pi(1-\exp[-\lambda(x)]^{\alpha}))](1 + e^{-(-\cot[\pi\Psi^{\alpha}])})^{-1}$$

$$\frac{f_X(x)}{f_Y(x)} = \alpha_1 L(X)^{\alpha_1-\alpha_2}\sin\left(\pi L(X)^{\alpha_2}\right)e^{\cot(\pi L(X)^{\alpha_1})}\,\frac{\left(e^{\cot(\pi L(X)^{\alpha_1})}+1\right)\csc\left(\pi L(X)^{\alpha_1}\right)}{\alpha_2 e^{\cot(\pi L(X)^{\alpha_2})}\left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right)}$$

$$\begin{aligned}\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} &= \pi\alpha_1 L(X)^{\alpha_1-1}L(X)\left(-\cot\left(\pi L(X)^{\alpha_1}\right) - \frac{e^{\cot(\pi L(X)^{\alpha_1})}\csc^2\left(\pi L(X)^{\alpha_1}\right)\left(e^{\cot(\pi L(X)^{\alpha_1})}\right)'\left(e^{\cot(\pi L(X)^{\alpha_1})}+1\right)}{e^{\cot(\pi L(X)^{\alpha_1})}\left(e^{\cot(\pi L(X)^{\alpha_1})}+1\right)}\right)\\
&\quad + \pi\alpha_2 L(X)^{\alpha_2-1}L(X)\left(\frac{e^{\cot(\pi L(X)^{\alpha_2})}\csc^2\left(\pi L(X)^{\alpha_2}\right)\left(e^{\cot(\pi L(X)^{\alpha_2})}\right)'\left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right)}{e^{\cot(\pi L(X)^{\alpha_2})}\left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right)}\right)\\
&\quad + \pi\alpha_2 L(X)^{\alpha_2-1}L(X)\left(\cot\left(\pi L(X)^{\alpha_2}\right)\right)\\
&\quad + \pi\alpha_1 L(X)^{\alpha_1-1}L(X)\left(-e^{\cot(\pi L(X)^{\alpha_1})}\right)\frac{\left(\csc^2\left(\pi L(X)^{\alpha_1}\right)\right)\left(e^{\cot(\pi L(X)^{\alpha_1})}+1\right)}{e^{\cot(\pi L(X)^{\alpha_1})}\left(e^{\cot(\pi L(X)^{\alpha_1})}+1\right)}\\
&\quad - \left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right)\frac{\left(\pi\alpha_2 L(X)^{\alpha_2-1}L(X)\left(-e^{\cot(\pi L(X)^{\alpha_2})}\right)\csc^2\left(\pi L(X)^{\alpha_2}\right)\right)}{e^{\cot(\pi L(X)^{\alpha_2})}\left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right)}\\
&\quad + \frac{(\alpha_1-\alpha_2)\,L(X)}{L(X)}\end{aligned}$$

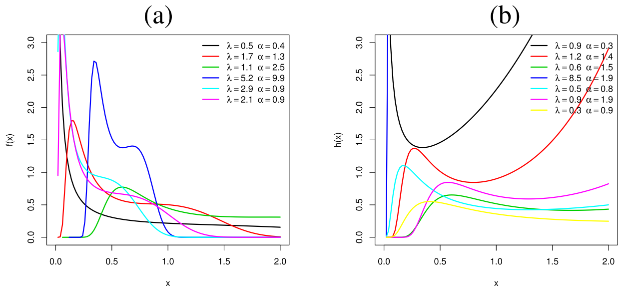


FIGURE 1. The LCEE (a) density (b) hazard function plots.

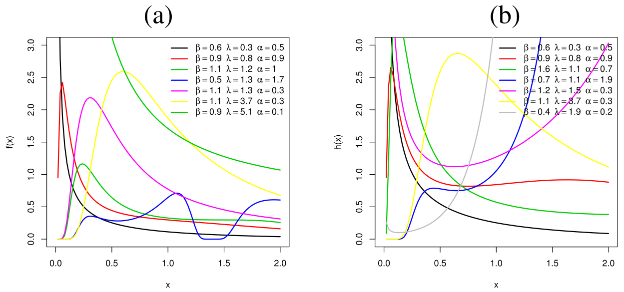


FIGURE 2. The LCEGa (a) density (b) hazard function plots.

B. THE LOGISTIC COT EXPONENTIATED GAMMA (LCEGa) DISTRIBUTION

Exercising the gamma distribution as baseline in Eq. (3), the main functions of LCEGa are produced.

$$N(x) = \left(1 + e^{-(-\cot[\pi(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)})^{\alpha}])}\right)^{-1}$$

$$n(x) = \pi\,\alpha\,\frac{\beta^{\lambda}}{\Gamma(\lambda)}\,x^{\lambda-1}\,\mathrm{e}^{-\beta x}\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{(\alpha-1)} \left[\csc^2(\pi\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{\alpha})\right]\left(e^{\cot(\pi L(X)^{\alpha_2})}+1\right) e^{-\left(-\cot\left[\pi\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{\alpha}\right]\right)}\left(1+e^{-(-\cot\left[\pi\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{\alpha}\right])}\right)^{-2}$$

$$h(x) = \pi\,\alpha\,\frac{\beta^{\lambda}}{\Gamma(\lambda)}\,x^{\lambda-1}\,\mathrm{e}^{-\beta x}\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{(\alpha-1)} \left[\csc^2(\pi\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{\alpha})\right]\left(1+e^{-(-\cot\left[\pi\left(\frac{\gamma(\lambda,\beta x)}{\Gamma(\lambda)}\right)^{\alpha}\right])}\right)^{-1}$$

C. THE LOGISTIC COT EXPONENTIATED Weibull(LCEW) DISTRIBUTION

Disbursing Weibull as base model with cdf $L(X) = 1 - \exp[-\lambda(x)^{\beta}]$ and density $L(X) = \lambda\beta x^{\beta-1}\exp[-\lambda(x)^{\beta}]$ The two-parameter LCEW having the following main functions is produced.

$$N(x) = (1 + e^{-(-\cot[\pi\Phi^{\alpha}])})^{-1},\ \ \Phi = (1 - \exp[-\lambda(x)^{\beta}]). \tag{14}$$

$$n(x) = \pi\alpha\,(\lambda\beta\,x^{\beta-1}\exp[-\lambda(x)^{\beta}])\Phi^{(\alpha-1)} [\csc(\pi\,\Phi^{\alpha})] e^{-(-\cot[\pi\,\Phi^{\alpha}])}(1+e^{-(-\cot[\pi\,\Phi^{\alpha}])})^{-2}$$

$$h(x) = \pi\alpha\,(\lambda\beta\,x^{\beta-1}\exp[-\lambda(x)^{\beta}])\Phi^{(\alpha-1)} [\csc(\pi\,\Phi^{\alpha})](1+e^{-(-\cot[\pi\,\Phi^{\alpha}])})^{-1} \tag{15}$$

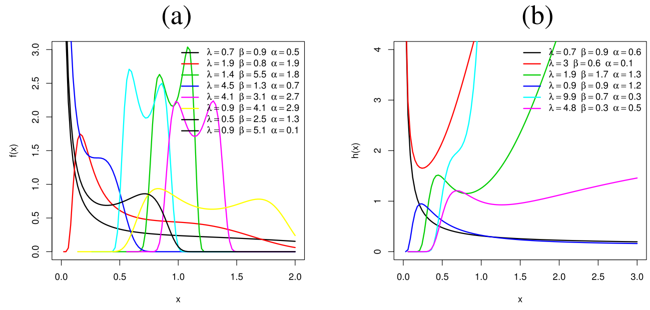


FIGURE 3. The LCEW's plots for (a) density and (b) hazard rate function.

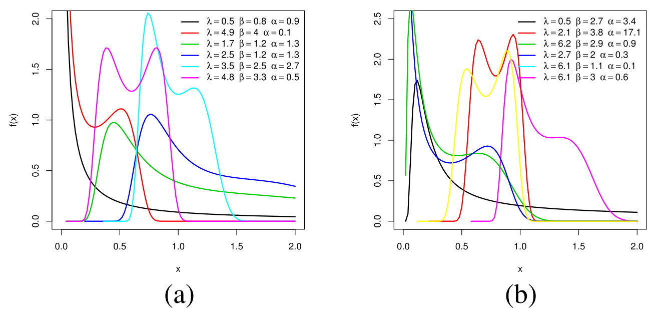


FIGURE 4. The LCEB's plots for (a) density (b) density.

V. THE LCEB DISTRIBUTION

Here, a new distribution logistic cot exponentiated Burr(LCEB) is proposed. Firstly, its cdf, pdf, and hazard functions are derived then graphs of the pdf and hazard are presented with possible shapes to testify to the flexibility present in the model.

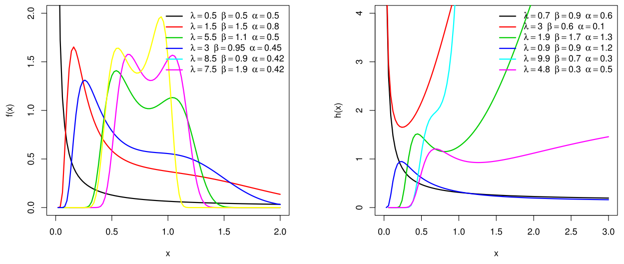
A. METHODOLOGY

$L(X)$ be the Burr cdf with $\lambda > 0$ and $\beta > 0$, $L(X) = 1 - \left(1+x^{\lambda}\right)^{-\beta}$, $x > 0$ $\lambda, \beta > 0$. with density $L(X) = \lambda\beta(x)^{\lambda-1}(1+(x)^{\lambda})^{-(\beta+1)}$ it follows three-parameter LCEB whose cdf, pdf, and hazard function, respectively, are:

$$N(x) = \left(1 + e^{-(-\cot(\pi\,\Delta^{\alpha}))}\right)^{-1},\ \ \Delta = 1 - \left(1+x^{\lambda}\right)^{-\beta}.$$

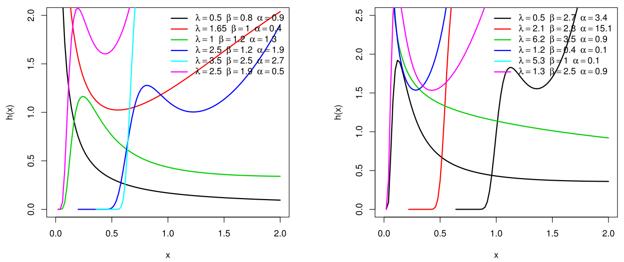
$$n(x) = \pi\,\alpha\,(\lambda\beta(x)^{\lambda-1}(1+(x)^{\lambda})^{-(\beta+1)})\,\Delta^{(\alpha-1)} \left(\csc^2(\pi\,\Delta^{\alpha})\right) e^{-(-\cot(\pi\,\Delta^{\alpha}))} \left(1 + e^{-(-\cot(\pi\,\Delta^{\alpha}))}\right)^{-2}$$

$$h(x) = \pi\,\alpha\,(\lambda\beta(x)^{\lambda-1}(1+(x)^{\lambda})^{-(\beta+1)})\,\Delta^{(\alpha-1)} \left(\csc^2(\pi\,\Delta^{\alpha})\right) \left(1 + e^{-(-\cot(\pi\,\Delta^{\alpha})}\right)^{-1} \tag{16}$$



(a) (b)

FIGURE 5. The LCEB's plots for (a) density and (b) hazard rate.



(a) (b)

FIGURE 6. The LCEB's plots for (a) hazard (b) hazard rate.

B. CHARACTERISTICS OF LCEB DISTRIBUTION

This section deduces the LCEB's reliability properties, as well as its residual and reverses residual life, quantile function, moments, generating functions, entropies, order statistics, and other relevant mathematical aspects.

1) RELIABILITY MEASURES

$$S(x) = \left(1 + exp(\cot(\pi\,\Delta^{\alpha}))\right)\left[1 - \left(1 + exp(\cot(\pi\,\Delta^{\alpha}))\right)\right]^{-1}$$

$$h(x) = \pi\,\alpha\,(\lambda\beta(x)^{\lambda-1}(1+(x)^{\lambda})^{-(\beta+1)}) \frac{\Delta^{(\alpha-1)}\left(\csc^2(\pi\,\Delta^{\alpha})\right)}{\left(1+e^{-(-\cot(\pi\,\Delta^{\alpha}))}\right)}$$

2) QUANTILE FUNCTION, SIMULATION, SKEWNESS, AND KURTOSIS

The qf of LCEB is:

$$X = D_F(t) = \left[\left(1 - \left(\frac{1}{\pi}\cot^{-1}\left(\frac{(1-t)}{t}\right)\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{-\beta}} - 1\right]^{\frac{1}{\lambda}} \tag{17}$$

The median of the LCEB (λ, β, α) distribution is given by

$$\begin{aligned} Med &= Q_F(0.5) \\ &= \left[\left(1 - \left(\frac{1}{\pi}\cot^{-1}\left(\frac{(1-0.5)}{0.5}\right)\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{-\beta}} - 1\right]^{\frac{1}{\lambda}} \end{aligned}$$

3) RESIDUAL AND REVERSE RESIDUAL LIFE

The LCEB random variable X's residual lifetime, indicated by the symbol $R_t(x)$, is obtained as shown at the bottom of the next page. In addition, many scholars have recently been interested in the reversed hazard rate function $\bar{R}_t(x)$. Shaked and Shanthikumar (1), for example. is created as shown at the bottom of the next page.

4) SUB-MODELS OF LCEB

TABLE 4. Some sub-models of LCEB distribution.

S.No.	λ	β	α	Reduced distribution
1	1	-	-	Logistic cot exponentiated Lomax(Pareto type 2)distribution
2	-	1	-	Logistic cot exponentiated Fisk distribution
3	-	-	1	Logistic cot Burr distribution
4	1	-	1	Logistic cot Lomax(Pareto type 2)distribution
5	-	1	1	Logistic cot Fisk distribution

VI. LCEB AND STATISTICAL INFERENCE

A. METHODS OF PARAMETER ESTIMATION

Let $x_1, \ldots, x_n$ represent a sample of size n taken from the LCEB distribution represented by (16). The vector of parameters $\mathbf{\Theta} = (\alpha,\ \lambda\,\beta)^{\top}$ may be modeled using the log-likelihood function as described in the following sentence.

$$\begin{aligned} \ell = &\sum_{i=1}^{n} \log\left(\csc^2\left(\pi\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)\right)^{\alpha}\right) \\ &+ \sum_{i=1}^{n} \cot\left(\pi^{\alpha}\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)^{\alpha}\right) \\ &+ \sum_{i=1}^{n} \log\left(e^{\cot\left(\pi^{\alpha}\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)^{\alpha}\right)}+1\right) \end{aligned} \tag{18}$$

The score vector $U(\Theta)$'s components are provided by

$$U_{\alpha} = \sum_{i=1}^{n} \log\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right) + \frac{n}{\alpha} \tag{19}$$

$$\begin{aligned} U_{\beta} =&(\alpha-1)\sum_{i=1}^{n} \frac{\left(x_i^{\lambda}+1\right)^{-\beta}\log\left(x_i^{\lambda}+1\right)}{1-\left(x_i^{\lambda}+1\right)^{-\beta}} \\ &- \sum_{i=1}^{n} \log\left(x_i^{\lambda}+1\right) + \frac{n}{\beta} \end{aligned} \tag{20}$$

$$\begin{aligned} U_{\lambda} =&(\alpha-1)\sum_{i=1}^{n} \frac{\beta x_i^{\lambda}\log\left(x_i\right)\left(x_i^{\lambda}+1\right)^{-\beta-1}}{1-\left(x_i^{\lambda}1\right)^{-\beta}} \\ &-(\beta+1)\sum_{i=1}^{n}\frac{x_i^{\lambda}\log\left(x_i\right)}{x_i^{\lambda}+1} + \sum_{i=1}^{n}\log\left(x_i\right) + \frac{n}{\lambda} \end{aligned} \tag{21}$$

The MLEs of the model parameters are produced by setting these aligns to zero and concurrently solving them. Following are the results of the observed Fisher information matrix:

$$U_{\alpha\,\alpha} = -\frac{n}{\alpha^2}$$

$$\begin{aligned} U_{\beta\,\beta} = -\frac{n}{\beta^2} + (\alpha-1)\sum_{i=1}^{n}\Bigg(&-\frac{\left(x_i^{\lambda}+1\right)^{-2\beta}\log^2\left(x_i^{\lambda}+1\right)}{\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)^2} \\ &-\frac{\left(x_i^{\lambda}+1\right)^{-\beta}\log^2\left(x_i^{\lambda}+1\right)}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}\Bigg) \end{aligned}$$

$$U_{\lambda,\lambda} = (\alpha-1)\sum_{i=1}^{n}\Bigg(-\frac{\beta^2 x_i^{2\lambda}\log^2\left(x_i\right)\left(x_i^{\lambda}+1\right)^{-2\beta-2}}{\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)^2}$$

$$+\frac{\beta x_i^{\lambda}\log^2(x_i)\left(x_i^{\lambda}+1\right)^{-\beta-1}}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}\Bigg)-\frac{n}{\lambda^2}$$
$$+(\alpha-1)\sum_{i=1}^{n}\beta x_i^{\lambda}\log(x_i)$$
$$\frac{\left(x_i^{\lambda}(-\log(x_i))\left(x_i^{\lambda}+1\right)^{-\beta-2}-\beta x_i^{\lambda}\log(x_i)\left(x_i^{\lambda}+1\right)^{-\beta-2}\right)}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}$$
$$-(\beta+1)\sum_{i=1}^{n}\left(\frac{x_i^{\lambda}\log^2(x_i)}{x_i^{\lambda}+1}-\frac{x_i^{2\lambda}\log^2(x_i)}{\left(x_i^{\lambda}+1\right)^2}\right)$$
$$U_{\alpha,\beta}=\sum_{i=1}^{n}\frac{\left(x_i^{\lambda}+1\right)^{-\beta}\log\left(x_i^{\lambda}+1\right)}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}$$
$$U_{\alpha,\lambda}=\sum_{i=1}^{n}\frac{\beta x_i^{\lambda}\log(x_i)\left(x_i^{\lambda}+1\right)^{-\beta-1}}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}$$
$$U_{\beta,\lambda}=(\alpha-1)\sum_{i=1}^{n}\left(\frac{x_i^{\lambda}\log(x_i)\left(x_i^{\lambda}+1\right)^{-\beta-1}}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}\right.$$
$$\left.-\frac{\beta x_i^{\lambda}\log(x_i)\left(x_i^{\lambda}+1\right)^{-\beta-1}\log\left(x_i^{\lambda}+1\right)}{1-\left(x_i^{\lambda}+1\right)^{-\beta}}\right)$$
$$-\sum_{i=1}^{n}\frac{x_i^{\lambda}\log(x_i)}{x_i^{\lambda}+1}-(\alpha-1)$$
$$\sum_{i=1}^{n}\left(\frac{\beta x_i^{\lambda}\log(x_i)\left(x_i^{\lambda}+1\right)^{-2\beta-1}\log\left(x_i^{\lambda}+1\right)}{\left(1-\left(x_i^{\lambda}+1\right)^{-\beta}\right)^2}\right).$$

B. SIMULATIONS

The accuracy of the maximum likelihood technique for estimating the LCEB parameters is evaluated with the use of Monte Carlo simulations, which may be found in the aforementioned section. The simulation study consists of a total of one thousand iterations, each of which uses a different combination of sample sizes and parameter settings: $n = 50, 100, 200, 300$, and 500. I: $\alpha = 0.1, \beta = 0.3$ and $\lambda = 0.2$, II: $\alpha = 0.25, \beta = 0.75$ and $\lambda = 0.5$, and III: $\alpha = 0.5, \beta = 0.35$ $\lambda = 0.25,$. The mean square errors (MSEs), average widths (AWs), and average biases (Biases) of the 95% confidence intervals for the parameters *alpha* and *lambda* are all provided in Table 5 for various sample sizes. These data indicate that the MLEs provide an accurate prediction of the characteristics of the LCEB distribution. The biases, MSEs, L.bounds, and U.bounds of X will all decrease as the sample size n increases. In addition, the confidence limits' critical points (CPs) are located within a close proximity to the nominal 95 percent values. As a consequence of this, it is feasible to estimate the model parameters and provide confidence ranges for them by utilising the MLEs and the discoveries on their asymptotic behaviour.

TABLE 5. Monte Carlo simulation results: Biasedness, Mean Square Error (MSEs), Lower limit and Upper limit.

Parameter	n	Biasedness	Mean Square Error	Lower limit	Upper limit
		I			
α	50	1.017	26.223	2.802	5.019
	100	0.507	0.410	0.105	1.214
	200	0.438	0.250	0.198	0.881
	300	0.423	0.216	0.259	0.789
	500	0.423	0.202	0.322	0.728
β	50	15.409	1556.516	43.961	70.310
	100	12.383	788.318	22.326	41.036
	200	12.224	1000.469	7.924	25.113
	300	11.406	159.326	6.585	17.868
	500	11.346	157.775	7.898	16.111
λ	50	-0.118	0.018	0.029	0.167
	100	-0.124	0.018	0.035	0.135
	200	-0.128	0.019	0.038	0.107
	300	-0.132	0.019	0.043	0.095
	500	-0.136	0.019	0.045	0.084
		II			
α	50	0.993	69.333	4.157	6.640
	100	0.423	0.330	0.113	1.338
	200	0.370	0.185	0.226	1.014
	300	0.342	0.144	0.290	0.895
	500	0.330	0.122	0.357	0.803
β	50	15.749	743.979	34.021	61.434
	100	13.584	262.014	12.488	33.726
	200	12.047	163.492	6.853	20.851
	300	11.916	160.360	8.047	19.500
	500	11.379	132.126	8.322	15.936
λ	50	-0.160	0.050	0.093	0.719
	100	-0.148	0.036	0.117	0.619
	200	-0.158	0.032	0.177	0.508
	300	-0.156	0.030	0.211	0.479
	500	-0.161	0.028	0.241	0.437
		III			
α	50	1.362	31.992	5.770	9.494
	100	0.511	2.827	0.559	2.512
	200	0.279	0.194	0.223	1.348
	300	0.238	0.1031	0.325	1.151
	500	0.216	0.071	0.415	1.018
β	50	3.003	10.831	0.996	5.901
	100	2.697	7.975	1.549	4.558
	200	2.517	6.626	1.890	3.845
	300	2.491	6.368	2.064	3.617
	500	2.468	6.185	2.227	3.408
λ	50	0.023	0.018	0.055	0.567
	100	0.024	0.010	0.091	0.465
	200	0.029	0.005	0.151	0.407
	300	0.027	0.003	0.175	0.379
	500	0.027	0.002	0.199	0.355

VII. DATA ANALYSIS TO THE PROPOSED DISTRIBUTION

This section covered the analysis of two real-life data sets using many models, focusing on the LCEB model.

$$R_t(x)=\left(1+e^{-(-\cot(\pi(1-(1+x+t^{\lambda})^{-\beta})^{\alpha}))}\right)\left(1-\left(1+e^{-(-\cot(\pi(1-(1+x^{\lambda})^{-\beta})^{\alpha}))}\right)\right)$$
$$\left[\left(1-\left(1+exp(\cot(\pi(1-\left(1+x+t^{\lambda}\right)^{-\beta})^{\alpha}))\right)\right)\left(1+exp(\cot(\pi(1-\left(1+x^{\lambda}\right)^{-\beta})^{\alpha}))\right)\right]^{-1}$$

$$\bar{R}_t(x)=\frac{\left(1+e^{-(-\cot(\pi(1-(1+x-t^{\lambda})^{-\beta})^{\alpha}))}\right)\left(1-\left(1+e^{-(-\cot(\pi(1-(1+x^{\lambda})^{-\beta})^{\alpha}))}\right)\right)}{\left(1-\left(1+e^{-(-\cot(\pi(1-(1+x-t^{\lambda})^{-\beta})^{\alpha}))}\right)\right)\left(1+e^{-(-\cot(\pi(1-(1+x^{\lambda})^{-\beta})^{\alpha}))}\right)}$$

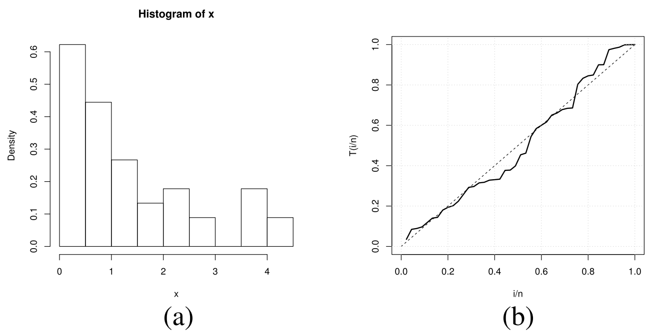


FIGURE 7. For data set 1,(a) Histogram (b) TTT plot.

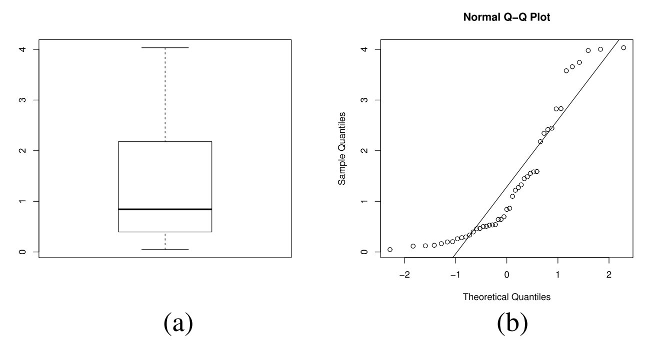


FIGURE 8. For data sets 1 (a) Boxplot (b) Normal Q-Q plot.

A. THE FIRST DATA SET

The first data set, which was compiled by Bekker et al. (2000), includes the survival rates (measured in years) of a group of patients who were treated with chemotherapy. The following data set is presented: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The histogram of data set 1 is shown in Figures 7 (a), which also displays the TTT plot's indication of a heavy right tail in Figure 7 (b). Figure 8 (a) displays the boxplot of data set 1, and Figure 8 (b) displays the Q-Q plot (b).

B. THE SECOND DATA SET

The second data are obtained from Gamma Frechet JSTA May 2013 waiting times between 64 consecutive eruptions of the Kiama Blowhole (Pinhoet al., 2012), which are 83, 51, 87, 60, 28, 95, 8, 27, 15, 10, 18, 16, 29, 54, 91, 8, 17, 55, 10, 35, 47, 77, 36, 17, 21, 36, 18, 40, 10, 7, 34, 27, 28, 56, 8, 25, 68, 146, 89, 18, 73, 69, 9, 37, 10, 82, 29, 8, 60, 61, 61, 18, 169, 25, 8, 26, 11, 83, 11, 42, 17, 14, 9, 12.

The summary statistics can be easily calculated, so we omitted them Figures 9 (a) show a heavy right tail on the TTT plot and the histogram of data set 2, which both indicate a right tail. Shows a heavy right tail on the TTT plot and the histogram of data set 2, indicating a right tail in Figure 9 (b).

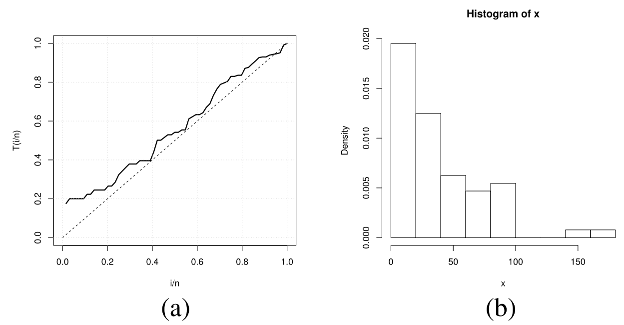


FIGURE 9. For data sets 2 (a) TTT plot (b) Histogram plot.

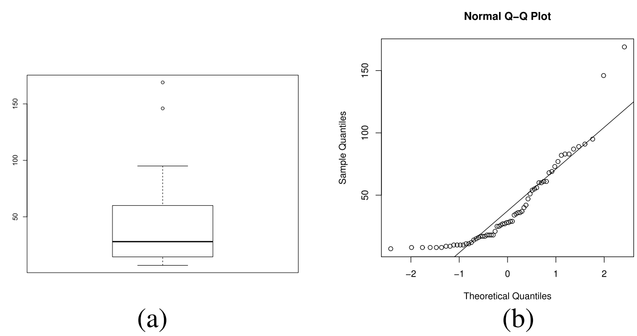


FIGURE 10. For data sets 2 (a) Boxplot (b) Normal Q-Q plot.

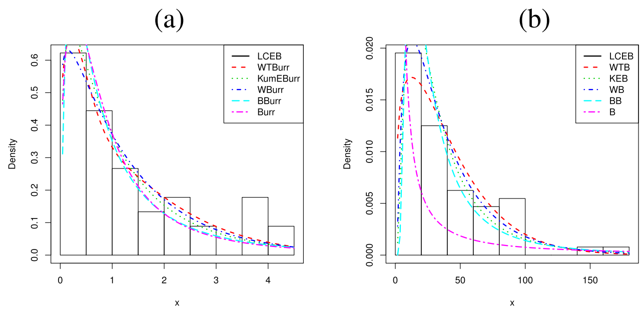


FIGURE 11. (a) Estimated pdfs of data set 1, (b)Estimated pdfs of data set 2.

The figure displays the box plot that was generated from data set 2 can be found in 10 (a) and the QQ plot in Figure 10 (b).

1) RESULTS CONCLUDED FROM THE DATA ANALYSIS

We calculate certain well-known measures of goodness-of-fit statistics, including the log-likelihood function assessed at the MLEs ($\hat{\ell}$), Anderson-Darling ($A*$), and Cramer-von Mises ($W*$), to investigate data sets 1 and 2 and compare the fitted models. in [16], the statistics $A*$ and $W*$ are discussed in depth. With the aid of the R-software, the necessary computations are performed. The better fit is shown by lower $A*$, $W*$.

The model parameters MLEs and its standard errors (in parentheses) are listed in Table 6 for Data Sets 1 and 2.

The LCEB model offers a better match than other models, according to all of the Table 7 findings.

TABLE 6. MLEs and their standard errors (in parentheses) for the data sets 1 and 2.

Distribution	λ	β	α	σ	θ
	Data set 1				
LCEB	1.8339 (0.9592)	0.6313 (0.5179)	0.4959 (0.3138)	- -	- -
WTBurr	0.7513 (0.2149)	0.4351 (0.5089)	2.7916 (3.0284)	0.9562 (0.2871)	- -
KumEBurr	7.5088 (77.2000)	94.6498 (252.3350)	0.1696 (0.4342)	2.2587 (7.0380)	2.7257 (27.8501)
WBurr	2.7747 (17.0895)	2.2049 (4.7452)	0.5377 (1.1012)	0.6294 (1.0139)	- -
BBurr	14.9386 (66.2553)	31.7049 (138.4791)	0.3218 (0.7179)	0.5806 (2.2323)	- -
Burr	1.4215 (0.1791)	1.2435 (0.1945)	- -	- -	- -
	Data set 2				
LCEB	1.3106 (2.4601)	0.8928 (1.7927)	26.1013 (19.9295)	- -	- -
WTBurr	0.0095 (0.0042)	0.2406 (0.0605)	2.8160 (0.4340)	1.8664 (0.3721)	- -
KumEBurr	26.9917 (109.0143)	7.2839 (15.9483)	0.1501 (0.2083)	5.8259 (7.6099)	26.0711 (105.2834)
WBurr	2.0773 (18.0927)	3.4357 (1.6967)	2.6137 (15.7114)	0.0612 (0.4749)	- -
BBurr	15.8442 (3.5292)	12.9804 (24.0289)	3.1139 (5.0624)	0.0786 (0.1218)	- -
Burr	6.2127 (25.9616)	0.0481 (0.2009)	- -	- -	- -

TABLE 7. The statistics $\hat{\ell}$, A^*, W^*, for the data sets 1 and 2.

Distribution	$\hat{\ell}$	AIC	BIC	A^*	W^*
	Data set 1				
LCEB	**56.97902**	**120.5434**	**125.378**	**0.3004**	**0.0357**
WTBurr	56.9856	121.9712	129.1979	0.3988	0.0555
KumEBurr	58.0709	126.1419	135.1752	0.4300	0.0618
WBurr	58.0536	124.1072	131.3338	0.5047	0.0749
BBurr	58.8978	125.7957	133.0223	0.4535	0.0622
Burr	59.8167	123.6335	127.2468	0.5342	0.0774
	Data set 2				
LCEB	**288.0167**	**582.0334**	**588.5100**	**0.4767**	**0.0681**
WTBurr	296.9266	601.8533	610.4888	1.0062	0.1467
KumEBurr	294.0857	598.1713	608.9657	0.8207	0.1119
WBurr	293.8743	595.7486	604.3841	0.7572	0.1043
BBurr	293.9108	595.8217	604.4572	0.9394	0.1309
Burr	355.3914	714.7829	719.1007	0.9263	0.1288

(a) (b)

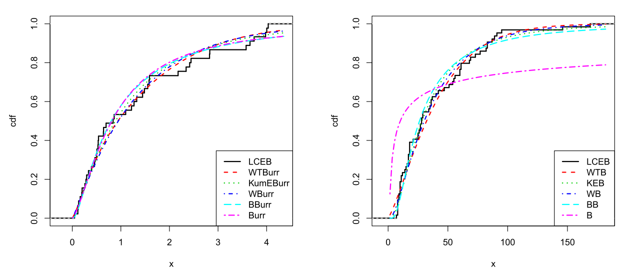


FIGURE 12. (a) Estimated cdfs of data set 1, (b) Estimated cdfs of data set 2.

As we can easily see from Figures 11, and 12, that proves to be the greatest among competitors and visually appealing. Not to mention, the LCEB model can provide greater goodness-of-fits to more complex models, with three parameters or more.

VIII. CONCLUDING REMARKS

The literature has extensively examined the generalized continuous univariate distributions. The LCEG family of distributions, which includes a four-parameter LCEB distribution with monotone and non-monotone hazard rates, is a novel class of distributions that we suggest. We examine a few of the new family's structural characteristics. In order to provide an accurate estimate of the model parameters, the maximum likelihood method is used. A Monte Carlo simulation analysis is given to confirm the accuracy of the estimations. We test which distribution fits these data sets best using a variety of goodness-of-fit measures. We have come to the conclusion that these one-of-a-kind models often provide better matches than competing models. We believe that the family that is proposed, along with the models that it generates, will see more use across a variety of domains.

APPENDIX A

A. SHAPES OF THE DENSITY FUNCTION

Forms of the density function may be characterized mathematically, and the roots of the following equation represent the critical points of the LCEG density function:

$$\frac{d\ \log n(x)}{dx} = \frac{g'(x)}{L(X)} + \frac{(\alpha-1)L(X)}{L(X)} - \pi\,\alpha\,L(X)\, L(X)^{(\alpha-1)}\left[2\cot(\pi(L(X))^{\alpha}) + (\csc(\pi(L(X))^{\alpha}))^2 - 2\frac{[\csc(\pi(L(X))^{\alpha})]^2\, e^{\cot(\pi(L(X))^{\alpha})}}{1+e^{\cot(\pi(L(X))^{\alpha})}}\right] = 0 \tag{22}$$

and there may be more than one root to (22). Let $\lambda(x) = d^2\ \log[n(x)]/d\,x^2$ can be find very easily. As shown at the top of the next page.

B. SHAPES OF THE HAZARD FUNCTION

The critical points of the hrf $h(x)$ are obtained from the following align:

$$\frac{g'(x)}{L(X)} + \frac{(\alpha-1)L(X)}{L(X)} - 2\pi\alpha L(X)^{\alpha-1}L(X)\cot\left(\pi L(X)^{\alpha}\right) + \frac{\pi\alpha L(X)^{\alpha-1}L(X)e^{\cot(\pi L(X)^{\alpha})}\csc^2\left(\pi L(X)^{\alpha}\right)}{e^{\cot(\pi L(X)^{\alpha})}+1} = 0 \tag{23}$$

and there may be more than one root to (23). Let $\lambda(x) = d^2\ \log[n(x)]/d\,x^2$. We have

$$\begin{aligned}\lambda(x) &= \frac{g''(x)}{L(X)} - \frac{g'(x)^2}{L(X)^2} + \frac{(\alpha-1)g'(x)}{L(X)} \\ &\quad - 2\pi\alpha L(X)^{\alpha-1}g'(x)\cot\left(\pi L(X)^{\alpha}\right) \\ &\quad + \frac{\pi\alpha L(X)^{\alpha-1}g'(x)e^{\cot(\pi L(X)^{\alpha})}\csc^2\left(\pi L(X)^{\alpha}\right)}{e^{\cot(\pi L(X)^{\alpha})}+1} \\ &\quad + 2\pi^2\alpha^2 L(X)^{2\alpha-2}L(X)^2\csc^2\left(\pi L(X)^{\alpha}\right) \\ &\quad + \frac{\pi^2\alpha^2 L(X)^{2\alpha-2}L(X)^2 e^{2\cot(\pi L(X)^{\alpha})}\csc^4\left(\pi L(X)^{\alpha}\right)}{\left(e^{\cot(\pi L(X)^{\alpha})}+1\right)^2} \\ &\quad - \frac{\pi^2\alpha^2 L(X)^{2\alpha-2}L(X)^2 e^{\cot(\pi L(X)^{\alpha})}\csc^4\left(\pi L(X)^{\alpha}\right)}{e^{\cot(\pi L(X)^{\alpha})}+1} \\ &\quad - \left(e^{\cot(\pi L(X)^{\alpha})}+1\right)^{-1}\big[2\pi^2\alpha^2 L(X)^{2\alpha-2}L(X)^2 e^{\cot(\pi L(X)^{\alpha})}\end{aligned}$$

$$\begin{aligned}\lambda =&\frac{g''(x)}{L(X)}-\frac{g'(x)^2}{L(X)^2}+\frac{(\alpha-1)g'(x)}{L(X)}\\&-2\pi\alpha L(X)^{\alpha-1}g'(x)\cot\left(\pi L(X)^{\alpha}\right)\\&-\pi\alpha L(X)^{\alpha-1}g'(x)\csc^2\left(\pi L(X)^{\alpha}\right)\\&+\frac{2\pi\alpha L(X)^{\alpha-1}g'(x)e^{\cot(\pi L(X)^{\alpha})}\csc^2(\pi L(X)^{\alpha})}{e^{\cot(\pi L(X)^{\alpha})}+1}\\&+2\pi^2\alpha^2L(X)^{2\alpha-2}L(X)^2\csc^2\left(\pi L(X)^{\alpha}\right)\\&+\frac{2\pi^2\alpha^2L(X)^{2\alpha-2}L(X)^2e^{2\cot(\pi L(X)^{\alpha})}\csc^4(\pi L(X)^{\alpha})}{\left(e^{\cot(\pi L(X)^{\alpha})}+1\right)^2}\\&-\frac{2\pi^2\alpha^2L(X)^{2\alpha-2}L(X)^2e^{\cot(\pi L(X)^{\alpha})}\csc^4(\pi L(X)^{\alpha})}{e^{\cot(\pi L(X)^{\alpha})}+1}\\&+2\pi^2\alpha^2L(X)^{2\alpha-2}L(X)^2\cot\left(\pi L(X)^{\alpha}\right)\csc^2\left(\pi L(X)^{\alpha}\right)\\&-\frac{4\pi^2\alpha^2L(X)^{2\alpha-2}L(X)^2e^{\cot(\pi L(X)^{\alpha})}\cot(\pi L(X)^{\alpha})\csc^2(\pi L(X)^{\alpha})}{e^{\cot(\pi L(X)^{\alpha})}+1}\\&-\frac{(\alpha-1)L(X)^2}{L(X)^2}-2\pi(\alpha-1)\alpha L(X)^{\alpha-2}L(X)^2\cot\left(\pi L(X)^{\alpha}\right)\\&-\pi(\alpha-1)\alpha L(X)^{\alpha-2}L(X)^2\csc^2\left(\pi L(X)^{\alpha}\right)\\&+\frac{2\pi(\alpha-1)\alpha L(X)^{\alpha-2}L(X)^2e^{\cot(\pi L(X)^{\alpha})}\csc^2(\pi L(X)^{\alpha})}{e^{\cot(\pi L(X)^{\alpha})}+1}\end{aligned}$$

$$\begin{aligned}&\cot\left(\pi L(X)^{\alpha}\right)\csc^2\left(\pi L(X)^{\alpha}\right)\Big]\\&-\frac{(\alpha-1)L(X)^2}{L(X)^2}-2\pi(\alpha-1)\alpha L(X)^{\alpha-2}L(X)^2\cot\left(\pi L(X)^{\alpha}\right)\\&+\frac{\pi(\alpha-1)\alpha L(X)^{\alpha-2}L(X)^2e^{\cot(\pi L(X)^{\alpha})}\csc^2(\pi L(X)^{\alpha})}{e^{\cot(\pi L(X)^{\alpha})}+1}\end{aligned}$$

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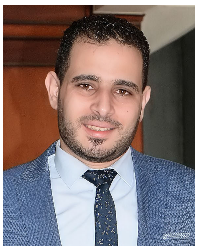

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