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RESEARCH ARTICLE

State Observer Under Multi-Rate Sensing Environment and Its Design Using l_2-Induced Norm

HIROSHI OKAJIMA[1], (Member, IEEE), YOHEI HOSOE[2], (Member, IEEE), AND TOMOMICHI HAGIWARA[2], (Senior Member, IEEE)

[1]Graduate School of Science and Technology, Kumamoto University, Kumamoto 860-0862, Japan
[2]Graduate School of Engineering, Kyoto University, Kyoto 606-8501, Japan

Corresponding author: Hiroshi Okajima (okajima@cs.kumamoto-u.ac.jp)

ABSTRACT Since the sensing period of a signal generally depends on the used sensor, it may be different from another, even in a single control system if it involves multiple sensors. This paper investigates a design problem of state observers for linear time-invariant discrete-time plants under a multi-rate sensing environment. The sensing periods of the sensors in the plant are assumed as mutually rational ratios. First, we characterize a state observer for a plant with multi-rate sensing as a periodically time-varying state observer. Then, we discuss the l_2 performance analysis of a state estimation error with the given periodically time-varying state observer. A linear matrix inequality (LMI) condition is provided for the analysis. By extending the LMI condition for analysis, we also provide that for multi-rate observer synthesis. Finally, we numerically illustrate the effectiveness of the proposed multi-rate state observer. Even if all the sensors have the same period, the sensing timing is not unique. Therefore, we numerically analyze whether the performance changes when the observation timing between multiple sensors is different.

INDEX TERMS Multi-rate observation, state observer, linear matrix inequality, l_2-induced norm, periodically time-varying system, multiple sensors.

I. INTRODUCTION

Practical control systems are usually constructed with multiple components, such as sensors and actuators. Then, even in a single control system, its components generally have different specs and may operate with different sampling periods. For example, in the track-following system in hard disk drives or optical disk drives, the sampling rate of the position error sensing is limited, and hence, the observation period of the tracking error is longer than the control period of the head arm. Systems involving visual sensors, AD converters and communication channels would also be examples in which the sensing period may be significantly longer than the control period. For this kind of systems, multi-rate control has been conventionally studied. In [1], multi-rate sampled-data stabilization for systems with time delay was studied for the case when the sampling rate of control is made faster than that of observation. A different multi-rate control scheme is proposed in [2], where multi-rate sampled-data measurements are used to preserve the stability achieved by a slow sampled-data controller. Methods of perfect tracking control on a multi-rate feedforward system are investigated for motors and electric vehicles in [3], [4], and [5]. Multi-rate control methods for systems with asynchronous measurements have also been studied (see, e.g., [6] and [7]).

Although the above earlier studies all focus on the difference between the period of observation (i.e., sensing) and that of control, a similar difference for multiple sensors is also important to deal with. In recent years, autonomous robots and automatic driving have been actively studied, and precision of the simultaneous localization and mapping (SLAM) [8], [9] is essential for their realization. SLAM has been performed using a variety of sensors, such as camera sensors, 3D-LiDAR sensors, and inertial measurement units (IMU),

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but the feasible sensing period is different in each sensor type. It is required to realize appropriate "sensor fusion" for high localization and mapping performances. If we view a plant equipped with this kind of sensors as a single-rate system and consider controlling it, the corresponding control performance becomes deteriorated because the virtual common period taken for all the components in the system has to be excessively long.

Standard classes of observers have long been widely studied in control fields [10], [11], [12], [13]. The observer-based multi-rate control system can be realized in the following two steps. In the first step, the state is estimated for each underlying discrete-time, including such a timing when some of the sensors do not observe the corresponding signals. In the second step, the control inputs of the plant are calculated using the estimated state by the multi-rate observer in the first step. Since the state at each underlying discrete-time is estimated by the multi-rate observer, a single-rate state feedback controller can be applied in the second step. This paper focuses on the synthesis of multi-rate state observers corresponding to the aforementioned first step. We develop a periodically time-varying state observer design method based on l_2-induced norm evaluations for systems with multiple observation periods with mutually rational ratios and with somewhat asynchronous timings. Observer problems with multi-rate sensing have been addressed in various studies [14], [15], [16]. In [14], the multi-rate design of a sliding-mode observer is considered where the observer processing rate is higher than the control update rate. In [15], a continuous-time state feedback controller is designed using a discretized high-gain observer for nonlinear systems. In [16], as a method of performing control with a smaller sampling period, a dual-rate observer for a system with a slow observation period of the sensor has been proposed to estimate the state with the same period as the control period. In this method, the observation rate of the output is N times slower than that of the input, and the observer is treated in the framework of a periodically time-varying system by applying the lifting method [17], [18]. An observer-based controller design problem is dealt with in [19] for a class of networked control systems with multi-rate sampling. In [20], a moving horizon estimation method of state has been studied for the case when the sampling rates of the sensors are not uniform. Other various studies about multi-rate observers are investigated in many aspects (nonlinear systems [21], continuous systems [22], asynchronous measurement [23]). It is significant to carry out research from the viewpoint of realizing a control system with a short sampling period for control. As related works about multi-rate systems, there exist various published research works about analysis and synthesis methods of periodically time-varying systems (see, e.g., [24], [25], and [26]). In [27], an analysis method of periodically time-varying sampled-data controllers based on L_p performance is considered for continuous-time systems. The L_2-induced norm of periodic linear switched systems under fast switching is provided in [28].

The different point to the previous studies about multi-rate state observers is that this paper guarantees performance about state estimation errors in the sense of the l_2-induced norm. The proposed observer structure is a simple periodically time-varying observer and has a simple structure that is easy to implement on a low-spec computer. To describe such an observer, we introduce a set of periodically time-varying observer gains. The design method of these gains is derived based on a periodically time-varying energy supply function.

This paper is organized as follows: First, we propose a multi-rate state observer structure for the given multiple-input multiple-output systems with different observation periods and timings described as periodically time-varying systems. Second, we introduce periodically-time-varying matrices for viewing the multi-rate system as a time-varying system. Furthermore, the influence of process noise and observation noise on the state estimation value is evaluated by the l_2-induced norm about a state estimation error. To this end, we consider the error system for the proposed periodically time-varying state observer and the plant. By using a time-varying energy supply function, the analysis method of the l_2-induced norm from the observation noise and the process noise to the estimation error is represented as an LMI problem. Furthermore, we also provide an LMI condition for multi-rate observer synthesis by extending the LMI condition for analysis. Finally, we evaluate the performance achieved by the proposed observer using numerical examples.

This paper could be regarded as providing advanced arguments of those developed in an article written in Japanese [29]. The article [29] handles time-varying state observer with cycling method [18], which makes it possible to view the given time-varying system virtually as a linear time-invariant system. The state observer is then designed within the framework of the linear time-invariant system and is evaluated by the l_2 induced norm from the disturbance to the state estimation error. The size of the associated LMIs obtained in this direction, however, depends on the period of the multi-rate observer and tends to be quite large. On the other hand, this paper presents a periodically time-varying energy supply function to directly handle the l_2 optimization problem for the time-varying state observer. This contributes to suppressing the LMI size to be very large. In addition, the evaluation signal in the l_2 induced norm analysis/synthesis is not limited to state estimation errors in this paper. Various types of numerical simulations are shown. In particular, this paper explicitly notes on the fact that the mutual sensing timing of multiple sensors is not unique even if their sensing periods are fixed, and the effect of the observation timing of multiple sensors is illustrated by numerical simulations in this paper.

Notation: The set of real numbers is denoted by $\mathbf{R}$, and the set of positive integers is denoted by N. The l_2-induced norm of the discrete-time system G with input u and output y is defined by

$$\|G\|_{l_2/l_2} = \sup_{u \in l_2} \frac{\|y\|_2}{\|u\|_2}, \tag{1}$$

where $\|\cdot\|_2$ represents the l_2 norm of the signal.

II. PROBLEM FORMULATION

A. PLANT AND OBSERVED OUTPUTS WITH DIFFERENT SAMPLING PERIODS

Let us consider the (underlying single-rate) discrete-time linear time-invariant multi-input multi-output (MIMO) plant P denoted by

$$x(k+1) = Ax(k) + B_u u(k) + B_d d(k), \tag{2}$$
$$y_r(k) = Cx(k), \tag{3}$$

where the nonnegative integer k denotes time, $x(k) \in \mathbf{R}^n$ is the state vector, $u(k) \in \mathbf{R}^{m_u}$ is the input vector, $d(k) \in \mathbf{R}^{m_d}$ is the process noise, and $y_r(k) \in \mathbf{R}^q$ is the vector of the outputs to which sensors are assumed to be attached. It follows that $A \in \mathbf{R}^{n \times n}$, $B_u \in \mathbf{R}^{n \times m_u}$, $B_d \in \mathbf{R}^{n \times m_d}$, $C \in \mathbf{R}^{q \times n}$. The pair (C, A) is always assumed to be observable in this paper.

On the other hand, this paper assumes such limitations on the sensor devices that the i-th entry of $y_r(k)$ is periodically measured with the sensing period $N_i \in N$ for each $i = 1, \ldots, q$. Thus, the sensing of the underlying single-rate system is with multiple rates.

To describe the multiple rate sensing in more detail and to describe the observation timing more explicitly, we introduce the periodically time-varying matrices S_k $(k = 0, 1, \ldots)$ given by

$$S_k = \mathrm{diag}[s_1(k), \ldots, s_q(k)], \tag{4}$$

with the period N, where N is the least common multiple of N_i $(i = 1, \ldots, q)$. Here, the elements $s_i(k)$, $i = 1, \ldots, q$ are defined to take either 1 or 0 as follows: $s_i(k) = 1$ if the i-th component of $y_r(k)$ is observed at time k, while $s_i(k) = 0$ otherwise. The period of S_k as a whole is N, but that of $s_i(k)$ is N_i for each i. As an example, consider Fig. 1 for the case where the plant P is a two-output system ($D := \mathrm{diag}[D_{11}, D_{12}]$ and $w(k) := [w_1(k), w_2(k)]^T$ will be explained later). Then assume that the observation periods of the two observed outputs are $N_1 = 3$ and $N_2 = 6$. Their least common multiple is $N = 6$, which becomes the system period. If both of the two components of $y_r(k)$ are assumed to be measured at the initial time $k = 0$ (as indicated in Fig. 1 with $\theta_1 = \theta_2 = 0$), then $s_i(k)$, $i = 1, 2$ are given as follows:

$$s_1(k) = \begin{cases} 1, & k = 0, 3 \\ 0, & k = 1, 2, 4, 5 \end{cases} \tag{5}$$

$$s_2(k) = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, 3, 4, 5 \end{cases} \tag{6}$$

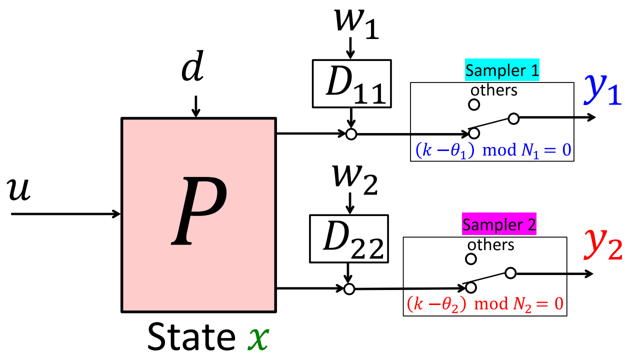

FIGURE 1. Plant and measured outputs with different sensing periods.

This implies that

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$$S_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

It also may be the case that we have no instant at which the two components of $y_r(k)$ are measured in a synchronous fashion. Indeed, the case with $\theta_1 = 0$, $\theta_2 = 1$ in Fig. 1 is such an example. Even in that case, however, we can describe the whole system by using the following $s_i(k)$.

$$s_1(k) = \begin{cases} 1, & k = 0, 3 \\ 0, & k = 1, 2, 4, 5 \end{cases} \tag{9}$$

$$s_2(k) = \begin{cases} 1, & k = 1 \\ 0, & k = 0, 2, 3, 4, 5 \end{cases} \tag{10}$$

This implies that

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{11}$$

$$S_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12}$$

More generally, once the mutual timing of the actions of the multiple sensors is determined, we can describe the observed output of the system (2) at time k by

$$y(k) = C_k x(k) + D_k w(k), \tag{13}$$

where $w(k) \in \mathbf{R}^q$ denotes the noise vector that would affect the measurement of $y_r(k)$ by $Dw(k)$, and the N-periodic matrices $C_k \in \mathbf{R}^{q \times n}$ and $D_k \in \mathbf{R}^{q \times q}$ are defined as

$$C_k = S_k C, \quad D_k = S_k D. \tag{14}$$

It would be natural that we assume $D \in \mathbf{R}^{q \times q}$ is diagonal.

Remark 1: Suppose that we are given the information that the i-th component of $y(k)$ is zero for some k. This information alone cannot determine whether the measurement of the i-th component of $y_r(k)$ has indeed occurred and is zero, or this component was not measured at time k. However, this causes no problems in the following arguments because the equation for the observer introduced in the following subsection involves not only $y(k)$ given by (13) but also the same matrices $S_k (k = 0, 1, \cdots)$ as in (14).

S_k is assumed to be given and construct a periodically time-varying observer using the observed output (13).

B. CONFIGURATION OF PERIODICALLY TIME-VARYING STATE OBSERVER

A basic idea of a state observer is to include the model of the plant and produce an estimated value $\hat{x}$ of the plant state x by emulating the behavior of the plant P. To proceed with our state observer design for multi-rate plants consisting of (2) and (13) with this idea, we confine ourselves to the structure given by

$$x_{ob}(k+1) = (A - L_k S_k C)x_{ob}(k) + B_u u(k) + L_k S_k y(k), \qquad (15)$$

where $L_k, k = 0, 1, \ldots$ are N-periodic observer gains. The remaining part of this paper is devoted to establishing a method for designing these gains in such a way that $x_{ob}(k)$ tends to the plant state $x(k)$ as $k \to \infty$. The proposed observer (15) is obviously a time-varying system.

Note that (15) with $S_0 = I, S_1 = 0, \ldots, S_{N-1} = 0$ matches the dual-sampling-rate observer in [16].

C. ERROR SYSTEM WITH PERIODICALLY TIME-VARYING STATE OBSERVER

The estimation error of the state is defined by $e(k) = x(k) - x_{ob}(k)$ for the observer (15). The error system describing the behavior of this error is given by

$$e(k+1) = A_{ek}e(k) - L_k S_k D w(k) + B_d d(k), \qquad (16)$$

where the N-periodic matrices A_{ek} are defined by $A_{ek} = A - L_k S_k C$. The N-periodic observer gain L_k should be designed so that the influence of $d(k)$ and $w(k)$ on $e(k)$ is suppressed in the above error system, while ensuring the stability of the matrix $A_e := \prod_{k=1}^{N} A_{ek}$ $(= A_{eN} \ldots A_{e1})$. More precisely, we assume that

$$d_* = [d^T, w^T]^T \in l_2, \qquad (17)$$

and consider the evaluation output

$$z(k) = We(k), \qquad (18)$$

where W is a weighting matrix. Then, for the discrete-time system from d_* to z denoted by G_z, we aim at stabilizing G_z and minimizing its l_2-induced norm through the optimal design of the N-periodic gain L_k of the state observer (15). We can evaluate the l_2 induced norm from the disturbance d_* to the state estimation error e when we set $W = I$ in (18).

III. OPTIMAL DESIGN OF TIME-VARYING GAINS L_k MINIMIZING THE l_2-INDUCED NORM

In this section, the design method of the observer gains L_k for (15) is provided based on the energy supply function. As a first step, an analysis method of the l_2-induced norm from d_* to z is characterized by the following theorem.

Theorem 1: Suppose that the error system (16) and the weighting matrix W in (18) are given. For given $\gamma > 0$, the following condition (i) holds if condition (ii) holds.

(i) The matrix A_e is Schur stable, and $\|G_z\|_{l_2/l_2} < \gamma$.
(ii) There exist N-periodic matrices $P_k > 0$ satisfying

$$\Theta_k > 0 \qquad (19)$$

for all $k = 0, \ldots, N-1$, where Θ_k is defined as

$$\Theta_k = \begin{bmatrix} P_{k-1} - A_{ek}^T P_k A_{ek} - W^T W & -A_{ek}^T P_k B_d & A_{ek}^T P_k L_k S_k D \\ -(A_{ek}^T P_k B_d)^T & \gamma^2 I - B_d^T P_k B_d & B_d^T P_k L_k S_k D \\ (A_{ek}^T P_k L_k S_k D)^T & (B_d^T P_k L_k S_k D)^T & \gamma^2 I - (L_k S_k D)^T P_k L_k S_k D \end{bmatrix}. \qquad (20)$$

with $P_{-1} = P_{N-1}$.

Proof 1: The (1, 1) block of Θ_k in (20) is positive-definite by (19). Therefore, the following condition holds for $k = 0, \ldots, N-1$.

$$P_{k-1} - A_{ek}^T P_k A_{ek} > 0 \qquad (21)$$

This implies Schur stability of $A_e := \prod_{k=1}^{N} A_{ek}$ by the result in [30].

Next, taking $\xi(k) = [e(k)^T, d(k)^T, w(k)^T]^T$, the following inequality holds from (19) for a sufficiently small $\varepsilon > 0$.

$$\xi(k)^T \Theta_k \xi(k) \geq \varepsilon \xi(k)^T \xi(k) \qquad (22)$$

By a direct computation of the left-hand side of (22) with (16) and (17), the following inequality holds for $k = 0, \ldots, N-1$.

$$e(k)^T P_{k-1} e(k) - e(k+1)^T P_k e(k+1) + \gamma^2 d_*(k)^T d_*(k) - e(k)^T W^T W e(k) \geq \varepsilon \xi(k)^T \xi(k) \qquad (23)$$

Note that S_k and L_k are given by N-periodic parameters, and (23) also holds for $k = N, N+1, \cdots$. Here, the N-periodically time-varying (in k) function

$$V_k(\chi) = \chi^T P_{k-1} \chi, \qquad (24)$$

is a positive definite function for each k, and thus can be taken as an energy supply function. From (24) and (23), the inequality

$$V_k(e(k)) - V_{k+1}(e(k+1)) + \gamma^2 d_*(k)^T d_*(k) - z(k)^T z(k) \geq \varepsilon \xi(k)^T \xi(k), \qquad (25)$$

holds for each k, where $z(k)$ is the evaluation output in (18). In (25), $V_k(e(k)) - V_{k+1}(e(k+1))$ can be regarded as a dissipation function [31].

Summing up both sides of (25) from $k=0$ to $k=K$ for $K>0$ leads to

$$V_0(e(0)) - V_{K+1}(e(K+1)) + \gamma^2 \sum_{k=0}^{K} d_*(k)^T d_*(k) - \sum_{k=0}^{K} z(k)^T z(k) \geq \varepsilon \sum_{k=0}^{K} \xi(k)^T \xi(k) \quad (26)$$

In the definition of the l_2-induced norm, the initial plant state $x(0)$ and observer state $x_{ob}(0)$ are assumed to be zero. Hence the initial error $e(0)$ is zero and thus $V_0(e(0)) = 0$ holds. Hence, for any given noise $d_* \in l_2$,

$$(\gamma^2 - \varepsilon) \sum_{k=0}^{K} d_*(k)^T d_*(k) - \sum_{k=0}^{K} z(k)^T z(k) \geq V_{K+1}(e(K+1)) \geq 0 \quad (27)$$

Consequently, by letting $K \to \infty$, we see that the l_2-induced norm of G_z is characterized by the following inequality.

$$\sup_{d_* \in l_2} \frac{\|z\|_2}{\|d_*\|_2} \leq (\gamma^2 - \varepsilon)^{1/2} < \gamma \quad (28)$$

□

The above theorem enables us to analyze the state estimation performance in terms of the l_2-induced norm of G_z by minimizing γ under condition (ii). LMIs in condition (ii) are easy to analyze numerically by using standard SDP-solver. We then extend this idea for the design problem of the observer gains L_k. When L_k are viewed as decision variables in (19), variable products exist because P_k are also decision variables. However, this can be resolved by using the traditional change of variables:

$$Y_k = P_k L_k \quad (29)$$

Then, our synthesis method of L_k based on the minimization of the l_2-induced norm from d_* to z can be summarized as in the following theorem.

Theorem 2: Suppose that the plant described by (2) and (13) and the weighting matrix W in (18) are given. For given $\gamma > 0$, the following condition (i) holds if condition (ii) holds.

(i) There exist $L_k \in \mathbf{R}^{n \times q}$, $k = 1, \ldots, N$ such that $A_e := \prod_{k=1}^{N} A_{ek}$ is Schur stable and $\|G_z\|_{l_2/l_2} < \gamma$.

(ii) There exist the matrices $P_k > 0$, Y_k, $k = 0, \ldots, N-1$ satisfying

$$\hat{\Theta}_k > 0 \quad (30)$$

for all $k = 0, \ldots, N-1$, where $\hat{\Theta}_k$ is defined as

$$\hat{\Theta}_k = \begin{bmatrix} P_k & P_kA - Y_kS_kC & P_kB_d & -Y_kS_kD \\ (P_kA - Y_kS_kC)^T & P_{k-1} - W^TW & 0 & 0 \\ (P_kB_d)^T & 0 & \gamma^2 I & 0 \\ -(Y_kS_kD)^T & 0 & 0 & \gamma^2 I \end{bmatrix}$$

with $P_{-1} = P_{N-1}$.

In particular, if (ii) holds, then

$$L_k = P_{k \bmod N}^{-1} Y_{k \bmod N} \quad (31)$$

are the time-varying gain such that $\|G_z\|_{l_2/l_2} < \gamma$ is satisfied.

Proof 2: Take $L_k = P_k^{-1} Y_k$ with the solution of the LMI (30). Then, the inequality

$$\check{\Theta}_k > 0 \quad (32)$$

holds for each $k = 0, \ldots, N-1$, where $\check{\Theta}_k$ is given as follows.

$$\check{\Theta}_k = \begin{bmatrix} P_k & P_kA_{ek} & P_kB_d & -P_kL_kS_kD \\ (P_kA_{ek})^T & P_{k-1} - W^TW & 0 & 0 \\ (P_kB_d)^T & 0 & \gamma^2 I & 0 \\ -(P_kL_kS_kD)^T & 0 & 0 & \gamma^2 I \end{bmatrix}$$

By using Schur complement with the following matrix decomposition of $\check{\Theta}_k$, we can prove that (32) is equivalent to (19).

$$\check{\Theta}_k = \left[\begin{array}{c|ccc} P_k & P_kA_{ek} & P_kB_d & -P_kL_kS_kD \\ \hline (P_kA_{ek})^T & P_{k-1} - W^TW & 0 & 0 \\ (P_kB_d)^T & 0 & \gamma^2 I & 0 \\ -(P_kL_kS_kD)^T & 0 & 0 & \gamma^2 I \end{array}\right]$$

Hence, (19) holds for all $k = 0, \ldots, N-1$. This, together with Theorem 1, implies that at least the above L_k is one such time-varying gain such that A_e becomes Schur stable and the corresponding G_z satisfies $\|G_z\|_{l_2/l_2} < \gamma$. This completes the proof. □

The time-varying observer gains L_k can be obtained by minimizing γ based on the inequality condition of (ii) of Theorem 2. We can see that there are no variable products in (30), and it is possible to solve it as an LMI problem. By using such obtained L_k, $\|G_z\|_{l_2/l_2} < \gamma$ is guaranteed. The minimization problem of γ with condition (ii) in Theorem 2 is also solvable by using standard SDP-solver. Thus, by designing L_k based on Theorem 2, we can obtain a multi-rate state observer with less influence of d_* on z in the meaning of the l_2-induced norm.

IV. SIMULATIONS

A. DESIGN EXAMPLE OF MULTI-RATE STATE OBSERVER

We illustrate the effectiveness of the multi-rate state observer by simulations. The parameters for the simulation are assumed as $m_u = 2$, $m_d = 1$, $n = 3$ and $q = 2$. Then, the plant parameters A, B_u, B_d, C and D are given by:

$$A = \begin{bmatrix} 0.95 & 0.5 & 0.2 \\ -0.1 & 0.9 & -0.2 \\ 0 & 0.1 & 0.85 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 0.5 & 2 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.1 \\ 0.3 \\ 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.5 & 0 \\ 1 & 2.5 & 0.2 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

In this subsection, N_1 and N_2 are given by 2 and 3, respectively. The least common multiple of N_1 and N_2 is 6 and we set the periodically time-varying state observer with a period of $N = 6$. In addition, $\theta_1 = \theta_2 = 0$ is assumed. Then, the structures of S_k are given as follows.

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$S_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We solve the minimization problem of γ with the conditions in Theorem 2-(ii) and obtain the time-varying observer gain L_k. A weight W is set as an identity matrix $W = I$ for Theorem 2. The number of the decision variables of LMI is 73 in total, and when (30) is specifically written down, the matrix sizes of the coalition LMIs for each k are 9×9 in the simulation setting. By minimizing γ in the LMI condition of the Theorem 2 with MATLAB function "*mincx*", $\gamma = 1.33 =: \gamma_{prop}$ is obtained, and the observer gains are given as follows:

$$\begin{aligned} L_k &= P^{-1}_{k\,mod\,6} Y_{k\,mod\,6} \\ P_0^{-1}Y_0 &= \begin{bmatrix} 0.228 & 0.294 \\ -0.125 & 0.267 \\ 0.084 & 0.110 \end{bmatrix}, P_1^{-1}Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ P_2^{-1}Y_2 &= \begin{bmatrix} 0.356 & 0 \\ 0.099 & 0 \\ 0.160 & 0 \end{bmatrix}, P_3^{-1}Y_3 = \begin{bmatrix} 0 & 0.341 \\ 0 & 0.245 \\ 0 & 0.128 \end{bmatrix}, \\ P_4^{-1}Y_4 &= \begin{bmatrix} 0.227 & 0 \\ -0.065 & 0 \\ 0.101 & 0 \end{bmatrix}, P_5^{-1}Y_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \quad (33)$$

For comparison, we consider the three different results for the dual-rate observer [16]. It is possible to design observer gain to optimize the l_2-induced norm because it can be transformed into LTI systems using the lifting technique. First, if we use only one output y_1 to estimate the state of the plant, it can be regarded as a dual-rate system with $N = N_1 = 2$(Case A). The optimal value by designing L_k is given as $\gamma_{y1} = 1.67$. On the other hand, if we use y_2 for state estimation(Case B), $\gamma_{y2} = 1.68$ is obtained by the best observer gain. Alternately, when we interpret that the sensor outputs y_1 and y_2 can be observed only for the least common multiple of N_1 and N_2, i.e. the output is regarded as 6-periodic sensor(Case C), $\gamma_{N-period} = 2.10$ is obtained by the best observer gain. Therefore, it has been confirmed from these numerical example that the state estimation performance of the proposed method ($\gamma_{prop} = 1.33$) is better than the three cases of the dual-rate observers.

Next, we simulate the time response of the estimated state to verify the effectiveness of the proposed observer. The initial states of the plant and the observer are assumed as $x(0) = x_{ob}(0) = 0$. The values of $d(k)$, $w_1(k)$ and $w_2(k)$ at each time are selected from random value from the standard normal distribution with average values $\mu_d = 0$, $\mu_{w1} = \mu_{w2} = 0$, and its standard deviations for each case are $\sigma_d = 0.2$, $\sigma_{w1} = \sigma_{w2} = 0.2$. Note that we set $d_u(k) = d_{y1}(k) = d_{y2}(k) = 0$ after $k = 500$.

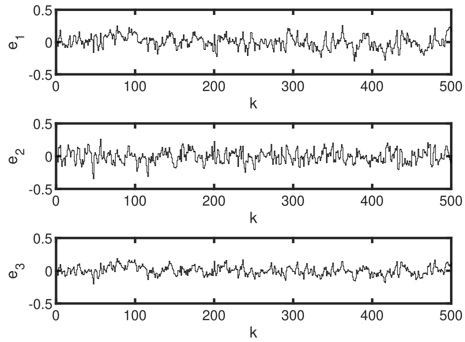

FIGURE 2. Estimated error of proposed method.

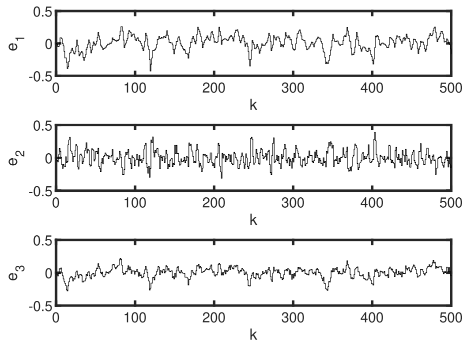

FIGURE 3. Estimated error of dual-rate observer using y_1 (Case A).

Fig. 2 shows the case with the proposed method, Fig. 3 shows the case where only y_1 can be observed (Case A), Fig. 4 shows the case where only y_2 can be observed (Case B) and Fig. 5 shows the case that the sensor is regarded as 6-periodic sensor (Case C).

The state estimation error of the proposed method is the lowest in these figures. The simulation results indicate that the use of a larger number of output signals gives better state estimation performance. From the simulation results, we evaluate a ratio from the noise to the state estimation error through

$$g_{sim} = \frac{\sqrt{\sum_{k=0}^{600} e(k)^T e(k)}}{\sqrt{\sum_{k=0}^{600} d(k)^T d(k)}} \quad (34)$$

as an estimate of the l_2 induced norm from d to e, where the summations are truncated at $k = 600$ since $e(600) \simeq 0$ can be seen because A_e is Schur stable and $d(k)$ equals zero after $k = 500$. For each method, the ratio values are $g_{sim,prop} = 0.428$, $g_{sim,y_1} = 0.735$ and $g_{sim,y_2} = 0.463$, $g_{sim,6-period} = 0.594$. We can find that $g_{sim} < \gamma$ for all methods. We can confirm that $g_{sim,prop}$ is smallest in these methods.

The results of the analysis of the performance for various combinations of N_1 and N_2 are shown in Table 1. We assume $\theta_1 = \theta_2 = 0$ and set $S_k(k)$ so that $S_0 = I$ in all combinations. From Table 1, we can confirm that the higher the frequency of observation, i.e., the shorter the period of observation, the better the state estimation performance.

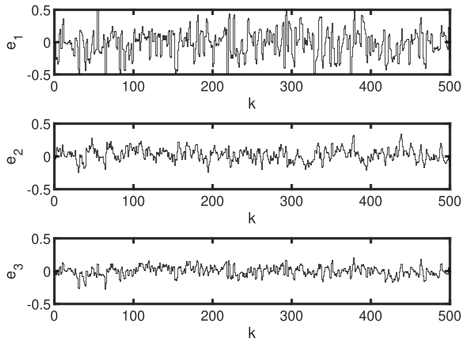

FIGURE 4. Estimated error of dual-rate observer using y_2 (Case B).

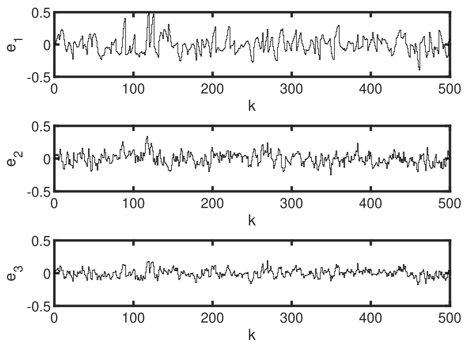

FIGURE 5. Estimated error of dual-rate observer using y_1 and y_2 as 6-periodic sensor (Case C).

B. EFFECT BY DIFFERENT MUTUAL OBSERVATION TIMING COMBINATIONS OF SENSORS

In this subsection, three different types of observation timing combinations are compared to evaluate the effects by selecting different observation timing. Two observed outputs exist and $N_1 = N_2 = 3$ is assumed. It is easy to analyze different observation timing because all we have to do is to set appropriate S_k for analyzing the system with various observation timing combinations. We confirm whether observation timing affects state estimation performance or not. Plant parameter matrices A, B, C, and D are the same as the former subsection. It means that the sampling periods of the sensor output y_1 and the sensor output y_2 are the same. In this setting, we can consider the observation timing as three cases. Case 1 is that observation timing is the same for two outputs i.e. $\theta_1 = \theta_2 = 0$. Case 2 and Case 3 are that the observation timing is different. In Case 2 and Case 3, $\theta_1 = 0, \theta_2 = 1$, $\theta_1 = 0$ and $\theta_2 = 2$ are assumed, respectively. Then, S_k $(k = 0, 1, 2)$ for each cases are given as follow:

$$\textbf{Case1}: S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{35}$$

$$\textbf{Case2}: S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{36}$$

$$\textbf{Case3}: S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{37}$$

TABLE 1. Performance by various sensing periods.

N_1,N_2	1, 1	1, 2	1, 3	2, 1	3, 1	3, 2	2, 3	3, 3
γ_{prop}	1.00	1.10	1.13	1.14	1.19	1.36	1.33	1.43

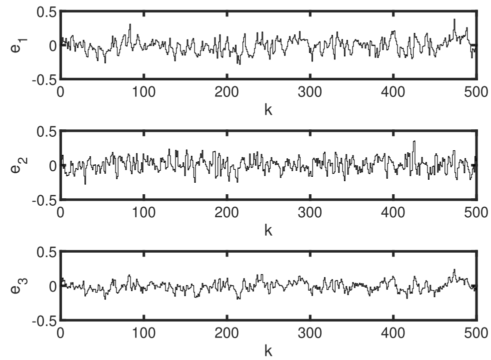

FIGURE 6. Estimation error of Case 1.

We solve the minimization problem of γ with condition (ii) in Theorem 2. Then, we obtain periodically time-varying gains L_k for the three cases. In Case 1, the l_2-induced norm is given as $\gamma_{c1} = 1.43$ by solving the LMI in (30) with (35). The time-varying gains for Case 1 are given by:

$$\begin{aligned} L_k &= P^{-1}_{k\,mod\,3} Y_{k\,mod\,3} \\ P_0^{-1}Y_0 &= \begin{bmatrix} 0.276 & 0.248 \\ -0.166 & 0.307 \\ 0.113 & 0.076 \end{bmatrix}, \; P_1^{-1}Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ P_2^{-1}Y_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

Next, we consider Case 2 and the l_2-induced norm is given as $\gamma_{c2} = 1.41$. The time-varying gains for Case 2 are given by:

$$\begin{aligned} L_k &= P^{-1}_{k\,mod\,3} Y_{k\,mod\,3} \\ P_0^{-1}Y_0 &= \begin{bmatrix} 0.348 & 0 \\ 0.083 & 0 \\ 0.155 & 0 \end{bmatrix}, \; P_1^{-1}Y_1 = \begin{bmatrix} 0 & 0.315 \\ 0 & 0.265 \\ 0 & 0.111 \end{bmatrix}, \\ P_2^{-1}Y_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{39}$$

Finally, we consider Case 3 and the l_2-induced norm is given as $\gamma_{c3} = 1.42$. The time-varying gains for Case 3 are given by:

$$\begin{aligned} L_k &= P^{-1}_{k\,mod\,3} Y_{k\,mod\,3} \\ P_0^{-1}Y_0 &= \begin{bmatrix} 0.246 & 0 \\ -0.099 & 0 \\ 0.111 & 0 \end{bmatrix}, \; P_1^{-1}Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ P_2^{-1}Y_2 &= \begin{bmatrix} 0 & 0.342 \\ 0 & 0.248 \\ 0 & 0.119 \end{bmatrix}. \end{aligned} \tag{40}$$

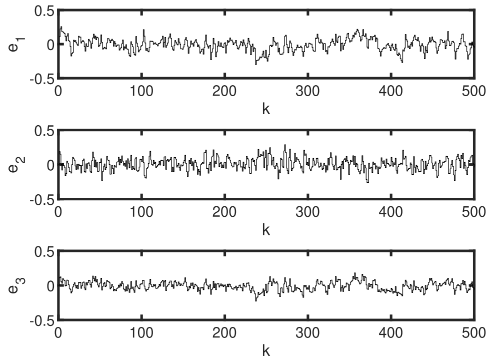

FIGURE 7. Estimation error of Case 2.

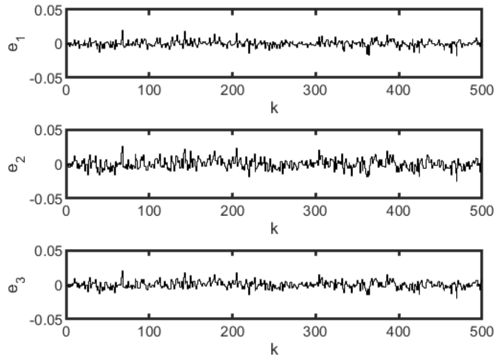

FIGURE 8. Estimation error of Case 3.

We can find that the value γ is different by changing the observation timing of the observer system. The smallest γ is obtained in Case 2. The numerical simulation results of the state estimation errors with Case 1, Case 2, and Case 3 are shown in Fig. 6, 7 and 8, respectively. We also find that gain parameters are different by changing the observation timing. From these results, we can see that how to give observation timing is important for minimizing the estimation error.

C. OBSERVER DESIGN FOR UNSTABLE PLANT

In this section, we briefly discuss observer design for an unstable plant. Simulation setting is exactly the same as in section IV-A except for the matrix A. A in this section is given as follow.

$$A = \begin{bmatrix} 0.95 & 0.5 & 0.2 \\ -0.1 & 0.9 & -0.2 \\ 0 & 0.1 & 0.85 \end{bmatrix} \tag{41}$$

In this case, the poles of A are $(1.031, 1.085 \pm 0.240\, i)$, and the plant is unstable. We solve the minimization problem of γ with the conditions in Theorem 2-(ii) and obtain the time-varying observer gain L_k as follows.

$$\begin{aligned} L_k &= P^{-1}_{k\ mod\ 6} Y_{k\ mod\ 6} \\ P_0^{-1}Y_0 &= \begin{bmatrix} -0.943 & 1.424 \\ 2.944 & -1.887 \\ -2.058 & 2.086 \end{bmatrix}, \ P_1^{-1}Y_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\ P_2^{-1}Y_2 &= \begin{bmatrix} 0.142 & 0 \\ 1.490 & 0 \\ -0.458 & 0 \end{bmatrix}, \ P_3^{-1}Y_3 = \begin{bmatrix} 0 & 0.257 \\ 0 & 0.964 \\ 0 & 0.110 \end{bmatrix}, \\ P_4^{-1}Y_4 &= \begin{bmatrix} -0.421 & 0 \\ 2.291 & 0 \\ -1.170 & 0 \end{bmatrix}, \ P_5^{-1}Y_5 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{42}$$

Then, the poles of A_e are given as $(0.1270, -0.0000, -0.0002)$ and we can see that the error system (16) is stabilized by L_k. We can confirm the multi-rate observer can be designed by Theorem 2 for the unstable plant.

V. CONCLUSION

In this paper, we proposed a design method of a periodically time-varying state observer for multi-rate systems. By introducing N-periodic matrices S_k, the multi-rate observer is regarded as a periodically time-varying system that is easy to analyze and design. Furthermore, by using the proposed time-varying energy supply function, a design method of state observer gains in the sense of the l_2-induced norm is provided as an LMI optimization problem. The proposed method can easily design state observer gains for systems that include multiple outputs with various observation periods and timing.

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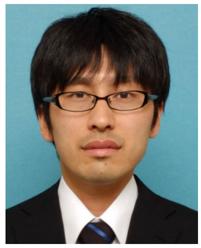

HIROSHI OKAJIMA (Member, IEEE) received the M.E. and Ph.D. degrees from Osaka University, Japan, in 2004 and 2007, respectively. He is currently an Associate Professor with Kumamoto University, Japan. His research interests include tracking control, analysis of non-minimum phase systems, and data quantization for networked systems. He is a member of SICE and ISCIE.

YOHEI HOSOE (Member, IEEE) was born in Gifu, Japan, in January 1987. He received the B.E., M.E., and Ph.D. degrees in electrical engineering from Kyoto University, Kyoto, Japan, in 2009, 2011, and 2013, respectively. Since 2013, he has been with the Department of Electrical Engineering, Kyoto University, where he has been a Junior Associate Professor, since 2020. His research interests include robust control theory, stochastic systems, and periodic systems.

TOMOMICHI HAGIWARA (Senior Member, IEEE) was born in Osaka, Japan, in March 1962. He received the B.E., M.E., and Dr.E. (Ph.D.) degrees in electrical engineering from Kyoto University, Kyoto, Japan, in 1984, 1986, and 1990, respectively. Since 1986, he has been with the Department of Electrical Engineering, Kyoto University, where he has been a Professor, since 2001. His research interests include dynamical system theory and control theory, such as analysis and design of sampled-data systems, time-delay systems, and robust control systems.