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RESEARCH ARTICLE

Adaptive Properties of a Ferromagnetic Single-Domain Grain in Alternating Magnetic Fields

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ABSTRACT A general self-consistent theory of adaptive properties of a ferromagnetic single-domain grain in alternating magnetic fields is constructed. Thresholds of new family of parametric instabilities of all orders are calculated. It is shown that the level of excitation of the magnetic moment and the phase of forced oscillations with respect to the pump field serve as a convenient tool for describing emerging nonequilibrium states. An analysis was made of the overthreshold excited state determined by third- and fourth-order nonlinear interactions in terms of deviations of the ferromagnetic moment. Adaptive nonequilibrium states describe energy flows from the pump field to the thermal bath and are characterized by non-linear damped oscillations. Near the frequency of these oscillations, the effect of mixing of weak RF signals with the microwave pump field arises, as well as the effect amplification of the RF modulation of the pump field. Using mathematical analogies, the developed theory can be transferred to other physical systems.

INDEX TERMS Magnetic single-domain particle, parametric resonance, signal mixing, signal amplification, adaptive state.

I. INTRODUCTION

The adaptive states of nonequilibrium magnetic systems under conditions of their excitation by external alternating magnetic fields and, in particular, the responses of these states to weak signals are of interest for searching for possibilities for the development of active devices operating on principles different from semiconductor electronics. Interest in the nonequilibrium dynamic effects, which can be used to develop magnetic devices, is increasing every year. The main objects of study are magnetic films, heterostructures, micro- and nanoparticles [1], [2], [3], [4], [5], [6], [7].

In this paper, we theoretically study the adaptive properties of a ferromagnetic single-domain grain in microwave and RF fields. The main goal of our study is to show that the excited nonequilibrium states of such a system can be used to develop devices capable of mixing and amplifying weak signals. For simplicity, we consider the excitation of a uniform precession of the magnetic moment of a grain in an effective magnetic field. It is assumed that the spectrum of inhomogeneous magnetic excitations, due to the smallness of the system, is in the region of much higher frequencies. Our analysis leads us to the mathematical model of a harmonic oscillator with a nonlinearity that passes into an adaptive state in an alternating field when a parametric resonance of the first or higher order is excited.

The parametric resonance of a harmonic oscillator occurs when the frequency of the oscillator is periodically modulated and is characterized by a threshold amplitude modulation above which occurs exponentially fast growth of the amplitude of excited oscillations. This phenomenon manifests itself in many completely different areas and environments, such as: in mechanics [8], [9], [10], electronics [11], [12], [13], ion traps [14], atomic microscopy [15], DNA [16],

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Bose-Einstein condensate [17], nanoparticles [18], etc. In this work, we will use the general method developed by us in [19] for parametric resonances of a harmonic oscillator. This method is based on the use of complex amplitudes of the oscillator, an analogue of the creation and annihilation operators in quantum mechanics. The rationale for this method with the natural connection between classical and quantum mechanics was given in [20]. This universal Hamilton's method greatly simplifies the rather cumbersome analysis of parametric resonance and makes it possible to calculate thresholds of all orders, as well as abovethreshold excited states. Thus, we construct a general theory of adaptive states of a parametrically excited oscillator, applicable both to magnetic dynamics and to other physical systems. It should be noted that the Hamilton's method with complex variables has been successfully used in the spin-wave dynamics of magnetic systems [7], [21], [22], [23]. The complex variable method was also used in nonlinear auto-oscillator theory of microwave generation by spin polarized current [24].

The research in the parametric excitation of the uniform precession of the magnetic moment of a ferromagnetic grain was previously applied to the so-called process of microwave-assisted switching and studied analytically (e.g., [25], [26]) and by numerical simulations (e.g., [27], [28]). Although the magnetization reversal process is not considered in our work, our results can simplify the approach to this problem. We will study analytically the parametric resonances of a homogeneous precession, as well as the properties of emerging above-threshold excited states. The response of these energy flow states to alternating magnetic fields, as will be shown below, has a specific nonlinear resonance. The frequency of this resonance depends on the properties of the system and the pumping source. We will explore the possibilities of this non-linear resonance for signal mixing and amplification.

First, we formulate the model, write down the complex variables and dynamic equations for them. Then we calculate the thresholds of parametric resonances of all orders in the resonance approximation and make estimates. We will show that the transition to quadratic variables reduces the problem to two physically understandable variables, the excitation intensity and the mismatch phase of the forced oscillations from the pump field. In the second part of the paper, the above-threshold state of the excited system is considered taking into account the fourth-order diagonal nonlinear term. The stability of this state and the effects of signal mixing and signal amplification in the region of nonlinear resonance are studied. The article ends with a discussion. The appendix describes a canonical transformation to obtain the diagonal non-linear term from a set of non-secular non-linear terms.

II. MODEL

Consider a single-domain ferromagnetic grain with uniaxial anisotropy along z axis in a magnetic field $\mathbf{H}_0$ deflected from the anisotropy axis (Fig.1). An alternating magnetic field $\mathbf{h}(t)$ is also applied to this grain. The energy $\mathcal{E}$ of the grain can be written as

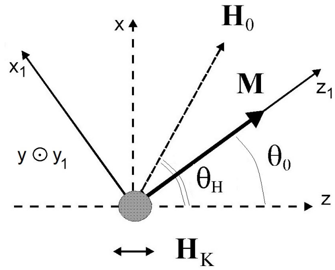

FIGURE 1. Geometry of the model.

$$\mathcal{E}/V = K_u \sin^2\theta_0 - M_s H_0 \cos(\theta_H - \theta_0) - \mathbf{M}\cdot\mathbf{h}(t). \quad (1)$$

Here V is the grain volume, K_u is the uniaxial anisotropy constant, θ_0 is the angle between the magnetization $\mathbf{M} = M_s\mathbf{m}$ ($|\mathbf{M}| = M_s$) and the axis of anisotropy, θ_H is the angle between the magnetic field and the axis of anisotropy. The equilibrium magnetization state is defined by the condition $\partial\mathcal{E}/\partial\theta_0 = 0$ leading to the equation $H_K \sin 2\theta_0 = 2H_0 \sin(\theta_H - \theta_0)$, which can be written as

$$\theta_H = \theta_0 + \arcsin\left(\frac{\sin 2\theta_0}{2(H_0/H_K)}\right), \quad (2)$$

where $H_K = 2K_u/M_s$ is the anisotropy field. We see that the equilibrium orientation θ_0 of the magnetization depends only on the direction of the magnetic field θ_H and the ratio.H_0/H_K. Dependences of θ_0 on θ_H for several H_0/H_K are shown in Fig.2.

It is convenient to introduce axes associated with the equilibrium direction of magnetization by the transformation

$$x = x_1\cos\theta_0 + z_1\sin\theta_0, \quad m^x = m^{x_1}\cos\theta_0 + m^{z_1}\sin\theta_0, \quad (3a)$$

$$z = -x_1\sin\theta_0 + z_1\cos\theta_0, \quad m^z = -m^{x_1}\sin\theta_0 + m^{z_1}\cos\theta_0, \quad (3b)$$

$$y = y_1, \quad m^y = m^{y_1}. \quad (3c)$$

Thus the energy (1) can be written as

$$\begin{aligned}\mathcal{E}/\varkappa = \frac{\gamma H_K}{2}\left[1 - \left(-m^{x_1}\sin\theta_0 + m^{z_1}\cos\theta_0\right)^2\right] \\ - \gamma H_0\left[\left(-m^{x_1}\sin\theta_0 + m^{z_1}\cos\theta_0\right)\cos\theta_H \right. \\ \left. + \left(m^{x_1}\cos\theta_0 + m^{z_1}\sin\theta_0\right)\sin\theta_H\right] - \gamma\mathbf{m}\cdot\mathbf{h}(t).\end{aligned} \quad (4)$$

where $\varkappa = M_s V/\gamma$ is a dimensional (energy$\times$ time) constant.

A. COMPLEX VARIABLES

We can describe the magnetization dynamics in terms of complex variables a^* and a (classical analog of creation

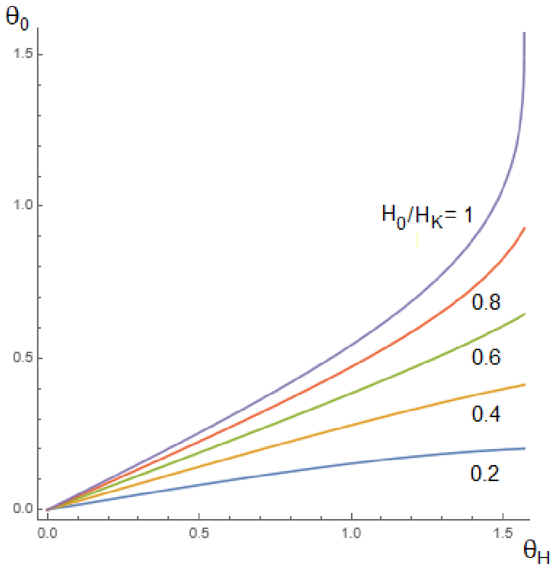

FIGURE 2. Dependence of θ_0 on θ_H for several H_0/H_K.

and annihilation operators in quantum mechanics) using the Holstein-Primakoff transformation in the form:

$$m^{z_1} = 1 - a^*a, \tag{5a}$$
$$m^+ = m^{x_1} + im^{y_1} = \sqrt{2 - a^*a}a, \tag{5b}$$
$$m^- = m^{x_1} - im^{y_1} = a^*\sqrt{2 - a^*a}. \tag{5c}$$

Hamilton's equations for complex amplitudes a^*, a have the form [7], [20]:

$$i\frac{d}{dt}a = \frac{\partial \mathcal{E}/\varkappa}{\partial a^*}, \quad i\frac{d}{dt}a^* = -\frac{\partial \mathcal{E}/\varkappa}{\partial a}. \tag{6}$$

It should be noted that this form of the equations of motion fully corresponds to the rotation of the magnetic moment $d\mathbf{M}/dt = \mathbf{M} \times \gamma \mathbf{H}_{eff}$ in the effective magnetic field $\mathbf{H}_{eff} = -\partial(\mathcal{E}/\varkappa)/\partial \mathbf{M}$. In what follows, we consider cases of complex amplitudes small in absolute value $|a| \ll 1$; therefore, we restrict ourselves to the following expansions for the transverse components in (5b) and (5c):

$$m^{x_1} \simeq \frac{a + a^*}{\sqrt{2}} - \frac{a^*aa + a^*a^*a}{4\sqrt{2}}, \tag{7a}$$
$$m^{y_1} \simeq i\frac{a^* - a}{\sqrt{2}} - i\frac{a^*a^*a - a^*aa}{4\sqrt{2}}. \tag{7b}$$

The energy (Hamiltonian) can be represented as the sum $\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \ldots$, where $\mathcal{E}_j$ is a term of jth order in complex variables. $\mathcal{E}_0$ denotes the energy at equilibrium. The linear term describes the interaction with the alternating field $[-m^{x_1}h^{x_1}(t) - m^{y_1}h^{y_1}(t)]$:

$$\mathcal{E}_1/\varkappa = -\gamma h^{x_1}(t)\frac{a + a^*}{\sqrt{2}} - i\gamma h^{y_1}(t)\frac{a^* - a}{\sqrt{2}}. \tag{8}$$

The quadratic form can be written as

$$\mathcal{E}_2/\varkappa = \left[\mathcal{A} + \gamma h^{z_1}(t)\right]a^*a + \frac{\mathcal{B}}{2}(aa + a^*a^*), \tag{9}$$

where

$$\mathcal{A} = \gamma H_K[1 - (3/2)\sin^2\theta_0] + \gamma H_0\cos(\theta_H - \theta_0),$$
$$\mathcal{B} = -(\gamma H_K/2)\sin^2\theta_0. \tag{10}$$

The nondiagonal terms aa and a^*a^* in time independent part of (9) can be eliminated by the linear canonical transformation (see, e.g., [7]) to new complex variables b^* and b:

$$a = ub + vb^*, \quad a^* = ub^* + vb,$$
$$u = \sqrt{\frac{\mathcal{A} + \omega_0}{2\omega_0}}, \quad v = -\frac{\mathcal{B}}{|\mathcal{B}|}\sqrt{\frac{\mathcal{A} - \omega_0}{2\omega_0}}. \tag{11}$$

Here

$$\omega_0 = \sqrt{\mathcal{A}^2 - \mathcal{B}^2}$$
$$= \gamma H_K\{[\cos 2\theta_0 + (H_0/H_K)\cos(\theta_H - \theta_0)]$$
$$\times [\cos^2\theta_0 + (H_0/H_K)\cos(\theta_H - \theta_0)]\}^{1/2} \tag{12}$$

is the frequency of ferromagnetic resonance. Thus we obtain the energy terms (8) and (9) become correspondingly:

$$\mathcal{E}_1/\varkappa = -\gamma h^{x_1}(t)\,(u + v)\,\frac{b + b^*}{\sqrt{2}} - i\gamma h^{y_1}(t)\,(u - v)\,\frac{b^* - b}{\sqrt{2}}, \tag{13}$$

and

$$\mathcal{E}_2/\varkappa = \left[\omega_0 + \frac{\mathcal{A}}{\omega_0}\gamma h^{z_1}(t)\right]b^*b - \frac{\mathcal{B}}{2\omega_0}\gamma h^{z_1}(t)(bb + b^*b^*). \tag{14}$$

In the absence of external fields, the condition $\mathcal{E}_1/\varkappa = 0$ corresponds to equilibrium, and the quadratic term describes a harmonic oscillator. The role of anharmonic terms in energy (third and fourth orders in complex variables) will be considered later.

B. DYNAMIC EQUATIONS

Hamilton's equations for complex amplitudes b^*, b have the form as (6):

$$i\frac{d}{dt}b = \frac{\partial \mathcal{E}/\varkappa}{\partial b^*}, \quad i\frac{d}{dt}b^* = -\frac{\partial \mathcal{E}/\varkappa}{\partial b}. \tag{15}$$

The losses in these equations can be accounted if we supplement these equations with the damping (see, e.g., [7]) η. Taking into account linear (13) and the quadratic form (14), one gets

$$i\left(\frac{d}{dt} + \eta\right)b = \omega_0\left[1 + \frac{\mathcal{A}\gamma h^{z_1}(t)}{\omega_0^2}\right]b - \frac{\mathcal{B}\gamma h^{z_1}(t)}{\omega_0}b^* - \xi(t),$$
$$i\left(\frac{d}{dt} + \eta\right)b^* = -\omega_0\left[1 + \frac{\mathcal{A}\gamma h^{z_1}(t)}{\omega_0^2}\right]b^* + \frac{\mathcal{B}\gamma h^{z_1}(t)}{\omega_0}b + \xi^*(t), \tag{16a}$$

where

$$\xi(t) = \frac{\gamma}{\sqrt{2}}\left[h^{x_1}(t)(u+v) + ih^{y_1}(t)(u-v)\right]. \tag{17}$$

One can apply a change of complex variables as follows

$$b = \widetilde{b}\exp\left[-i\frac{\mathcal{A}\gamma}{\omega_0}\int_0^t h^{z_1}(t')dt'\right] \tag{18}$$

and obtain

$$i\left(\frac{d}{dt}+\eta\right)\widetilde{b} = \omega_0\widetilde{b} - \mathcal{B}F(t)\widetilde{b}^* - \widetilde{\xi}(t), \tag{19a}$$

$$i\left(\frac{d}{dt}+\eta\right)\widetilde{b}^* = -\omega_0\widetilde{b}^* + \mathcal{B}F^*(t)\widetilde{b} + \widetilde{\xi}^*(t), \tag{19b}$$

where

$$F(t) = \frac{\gamma h^{z_1}(t)}{\omega_0}\exp\left[i\frac{2\mathcal{A}\gamma}{\omega_0}\int_0^t h^{z_1}(t')dt'\right], \tag{20}$$

$$\widetilde{\xi}(t) = \xi(t)\exp\left[i\frac{\mathcal{A}\gamma}{\omega_0}\int_0^t h^{z_1}(t')dt'\right]. \tag{21}$$

From the equations obtained, we see two fundamentally different types of action of external fields on the system. Linear coupling of the harmonic oscillator with an alternating field through $\widetilde{\xi}(t)$ effective only if this field oscillates with a frequency close to the frequency of a ferromagnetic resonance ω_0 within the linewidth of this resonance $\eta \ll \omega_0$. In this case, the amplitude of the oscillations increases linearly. Whereas term with $F(t)$, which contains the field component $h^{z_1}(t)$ parallel to the equilibrium magnetization, as we will see below, leads to a large family of parametric resonances near frequencies $\omega \simeq 2\omega_0/n$, where $n = 1, 2, 3, \ldots$. When these parametric resonances are excited, the oscillation amplitude grows exponentially. This nontrivial family of solutions has not yet been studied and is therefore of great interest. In what follows, we restrict ourselves to the field $h^{z_1}(t)$ and omit the $\widetilde{\xi}(t)$ term (strictly mathematically it means $h^{x_1}(t) = h^{y_1}(t) = 0$).

III. PARAMETRIC RESONANCES

Consider a periodic time dependence $h^{z_1}(t) = h^{z_1}(t+T)$, where T is a period. Without loss of generality we can also consider $h^{z_1}(t) = h^{z_1}(-t)$. From (20) we can obtain

$$F(t) = \sum_{n=-\infty}^{\infty} F_n \exp(in\omega t),$$

$$F_n = F_n^* = \frac{1}{T}\int_0^T F(t)\exp(-in\omega t)\,dt,$$

where $\omega = 2\pi/T$. Thus, we see that the external influence in the (19) is a set of oscillating exponentials. It is convenient to use slow variables c and c^* with respect to one of the frequencies

$$\widetilde{b} = c\exp\left(-i\frac{n\omega t}{2}\right), \quad \widetilde{b}^* = c^*\exp\left(i\frac{n\omega t}{2}\right). \tag{22}$$

As a result, the dynamic equations (19) take the following form

$$i\frac{dc}{dt} = \left(\omega_0 - \frac{n\omega}{2} - i\eta\right)c - \mathcal{B}\left[F_n + \mathcal{F}_n(t)\right]c^*, \tag{23a}$$

$$i\frac{dc^*}{dt} = -\left(\omega_0 - \frac{n\omega}{2} + i\eta)\right)c^* + \mathcal{B}\left[F_n + \mathcal{F}_n(t)\right]c, \tag{23b}$$

where

$$\mathcal{F}_n(t) = \mathcal{F}_n^*(t) = \sum_{\nu\neq n} F_\nu \exp\left[i\nu - n)\omega t\right]. \tag{24}$$

A. RESONANCE APPROXIMATION

It is easy to see that in the (23) the only term with an explicit time dependence is $\mathcal{F}_n(t)$. If $|\omega_0 - n\omega/2| \ll \omega$, then the function $\mathcal{F}_n(t)$ consists of rapidly oscillating terms compared to the slow dynamics of the rest of the system. The general approach to solving such problems consists of two parts. At first, we consider only slow dynamics of c and c^* variables which describe the behavior of the system in the vicinity of the resonance $\omega_0 \simeq n\omega/2$ (so-called, resonance approximation) and neglect the role of the rapidly oscillating terms (their average is assumed to be zero). In the second part, small corrections to the resonant motion from non-resonant terms can be taken into account using the averaging method [29]. Usually, the second part of the solution of the problem is rather cumbersome calculations that give only small corrections to the resonant motion. For this reason, below we restrict ourselves to the resonance approximation, which contains the basic physics of the phenomena under study. Thus from (23) one has

$$i\frac{d}{dt}\begin{pmatrix} c \\ c^* \end{pmatrix} = \begin{pmatrix} \omega_0 - \frac{n\omega}{2} - i\eta & -\mathcal{B}F_n \\ \mathcal{B}F_n & -\omega_0 + \frac{n\omega}{2} - i\eta \end{pmatrix}\begin{pmatrix} c \\ c^* \end{pmatrix}. \tag{25}$$

For the Lyapunov's exponent: $c, c^* \propto e^{\lambda t}$ one obtains

$$\lambda_\pm = -\eta \pm \sqrt{(\mathcal{B}F_n)^2 - \left(\omega_0 - \frac{n\omega}{2}\right)^2}. \tag{26}$$

Thus the threshold of parametric resonance ($\lambda = 0$) is

$$(\mathcal{B}F_n)^2 = \left(\omega_0 - \frac{n\omega}{2}\right)^2 + \eta^2, \tag{27}$$

which is minimal at the exact resonance condition $\omega_0 = n\omega/2$:

$$|\mathcal{B}F_n| = \eta. \tag{28}$$

Formulas (27) and (28) represent a general solution in a resonance approximation for the parametric resonance $\omega \simeq 2\omega_0/n$ in a periodic field.

B. EXAMPLE: COSINE FUNCTION

Let us consider $h^{z_1}(t) = h\cos(\omega t)$, then one has $\int_0^t h^{z_1}(t')dt' = (h/\omega)\sin(\omega t)$ and

$$F_n = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{\gamma h}{\omega_0}\cos(\omega t)\exp\left[i\frac{2\mathcal{A}\gamma h}{\omega_0\omega}\sin(\omega t) - in\omega t\right]dt$$
$$= \frac{\gamma h}{2\omega_0}\left[J_{n-1}\left(\frac{2\mathcal{A}\gamma h}{\omega_0\omega}\right) + J_{n+1}\left(\frac{2\mathcal{A}\gamma h}{\omega_0\omega}\right)\right]. \tag{29}$$

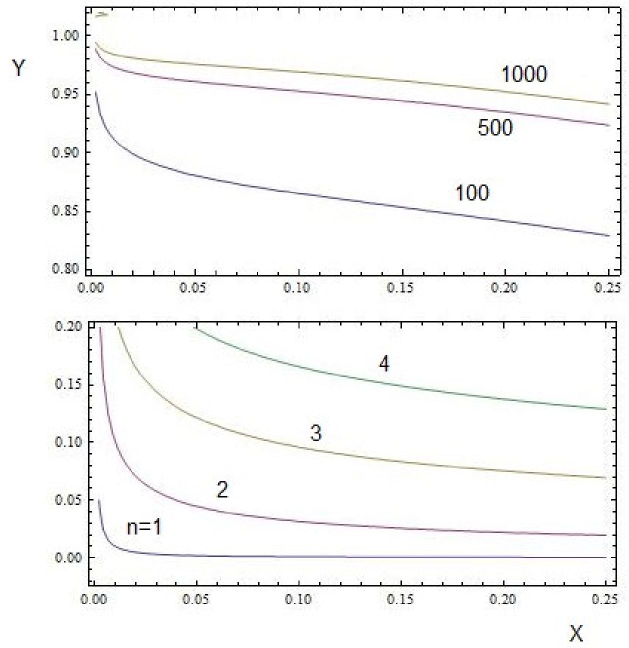

FIGURE 3. Relative threshold $Y = \gamma h_c^{(n)}/\omega_0$ versus relative coupling $X = |\mathcal{B}|/\omega_0$ at different orders n of parametric resonance and $Q = 10^4$.

At the resonance $\omega_0 = n\omega/2$ from Eqs.(28) and (29) we get the equation for critical amplitude $h_c^{(n)}$

$$\gamma h_c^{(n)} \left| J_{n-1}\left(\frac{n\mathcal{A}}{\omega_0^2}\gamma h_c^{(n)}\right) + J_{n+1}\left(\frac{n\mathcal{A}}{\omega_0^2}\gamma h_c^{(n)}\right)\right| = \frac{2\eta\omega_0}{|\mathcal{B}|}. \tag{30}$$

This formula can be rewritten in the dimensionless form:

$$Y\left|J_{n-1}\left(n\sqrt{1+X^2}Y\right) + J_{n+1}\left(n\sqrt{1+X^2}Y\right)\right| = \frac{1}{XQ}, \tag{31}$$

where $Y = \gamma h_c^{(n)}/\omega_0$, $X = |\mathcal{B}|/\omega_0$ and $Q = \omega_0/2\eta$. Figure 3 demonstrates the relative critical amplitudes Y dependence on relative coupling X for various n at $Q = 10^4$.

If $nY \ll 1$ and $X \ll 1$, one can use the Bessel functions expansion in (31) and obtain the following solutions

$$\gamma h_c^{(1)}/\omega_0 \simeq 2\eta/|\mathcal{B}|,$$
$$\gamma h_c^{(2)}/\omega_0 \simeq \sqrt{2\eta/|\mathcal{B}|}, \tag{32}$$
$$\frac{\gamma h_c^{(3)}}{\omega_0} \simeq \frac{2}{3}\left(\frac{6\eta}{|\mathcal{B}|}\right)^{1/3}.$$

Let's make numerical estimates. Consider a grain with $H_K \approx 50$ Oe (e.g., permalloy [30]) in a field of $H_0 = 35.4$ Oe perpendicular to uniaxial anisotropy ($\theta_H = 90^o$ and $\theta_0 = 45^o$). Then $\omega_0 = 99.1$ MHz and $h_c^{(1)} \simeq 0.01$ Oe, $h_c^{(2)} \simeq 0.6$ Oe and $h_c^{(3)} \simeq 2.24$ Oe at $Q = \omega_0/2\eta = 10^4$. For Co grain at the same angles and quality factors ($H_K \approx 6.43$ kOe, $H_0 = 4.55$ kOe, $\omega_0 = 12.7$ GHz) one obtains $h_c^{(1)} \simeq 1.29$ Oe.

IV. QUADRATIC VARIABLES

It is interesting to rewrite (23) using the following variables: $\sigma = c^2$, $\sigma^* = (c^*)^2$ and $N = c^*c$. After simple calculations one obtains

$$\frac{i}{2}\frac{d\sigma}{dt} = \left(\omega_0 - \frac{n\omega}{2} - i\eta\right)\sigma - \mathcal{B}\left[F_n + \mathcal{F}_n(t)\right]N, \tag{33a}$$

$$\frac{i}{2}\frac{d\sigma^*}{dt} = -\left(\omega_0 - \frac{n\omega}{2} + i\eta)\right)\sigma^* + \mathcal{B}\left[F_n + \mathcal{F}_n(t)\right]N, \tag{33b}$$

$$\frac{dN}{dt} = -2\eta N + i\mathcal{B}\left[F_n + \mathcal{F}_n(t)\right](\sigma^* - \sigma). \tag{33c}$$

Combining the obtained equations, one can obtain the following relation:

$$\left(\frac{d}{dt} - 4\eta\right)(N^2 - |\sigma|^2) = 0. \tag{34}$$

It follows from this equation that the possible difference between N and $|\sigma|$ decreases exponentially fast $\propto \exp(-4\eta t)$ and can be neglected at times $t > 1/4\eta$. Thus, in general, we can consider solutions when $|\sigma| = N$, for example,

$$\sigma = iN\exp(-i\phi) \tag{35}$$

where ϕ is a real variable, a phase. The equations (33a- 33c) become

$$\left\{\frac{d\phi}{dt} - 2\left(\omega_0 - \frac{n\omega}{2}\right) + 2\mathcal{B}[F_n + \mathcal{F}_n(t)]\sin\phi\right\}N = 0, \tag{36a}$$

$$\frac{dN}{dt} + 2\eta N - 2\mathcal{B}\left[F_n + \mathcal{F}_n(t)\right]N\cos\phi = 0. \tag{36b}$$

If we restrict ourselves to the resonance approximation (neglecting $\mathcal{F}_n(t)$), then from these equations we obtain the following stationary solutions

$$\sin\phi_0 = \left(\omega_0 - \frac{n\omega}{2}\right)/\mathcal{B}F_n, \tag{37a}$$
$$\cos\phi_0 = \eta/\mathcal{B}F_n. \tag{37b}$$

They obviously reduce to the equation (27) describing the threshold of parametric resonance in the resonance approximation.

Suppose that $\mathcal{F}_n(t)$ is small, then we can consider (36a) and (36b) in a linear approximation in the deviations of $\delta\phi$ and δN from stationary solution (37). However, as follows from (36b), the linear deviation $d\delta N/dt = 0$ and the δN has no time dependence. The equation for $\delta\phi$, which follows from (36a), has a solution oscillating in the vicinity of ϕ_0, which does not affect the parametric instability threshold (27). Thus, the formula (27) obtained in the resonance approximation is preserved up to the second order of smallness of $\mathcal{F}_n(t)$.

V. ACCOUNT OF NONLINEARITY

The nonlinear energy terms of the system (4) have the form

$$\mathcal{E}_3/\varkappa = -\frac{\gamma H_K}{2\sqrt{2}}\sin 2\theta_0(a^*aa + a^*a^*a) \tag{38}$$

and

$$\mathcal{E}_4/\varkappa = \frac{\gamma H_K}{8}\sin^2\theta_0(a^*aaa + a^*a^*a^*a) + \frac{\gamma H_K}{4}(1-3\cos^2\theta_0)a^*a^*aa. \quad (39)$$

After performing a linear canonical transformation (11), we get

$$\mathcal{E}_3/\varkappa = \Psi_1(bbb + b^*b^*b^*) + \Psi_2(b^*bb + b^*b^*b), \quad (40)$$
$$\mathcal{E}_4/\varkappa = \Phi_0 b^*b^*bb, \quad (41)$$

where

$$\Psi_1 = -\frac{\gamma H_K}{2\sqrt{2}}uv(u+v)\sin 2\theta_0, \quad (42a)$$
$$\Psi_2 = -\frac{\gamma H_K}{2\sqrt{2}}[u^3 + v^3 + 2uv(u+v)]\sin 2\theta_0, \quad (42b)$$

$$\Phi_0 = \frac{\gamma H_K}{4}\{3uv(u^2+v^2)\sin^2\theta_0 + [(u^2+v^2)^2 + 2(uv)^2](1-3\cos^2\theta_0)\}. \quad (43)$$

The next step is to eliminate non-secular terms (40) using the non-linear canonical transformation described in the Appendix. As a result, third-order nonlinearities are reduced to the fourth order with a changed anharmonicity value.

$$\mathcal{E}_4/\varkappa = \widetilde{\Phi}_0 b^*b^*bb, \quad (44)$$

where

$$\widetilde{\Phi}_0 = \Phi_0 - 3\frac{|\Psi_1|^2 + |\Psi_2|^2}{\omega_0}. \quad (45)$$

Dependences $\widetilde{\Phi}_0/H_K$ at several H_0/H_K are shown in Fig.4.

VI. OVERTHRESHOLD STATE

It is convenient to study the threshold dynamics of the magnetic moment using quadratic variables, with the help of which in the resonance approximation the problem is reduced to

$$\left[\frac{d\phi}{dt} - 2\left(\omega_0 + 2\widetilde{\Phi}_0 N - \frac{n\omega}{2}\right) + 2\mathcal{B}F_n\sin\phi\right]N = 0, \quad (46a)$$

$$\frac{dN}{dt} + 2\eta N - 2\mathcal{B}F_n N\cos\phi = 0. \quad (46b)$$

It is easy to see that the only difference between these equations and those obtained earlier (36a,36b) is in the $2\widetilde{\Phi}_0 N$ term, which changes the resonance conditions. Stationary state is reduced to

$$\sin\phi_0 = \left(\omega_0 + 2\widetilde{\Phi}_0 N_0 - \frac{n\omega}{2}\right)/\mathcal{B}F_n, \quad (47a)$$
$$\cos\phi_0 = \eta/\mathcal{B}F_n, \quad (47b)$$

and

$$(\mathcal{B}F_n)^2 = \left(\omega_0 + 2\widetilde{\Phi}_0 N_0 - \frac{n\omega}{2}\right)^2 + \eta^2. \quad (48)$$

This relation is obviously a generalization of the formula (27). Equation (48) can be conveniently rewritten as

$$2\widetilde{\Phi}_0 N_0 = -\left(\omega_0 - \frac{n\omega}{2}\right) \pm \sqrt{(\mathcal{B}F_n)^2 - \eta^2}. \quad (49)$$

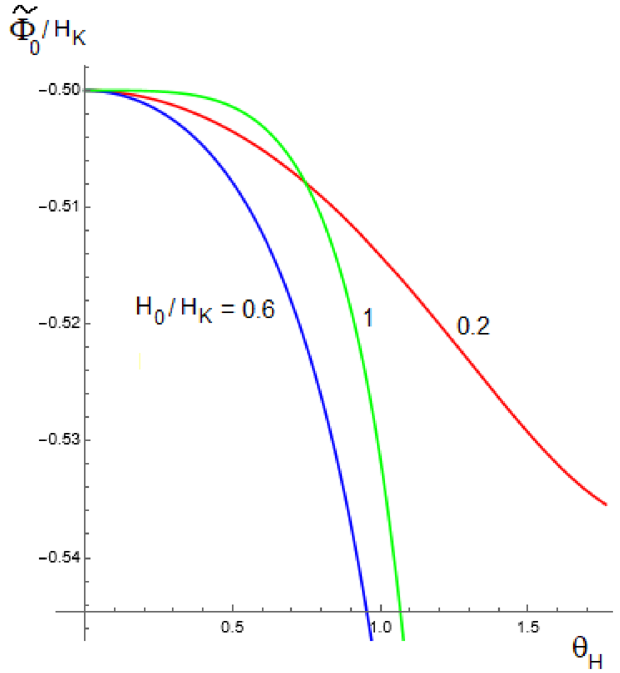

FIGURE 4. Dependence $\widetilde{\Phi}_0/H_K$ at several H_0/H_K.

A. STEADY STATE STABILITY

Let us now consider the stability of the resulting stationary state (47a), (47b), (48). To do this, we write the linearized equations for small deviations from equilibrium $\phi = \phi_0 + \delta\phi$, $N = N_0 + \delta N$:

$$\frac{d\delta\phi}{dt} + 2\eta\delta\phi - 4\widetilde{\Phi}_0\delta N = 0, \quad (50a)$$
$$\frac{d\delta N}{dt} + 2\left(\omega_0 + 2\widetilde{\Phi}_0 N_0 - \frac{n\omega}{2}\right)N_0\delta\phi = 0. \quad (50b)$$

Assuming that $\delta\phi, \delta N \propto \exp(\lambda t)$, we get the characteristic equation

$$\det\begin{pmatrix} \lambda + 2\eta & -4\widetilde{\Phi}_0 \\ 2N_0\left(\omega_0 + 2\widetilde{\Phi}_0 N_0 - n\omega/2\right) & \lambda \end{pmatrix} = 0, \quad (51)$$

or, in the explicit form:

$$\lambda_\pm = -\eta \pm \sqrt{\eta^2 - 8\widetilde{\Phi}_0 N_0\left(\omega_0 + 2\widetilde{\Phi}_0 N_0 - \frac{n\omega}{2}\right)}. \quad (52)$$

The most "dangerous" solution is with λ_+. At exact resonance $\omega_0 = n\omega/2$, we get

$$\lambda_+ = -\eta + \sqrt{5\eta^2 - 4(\mathcal{B}F_n)^2}. \quad (53)$$

It is easy to see that this formula describes the start of pumping with threshold (28). Then, with increasing pumping F_n, the stability of the above-threshold state reaches $-\eta$. At the pumping amplitude

$$|\mathcal{B}F_n| = \frac{\sqrt{5}}{2}\eta \quad (54)$$

the radical expression in (53) vanishes and then becomes negative at larger pumpings. Thus $\lambda_+ = -\eta + i\Omega_n$, other words, the non-equilibrium adaptive state of the system when deviating from flow equilibrium is characterized by oscillations with frequency

$$\Omega_n = 2\eta\sqrt{P_n - 5/4}. \tag{55}$$

and damping η. Here $P_n = (\mathcal{B}F_n/\eta)^2$ describes overcriticality on power.

VII. MIXING OF SIGNALS

The presence of natural oscillation in the adapted system makes it possible to linearly excite this oscillation by an external alternating magnetic field and thereby mix the oscillations of this field with forced oscillations at the pump frequency. Mathematically, this looks like a slow modulation ($\omega_m \ll \omega_0$) of the FMR frequency:

$$\frac{d\phi}{dt} - 2\left(\omega_0 + 2\widetilde{\Phi}_0 N + \zeta\gamma H_m \cos\omega_m t - \frac{n\omega}{2}\right) - 2\mathcal{B}F_n \sin\phi = 0, \tag{56a}$$

$$\frac{dN}{dt} + 2\eta N - 2\mathcal{B}F_n N \cos\phi = 0. \tag{56b}$$

Here ζ describes the projection of the modulation field $H_m \cos\omega_m t$ on the equilibrium direction of the magnetic moment. We will look for a solution in the form

$$\delta\phi = A_\phi \cos\omega_m t + B_\phi \sin\omega_m t, \tag{57}$$
$$\delta N = A_N \cos\omega_m t + B_N \sin\omega_m t.$$

Simple algebraic calculations at exact resonance $\omega_0 = n\omega/2$ lead to the following system of linear equations:

$$\begin{pmatrix} -\omega_m & 2\eta & 0 & -4\widetilde{\Phi}_0 \\ 2\eta & \omega_m & -4\widetilde{\Phi}_0 & 0 \\ 0 & 4\widetilde{\Phi}_0 N_0^2 & -\omega_m & 0 \\ 4\widetilde{\Phi}_0 N_0^2 & 0 & 0 & \omega_m \end{pmatrix} \begin{pmatrix} A_\phi \\ B_\phi \\ A_N \\ B_N \end{pmatrix} = \begin{pmatrix} 0 \\ 2\zeta\gamma H_m \\ 0 \\ 0 \end{pmatrix}. \tag{58}$$

The solution of these equations can be represented as

$$A_\phi = -\left(\frac{\zeta\gamma H_m}{\eta}\right) X_m^2/\Delta_n(X_m),$$
$$B_\phi = -\left(\frac{\zeta\gamma H_m}{\eta}\right) X_m \left(1 - P_n + X_m^2\right)/\Delta_n(X_m),$$
$$A_N = \left(\frac{\zeta\gamma H_m}{2\widetilde{\Phi}_0}\right) (1 - P_n) \left(1 - P_n + X_m^2\right)/\Delta_n(X_m),$$
$$B_N = -\left(\frac{\zeta\gamma H_m}{2\widetilde{\Phi}_0}\right) X_m (1 - P_n)/\Delta_n(X_m),$$

where $X_m = \omega_m/2\eta$ and

$$\Delta_n(X) = (P_n - 1)^2 - (2P_n - 3)X^2 + X^4.$$

The modulation depth of the absorbed signal is determined by the amplitude $\delta N_m = \sqrt{A_N^2 + B_N^2}$. Figure 5a shows the low frequency signal mixing efficiency as a function of the modulation frequency at two values of overcriticality.

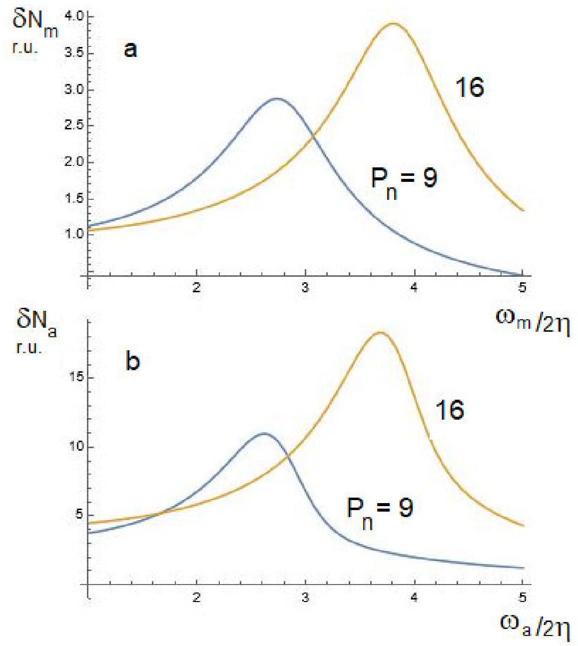

FIGURE 5. **a) Frequency dependence of the amplitude modulation of the absorbed microwave power. b) Frequency dependence of the amplitude modulation gain of the absorbed microwave power.**

The peaks approximately correspond to the Ω_n frequency at different overcriticalities.

VIII. EFFECT OF AMPLIFICATION

Let us now consider the possibility of amplifying the amplitude modulation depth $A_a \ll 1$ with a frequency $\omega_a \ll \omega_0$ applied to the pumping system. In this case, the system of dynamic equations for the ϕ and N can be written as

$$\Big[\frac{d\phi}{dt} - 2\left(\omega_0 + 2\widetilde{\Phi}_0 N - \frac{n\omega}{2}\right) + 2\mathcal{B}F_n (1 + A_a \cos\omega_a t)\sin\phi\Big]N = 0, \tag{59a}$$

$$\frac{dN}{dt} + 2\eta N - 2\mathcal{B}F_n (1 + A_a \cos\omega_a t) N \cos\phi = 0. \tag{59b}$$

We will also look for a solution in the form

$$\delta\phi = A_1 \cos\omega_a t + B_1 \sin\omega_a t, \tag{60}$$
$$\delta N = A_2 \cos\omega_a t + B_2 \sin\omega_a t,$$

and obtain

$$\begin{pmatrix} -\omega_a & 2\eta & 0 & -4\widetilde{\Phi}_0 \\ 2\eta & \omega_a & -4\widetilde{\Phi}_0 & 0 \\ 0 & 4\widetilde{\Phi}_0 N_0^2 & -\omega_a & 0 \\ 4\widetilde{\Phi}_0 N_0^2 & 0 & 0 & \omega_a \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -4\widetilde{\Phi}_0 N_0 A_a \\ 0 \\ 2\eta N_0 A_a \end{pmatrix}, \tag{61}$$

where

$$A_1 = -A_a\left(1 - P_n + 2X_a^2\right)\sqrt{P_n - 1}/\Delta_n(X_a),$$
$$B_1 = -A_aX_a\left(-P_n + X_a^2\right)\sqrt{P_n - 1}/\Delta_n(X_a),$$
$$A_2 = \left(\frac{\eta A_a}{2\widetilde{\Phi}_0}\right)(P_n - 1)^{3/2}\left(P_n - X_a^2\right)/\Delta_n(X_a),$$
$$B_2 = \left(\frac{\eta A_a}{2\widetilde{\Phi}_0}\right)X_a\left(1 + X_a^2\right)\sqrt{P_n - 1}/\Delta_n(X_a),$$

and $X_a = \omega_a/2\eta$.

An increase in the modulation depth $\delta N_m = \sqrt{A_N^2 + B_N^2}$ of the absorbed microwave signal can be seen in Fig. 5b.

IX. DISCUSSION

In this work, we have developed a detailed general theory of the adaptive properties of a ferromagnetic single-domain grain in alternating magnetic fields, which excites parametric resonances of a uniform precession of the magnetic moment. We used the resonance approximation within which the general formulas (27) and (30) were obtained for all orders ($n = 1, 2, 3, \ldots$) of parametric resonance. Numerical estimates show that the first few thresholds are achievable experimentally at frequencies in the vicinity $2\omega_0$, ω_0, $2\omega_0/3$. In addition, it was demonstrated that it is convenient to solve the problem of parametric excitation of the uniform precession of magnetization using quadratic variables c^2, $(c^*)^2$, and c^*c.

We have derived equations for quadratic complex variables for resonances $\simeq 2\omega_0/n$, $n = 1, 2, 3\ldots$, taking into account non-linear terms. The nonlinear terms in the energy of the system were previously transformed to a secular form (45) by a nonlinear canonical transformation described in the application. Thus, we have the opportunity to study emerging adaptive states that depend both on the linear and nonlinear properties of the system itself and on the magnitude of the pump field. It is interesting that the problem is reduced to two simple physical variables, to the excitation intensity N and to the phase ϕ, in which the forced oscillations of the magnetization differ from those of the pump field.

Although the magnetization reversal process is not considered in our work, our results imply the most efficient condition for the microwave-assisted reversal from the fourth-order nonlinearity. From Fig. 4 we can see that the smaller absolute value of $\left|\widetilde{\Phi}_0\right|/H_K$, leading to the restriction of parametric excitation, is at $\theta_H = 0$ (and, accordingly, at $\theta_H = \pi$). Hence it follows that the microwave assisted reversal will achieve the largest transverse magnetization amplitudes in a magnetic field $\mathbf{H}_0$ lying on the anisotropy axis and directed opposite to the equilibrium magnetization $\mathbf{M}$.

By analyzing the equations for variables quadratic in complex amplitudes, we succeeded in finding solutions for the stationary state (45), (46) and the stability conditions for this state (50). It turned out that near the stationary state, the behavior of the system is described by a damped nonlinear oscillation with frequency (53), which depends both on the properties of the system and on the pump power. Excitation of this oscillation of the adaptive system can be done in two ways. In the first case, an RF field is applied to the system, modulating the external field with a frequency close to that of the adaptive system. As a result, the forced oscillations of the adaptive system at the frequency of the pump field become modulated by the RF field. In other words, the system mixes the RF with the pump oscillations and enhances the modulation depth (see Fig. 5a). In the second case, a pump field modulated by a frequency close to that of the adaptive system is applied to the system. As can be seen from Fig. 5b, the modulation depth is enhanced (amplified) by the adaptive system.

Since the temperatures of magnetic ordering of ferromagnetic particles can be high, the effects of signal mixing and amplification on the considered adaptive states can be used to develop instruments and devices capable of operating under extreme conditions that are inaccessible to semiconductor technology.

It should be noted that the results obtained for a ferromagnetic particle with uniaxial anisotropy can be easily generalized to the case of particles with more complex anisotropy. To do this, it is only necessary to choose the correct equilibrium orientation of the magnetic moment (similar to Fig. 1) and bring the coordinates of the magnetic moment (similar to formulas (3)) to this geometry.

It should also be noted the general view of the developed theory, which is based on the dynamics of complex variables describing a ferromagnetic grain as a harmonic oscillator with nonlinearity. The Hamiltonian is the sum of Eqs. (14), (40) and (41) and can also be written for oscillations of a completely different nature. To describe the nonlinearities of the system, which determine the adaptive properties of the excited, overthreshold state, a canonical transformation has been used that significantly simplifies the structure of the resulting dynamic equations. The canonical transformation is constructed by analogy with the methods developed in quantum theory. This is a rotation of coordinates using a unitary matrix, in which off-diagonal non-linear elements vanish. Diagonal nonlinear terms of the fourth order in amplitude are responsible for the adaptive state of the excited system. The adaptive states of our system, by mathematical analogy, can be transferred to many other physical objects.

APPENDIX

X. CANONICAL TRANSFORMATION

Consider a Hamiltonian of the form

$$\mathcal{H}/\varkappa = \omega_0 b^*b + (\Psi_1 b^*b^*b^* + \Psi_2 b^*b^*b + \text{c.c.}) + \Phi_0 b^*b^*bb, \tag{62}$$

in which there is a quadratic term describing a harmonic oscillator with the frequency ω_0 and anharmonic terms of the third and fourth order in complex amplitudes. Bearing in mind that the time dependences of the amplitudes of the harmonic oscillator are $b \propto \exp(-i\omega_0 t)$ and $b^* \propto \exp(i\omega_0 t)$, we see that the quadratic term and the fourth-order term have no time dependence in the Hamiltonian (62), while the third-order anharmonicities oscillate. The wrong idea may arise that the terms of the third order can be omitted when taking into account the anharmonicity of the system. Here we show how, using the canonical transformation, one can eliminate the third-order terms and obtain their effective contribution to the fourth-order anharmonicity.

The idea of a canonical transformation follows from quantum mechanics, in which the new Hamiltonian can be obtained by a formal rotation in the operator representation (e.g., [7], [31])

$$\exp\left(\hat{\mathcal{R}}\varphi\right)\hat{\mathcal{H}}\exp\left(-\hat{\mathcal{R}}\varphi\right) = \hat{\mathcal{H}} + [\hat{\mathcal{H}}, \hat{\mathcal{R}}]\varphi + \frac{1}{2}[[\hat{\mathcal{H}}, \hat{\mathcal{R}}], \hat{\mathcal{R}}]\varphi^2 + \ldots = \hat{\mathcal{H}}(\varphi), \tag{63}$$

where $\hat{\mathcal{R}}$ is an antihermitian operator, φ is a formal dimensionless parameter and $[\hat{\mathcal{H}}, \hat{\mathcal{R}}] = \hat{\mathcal{H}}\hat{\mathcal{R}} - \hat{\mathcal{R}}\hat{\mathcal{H}}$ is a commutator. The new Hamiltonian is obtained by solving the differential equation

$$\frac{d}{d\varphi}\hat{\mathcal{H}}(\varphi) = [\hat{\mathcal{R}}, \hat{\mathcal{H}}(\varphi)], \quad \hat{\mathcal{H}}(0) = \hat{\mathcal{H}}. \tag{64}$$

This equation is solved with the purposefully chosen $\hat{\mathcal{R}}$, and the condition is imposed on the solution that for the "inconvenient" terms of the original Hamiltonian in the new Hamiltonian disappear.

For the classical case, the application of the above-described canonical transformation means the transition from the original complex variables a^* and a to the new variables b^* and b. In this case, we write the "rotated" Hamiltonian in the form

$$\mathcal{H}(\varphi)/\varkappa = \omega_0 b^* b + [\Psi_1(\varphi) b^* b^* b^* + \Psi_2(\varphi) b^* b^* b + \text{c.c.}] + \Phi_0(\varphi) b^* b^* b b + \ldots \tag{65}$$

To eliminate the third-order terms, we will use a transformation generator

$$\mathcal{R} = \left(\mathcal{R}_1 b^* b^* b^* + \mathcal{R}_2 b^* b^* b - \text{c.c.}\right) \tag{66}$$

and a classical "commutator"

$$[\mathcal{A}, \mathcal{B}] = \frac{\partial \mathcal{A}}{\partial b}\frac{\partial \mathcal{B}}{\partial b^*} - \frac{\partial \mathcal{B}}{\partial b}\frac{\partial \mathcal{A}}{\partial b^*}. \tag{67}$$

Substituting (65)-(67) into the Eq.(64), one gets easily solvable differential equations for the coefficients $\Psi_1(\varphi)$, $\Psi_2(\varphi)$, and $\Phi_0(\varphi)$:

$$\frac{d}{d\varphi}\Psi_1(\varphi) = -3\mathcal{R}_1\omega_0, \ \rightarrow \ \Psi_1(\varphi) = -3\mathcal{R}_1\omega_0\varphi + \Psi_1, \tag{68}$$

$$\frac{d}{d\varphi}\Psi_2(\varphi) = -\mathcal{R}_2\omega_0, \ \rightarrow \ \Psi_2(\varphi) = -\mathcal{R}_2\omega_0\varphi + \Psi_2, \tag{69}$$

$$\frac{d}{d\varphi}\Phi_0(\varphi) = -[9\mathcal{R}\Psi_1^*(\varphi) + 3\mathcal{R}_2\Psi_2^*(\varphi) + \text{c.c.}]. \tag{70}$$

Assuming that the coefficients at the third-order terms are equal to zero at $\varphi = 1$, from Eqs. (68) and (69) one gets the generator coefficients: $\mathcal{R}_1 = \Psi_1/3\omega_0$ and $\mathcal{R}_2 = \Psi_2/\omega_0$. Thus we eliminate the third order terms in complex variables and obtain the correction to the coefficient of the secular fourth order term:

$$\begin{aligned}\Phi_0(1) &= \Phi_0 - \left[9\mathcal{R}_1\int_0^1 \Psi_1^*(\varphi)d\varphi + 3\mathcal{R}_2\int_0^1 \Psi_2^*(\varphi)d\varphi + \text{c.c.}\right] \\ &= \Phi_0 - 3\frac{|\Psi_1|^2 + |\Psi_2|^2}{\omega_0}.\end{aligned} \tag{71}$$

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