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RESEARCH ARTICLE

An Output-Feedback Design Approach for Robust Stabilization of Linear Systems With Uncertain Time-Delayed Dynamics in Sensors and Actuators

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ABSTRACT In this paper, we propose a control approach for the robust stabilization of linear time-invariant (LTI) systems with non-negligible sensor and actuator dynamics subject to time-delayed signals. Our proposition is based on obtaining an augmented model that encompasses the plant, sensor, and actuator dynamics and also the time-delay dynamic effect. We make use of the Padé Approximation for modeling the time-delay impact on the feedback loop. Since the actual plant state variables are not available for feedback, the sensor outputs, which represent a subset of the augmented system state variables, are used for composing a static output-feedback control law. The robust controller gains are computed by means of a two-stage strategy based on linear matrix inequalities (LMI). For obtaining less conservative conditions we consider the use of homogeneous-polynomial Lyapunov functions (HPLF) – and other decision variables – of arbitrary degree. In our proposition, we also take into account the inclusion of a minimum decay rate criterion in order to improve closed-loop system transient response. Disturbance rejection is also addressed through extensions to $\mathscr{H}_2$ guaranteed cost minimization. The effectiveness of the proposed strategy is attested in the design of a controller for the lateral axis dynamics of an aircraft and other academic examples.

INDEX TERMS Robust control, time-delay, linear time-invariant systems, static output feedback, linear matrix inequalities.

In some situations, automatic control systems are composed of sensors and actuators with non-negligible dynamics as, for instance, in embedded controllers of modern light-weight aircraft [1]. Due to the intrinsic aeroelastic nature of such systems, we observe a strong interaction between the aircraft structure and its control and actuator systems. Such interconnection is referred in the specialized literature as aeroservoelasticity [2]. As a consequence, for properly representing the system in order to achieve desired aeroelastic characteristics, these additional dynamics must be considered in system modeling [3], [4], [5].

In face of such practical issue, one may note that the actual plant state variables are not available for composing the feedback loop, but only the sensors outputs, hence hindering the implementation of standard state-feedback control techniques. Therefore, the employment of additional sensors, which may also present non-negligible dynamics, might be demanded. Moreover, ignoring such parasitic dynamics may incur in performance loss and, in the worst case, compromise the closed-loop stability, as been long known [6], [7]. This fact motivated the development of studies for robust

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control designs that may address the problem of actuators and sensors dynamics. We may cite, for instance, contributions regarding observed-based design [8], [9], sliding mode control (SMC) [10], and output feedback control [11].

In this paper, we particularly discuss the output-feedback control and how it emerges as a convenient approach, which can be employed by considering an augmented system representation encompassing the plant, sensors, and actuators dynamics. Then, only the sensor output signals are used in the feedback loop, as presented in [12].

Output feedback is a relevant field of research in control theory on its own, especially concerning the static output-feedback (SOF) framework, which is still considered as a major open problem [13]. No closed solution is available, even in the case where the plant model is assumed to be accurately known. In fact, the SOF stabilization is an NP-hard problem [14]. Such challenging nature motivated the development of studies on this subject over the past decades, addressing several control problems, as the reader may see in the survey [15] and references within. Notably, most of the available contributions are based on severe restrictions over the problem variables or even on the system model matrices. It is not rare to find sufficient design conditions in the literature that were obtained through the imposition of restrictions on the system output matrix format [16], [17]. Conversely, in this paper we are focused on a design strategy based on a two-stage method [18], [19] that does not impose such constraints. We also consider that the output matrix can be dependent on an uncertain parameter, which is also a feature that is not available in many papers on the SOF control [20]. Furthermore, the SOF design consists in a more simple and direct solution when compared to observer-based control systems. Indeed, SOF is based on the computation of a single feedback gain matrix, which entails a simpler implementation setup in practice [15].

However, the problem gets even more involved when the sensor and actuator dynamics involve time-delay. Due to its relevance, the effects of time delay have been investigated in several areas of engineering, such as power [21], communication [22], and control systems [23]. In particular, the research on network communication delay has been flagged as a relevant issue for advanced aircraft data exchange systems [24]. Even more complicated problems arise in the case of uncertain delays. As a matter of fact, the huge amount of data flow in aircraft network buses implies in the uncertain behavior of such systems. Furthermore, the relationship between time delay and actuator dynamics have also raised interest to the development of research on the stability margin in fighter aircraft [25], for instance.

The practical relevance of the effects of time delay in dynamic systems motivated the research on modeling and control design strategies that are able to guarantee robustness over the above mentioned issues [26], [27]. Many of the available methods are related to predictor-like techniques [28], [29], [30], which are intended to compensate the delay effect through the transformation of the delayed system into a delay-free model through finite integrals over past control input values. Predictor-based control results can be found in the linear time-invariant scenario with [31] and without considering model uncertainty [32]. Methods for nonlinear systems can also be found, such as input delay and additive disturbance compensation [33], and dealing with arbitrarily large time delay [34].

Despite the fact that predictor-like techniques are consolidated for addressing input delay (*i.e.*, delay affecting the control input), they struggle to handle systems affected by state delay, as the problem gets considerably more complicated to be modeled in this particular framework. For instance, recent works on this subject [35] managed to consider dynamic actuators via backstepping control with input delay, but does not include state delay nor sensor dynamics in the control design. Also, the dependence on integral terms might be sensitive to parametric uncertainty and delay mismatches [36].

The sliding mode control (SMC) [37], [38], [39] is also an example of technique for dealing with time delay in control systems, being particularly known for its robustness characteristics. However, as a drawback, the presence of time delay has a severe destabilizing effect in conventional SMC systems. For a more complete background, we refer the reader to the survey paper [36].

A more simple yet interesting approach is based on the development of an approximation model of the delayed dynamics. In such method, the infinite-dimension delayed system is treated as a finite-dimensional one by means of the truncation of an infinite series given in terms of a rational polynomial [36]. The main downside of this approach is that in some cases the rational approximation must be of high order to obtain a good representation. However, finite-dimension approximation has led to important contributions, specifically for linear systems [21], [40], [41], [42].

Upon the presented background, we propose in this work a control design strategy for dealing with uncertain linear time-invariant (LTI) systems whose state information and control input signals are obtained and applied by means of sensors and actuators with non-negligible dynamics, differently from most of the available strategies in the literature. Moreover, we assume that the communication channels between sensors, actuators and controller are susceptible to a delay in time, which is also a novelty, since the literature is usually concerned with the effects of either input or state delay, and not the joint effect of both types of delay. Furthermore, investigations on the synthesis of controllers for systems with time delays and also non-negligible dynamics in sensors and actuators are relatively scarce. In our paper, such practical issues are handled by defining an augmented system, where the time-delay effect is modeled using the Padé Approximation [43], [44]. The resulting overall system encompasses the plant, sensors, actuators, and time-delay dynamic states. By assuming that only the sensor outputs are available for feedback, we employ a two-stage-based SOF design method defined in terms of a homogeneous-polynomial Lyapunov function (HPLF) [45], [46]. Employing the SOF control for addressing time-delayed systems with actuators and sensors dynamics through an augmented system is a simple and direct, yet innovative approach that, to the best of the authors' knowledge, has not been considered so far. The new proposed controller synthesis strategy is formulated in the linear matrix inequalities (LMI) framework for including the specification of a minimal performance index in terms of a lower bound on the closed-loop system decay rate, with the purpose of achieving enhanced transient performance. Robustness in terms of disturbance rejection is also taken into account by means of the closed-loop $\mathscr{H}_2$ norm minimization. The validity and efficacy of the proposed strategies are evaluated on illustrative aircraft control design examples.

In the sequence, we summarize the main technical contributions of the present paper:

- New system modeling for encompassing non-negligible dynamics in both sensors and actuators, and also the time delay over sensors measurements and control command signals via Padé approximation. The new system model consists in an augmented state-space representation that can be employed in an LTI–SOF design encompassing all the mentioned practical issues, differently from most of the available works on the subject.
- New LMI-based two-stage SOF controller synthesis strategy for robust stabilization and transient performance improvement of uncertain LTI systems in terms of a lower bound on the closed-loop system decay rate and $\mathscr{H}_2$ norm minimization, considering homogeneous-polynomial Lyapunov functions.
- Study of the influence of the homogeneous polynomial decision variables degree when considering the SOF control design through the two-stage method. Our investigation results show that despite the use of higher polynomial degree in the second stage might improve feasibility, it promotes a negative effect when considering parameter-dependent state-feedback design in the first stage.

The notation is fairly standard: $\mathbf{M} > 0$ ($\mathbf{M} < 0$) implies that $\mathbf{M}$ is a positive definite (negative definite) matrix; (^{T}) denotes the transpose of a matrix; $\mathbb{R}$ denotes the set of the real numbers; $\mathbf{M} + (\bullet)^T = \mathbf{M} + \mathbf{M}^T$; and ($\mathbf{M}^*$) represents the complex conjugate of matrix $\mathbf{M}$. Otherwise, when alone in a partitioned matrix, ($*$) denotes the opposite symmetric block.

I. PRELIMINARIES

In this section, we define the problem addressed in the present paper. Important definitions for the development of the proposed contributions are also presented.

A. PROBLEM STATEMENT

Consider the uncertain linear system described as

$$\dot{x}(t) = \mathbf{A}(\alpha)x(t) + \mathbf{B}(\alpha)z(t) \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a vector with system states and $z(t) \in \mathbb{R}^m$ is a vector with control input signals. The parameter-dependent matrices $\mathbf{A}(\alpha) \in \mathbb{R}^{n\times n}$ and $\mathbf{B}(\alpha) \in \mathbb{R}^{n\times m}$ belong to a polytopic domain $\mathscr{D}$ parametrized in terms of a vector of uncertain time-invariant parameters $\alpha = (\alpha_1, \ldots, \alpha_N)$ such as

$$\mathscr{D} = \left\{(\mathbf{A}, \mathbf{B})(\alpha) : (\mathbf{A}, \mathbf{B})(\alpha) = \sum_{r=1}^{N} \alpha_r(\mathbf{A}_r, \mathbf{B}_r), \alpha \in \wedge_N\right\}, \tag{2}$$

where $(\mathbf{A}_r, \mathbf{B}_r)$ denotes the r-th polytope vertex, and

$$\wedge_N = \left\{\alpha \in \mathbb{R}^n : \sum_{r=1}^{N}\alpha_r = 1; \alpha_r \geq 0; r = 1, \ldots, N\right\}. \tag{3}$$

The state information is measured through q sensors, with dynamics described by

$$\dot{v}_i(t) = a_{v,i}v_i(t) - a_{v,i}\left(\sum_{j=1}^{n} c_{i,j}x_j(t)\right), \tag{4}$$

where $v_i(t)$ are the sensor outputs, composing the vector $v(t) = \left[v_1(t) \cdots v_q(t)\right]^T$, $a_{v,i} < 0$ are time-invariant (but possibly uncertain) parameters for $i = 1, 2, \ldots, q$, and $c_{i,j}$ are known constants for $j = 1, 2, \ldots, n$.

Also, consider the existence of m actuators whose dynamics are described by

$$\dot{z}_k(t) = a_{z,k}z_k(t) - a_{z,k}\left(\sum_{l=1}^{p} d_{k,l}u_{D_l}(t)\right), \tag{5}$$

composing the vector of control signals $z(t) = [z_1(t) \cdots z_m(t)]^T$. Moreover, in (5), $u_{D_l}(t)$ are the actuator input commands, forming the vector $u_D(t) = \left[u_{D_1}(t) \cdots u_{D_p}(t)\right]^T$, $a_{z,k} < 0$ are time-invariant (but possibly uncertain) parameters for $k = 1, 2, \ldots, m$, and $d_{k,l}$ are known constants for $l = 1, 2, \ldots, p$.

The overall control system block diagram, represented in Figure 1, helps to illustrate the considered system and control structure.

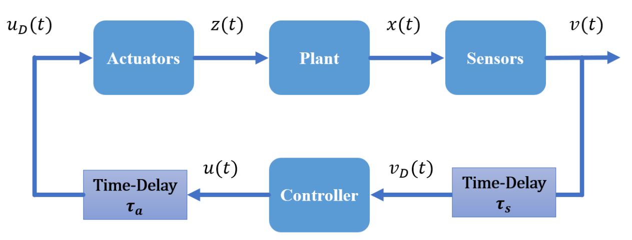

FIGURE 1. Closed-loop block diagram.

Note that we assume that each sensor output $v_i(t)$, used in feedback, is available for the controller with time delay, τ_{s_i}, in terms of a delayed sensor output signal $v_{D_i}(t)$. Likewise, each command signal produced by the controller, $u_l(t)$ is received by the actuator with time delay τ_{a_l}, in terms of a delayed actuator command signal $u_{D_l}(t)$.

Under these definitions, the problem addressed herein consists in designing a control law $u(t) = \mathbf{L}v_D(t)$ where $v_D(t) \in \mathbb{R}^q$ is the vector of time-delayed sensor outputs and $\mathbf{L} \in \mathbb{R}^{p\times q}$ is a gain matrix to be determined in order to ensure the closed-loop asymptotic stability of the overall system. Moreover, considering that the closed-loop system decay rate is defined as the highest positive real scalar σ such that

$$\lim_{t\to\infty} e^{\sigma t}\|x(t)\| = 0 \tag{6}$$

holds for all trajectories $x(t) \neq 0$ of the system states [47], the controller $\mathbf{L}$ must be designed in order to ensure that the decay rate σ is greater than a given lower bound γ.

Remark 1: Note that, for an stable system, $||x(t)||$ converges to zero as $t \to \infty$. In that sense, the system decay rate is related to the highest scalar $\sigma > 0$ such that the convergence of $||x(t)||$ to zero is faster than the growth of the exponential function $e^{\sigma t}$, as $t \to \infty$. Therefore, by the enforcing a lower bound γ, such that $\sigma > \gamma$ we can ensure that the actual decay rate is greater than γ. In these terms, γ represents a design parameter that establishes a performance criteria regarding the closed-loop system settling time. With a higher bound γ, we have a higher system decay rate, and thus a faster state convergence to zero. We refer the reader to works [47] and [48] for a more complete background on the decay rate definition.

B. IMPORTANT DEFINITIONS

The Finsler's Lemma is crucial for obtaining the synthesis conditions presented in this work and, therefore, it is formally stated in the sequence.

Lemma 1 [49]: Consider $w \in \mathbb{R}^n$, $\mathbf{S} \in \mathbb{R}^{n\times n}$, and $\mathbf{R} \in \mathbb{R}^{m\times n}$ with rank $(\mathbf{R}) < n$, and $\mathbf{R}^\perp$ with columns forming a basis for the null space of $\mathbf{R}$ (i.e. $\mathbf{R}\mathbf{R}^\perp = 0$).

Then, the following conditions are equivalent:

(i) $w^T\mathbf{S}w < 0, \quad \mathbf{R}w = 0, \quad \forall\ w \neq 0$,

(ii) ${\mathbf{R}^\perp}^T\mathbf{S}\mathbf{R}^\perp < 0$,

(iii) $\exists \eta \in \mathbb{R} : \mathbf{S} - \eta\mathbf{R}^T\mathbf{R} < 0$,

(iv) $\exists \mathbf{X} \in \mathbb{R}^{n\times m} : \mathbf{S} + \mathbf{X}\mathbf{R} + \mathbf{R}^T\mathbf{X}^T < 0$,

where η and $\mathbf{X}$ are additional variables (or multipliers).

The employment of Lemma 1 in our work is based on deriving an equivalent stability and performance certificate for (1). Such alternative representation allows for circumventing non-convex constraints, and then leading to tractable convex conditions. For further reference in the text, Lemma 2 presents stability and performance certificate regarding the minimum system decay rate.

Lemma 2: A sufficient condition for the robust stability of $\mathbf{A}(\alpha)$ is that there exist a positive-definite symmetric matrix $\mathbf{P}(\alpha)$ and matrices $\mathbf{F}(\alpha)$ and $\mathbf{G}(\alpha)$ such that

$$\begin{bmatrix} \mathbf{A}(\alpha)^\mathbf{T}\mathbf{F}(\alpha)^\mathbf{T} + (\bullet)^\mathbf{T} + 2\gamma\mathbf{P}(\alpha) & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^\mathbf{T} + \mathbf{G}(\alpha)\mathbf{A}(\alpha) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^\mathbf{T} \end{bmatrix} < 0, \tag{7}$$

holds for every $\alpha \in \wedge_N$. Additionally, the system decay rate has a lower bound γ, i.e., max $\sigma > \gamma$ in (6).

Proof: Note that (7) can be rewritten as

$$\begin{bmatrix} \mathbf{F}(\alpha) \\ \mathbf{G}(\alpha) \end{bmatrix} \begin{bmatrix} \mathbf{A}(\alpha) & -\mathbf{I} \end{bmatrix} + (\bullet)^T + \begin{bmatrix} 2\gamma\mathbf{P}(\alpha) & * \\ \mathbf{P}(\alpha) & \mathbf{0} \end{bmatrix} < 0, \tag{8}$$

which corresponds to condition (*iv*) of Lemma 1 with

$$\mathbf{X} = \begin{bmatrix} \mathbf{F}(\alpha) \\ \mathbf{G}(\alpha) \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} 2\gamma\mathbf{P}(\alpha) & * \\ \mathbf{P}(\alpha) & \mathbf{0} \end{bmatrix}, \ \mathbf{R}^T = \begin{bmatrix} \mathbf{A}(\alpha)^\mathbf{T} \\ -\mathbf{I} \end{bmatrix}. \tag{9}$$

Hence, by defining $w = \begin{bmatrix} x(t)^T & \dot{x}(t)^T \end{bmatrix}^T$ in condition (i) of Lemma 1, we have that (7) implies[1]

$$\begin{aligned} w^T\mathbf{S}w &= \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} 2\gamma\mathbf{P}(\alpha) & * \\ \mathbf{P}(\alpha) & \mathbf{0} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \\ &= \dot{x}^T\mathbf{P}(\alpha)x + x^T\mathbf{P}(\alpha)\dot{x} + 2\gamma x^T\mathbf{P}(\alpha)x < 0, \end{aligned} \tag{10}$$

with $\mathbf{R}w = 0$ since, $\dot{x}(t) = \mathbf{A}(\alpha)x(t)$. By defining $V(x) = x^T\mathbf{P}(\alpha)x$, (10) becomes $\dot{V}(x) < -2\gamma V(x)$, *i.e.*, the Lyapunov's constraint for stability and minimum decay rate $\gamma > 0$ [47]. ■

II. PROPOSED STRATEGY

In this section, we present the contributions of our work which are related to a system representation that allows for encompassing plant and additional dynamics in a single model, and a control synthesis strategy based on LMIs for designing an SOF controller that guarantees closed-loop stability and minimum decay rate.

A. SYSTEM AUGMENTATION

For dealing with this control problem, we propose the definition of an augmented system that encompass the plant, actuators, and sensors dynamics, and also the time delay effect. To this end, we first consider that the time delay is modeled using the Padé approximation [43].

When analyzed in the frequency domain, a time delay τ can be represented by the transfer function $e^{-\tau s}$. Using the Padé method, $e^{-\tau s}$ can be approximated by a rational polynomial function $R(s)$ as

$$e^{-\tau s} \approx R(s) = \frac{b_0 + b_1\tau s + \cdots + b_c(\tau s)^c}{a_0 + a_1\tau s + \cdots + a_k(\tau s)^k}, \tag{11}$$

where usually $c = k$, and k denotes the order of the approximated model.

Considering that every sensor output signal $v_i(t)$, $i = 1, \ldots, q$, is received by the controller with time delay $\tau = \tau_s$ (possibly uncertain), the function $R(s)$ in (11) can be transformed into an equivalent state-space model of order $k = k_s$ that represents the delay effect on the sensor output signal by using the following realization [43]:

$$\begin{cases} \dot{\delta}_{sd_i}(t) = \mathbf{A}_{sd}(\alpha)\delta_{sd_i}(t) + \mathbf{B}_{sd}v_i(t) \\ v_{D_i}(t) = \mathbf{C}_{sd}(\alpha)\delta_{sd_i}(t) + \mathbf{D}_{sd}v_i(t) \end{cases}, \tag{12}$$

[1] Note that in (10) the time dependence (t) was omitted for shortening the notation.

where $\delta_{sd_i}(t) \in \mathbb{R}^{k_s}$ is the vector of phase variables, $v_{D_i}(t)$ is the i-th time-delayed sensor output, and

$$\mathbf{A}_{sd_i}(\alpha) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{-a_0\tau_s^{-k_s}}{a_{k_s}} & \frac{-a_1\tau_s^{-k_s+1}}{a_{k_s}} & \frac{-a_2\tau_s^{-k_s+2}}{a_{k_s}} & \cdots & \frac{-a_{k_s-1}\tau_s^{-1}}{a_{k_s}} \end{bmatrix}, \tag{13}$$

$$\mathbf{B}_{sd_i} = \begin{bmatrix} 0 \; 0 \; 0 \cdots 1 \end{bmatrix}^T, \tag{14}$$

$$\mathbf{C}_{sd_i}(\alpha) = \frac{1}{a_{k_s}^2}\Big[(a_{k_s}b_0 - a_0 b_{k_s})\tau_s^{-k_s} \; (a_{k_s}b_1 - a_1 b_{k_s})\tau_s^{-k_s+1} \cdots (a_{k_s}b_{k_s-1} - a_{k_s-1}b_{k_s})\tau_s^{-1}\Big], \tag{15}$$

$$\mathbf{D}_{sd_i} = \frac{b_{k_s}}{a_{k_s}}, \tag{16}$$

with

$$a_j = \frac{(c_s + k_s - j)!k_s!}{j!(k_s - j)!}, \quad b_f = (-1)^f \frac{(c_s + k_s - f)!c_s!}{f!(c_s - f)!}, \tag{17}$$

for $j = 1, \ldots, k_s$, and $f = 1, \ldots, c_s$, where c_s is the numerator degree for the sensor delay model approximation in (11).

In that sense, we can model the sensor dynamics affected by a time delay $\tau = \tau_s$ by defining an augmented vector $s(t) \in \mathbb{R}^{q(1+k_s)}$ defined as

$$s(t) = \left[v(t)^T \;\; \delta_{sd}(t)^T\right]^T,$$

with $\delta_{sd}(t) = \left[\delta_{sd_1}(t) \cdots \delta_{sd_{k_s}}(t)\right]^T$, which combines the sensor dynamics (4) subject to a time-delay effect (12), yielding the following augmented state-space model

$$\begin{cases} \dot{s}(t) = \mathbf{A}_s(\alpha)s(t) + \mathbf{B}_s(\alpha)x(t) \\ v_D(t) = \mathbf{C}_s(\alpha)s(t) \end{cases}, \tag{18}$$

where

$$\mathbf{A}_s(\alpha) = \begin{bmatrix} \mathbf{A}_v(\alpha) & \mathbf{0}_{q\times qk_s} \\ \mathbf{B}_{sd} & \mathbf{A}_{sd}(\alpha) \end{bmatrix}, \quad \mathbf{B}_s(\alpha) = \begin{bmatrix} -\mathbf{A}_v(\alpha)\mathbf{C} \\ \mathbf{0}_{qk_s\times n} \end{bmatrix},$$
$$\mathbf{C}_s(\alpha) = \begin{bmatrix} \mathbf{D}_{sd} & \mathbf{C}_{sd}(\alpha) \end{bmatrix}$$

with

$$\mathbf{A}_v(\alpha) = \text{diag}\left\{a_{v,1}, a_{v,2}, \ldots, a_{v,q}\right\},$$
$$\mathbf{C} = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{q,1} & c_{q,2} & \cdots & c_{q,n} \end{bmatrix},$$

and

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{sd}(\alpha) = \text{diag}\left\{(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{sd_1}, (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{sd_2}, \ldots, (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{sd_q}\right\}, \tag{19}$$

with $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{sd_i}$ as in (13) - (16) (with the subindex "s" referring to the sensor time-delay), implying in $\mathbf{A}_{sd} \in \mathbb{R}^{qk_s\times qk_s}$, $\mathbf{B}_{sd} \in \mathbb{R}^{qk_s\times q}$, $\mathbf{C}_{sd} \in \mathbb{R}^{q\times qk_s}$, and $\mathbf{D}_{sd} \in \mathbb{R}^{q\times q}$.

Similarly, we can model the actuator dynamics affected by time-delayed command signals by considering a k_a-th order Padé approximation, incorporated in an augmented vector $a(t) \in \mathbb{R}^{m(1+k_a)}$

$$a(t) = \left[z(t)^T \;\; \delta_{ad}(t)^T\right]^T,$$

with $\delta_{ad}(t) = \left[\delta_{ad_1}(t) \cdots \delta_{ad_{k_a}}(t)\right]^T$, combining the actuator dynamics (5) subject to a time-delay effect modeled as in (12) (now with $k = k_a$ and $\tau = \tau_a$), yielding the following augmented state-space model

$$\begin{cases} \dot{a}(t) = \mathbf{A}_a(\alpha)a(t) + \mathbf{B}_a(\alpha)u_D(t) \\ z(t) = \mathbf{C}_a(\alpha)a(t) \end{cases}, \tag{20}$$

where

$$\mathbf{A}_a(\alpha) = \begin{bmatrix} \mathbf{A}_z(\alpha) & -\mathbf{A}_z(\alpha)\mathbf{D}\mathbf{C}_{ad} \\ \mathbf{0}_{mk_a\times m} & \mathbf{A}_{ad}(\alpha) \end{bmatrix},$$
$$\mathbf{B}_a(\alpha) = \begin{bmatrix} -\mathbf{A}_z(\alpha)\mathbf{D}\mathbf{D}_{ad} \\ \mathbf{B}_{ad} \end{bmatrix},$$
$$\mathbf{C}_a(\alpha) = \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{0}_{m\times mk_a} \end{bmatrix}$$
$$\mathbf{A}_z(\alpha) = \text{diag}\left\{a_{z,1}, a_{z,2}, \ldots, a_{z,m}\right\},$$
$$\mathbf{D} = \begin{bmatrix} d_{1,1} & d_{1,2} & \cdots & d_{1,p} \\ d_{2,1} & d_{2,2} & \cdots & d_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m,1} & d_{m,2} & \cdots & d_{m,p} \end{bmatrix},$$

and

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{ad} = \text{diag}\left\{(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{ad_1}, (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{ad_2}, \ldots, (\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{ad_m}\right\}, \tag{21}$$

with $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})_{ad_i}$ as in (13) - (16) (with the subindex "a" referring to the actuator time-delay), implying in $\mathbf{A}_{ad} \in \mathbb{R}^{mk_a\times mk_a}$, $\mathbf{B}_{ad} \in \mathbb{R}^{mk_a\times m}$, $\mathbf{C}_{ad} \in \mathbb{R}^{m\times mk_a}$, and $\mathbf{D}_{ad} \in \mathbb{R}^{m\times m}$.

Remark 2: In this work, for simplifying the notation and without loss of generality, we considered that every sensor output signal is subject to the same amount of time delay τ_s, likewise assumed for the actuator commands, with a time delay τ_a. Therefore, the matrices in (12) will be the same for each of the q sensor output signals and each of the m actuator commands, respectively. However, note that a more general approach can be directly employed by assuming that the time delay τ, and also its approximation model order k, are different for every considered signal, defining parameters such as τ_{s_i} and k_{s_i}, $i = 1, \ldots, q$, and τ_{a_l} and k_{s_l}, for $l = 1, \ldots, p$.

Next, to incorporate the time-delayed sensor and actuator dynamics in the LTI system (1), we promote a system augmentation, by defining the augmented state vector $w(t) \in \mathbb{R}^{n+q(k_s+1)+m(k_a+1)}$ as

$$w(t) = \left[x(t)^T \;\; s(t)^T \;\; a(t)^T\right]^T. \tag{22}$$

Then, we have the following augmented state-space representation

$$\begin{cases} \dot{w}(t) = \bar{\mathbf{A}}(\alpha)w(t) + \bar{\mathbf{B}}(\alpha)u(t) \\ y(t) = \bar{\mathbf{C}}(\alpha)w(t) \end{cases}, \tag{23}$$

where

$$\bar{\mathbf{A}}(\alpha) = \begin{bmatrix} \mathbf{A}(\alpha) & \mathbf{0}_{n\times q(ks+1)} & \mathbf{B}(\alpha)\mathbf{C_a}(\alpha) \\ \mathbf{B_s}(\alpha) & \mathbf{A_s}(\alpha) & \mathbf{0}_{q(1+k_s)\times m(1+k_a)} \\ \mathbf{0}_{m(1+k_a)\times n} & \mathbf{0}_{m(1+k_a)\times q(1+k_s)} & \mathbf{A_a}(\alpha) \end{bmatrix},$$

$$\bar{\mathbf{B}}(\alpha) = \begin{bmatrix} \mathbf{0}_{n\times p} \\ \mathbf{0}_{q(k+1)\times p} \\ \mathbf{B_a}(\alpha) \end{bmatrix},$$

and

$$\bar{\mathbf{C}}(\alpha) = \begin{bmatrix} \mathbf{0}_{q\times n} & \mathbf{C_s}(\alpha) & \mathbf{0}_{q\times m(1+k_a)} \end{bmatrix}.$$

The output vector $y(t)$ corresponds to the delayed-sensor output $v_D(t)$ in (18), which is available for feedback, in contrast to the actual system state vector $x(t)$. Therefore, the aforementioned problem may be addressed as a static output-feedback control design with $u(t) = \mathbf{L}v_D(t) = \mathbf{L}y(t)$.

Remark 3: At this point, we give emphasis to the first main contribution of this paper. This new modeling strategy, different from past results on this topic [12], is able to represent, in a single set of matrices, not only the dynamics associated to sensors and actuators but also the effect of delay in the communication channels that deliver the information generated in the sensors output and received in the actuators input, respectively, enabling to address more complex and general control problems.

Remark 4: It is also important to note that the polytopic approach enables our strategy to easily cope with uncertainties on sensors and/or actuators parameters, simply by considering them as additional uncertain parameters along with the plant uncertainties. The same procedure can be employed to consider uncertain time delays τ_s and/or τ_a. This is possible since these parameters will be part of the overall system matrices, generating an augmented polytope encompassing plant, sensor, actuator and delay uncertainties.

Remark 5: Our method considers the Padé approximation for modeling the time delay effect over the system dynamics. The approximation error can be reduced by choosing a higher-order rational polynomial function (11). With higher values of k_s we obtain a better approximation on the exact dynamic effect of the delay $e^{-\tau s}$. Note that the order of the delay approximation k_s is directly incorporated in our proposed system augmented model. Of course, the direct trade-off is that a higher-order state-space model is needed, as the parameter k_s will define the dimension of the sensor delay model matrices (13)-(16), and similarly in the actuator delay model given in (21).

B. CONTROL DESIGN

For the SOF controller design, we consider the use of a two-stage SOF controller synthesis strategy, based on the pioneer works of [18] and [19].

The two-stage method employment in our work consists in first computing a state-feedback gain $\mathbf{K}(\alpha)$ such that

$$\dot{w}(t) = (\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha))w(t),$$

i.e. the augmented system (23), is robustly stable in closed-loop with $u(t) = \mathbf{K}(\alpha)w(t)$. In the sequence, this gain matrix $\mathbf{K}(\alpha)$ is fed to a second-stage controller syntheses, in which the desired SOF stabilizing robust gain $\mathbf{L}$ is effectively computed.

1) FIRST-STAGE DESIGN

The first-stage state-feedback design can be performed using any available strategy in the literature. In this work, we consider well-known conditions [47], based on the existence of matrices $\mathbf{W} = \mathbf{W}' > 0$ and $\mathbf{Z}(\alpha)$ such that

$$\bar{\mathbf{A}}(\alpha)\mathbf{W} + \mathbf{W}\bar{\mathbf{A}}(\alpha)^{\mathbf{T}} + \bar{\mathbf{B}}(\alpha)\mathbf{Z}(\alpha) + \mathbf{Z}(\alpha)^{\mathbf{T}}\bar{\mathbf{B}}(\alpha)^{\mathbf{T}} + 2\beta\mathbf{W} < 0 \tag{24}$$

holds for every $\alpha \in \wedge_N$. In the synthesis conditions, $\mathbf{K}(\alpha) = \mathbf{W}^{-1}\mathbf{Z}(\alpha)$ guarantees the robust state-feedback stabilization of $\dot{w}(t) = (\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha))w(t)$ with lower bound β on the closed-loop system decay rate.[2]

Clearly, the conditions presented in (24) are of infinite-dimension, as they are dependent on the uncertain parameter α. Therefore, some manipulation over these constraints have to be performed to obtain an equivalent finite-dimension problem. We leave this issue to be properly discussed more ahead in the text, since the second-stage synthesis conditions are also presented in terms of parameter-dependent LMIs.

2) SECOND-STAGE DESIGN

In this paper, we propose a generalization of the SOF synthesis conditions proposed in [50] and adopted in [12] for computing SOF gains for the stabilization of the augmented system encompassing sensor and actuator dynamics. The results in [50] are achieved by considering that the LMI decision matrices have polytopic dependence on the uncertain parameter α. In this work, we assume that the decision variables have a homogeneous-polynomial dependence on α of arbitrary degree g [45], and also extend the synthesis conditions for enabling the enforcement of a minimum decay rate criterion.

For applying such strategy, we first formally enunciate in Theorem 1 a parameter-dependent LMI condition set that encompasses the results presented in [50].

Theorem 1: Assuming that there exists a state-feedback gain $\mathbf{K}(\alpha)$ such that $(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha))$ is asymptotically stable, then there exists a stabilizing static output-feedback gain $\mathbf{L}$ such that $\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha)$ is asymptotically stable, considering a decay rate greater than or equal to $\gamma > 0$, if there exist a symmetric parameter-dependent matrix

[2]Note that here the lower bound on the closed-loop decay rate was denoted as β instead of γ, as defined in Section I. Due to the sufficient nature of the two-stage method, different design constraints can be enforced in each stage, as discussed at the end of this section.

$\mathbf{P}(\alpha) > 0$, *parameter-dependent matrices* $\mathbf{F}(\alpha)$, $\mathbf{G}(\alpha)$, *and matrices* $\mathbf{H}$ *and* $\mathbf{J}$ *such that*

$$\begin{bmatrix} (\mathbf{F}(\alpha)\bar{\mathbf{A}}(\alpha) + \mathbf{F}(\alpha)\bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha)) + (\bullet)^{\mathbf{T}} + 2\gamma\mathbf{P}(\alpha) & * & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{G}(\alpha)\bar{\mathbf{A}}(\alpha) + \mathbf{G}(\alpha)\bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} & * \\ \bar{\mathbf{B}}(\alpha)^{\mathbf{T}}\mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{J}\bar{\mathbf{C}}(\alpha) - \mathbf{H}\mathbf{K}(\alpha) & \bar{\mathbf{B}}^{\mathbf{T}}(\alpha)\mathbf{G}(\alpha)^{\mathbf{T}} & -\mathbf{H} - \mathbf{H}^{\mathbf{T}} \end{bmatrix} < 0. \quad (25)$$

In the synthesis condition, the robust static output-feedback gain is given by $\mathbf{L} = \mathbf{H}^{-1}\mathbf{J}$.

Proof: Readily note that (25) implies in $\mathbf{H}$ being invertible [47]. In the sequence, applying a transformation on (25) with $\mathbf{T}(\alpha)$ and $\mathbf{T}(\alpha)^T$ [19], where

$$\mathbf{T}(\alpha) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{S}(\alpha)^{\mathbf{T}} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad (26)$$

by employing simple algebraic manipulation one can obtain

$$\begin{bmatrix} \Psi_{\mathbf{1,1}}(\alpha) & \Psi_{\mathbf{1,2}}(\alpha) \\ * & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} \end{bmatrix} < 0, \quad (27)$$

where

$$\Psi_{\mathbf{1,1}}(\alpha) = \Big[(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)(\mathbf{K}(\alpha) + \mathbf{S}(\alpha)))^{\mathbf{T}}\mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{S}(\alpha)^{\mathbf{T}}(\mathbf{J}\bar{\mathbf{C}}(\alpha) - \mathbf{H}(\mathbf{K}(\alpha) + \mathbf{S}(\alpha)))\Big] + (\bullet)^{\mathbf{T}} + 2\gamma\mathbf{P}(\alpha), \quad (28)$$

and

$$\Psi_{\mathbf{1,2}}(\alpha) = \mathbf{P}(\alpha) - \mathbf{F}(\alpha) + (\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)(\mathbf{K}(\alpha) + \mathbf{S}(\alpha)))^{\mathbf{T}}\mathbf{G}^{\mathbf{T}}(\alpha) \quad (29)$$

By defining $\mathbf{S}(\alpha) = \mathbf{H}^{-1}\mathbf{J}\bar{\mathbf{C}}(\alpha) - \mathbf{K}(\alpha)$, and $\mathbf{L} = \mathbf{H}^{-1}\mathbf{J}$ in (28) and (29), and performing simple multiplication distribution, we have that (27) becomes

$$\begin{bmatrix} \Big[(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha))^{\mathbf{T}}\mathbf{F}(\alpha)^{\mathbf{T}}\Big] + (\bullet)^{\mathbf{T}} + 2\gamma\mathbf{P}(\alpha) & \mathbf{P}(\alpha) - \mathbf{F}(\alpha) + (\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha))^{\mathbf{T}}\mathbf{G}^{\mathbf{T}}(\alpha) \\ * & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} \end{bmatrix} < 0, \quad (30)$$

which is a sufficient condition for the robust stabilization with a lower bound γ on the system decay rate according to Lemma 2, with $\dot{w}(t) = (\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha))w(t)$. ■

The result presented in Theorem 1 deserves some remarks. Firstly, it is worth noting that the SOF gain matrix $\mathbf{L}$ is not parameter-dependent, even though Theorem 1 involves matrix variables that depend on the parameter vector α. Secondly, as those in (24), the LMI conditions given in (25) are of infinite dimension. In order to make them computationally tractable, we need to convert them into a finite set of LMI conditions, by imposing some particular structure to the decision variables. Following previous works on the subject, by assuming that the parameter-dependent matrices are modeled as homogeneous polynomials of sufficiently large degree g on the uncertain parameter α, we may obtain a finite set of LMIs with no loss of generality [51]. This means that the higher the degree g considered for the polynomial matrices, the lesser is the conservatism introduced in the constraint formulation. For a sufficient large g, the obtained finite set of LMI will exactly represent the constraints in (25).

For an arbitrary degree g considered for the homogeneous polynomial matrices in (25), we can obtain a finite set of LMIs in order to solve the control design problem using semidefinite programming tools. For the particular case of $g = 1$, we have a polytopic parameter-dependent Lyapunov function (PDLF) such as

$$\mathbf{P}(\alpha) = \alpha_1\mathbf{P_1} + \alpha_2\mathbf{P_2} + \cdots + \alpha_N\mathbf{P}_N.$$

In this case, sufficient conditions for the LMIs in Theorem 1 can be obtained by checking a finite set of LMI constraints over the vertices of the polytopic parameter-dependent matrices. This result is formally stated in the following corollary.

Corollary 1: By assuming that $\mathbf{P}(\alpha)$, $\mathbf{F}(\alpha)$, *and* $\mathbf{G}(\alpha)$ *in Theorem 1 are homogeneous-polynomial matrices of degree* $g = 1$, *as well as the state-feedback first-stage gain matrix* $\mathbf{K}(\alpha)$, *then a sufficient condition for (25) to hold is that there exist symmetric matrices* $\mathbf{P_i} > 0$, *and matrices* $\mathbf{F_i}$, $\mathbf{G_i}$, $\mathbf{H}$, *and* $\mathbf{J}$ *such that*

$$\begin{bmatrix} (\mathbf{F_i}\bar{\mathbf{A}}_\mathbf{i} + \mathbf{F_i}\bar{\mathbf{B}}_\mathbf{i}\mathbf{K_i}) + (\bullet)^{\mathbf{T}} + 2\gamma\mathbf{P_i} & * & * \\ \mathbf{P_i} - \mathbf{F_i}^{\mathbf{T}} + \mathbf{G_i}\bar{\mathbf{A}}_\mathbf{i} + \mathbf{G_i}\bar{\mathbf{B}}_\mathbf{i}\mathbf{K_i} & -\mathbf{G_i} - \mathbf{G}_\mathbf{i}^{\mathbf{T}} & * \\ \bar{\mathbf{B}}_\mathbf{i}^{\mathbf{T}}\mathbf{F_i}^{\mathbf{T}} + \mathbf{J_i}\bar{\mathbf{C}}_\mathbf{i} - \mathbf{H}\mathbf{K_i} & \bar{\mathbf{B}}^{\mathbf{T}_\mathbf{i}}\mathbf{G}_\mathbf{i}^{\mathbf{T}} & -\mathbf{H} - \mathbf{H}^{\mathbf{T}} \end{bmatrix} < 0 \quad (31)$$

holds for $i = 1, 2, \ldots, N$,

$$\begin{bmatrix} \Xi_{11}^{ij} & * & * \\ \Xi_{21}^{ij} & -2(\mathbf{G_i} + \mathbf{G}_\mathbf{i}^{\mathbf{T}}) - (\mathbf{G_j} + \mathbf{G}_\mathbf{j}^{\mathbf{T}}) & * \\ \Xi_{31}^{ij} & \bar{\mathbf{B}}_\mathbf{i}^{\mathbf{T}}(\mathbf{G}_\mathbf{i}^{\mathbf{T}} + \mathbf{G}_\mathbf{j}^{\mathbf{T}}) + \bar{\mathbf{B}}_\mathbf{j}^{\mathbf{T}}\mathbf{G}_\mathbf{i}^{\mathbf{T}} & -3(\mathbf{H} + \mathbf{H}^{\mathbf{T}}) \end{bmatrix} < 0, \quad (32)$$

with

$$\Xi_{11}^{ij} = \Big[\bar{\mathbf{A}}^{\mathbf{T}_\mathbf{i}}(\mathbf{F}_\mathbf{i}^{\mathbf{T}} + \mathbf{F}_\mathbf{j}^{\mathbf{T}}) + \bar{\mathbf{A}}^{\mathbf{T}_\mathbf{j}}\mathbf{F}_\mathbf{i}^{\mathbf{T}} + \mathbf{K}_\mathbf{i}^{\mathbf{T}}(\bar{\mathbf{B}}_{\mathbf{i}^\mathbf{T}}\mathbf{F}_\mathbf{j}^{\mathbf{T}} + \bar{\mathbf{B}}_{\mathbf{j}^\mathbf{T}}\mathbf{F}_\mathbf{i}^{\mathbf{T}}) + \mathbf{K}_\mathbf{j}^{\mathbf{T}}\bar{\mathbf{B}}^{\mathbf{T}_\mathbf{i}}\mathbf{F}_\mathbf{i}^{\mathbf{T}}\Big] + (\bullet)^T + 2\gamma(2\mathbf{P_i} + \mathbf{P_j}), \quad (33)$$

$$\Xi_{21}^{ij} = 2\mathbf{P_i} + \mathbf{P_j} - (2\mathbf{F}_\mathbf{i}^{\mathbf{T}} + \mathbf{F}_\mathbf{j}^{\mathbf{T}}) + \mathbf{G_i}(\bar{\mathbf{A}}_\mathbf{i} + \bar{\mathbf{A}}_\mathbf{j}) + \mathbf{G_j}\bar{\mathbf{A}}_\mathbf{i} + \mathbf{G_i}(\bar{\mathbf{B}}_\mathbf{i}\mathbf{K_j} + \bar{\mathbf{B}}_\mathbf{j}\mathbf{K_i}) + \mathbf{G_j}\bar{\mathbf{B}}_\mathbf{i}\mathbf{K_i}, \quad (34)$$

and

$$\Xi_{31}^{ij} = \bar{\mathbf{B}}_{\mathbf{i}^\mathbf{T}}(\mathbf{F}_\mathbf{i}^{\mathbf{T}} + \mathbf{F}_\mathbf{j}^{\mathbf{T}}) + \bar{\mathbf{B}}_{\mathbf{j}^\mathbf{T}}\mathbf{F}_\mathbf{i}^{\mathbf{T}} + \mathbf{J}(2\bar{\mathbf{C}}_\mathbf{i} + \bar{\mathbf{C}}_\mathbf{j}) - \mathbf{H}(2\mathbf{K_i} + \mathbf{K_j}), \quad (35)$$

holds for $i, j = 1, 2, \ldots, N$ *and* $i \neq j$, *and*

$$\begin{bmatrix} \Xi_{11}^{ijk} & * & * \\ \Xi_{21}^{ijk} & \Xi_{22}^{ijk} & * \\ \Xi_{31}^{ijk} & \Xi_{23}^{ijk} & -6(\mathbf{H} + \mathbf{H}^{\mathbf{T}}) \end{bmatrix} < 0, \quad (36)$$

with

$$
\begin{aligned}
\Xi_{11}^{ijk} =& \Big[(\bar{\mathbf{A}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{A}}_\mathbf{j}^\mathbf{T})\mathbf{F}_\mathbf{k}^\mathbf{T} + (\bar{\mathbf{A}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{A}}_\mathbf{k}^\mathbf{T})\mathbf{F}_\mathbf{j}^\mathbf{T} + (\bar{\mathbf{A}}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{A}}_\mathbf{k}^\mathbf{T})\mathbf{F}_\mathbf{i}^\mathbf{T} + \\
& + (\mathbf{K}_\mathbf{i}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{j}^\mathbf{T} + \mathbf{K}_\mathbf{j}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T})\mathbf{F}_\mathbf{k}^\mathbf{T} + (\mathbf{K}_\mathbf{i}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{k}^\mathbf{T} + \mathbf{K}_\mathbf{k}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T})\mathbf{F}_\mathbf{j}^\mathbf{T} + \\
& + (\mathbf{K}_\mathbf{j}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{k}^\mathbf{T} + \mathbf{K}_\mathbf{k}^\mathbf{T}\bar{\mathbf{B}}_\mathbf{j}^\mathbf{T})\mathbf{F}_\mathbf{i}^\mathbf{T}\Big] + (\bullet)^T + 4\gamma(\mathbf{P}_\mathbf{i} + \mathbf{P}_\mathbf{j} + \mathbf{P}_\mathbf{k}), \\
\Xi_{21}^{ijk} =& 2(\mathbf{P}_\mathbf{i} + \mathbf{P}_\mathbf{j} + \mathbf{P}_\mathbf{k}) - 2(\mathbf{F}_\mathbf{i} + \mathbf{F}_\mathbf{j} + \mathbf{F}_\mathbf{k})^T \\
& + (\mathbf{G}_\mathbf{i} + \mathbf{G}_\mathbf{j})\bar{\mathbf{A}}_\mathbf{k} + (\mathbf{G}_\mathbf{i} + \mathbf{G}_\mathbf{k})\bar{\mathbf{A}}_\mathbf{j} + (\mathbf{G}_\mathbf{j} + \mathbf{G}_\mathbf{k})\bar{\mathbf{A}}_\mathbf{i} \\
& + \mathbf{G}_\mathbf{i}(\bar{\mathbf{B}}_\mathbf{j}\mathbf{K}_\mathbf{k} + \bar{\mathbf{B}}_\mathbf{k}\mathbf{K}_\mathbf{j}) + \mathbf{G}_\mathbf{j}(\bar{\mathbf{B}}_\mathbf{i}\mathbf{K}_\mathbf{k} + \bar{\mathbf{B}}_\mathbf{k}\mathbf{K}_\mathbf{i}) \\
& + \mathbf{G}_\mathbf{k}(\bar{\mathbf{B}}_\mathbf{i}\mathbf{K}_\mathbf{j} + \bar{\mathbf{B}}_\mathbf{j}\mathbf{K}_\mathbf{i}), \\
\Xi_{31}^{ijk} =& (\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{j}^\mathbf{T})\mathbf{F}_\mathbf{k}^\mathbf{T} + (\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{k}^\mathbf{T})\mathbf{F}_\mathbf{j}^\mathbf{T} + (\bar{\mathbf{B}}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{k}^\mathbf{T})\mathbf{F}_\mathbf{i}^\mathbf{T} \\
& + 2J(\bar{\mathbf{C}}_\mathbf{i} + \bar{\mathbf{C}}_\mathbf{j} + \bar{\mathbf{C}}_\mathbf{k}) - 2\mathbf{H}(\mathbf{K}_\mathbf{i} + \mathbf{K}_\mathbf{j} + \mathbf{K}_\mathbf{k}), \\
\Xi_{22}^{ijk} =& -2(\mathbf{G}_\mathbf{i} + \mathbf{G}_\mathbf{i}^\mathbf{T} + \mathbf{G}_\mathbf{j} + \mathbf{G}_\mathbf{j}^\mathbf{T} + \mathbf{G}_\mathbf{k} + \mathbf{G}_\mathbf{k}^\mathbf{T}), \\
\Xi_{23}^{ijk} =& (\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{j}^\mathbf{T})\mathbf{G}_\mathbf{k}^\mathbf{T} + (\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{k}^\mathbf{T})\mathbf{G}_\mathbf{j}^\mathbf{T} + (\bar{\mathbf{B}}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{k}^\mathbf{T})\mathbf{G}_\mathbf{i}^\mathbf{T},
\end{aligned}
$$

holds for $i = 1, 2, \ldots, N-2$, $j = i+1, \ldots, N-2$, *and* $k = j+1, \ldots, N$.

Proof: Note that by multiplying (31) by α_i and summing for $i = 1, \ldots, N$, by multiplying (32) by α_i and α_j, and summing for $i, j = 1, 2, \ldots, N$, $i \neq j$, and by multiplying (36) by α_i, α_j, and α_k, and summing for $i = 1, 2, \ldots, N-2$, $j = i+1, \ldots, N-2$, and $k = j+1, \ldots, N$, bearing in mind that $\alpha \in \wedge_N$, we directly obtain the parameter-dependent form (25). ■

It is important to observe that both Theorem 1 and, consequently, Corollary 1 encompass the conditions proposed in [50], showing that this previous work is a particular case of the LMI formulation proposed in the present work. Observe that by considering a robust first-stage gain matrix (i.e., $\mathbf{K}(\alpha) = \mathbf{K}$), the LMI conditions in Corollary 1 are reduced to Theorem 2 in [50], as they will no longer have a cross-product between three parameter-dependent matrices, and thus only sums in i and j will be needed. This result is stated in Corollary 2.

Corollary 2: By assuming that, in Theorem 1, $\mathbf{P}(\alpha)$, $\mathbf{F}(\alpha)$, *and* $\mathbf{G}(\alpha)$ *are homogeneous polynomials of degree* $g = 1$, *and that the state-feedback first-stage gain matrix is such that* $\mathbf{K}(\alpha) = \mathbf{K}$, *then a sufficient condition for (25) hold is that there exist symmetric matrices* $\mathbf{P}_\mathbf{i} > 0$, *and matrices* $\mathbf{F}_\mathbf{i}$, $\mathbf{G}_\mathbf{i}$, $\mathbf{H}$, *and* $\mathbf{J}$ *such that*

$$
\begin{bmatrix}
(\mathbf{F}_\mathbf{i}\bar{\mathbf{A}}_\mathbf{i} + \mathbf{F}_\mathbf{i}\bar{\mathbf{B}}_\mathbf{iK}) + (\bullet)^\mathbf{T} + 2\gamma\mathbf{P}_\mathbf{i} & * & * \\
\mathbf{P}_\mathbf{i} - \mathbf{F}_\mathbf{i}^\mathbf{T} + \mathbf{G}_\mathbf{i}\bar{\mathbf{A}}_\mathbf{i} + \mathbf{G}_\mathbf{i}\bar{\mathbf{B}}_\mathbf{iK} & -\mathbf{G}_\mathbf{i} - \mathbf{G}_\mathbf{i}^\mathbf{T} & * \\
\bar{\mathbf{B}}_{\mathbf{i}^\mathbf{T}\mathbf{F}_\mathbf{i}^\mathbf{T} + \mathbf{J}_\mathbf{i}}\bar{\mathbf{C}}_{\mathbf{i}-\mathbf{HK}} & \bar{\mathbf{B}}_\mathbf{i}^\mathbf{T}\mathbf{G}_\mathbf{i}^\mathbf{T} & -\mathbf{H} - \mathbf{H}^\mathbf{T}
\end{bmatrix} < 0 \tag{37}
$$

hold for $i = 1, 2, \ldots, N$,

$$
\begin{bmatrix}
\Xi_{11}^{ij} & * & * \\
\Xi_{21}^{ij} & -(\mathbf{G}_\mathbf{i} + \mathbf{G}_\mathbf{i}^\mathbf{T}) - (\mathbf{G}_\mathbf{j} + \mathbf{G}_\mathbf{j}^\mathbf{T}) & * \\
\Xi_{31}^{ij} & \bar{\mathbf{B}}_\mathbf{i}^\mathbf{T}\mathbf{G}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{j}^\mathbf{T}\mathbf{G}_\mathbf{i}^\mathbf{T} & -2(\mathbf{H} + \mathbf{H}^\mathbf{T})
\end{bmatrix} < 0, \tag{38}
$$

with

$$
\begin{aligned}
\Xi_{11}^{ij} =& \Big[\bar{\mathbf{A}}^{\mathbf{T}_\mathbf{i}\mathbf{F}_\mathbf{j}^\mathbf{T}} + \bar{\mathbf{A}}^{\mathbf{T}_\mathbf{j}\mathbf{F}_\mathbf{i}^\mathbf{T} + \mathbf{K}^\mathbf{T}}(\bar{\mathbf{B}}_\mathbf{i}^\mathbf{T}\mathbf{F}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{j}^\mathbf{T}\mathbf{F}_\mathbf{i}^\mathbf{T})\Big] \\
& + (\bullet)^T + 2\gamma(\mathbf{P}_\mathbf{i} + \mathbf{P}_\mathbf{j}), \qquad (39) \\
\Xi_{21}^{ij} =& \mathbf{P}_\mathbf{i} + \mathbf{P}_\mathbf{j} - (\mathbf{F}_\mathbf{i}^\mathbf{T} + \mathbf{F}_\mathbf{j}^\mathbf{T}) + \mathbf{G}_\mathbf{i}\bar{\mathbf{A}}_{\mathbf{j}+\mathbf{G}_\mathbf{j}}\bar{\mathbf{A}}_\mathbf{i} \\
& + \mathbf{G}_\mathbf{i}\bar{\mathbf{B}}_{\mathbf{jK}+\mathbf{G}_\mathbf{j}}\bar{\mathbf{B}}_\mathbf{iK}, \qquad (40)
\end{aligned}
$$

and

$$
\Xi_{31}^{ij} = \bar{\mathbf{B}}_\mathbf{i}^\mathbf{T}\mathbf{F}_\mathbf{j}^\mathbf{T} + \bar{\mathbf{B}}_\mathbf{j}^\mathbf{T}\mathbf{F}_\mathbf{i}^\mathbf{T} + \mathbf{J}(\bar{\mathbf{C}}_\mathbf{i} + \bar{\mathbf{C}}_\mathbf{j}) - \mathbf{H}(\mathbf{K}_\mathbf{i} + \mathbf{K}_\mathbf{j}), \tag{41}
$$

hold for $i = 1, 2, \ldots, N-1$ *and* $j = i+1, i+2, \ldots, N$.

Proof: Note that by multiplying (37) by α_i and summing for $i = 1, \ldots, N$, and by multiplying (38) by α_i and α_j, and summing for $i = 1, 2, \ldots, N-1$ and $j = i+1, i+2, \ldots, N$, bearing in mind that $\alpha \in \wedge_N$, we directly obtain the parameter-dependent form (25), with $\mathbf{K}(\alpha) = \mathbf{K}$. ■

As mentioned before, one can find a finite set of LMI conditions that ensure (25) by assuming that the decision variables are homogeneous-polynomial parameter-dependent matrices. Corollary 1 presents the sufficient conditions for (25) to hold for the case of $g = 1$. Progressively less conservative conditions might be obtained with higher order polynomials in α. However, deriving such finite set of LMI could be a laborious task, as can be seen from the complexity associated to the case of $g = 1$. Fortunately, one can employ computational packages available in the literature to computationally generate the finite set of LMI, as for instance the specialized parser ROLMIP [52], which is adopted in the present paper.

A final remark need to be made on how to define the degree of the polynomial variables in the first-stage design conditions (24). Note that the two-stage method consists of sufficient conditions, since the first-stage design is performed independently from the second stage, as long as the obtained feedback matrix $\mathbf{K}(\alpha)$ is a stabilizing one. Therefore, the designer can impose different restrictions in the first stage, either on the decay rate or on the degree of the polynomial variables. This means it is not mandatory to impose $\beta = \gamma$ in the design procedure, nor to specify the same degree on the decision variables $\mathbf{Z}(\alpha)$, $\mathbf{P}(\alpha)$, $\mathbf{F}(\alpha)$, or $\mathbf{G}(\alpha)$. Nevertheless, feasibility in the second stage is directly affected by these choices, as illustrated by the examples in the next section.

C. EXTENSION TO $\mathscr{H}_2$ CONTROL

Now consider the state-space realization

$$
\begin{aligned}
\dot{x}(t) &= \mathbf{A}(\alpha)x(t) + \mathbf{B}(\alpha)z(t) + \mathbf{B}_\mathbf{d}(\alpha)d(t) \\
\zeta(t) &= \mathbf{C}_\mathbf{z}(\alpha)x(t) + \mathbf{D}_\mathbf{z}(\alpha)z(t)
\end{aligned} \tag{42}
$$

where $\zeta(t) \in \mathbb{R}^{q_z}$ and $d(t) \in \mathbb{R}^{m_d}$ are the controlled output and disturbance input vectors, respectively. Additionally, $\mathbf{B}_\mathbf{d}(\alpha) \in \mathbb{R}^{n \times m_d}$ is the disturbance input matrix to the system dynamics, $\mathbf{C}_\mathbf{z}(\alpha) \in \mathbb{R}^{q_z \times n}$ is the controlled output matrix, and $\mathbf{D}_\mathbf{z}(\alpha) \in \mathbb{R}^{q_z \times m}$ is the control input direct transmission matrix.

Assuming that sensor and actuator dynamics are modeled as described in Section II, an augmented system can be derived such as

$$\begin{aligned} \dot{w}(t) &= \bar{\mathbf{A}}(\alpha)w(t) + \bar{\mathbf{B}}(\alpha)z(t) + \bar{\mathbf{B}}_{\mathbf{d}}(\alpha)d(t) \\ \zeta(t) &= \bar{\mathbf{C}}_{\mathbf{z}}(\alpha)w(t) + \bar{\mathbf{D}}_{\mathbf{z}}(\alpha)z(t) \\ y(t) &= \bar{\mathbf{C}}(\alpha)w(k), \end{aligned} \tag{43}$$

where $w(t)$ is defined as in (22) and $\bar{\mathbf{A}}(\alpha)$, $\bar{\mathbf{B}}(\alpha)$, and $\bar{\mathbf{C}}(\alpha)$ are given as in (23), while

$$\bar{\mathbf{B}}_{\mathbf{d}}(\alpha) = \begin{bmatrix} \mathbf{B}_{\mathbf{d}}(\alpha) \\ \mathbf{0}_{q(1+k_s)\times m_d} \\ \mathbf{0}_{m(1+k_a)\times m_d} \end{bmatrix}, \quad \bar{\mathbf{D}}_{\mathbf{z}}(\alpha) = \mathbf{D}_{\mathbf{z}}(\alpha)$$

and

$$\bar{\mathbf{C}}_{\mathbf{z}}(\alpha) = \begin{bmatrix} \mathbf{C}_{\mathbf{z}}(\alpha) & \mathbf{0}_{q_z\times q(1+k_s)} & \mathbf{0}_{q_z\times m(1+k_a)} \end{bmatrix}.$$

In such fashion, the augmented system (43) models the effect of an exogenous input signal $d(t)$ over the system dynamics. Moreover, it indicates the vector $\zeta(t)$, which consists of a linear combination of the original system state vector $x(t)$, defined according to the shape of the matrix $\mathbf{C}_{\mathbf{z}}(\alpha)$.

The $\mathscr{H}_2$ problem here considered consists in finding a robust controller $\mathbf{L}$ such that the augmented system (43) is asymptotically stable and also the the closed-loop $\mathscr{H}_2$ guaranteed cost is bounded by μ. The system decay rate is also imposed to have a lower bound γ. Such performance criteria are desired to be meet assuming a control law $u(t) = \mathbf{L}y(t)$.

For the matter considered in this work, the $\mathscr{H}_2$ guaranteed cost is defined as a positive scalar μ such that

$$\mu \geq ||\mathcal{H}(\alpha, s)||_2$$

where $||\mathcal{H}(\alpha, s)||_2$ is the system (43) $\mathscr{H}_2$ norm, defined as

$$||\mathcal{H}(\alpha, s)||_2^2 = \sup_{\alpha\in\wedge_N} \frac{1}{2\pi}\int_0^{+\infty} \mathrm{Tr}(\mathcal{H}(\alpha, j\omega)^*\mathcal{H}(\alpha, j\omega))d\omega$$

and

$$\mathcal{H}(\alpha, s) = \mathcal{C}(\alpha)(s\mathbf{I} - \mathcal{A}(\alpha)^{-1})\mathcal{B}(\alpha), \tag{44}$$

is the closed-loop system transfer matrix, defined over the complex variable s, here considered as the complex frequency variable $s = j\omega$, with

$$\mathcal{A}(\alpha) = \bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha), \quad \mathcal{B}(\alpha) = \bar{\mathbf{B}}_{\mathbf{d}}(\alpha), \quad \text{and,} \tag{45}$$

$$\mathcal{C}(\alpha) = \bar{\mathbf{C}}_{\mathbf{z}}(\alpha) + \bar{\mathbf{D}}_{\mathbf{z}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha). \tag{46}$$

To give basis to our next proposed results, we consider the following lemma, which represents a condition for minimizing the closed-loop $\mathscr{H}_2$ guaranteed cost as defined in this section.

Lemma 3 [53]: For a Hurwitz stable matrix $\mathcal{A}(\alpha)$, $||\mathcal{H}(\alpha, s)||_2^2 < \mu$ if and only if there exists parameter-dependent symmetric matrices $\mathbf{P}(\alpha) > 0$ and $\mathbf{Y}(\alpha) > 0$ such that

$$trace(\mathbf{Y}(\alpha)) < \mu^2 \tag{47}$$

$$\mathbf{Y}(\alpha) - \mathcal{B}(\alpha)^{\mathbf{T}}\mathbf{P}(\alpha)\mathcal{B}(\alpha) > 0 \tag{48}$$

$$\mathcal{A}(\alpha)^T\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathcal{A}(\alpha) + \mathcal{C}(\alpha)^T\mathcal{C}(\alpha) < 0, \tag{49}$$

Now, based on the considered two-stage procedure, we propose new sufficient LMI conditions for computing the robust $\mathscr{H}_2$ controller $\mathbf{L}$ based on the state-feedback controller $\mathbf{K}(\alpha)$ obtained in the previous stage, as enunciated in Theorem 2.

Theorem 2: Assuming that there exists a state-feedback gain $\mathbf{K}(\alpha)$ such that $(\bar{\mathbf{A}}(\alpha)+\bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha))$ is asymptotically stable, then there exists a stabilizing static output-feedback gain $\mathbf{L}$ such that $\bar{\mathbf{A}}(\alpha)+\bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha)$ is asymptotically stable, considering a decay rate greater than or equal to $\gamma > 0$, if there exist parameter-dependent symmetric matrices $\mathbf{P}(\alpha) > 0$ and $\mathbf{Y}(\alpha) > 0$, parameter-dependent matrices $\mathbf{F}(\alpha)$, $\mathbf{G}(\alpha)$, and matrices $\mathbf{H}$ and $\mathbf{J}$ such that are a solution to the following optimization problem:

$$min\ \nu$$

subject to

$$trace(\mathbf{Y}(\alpha)) \leq \nu, \tag{50}$$

$$\begin{bmatrix} \mathbf{Y}(\alpha) & \bar{\mathbf{B}}_{\mathbf{d}}(\alpha)^{\mathbf{T}}\mathbf{P}(\alpha) \\ \mathbf{P}(\alpha)\bar{\mathbf{B}}_{\mathbf{d}}(\alpha) & \mathbf{P}(\alpha) \end{bmatrix} > 0, \tag{51}$$

and

$$\begin{bmatrix} (\mathbf{F}(\alpha)\bar{\mathbf{A}}(\alpha) + \mathbf{F}(\alpha)\bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha)) + (\bullet)^T + 2\gamma\mathbf{P}(\alpha) & * & * & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{G}(\alpha)\bar{\mathbf{A}}(\alpha) + \mathbf{G}(\alpha)\bar{\mathbf{B}}(\alpha)\mathbf{K}(\alpha) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} & * & * \\ \bar{\mathbf{C}}_{\mathbf{z}}(\alpha) + \bar{\mathbf{D}}_{\mathbf{z}}(\alpha)\mathbf{K}(\alpha) & \mathbf{0} & -\mathbf{I} & * \\ \bar{\mathbf{B}}(\alpha)^{\mathbf{T}}\mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{J}\bar{\mathbf{C}}(\alpha) - \mathbf{H}(\alpha)\mathbf{K}(\alpha) & \bar{\mathbf{B}}(\alpha)^{\mathbf{T}}\mathbf{G}(\alpha)^{\mathbf{T}} & \bar{\mathbf{D}}_{\mathbf{z}}(\alpha) & -\mathbf{H}(\alpha) - \mathbf{H}(\alpha)^{\mathbf{T}} \end{bmatrix} < 0 \tag{52}$$

Then, at the optimal solution, system (43) in closed-loop with $u(t) = \mathbf{L}y(t)$, where $\mathbf{L} = \mathbf{H}^{-1}\mathbf{J}$, has a lower bound $\gamma > 0$ on the system decay rate and the closed-loop system $\mathscr{H}_2$ norm is bounded by μ such that $\mu = \sqrt{\nu} \geq ||\mathcal{H}(s)||_2$.

Proof: Assuming that (52) holds, we have that $\mathbf{H}$ is invertible. Now, by pre- and post-multiplying (52) by $\mathbf{U}(\alpha)$ and $\mathbf{U}(\alpha)^T$, where

$$\mathbf{U}(\alpha) = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{S}(\alpha)^{\mathbf{T}} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \tag{53}$$

where $\mathbf{S}(\alpha) = \mathbf{H}^{-1}\mathbf{J}\bar{\mathbf{C}}(\alpha) - \mathbf{K}(\alpha)$, we have after some algebraic manipulation:

$$\begin{bmatrix} \left(\mathbf{F}(\alpha)(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{H}^{-1}\mathbf{J}\bar{\mathbf{C}}(\alpha))\right) + (\bullet)^T + 2\gamma\mathbf{P}(\alpha) & * & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{G}(\alpha)\left(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{H}^{-1}\mathbf{J}\bar{\mathbf{C}}(\alpha)\right) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} & * \\ \bar{\mathbf{C}}_{\mathbf{z}}(\alpha) + \bar{\mathbf{D}}_{\mathbf{z}}(\alpha)\mathbf{H}^{-1}\mathbf{J}\bar{\mathbf{C}}(\alpha) & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0. \tag{54}$$

Defining $\mathbf{L} = \mathbf{H}^{-1}\mathbf{J}$ comes

$$\begin{bmatrix} \left(\mathbf{F}(\alpha)(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha))\right) + (\bullet)^{\mathbf{T}} + 2\gamma\mathbf{P}(\alpha) & * & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{G}(\alpha)\left(\bar{\mathbf{A}}(\alpha) + \bar{\mathbf{B}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha)\right) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} & * \\ \bar{\mathbf{C}}_{\mathbf{z}}(\alpha) + \bar{\mathbf{D}}_{\mathbf{z}}(\alpha)\mathbf{L}\bar{\mathbf{C}}(\alpha) & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0. \quad (55)$$

And, following the definition given in (45) and (46), we have

$$\begin{bmatrix} \mathbf{F}(\alpha)\mathcal{A}(\alpha) + (\bullet)^T + 2\gamma\mathbf{P}(\alpha) & * & * \\ \mathbf{P}(\alpha) - \mathbf{F}(\alpha)^{\mathbf{T}} + \mathbf{G}(\alpha)\mathcal{A}(\alpha) & -\mathbf{G}(\alpha) - \mathbf{G}(\alpha)^{\mathbf{T}} & * \\ \mathcal{C}(\alpha) & \mathbf{0} & -\mathbf{I} \end{bmatrix} < 0. \quad (56)$$

At this point, one should observe that the upper-left 2×2 block matrix in (56) represents the robust stability condition for $\dot{w}(t) = \mathcal{A}w(t)$, as seen in Lemma 2. Therefore, (56) is equivalently represented by

$$\begin{bmatrix} \mathcal{A}(\alpha)^T\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathcal{A}(\alpha) + 2\gamma\mathbf{P}(\alpha) & \mathcal{C}(\alpha)^T \\ \mathcal{C}(\alpha) & -\mathbf{I} \end{bmatrix} < 0. \quad (57)$$

Note that by applying the Schur complement on (57), we have

$$\mathcal{A}(\alpha)^T\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathcal{A}(\alpha) + 2\gamma\mathbf{P}(\alpha) + \mathcal{C}(\alpha)^T\mathcal{C}(\alpha) < 0$$
$$\Rightarrow \mathcal{A}(\alpha)^T\mathbf{P}(\alpha) + \mathbf{P}(\alpha)\mathcal{A}(\alpha) + \mathcal{C}(\alpha)^T\mathcal{C}(\alpha) < -2\gamma\mathbf{P}(\alpha) < 0,$$

as $\mathbf{P}(\alpha) > 0$ and $\gamma > 0$.

Finally, also by means of the Schur complement, we have that (51) is equivalent to $\mathbf{Y}(\alpha) - \mathcal{B}(\alpha)^{\mathbf{T}}\mathbf{P}(\alpha)\mathcal{B}(\alpha) > 0$. Therefore, according to Lemma 3, with $\nu = \mu^2$ we have $\mu = \sqrt{\nu} > ||\mathcal{H}(\alpha, s)||_2$, and by minimizing ν and consequently the trace of $\mathbf{Y}(\alpha)$, we minimize the system's $\mathscr{H}_2$ guaranteed cost. The proof is then finished. ■

It is important to stress that in the two-stage approach the first-stage gain matrix $\mathbf{K}(\alpha)$ can be designed using any stabilizing state-feedback control synthesis. In this paper, we consider the use of the conditions presented in (24). Investigating the efficiency of the proposed robust SOF $\mathscr{H}_2$ controller synthesis with different first-stage control design techniques are beyond the scope of this work.

Finally, as in Theorem 1, the LMI conditions in Theorem 2 are of infinite dimension. Therefore, the same procedure of defining a finite set of LMI by considering a homogeneous-polynomial structure for the decision variables using the specialized parser ROLMIP [52] is adopted for employing Theorem 2.

III. EXAMPLES

In this section, we present some examples in order to illustrate the application and benefits of our proposed approach for addressing the problem of robust stabilization of uncertain LTI systems subject to non-negligible sensor and actuator dynamics, and time delay, by means of the definition of an augmented system and the employment of a two-stage-based SOF control design. Also, we intend to show that the generalization of previous results proposed in this paper is indeed relevant for addressing complex SOF designs as the one exploited herein. The LMIs associated to the investigated problems are coded in the MATLAB software, with YALMIP interface [54], and the SDPT3 solver [55].

A. EXAMPLE 1

In this first example, we demonstrate the benefits of our proposed method. For that, we consider the control design of the lateral axis dynamics for an L-1011 aircraft. The system state-space model is adapted from [56] as

$$\dot{x}(t) = \begin{bmatrix} -2.980 & \theta & 0 & -0.034 \\ -\theta & -0.210 & 0.035 & -0.001 \\ 0 & 0 & 0 & 1 \\ 0.390 & -1.350 - 3\theta & 0 & -1.890 \end{bmatrix} x(t) + \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ -\theta \end{bmatrix} u(t), \quad (58)$$

where the four state variables $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ are the yaw rate, the sideslip angle, the bank angle and the roll rate, respectively. The control input $u(t)$ is the aileron deflection. Note that both system and input matrices are affected by an uncertain parameter θ, such that

$$-1.0 \leq \theta \leq -0.5$$

which represents the airspeed.

We assume that only the state variables $x_3(t)$ and $x_4(t)$ are measured on-line by means of two sensors with dynamics described as in (4), with

$$\mathbf{A_v} = \mathrm{diag}(-1, -1) \text{ and } \mathbf{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The control input $u(t)$ is applied through an actuator with dynamics as in (5), with

$$\mathbf{A_z} = -1 \text{ and } \mathbf{D} = 1.$$

The sensors and actuator information channels are subject to time delay in such way that the measured state information, $v(t)$, and actuator command, $u(t)$, present a time delay $\tau = \tau_s = \tau_a = 350$ ms (a realistic value considering Avionics Full Duplex Switched Ethernet (AFDX) aviation data buses [26]) before being delivered to the controller and to the system actuator, respectively.

1) SYSTEM MODELING

To apply the proposed method, we start by modeling the time delay effect using a Padé approximation of order 2. Therefore, regarding (17) with $k_s = k_a = 2$ and $\tau = \tau_s = \tau_a = 350$, we have that the state-space matrices of (12) for the delayed sensors and actuator are

- *Sensor Channel 1*

$$\mathbf{A_{sd_1}} = \begin{bmatrix} 0 & 1 \\ -59.2593 & -13.3333 \end{bmatrix}, \quad \mathbf{B_{sd_1}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\mathbf{C_{sd_1}} = \begin{bmatrix} 0 & -26.6667 \end{bmatrix}, \quad \mathbf{D_{sd_1}} = 1. \quad (59)$$

- *Sensor Channel 2*

$$\mathbf{A_{sd_2}} = \mathbf{A_{sd_1}}, \quad \mathbf{B_{sd_2}} = \mathbf{B_{sd_1}}, \mathbf{C_{sd_2}} = \mathbf{C_{sd_1}} \text{ and } \mathbf{D_{sd_2}} = \mathbf{D_{sd_1}}. \tag{60}$$

- Actuator Channel 1

$$\mathbf{A_{ad_1}} = \begin{bmatrix} 0 & 1 \\ -59.2593 & -13.3333 \end{bmatrix}, \quad \mathbf{B_{ad_1}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\mathbf{C_{ad_1}} = \begin{bmatrix} 0 & -26.6667 \end{bmatrix}, \quad \mathbf{D_{ad_1}} = 1. \tag{61}$$

Firstly, by considering that θ may only assume values within the given interval, we can represent the uncertain system (58) in terms of the convex combination of two vertices

- **Vertex 1**

$$\mathbf{A_1} = \begin{bmatrix} -2.980 & -1 & 0 & -0.034 \\ 1 & -0.210 & 0.035 & -0.001 \\ 0 & 0 & 0 & 1 \\ 0.390 & 1.650 & 0 & -1.890 \end{bmatrix}, \mathbf{B_1} = \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ 1.000 \end{bmatrix}$$

- **Vertex 2**

$$\mathbf{A_2} = \begin{bmatrix} -2.980 & -0.500 & 0 & -0.034 \\ 0.500 & -0.210 & 0.035 & -0.001 \\ 0 & 0 & 0 & 1 \\ 0.390 & 0.150 & 0 & -1.890 \end{bmatrix}, \mathbf{B_1} = \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ 0.500 \end{bmatrix}$$

defined according to the minimum and maximum values of θ, following the polytopic definition (2).

Given the two vertices $(\mathbf{A_1}, \mathbf{B_1})$, $(\mathbf{A_2}, \mathbf{B_2})$, as well as the sensor and actuator matrices, and the Padé delay model, a thirteenth-order augmented system is obtained as in (23).

2) CONTROL DESIGN

We now proceed to the design phase were we aim at computing a SOF gain $\mathbf{L}$ such that the augmented system $\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{L}\bar{\mathbf{C}}$ is asymptotic stable, relaying only on the available sensors output signal. For that, we apply the proposed two-stage HPLF-based LMI strategy for searching the desired stabilizing gain $\mathbf{L}$ with minimum decay rate specification.

To this end, we first design a stabilizing state-feedback controller by solving the LMI problem under the conditions given in (24). For this design, we consider a polynomial parameter-dependent variable $\mathbf{Z}(\alpha)$ with degree $g = 2$, and also impose a minimum first-stage decay rate specification $\beta = 0.02$. The obtained state-feedback controller is

$$\mathbf{K}(\alpha) = \alpha_1^2\mathbf{K}_1 + \alpha_1\alpha_2\mathbf{K}_2 + \alpha_2^2\mathbf{K}_3 \tag{62}$$

with

$$\mathbf{K}_1 = \big[-0.9624 \; -0.8310 \; -3.6782 \; -4.4970 \; 0.5890 \; 0.6020 \; 0.1047 \; -0.0117 \; -0.0141 \; 0.0199 \; -0.4611 \; -144.5650 \; 14.4716\big],$$
$$\mathbf{K}_2 = \big[-0.8498 \; -1.3383 \; -6.8547 \; -7.4811 \; 1.2836 \; 0.8902 \; 0.2097 \; -0.0216 \; -0.0279 \; 0.0378 \; -0.6510 \; -279.7175 \; 29.5492\big],$$
$$\mathbf{K}_3 = \big[0.0614 \; -0.76541 \; -3.4788 \; -2.8447 \; 0.7254 \; 0.2582 \; 0.1071 \; -0.0111 \; -0.0135 \; 0.0189 \; -0.5118 \; -142.6304 \; 14.3959\big].$$

In the sequence, we use $\mathbf{K}(\alpha)$ (in terms of the vertex matrices $\mathbf{K_1}, \mathbf{K_2}$, and $\mathbf{K_3}$) in the second-stage LMIs of Theorem 1, which are based on homogeneous-polynomial functions. We set the polynomial Lyapunov function $\mathbf{P}(\alpha)$ and auxiliary polynomial variables $\mathbf{F}(\alpha)$ and $\mathbf{G}(\alpha)$ to be of degree $g = 2$. In addition, by enforcing a minimum second-stage decay rate $\gamma = 0.2$, we find

$$\mathbf{L} = \begin{bmatrix} -0.5219 & -0.3148 \end{bmatrix}. \tag{63}$$

Remark 6: The decay rate bound β imposed in the first-stage design (and the resulting gain $\mathbf{K}(\alpha)$) directly impacts the feasibility in the second stage. For this particular example, by setting $\beta = 0.02$ we obtained feasibility in the second-stage with $\gamma = 0.2$. If desired, a search on β can be employed for obtaining an "optimal" maximum value for the second-stage decay rate bound γ, since with higher bounds we enforce faster transient responses [47].

3) PERFORMANCE ANALYSIS

We begin analyzing this result by emphasizing the simplicity of the obtained SOF gain. Since our method considers only the available measured system information (which, in this case, consists of the two sensors outputs), the designed feedback gain (63) is a 1×2 matrix. In a hypothetical full-state feedback implementation, the gain matrix would be of dimension 1×13, as obtained in the first-stage design (62), in order to encompass plant, sensors and actuators states, if they were possible to be measured. Moreover, such a control law would actually require the knowledge of the uncertain parameter vector α to obtain the gain matrix $\mathbf{K}(\alpha)$.

In Figure 2, we present the closed-loop time-response of system (58), with its dynamic sensors and actuators as given in (59)-(61), considering an initial condition $x(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, which represents the aircraft state after a gust perturbation [57]. The solid lines represent each of the four plant states. We can see that the SOF controller (63) enforced a stable behavior, even in with sensor and actuator delayed communication channels in $\tau = 350$ ms.

To illustrate the impact of neglecting the delay in the control design, we also plotted (dashed lines) the transient response of the closed-loop system with an SOF gain designed for an augmented system that only considers the sensors and actuator additional dynamics, as considered in [12].

Comparing both responses in this example, we can clearly see a degradation in the system performance, observed in terms of smaller damping during system transient, specially with $x_3(t)$ and $x_4(t)$ state variables (the ones used in the feedback loop).

For completing the analysis of our proposed method, we illustrate the impact of the minimum decay rate γ in the control design. For that, we present a comparison of the control design (63) – which considered a minimum decay rate constraint $\gamma = 0.2$ – with another design, carried out without imposing restrictions on the decay rate of the closed-loop system (i.e. $\gamma = 0$).

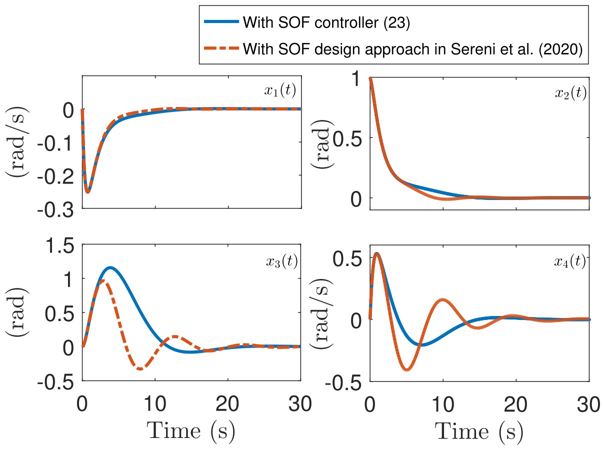

FIGURE 2. L-1011 lateral axis closed-loop dynamics with SOF design neglecting [12] and considering transport delay (SOF controller (63)).

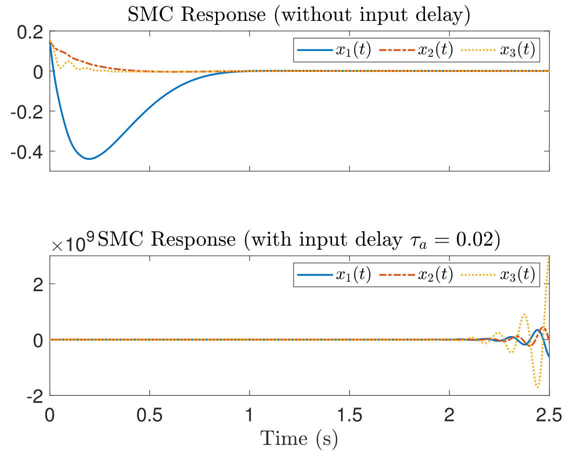

FIGURE 4. Simulation results obtained for Example 2 system considering a conventional SMC controller [58].

By employing the same procedure adopted in the synthesis of the SOF gain matrix (63), but now considering $\beta = \gamma = 0$ we obtain

$$\mathbf{L} = \begin{bmatrix} -0.2332 & -0.0502 \end{bmatrix}. \tag{64}$$

A comparison of the closed-loop responses obtained with the SOF gains (63) and (64) is presented in Figure 3 in terms of the state variables $x_3(t)$ and $x_4(t)$. One can clearly see that without imposing a minimum decay rate constraint, the closed-loop system exhibits a worse performance in terms of a longer settling-time.

Remark 7: A final yet important remark regarding Example 1 is that by considering a design approach via Corollary 1 or 2 we obtain some interesting results. If we consider a parameter-dependent first-stage design, by setting $\mathbf{Z}(\alpha)$ with degree $g = 1$ and use the obtained matrix gains in Corollary 1 we do not find a feasible solution for the same decay rate design parameters. Corollary 2 also fails in obtaining feasibility in the second-stage design. These results illustrate the benefits of considering a higher-degree Polya's relaxation associated to the HPLF approach, when compared to the polytopic method in [50], which will be more properly discussed in Example 3.

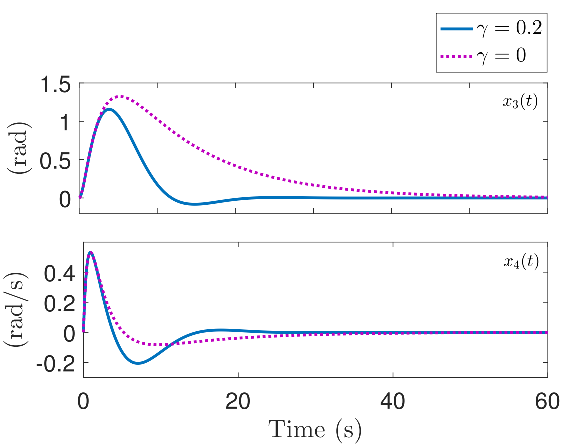

FIGURE 3. L-1011 lateral axis closed-loop dynamics with SOF design considering minimum decay rate $\gamma = 0.2$ and without minimum decay rate enforcement ($\gamma = 0$).

B. EXAMPLE 2

In this second example, we give more emphasis to the importance of considering the delay effect in the control design. To this end, we show that methods known for its robustness characteristics, such as conventional sliding mode control (SMC) techniques, suffers destabilization in the presence of time delay.

Consider an uncertain linear system such as (1), described in terms of the following vertex matrices:

- **Vertex 1**

$$\mathbf{A_1} = \begin{bmatrix} -0.277 & -32.980 & -5.432 \\ 0.365 & -0.319 & -9.490 \\ 0 & 0 & -5 \end{bmatrix}, \mathbf{B_1} = \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}$$

- **Vertex 2**

$$\mathbf{A_2} = \begin{bmatrix} -4.277 & -50 & -5.432 \\ 0.365 & -1.318 & -9.490 \\ 0 & 0 & -5 \end{bmatrix}, \mathbf{B_2} = \mathbf{B_1}$$

We consider the design of an sliding-mode controller in terms of the control law

$$u(t) = \mathbf{R}x(t) + \rho \frac{\mathbf{N}x(t)}{||\mathbf{M}x(t)|| + \delta},$$

where $\mathbf{R}$, $\mathbf{M}$, and $\mathbf{N}$ are constant matrices, and ρ a constant scalar. Such parameters are obtained through the employment of the classic SMC as described in [58], well-known for presenting robustness to several practical control issues. For the considered example, one might obtain:

$$\rho = 0.1, \quad \mathbf{R} = \begin{bmatrix} -0.58 & -7.96 & 17.09 \end{bmatrix},$$
$$\mathbf{N} = \begin{bmatrix} 0.1 & -1 & 0.1 \end{bmatrix}, \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} -0.1 & 1 & -0.1 \end{bmatrix}.$$

The constant parameter δ is a small scalar included for avoiding the chattering phenomenon, often present in sliding mode control structures [59]. For this example, we set $\delta = 10^{-2}$.

In a simulation considering an arbitrary initial condition $x_0 = \begin{bmatrix} 0.15 & 0.15 & 0.15 \end{bmatrix}^T$, the SMC controller is able to stabilize the considered system in closed-loop, yielding the transient response presented in Figure 4 (top).

However, when a time delay $\tau_a = 0.02$ is inserted in the control signal channel, such that $\dot{x} = \mathbf{A}(\alpha)x(t) + \mathbf{B}u(t - \tau_a)$, the same SMC controller is not able to maintain stability, as the simulation results for the same initial conditions show in Figure 4 (bottom).

In contrast, by employing our robust SOF method that encompasses the input signal delay in the controller design, we obtain a stabilizing controller

$$L = \begin{bmatrix} -0.0484 & -1.4265 & 2.2506 \end{bmatrix},$$

with sensor and actuator models matrices[3] $\mathbf{A_v} = diag(-100, -100, -100)$, $\mathbf{A_z} = -100$, and a 2^{nd} order delay Padé approximation model. The first- and second-stage LMI variables degrees are set as $degP = degF = degG = 2$, $degZ = 0$, and $\beta = \gamma = 0$. As we can see in Figure 5, in contrast to the SMC strategy, our method presents stability robustness with respect to the presence of time delay in the control input channel, since closed-loop transient when input delay is considered exhibits only small variations when compared to delay-free transient.

C. EXAMPLE 3

Now we aim at illustrating the efficacy of the $\mathscr{H}_2$ control strategy proposed in Theorem 2. To this, consider the uncertain continuous-time system borrowed from [60], defined in terms of polytope with vertex matrices:

- **Vertex 1**

$$\mathbf{A_1} = \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}, \ \mathbf{B_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{B_{d1}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\mathbf{C_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ \mathbf{C_{z1}} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \ \mathbf{D_{d1}} = 1$$

- **Vertex 2**

$$\mathbf{A_2} = \begin{bmatrix} 2 & -1 \\ 0 & -5 \end{bmatrix}, \ \mathbf{B_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{B_{d2}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\mathbf{C_2} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \ \mathbf{C_{z2}} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \ \mathbf{D_{d2}} = 2$$

Adapting the example for employing our proposed method, we also consider sensor and actuator with non-negligible dynamics, which are modeled as described in Section I, with $\mathbf{A_v} = -100$, $\mathbf{A_z} = -100$. We also assume that the information in both state and input channels are delayed in 30 ms, whose dynamic effects are modeled using a 2^{nd} order delay Padé approximation model.

Now, applying our two-stage procedure over the consequent augmented system, we first design a stabilizing state-feedback gain $\mathbf{K}(\alpha)$ using (24), with $\beta = 0$ and $degZ = 0$

[3]Note that we considered actuators and sensors with fast dynamics. With that, only the delay effect will have a significant impact in the controllers simulation responses, which is the objective in this comparison example.

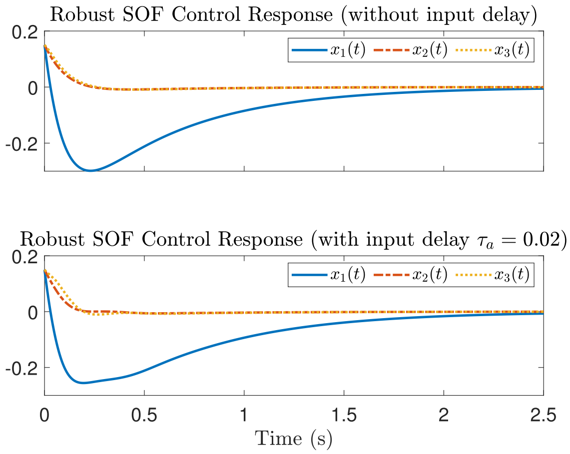

FIGURE 5. Simulation results obtained for Example 2 system considering a robust SOF controller using our proposed two-stage design via Theorem 1.

for obtaining:

$$\mathbf{K}(\alpha) = \begin{bmatrix} -1.9710 & -0.1149 & 0.9088 & -0.0047 \\ -4.7203 \times 10^{-5} & 0.8105 & -898.7575 & -7.8539 \end{bmatrix}.$$

Then, by solving the minimization problem stated in Theorem 2, with variables degrees set as $degP = degF = degG = degY = 1$ and $\gamma = 0$, we obtain a stabilizing robust SOF controller

$$\mathbf{L} = -4.3039,$$

which ensures a $\mathscr{H}_2$ guaranteed cost $\mu = 6.9858$. For comparison purposes, observe that in [60], the proposed $\mathscr{H}_2$ control strategy achieved a more conservative guaranteed cost $\mu = 8.0343$, even without the additional complexity associated to our approach, which considers delay and sensor/actuator non-negligible dynamics.

D. EXAMPLE 4

In this final example, we now aim at illustrating the benefits of considering our proposed HPLF-based two-stage robust SOF design strategy with minimum decay rate constraints for uncertain LTI systems with sensor and actuator time-delayed dynamics.

For that purpose, we performed a set of feasibility tests, which consisted in attempting to find a feasible solution for the control design of the L-1011 lateral axis dynamics (presented in Example 1) using the polytopic parameter-dependent strategy considered in [12] (Corollary 2) and the HPLF generalization proposed in this work (Theorem 1).

Note that for different ranges for the uncertain parameter θ and different values for the time delay τ we have a different control design problem. We consider that the uncertain parameter θ lies in the range specified by $-1.0 - \delta \leq \theta \leq -0.5 + \delta$, with $1.0 \leq \delta \leq 2.5$. At the same time, we consider a range of test values for the time-delay defined by $0.05 \leq \tau \leq 0.5$. Therefore, for each pair (δ, τ) in the specified ranges, we have a different problem in terms of uncertain parameter and time delay. Moreover, observe that for higher values of δ, we are assuming that the uncertain parameter belongs to a wider uncertainty range.

1) HPLF *vs.* Polytopic PDLF

In a first study, we assume that no minimum decay rate is enforced in the control designs in both stages (i.e. $\beta = \gamma = 0$). Then, for each pair (δ, τ) we seek to find a first-stage state-feedback gain[4] using (24), and then we feed the obtained controller information to the LMI problems stated in Corollary 2 and Theorem 1. In Figure 6 we present for which pairs (δ, τ) each strategy succeeded in finding a stabilizing robust SOF gain $\mathbf{L}$.

As we can clearly see, the HPLF strategy considered in this paper (Theorem 1) outperforms the polytopic PDLF approach used in [12], and proposed in [50] (Corollary 2), as Theorem 1 is able to provide a feasible solution for a larger number of problems. For this test, the variables $(\mathbf{P}, \mathbf{F}, \mathbf{G})(\alpha)$ in Theorem 1 where assumed to be polynomials with degree $g = 2$. In practical terms, the result presented in Figure 6 shows that Theorem 1 guarantees the robust SOF stabilization of the L-1011 lateral axis dynamics for a wider range of uncertainty on the airspeed parameter θ, by allowing the decision variables to be defined as homogeneous polynomials with degree higher than $g = 1$, as in [50].

In a second feasibility test, we now consider that a minimum decay rate is imposed in both stages of design. Repeating the same procedure as in the first test, the feasibility region in Figure 7 is obtained. Once again, we see that the HPLF strategy outperformed the polytopic PDLF approach used in [12]. However, a comparison with the results in Figure 6 reveals that the additional constraint of a minimum decay rates increases the difficulty of providing a stabilizing SOF gain. Indeed, the feasibility region in Figure 7 covers a smaller range of model uncertainty, which is defined by the value of δ.

2) INFLUENCE OF THE POLYNOMIAL MATRIX $\mathbf{K}(\alpha)$ DEGREE

For completing the analysis of the results proposed in our work, we present a study of the influence that the degree of the polynomial matrix $\mathbf{K}(\alpha)$, chosen in the first-stage design, exerts over the feasibility in the second stage.

New feasibility tests – on the same basis of the two previous ones – where conducted, by combining different choices for the polynomial degrees of the decision variables in both first and second stages. In the first stage, three values were tested for the degree *degZ* of the polynomial variable $\mathbf{Z}(\alpha)$, namely $degZ = 0$, $degZ = 1$, and $degZ = 2$. It is worth recalling that *degZ* also corresponds to the degree of $\mathbf{K}(\alpha)$,

[4]Since the second-stage design is sensitive to the state-feedback gain designed in the first stage, we perform both tests assuming that the first stage is executed considering the degree of $\mathbf{Z}(\alpha)$ to be equal to 0 ($degZ = 0$), enabling the use of the same gain $\mathbf{K}(\alpha) = \mathbf{K}$ in both second-stage synthesis conditions of Theorem 1 and Corollary 2 [50].

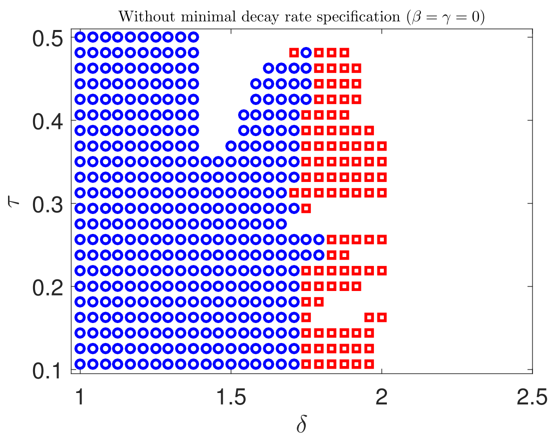

FIGURE 6. Feasibility region obtained for the L-1011 lateral axis SOF stabilization problem without imposing a minimum decay rate when applying the polytopic PDLF SOF design strategy [50] (○ - Corollary 2); and when using the HPLF extension proposed in Theorem 1 (○ and □).

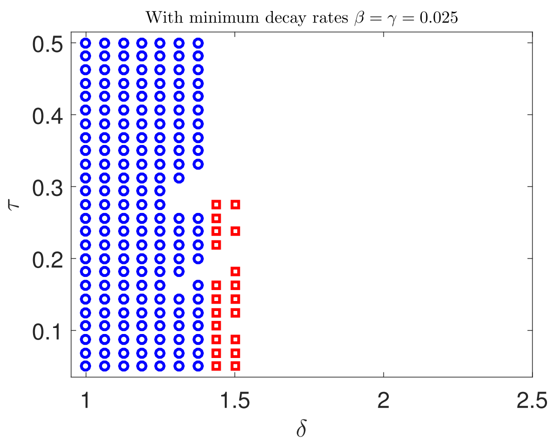

FIGURE 7. Feasibility region obtained for the L-1011 lateral axis SOF stabilization problem imposing a minimum decay rate ($\beta = \gamma = 0.025$) when applying the polytopic PDLF SOF design strategy [12] (○); and when using the HPLF extension proposed in Theorem 1 (○ and □).

as $\mathbf{K}(\alpha) = \mathbf{W}^{-1}\mathbf{Z}(\alpha)$. In the second stage, the polynomial variables $\mathbf{P}(\alpha)$, $\mathbf{F}(\alpha)$, and $\mathbf{G}(\alpha)$ were also chosen to be of the same degree *degPFG* (*i.e.* $degP = degF = degG = degPFG$), which was set to either 0, 1, or 2. For each pair $(degZ, degPFG) \in \{0, 1, 2\} \times \{0, 1, 2\}$, the feasibility of the LMI (1) in Theorem (25) was assessed. The results are presented in Figure 8.

As expected in light of the previous results, by allowing the decision variables to be homogeneous-polynomially dependent on the uncertain parameter we obtain a wider feasibility region, which increases with higher polynomial degrees.

However, the inverse result is observed regarding the polynomial degree established for the first-stage gain $\mathbf{K}(\alpha)$. As seen from both scenarios (with and without minimal decay rate specification), the best feasibility results in the second-stage design are obtained when the first-stage gain is set to be independent from the uncertain parameter. Moreover, the higher degree set for the polynomial gain matrix

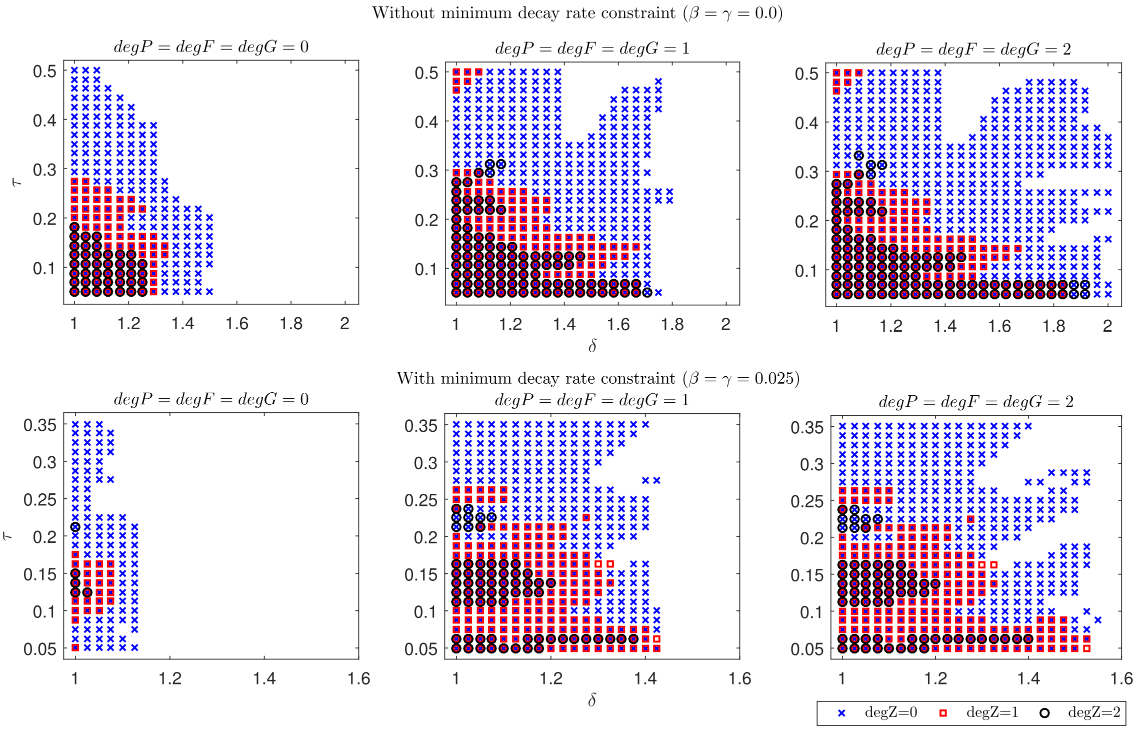

FIGURE 8. Feasibility regions obtained for the L-1011 lateral axis SOF stabilization problem for different choices of the polynomial degree of the decision variables associated to the first- and second-stage designs with (bottom charts) and without (top charts) minimum decay rate constraints.

$\mathbf{K}(\alpha)$, the smaller is the resulting feasibility region in the second stage.

IV. CONCLUSION

The proposed strategy is able to address additional sensors and actuator non-negligible dynamics subject to time delay by means of an SOF control design applied to an augmented system representation. The simulation results show the importance of considering such practical issues in the control design, attesting to the relevance of the synthesis method presented in this work. In practical terms, the results attest for the potential of the proposed approach to be applied in the control design for others attitude angles in aircraft.

By confronting the results observed in Example 1, we see that a minimum decay rate specification is valuable to improve the closed-loop dynamics. However, enforcing this specification may be a challenging issue, as the feasibility of the LMIs involved in the synthesis of the robust controller may be compromised. From Example 2, we see that even techniques well-known for presenting robustness to several practical control issues are susceptible to present undesired dynamic characteristics when neglecting the presence of time-delay effects. Furthermore, the proposed extensions for dealing with disturbance rejection via $\mathscr{H}_2$ norm optimization shows to achieve better results when compared to other strategies available in literature, as exposed in Example 3.

At last, it is important to highlight that the polytopic system modeling strategy considered in our paper not only enables the designer to consider uncertain sensor and actuator parameters, and uncertain time delays, but also that these parameters can be time-varying. Therefore, the proposed method can be directly applied to address linear parameter-varying (LPV) systems through a gain-scheduling control design. Additionally, considering the practical relevance of the nonlinear systems case, the development of an extension of the proposed LMI conditions to be applied to fuzzy Takagi-Sugeno models are in progress.

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