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RESEARCH ARTICLE

A Simple Procedure for the Creation of Stability Charts in Delay Parameter Space for a Class of LTI Systems With Two Delays

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ABSTRACT The central task of the stability robustness with respect to delay uncertainties lies in creating the whole stability chart (delay map) in the delay parameter space. In this paper, we present a very simple frequency-sweeping procedure for creating stability chart in the delay parameter plane for a class of linear time-invariant (LTI) two-delay systems with a delay crossing talk. Actually, the procedure based on using Rekasius pseudo-delay substitution and discriminant of quadratic polynomial to characterize the crossing frequency set. The exact and exhaustive determination of crossing frequency intervals involves only finding all positive real roots of a real-coefficient polynomial. With the availability of the entire crossing frequency set, the famous cluster treatment of characteristic roots (CTCR) paradigm is applied to construct complete stability chart in the delay plane. A by-product of the proposed procedure is the revelation of the non-zero finite frequencies corresponding to infinite pseudo-delays. These frequencies divide the crossing frequency intervals into subintervals over each of which the frequency-sweeping technique provides continuous stability crossing curves in the domain of pseudo delays. For illustration and validation, two examples are provided.

INDEX TERMS Delay map, frequency sweeping, stability crossing curves, time-delay systems.

I. INTRODUCTION

Time delays arise naturally in various feedback control systems due to the fact that the time it takes to transport mass and energy, to transmit information, to measure process variables, and to execute control laws. Also, time delays are often intentionally introduced into a control loop to stabilize a system and/or to eliminate the influences of high-frequency noises. Since the primary issue in the design of feedback controllers is to ensure the stability of the control system, the stability analysis of time-delay systems have been received continuous attention for more than six decades as evidenced by a large volume of papers and books on this topic, see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9] and the references therein.

In general, the stability analysis of time-delay systems can be carried out in the time domain or in the frequency domain. For a broad class of nonlinear, stochastic, sampled data, or time-variant systems with fixed- or time-varying delays [10], [11], [12], [13], [14], [15], [16], [17], the stability analysis is usually performed in the time domain by using the Lyapunov theory or its variants [4]. For the class of linear time-invariant continuous-time systems with time delays, the time-domain state-space model is often discretized for the stability analysis via the frequency-domain approach [9], [18], [19]. The stability of the most fundamental continuous linear time-invariant (LTI) TDSs with fixed delays is preferably studied in the frequency domain using the spectrum analysis, see [20] and the references therein.

Since the presence of time delays, irrespective of inherently presented or intentionally inserted, may stabilize or destabilize a system, many researchers have devoted their efforts to investigate the delay-induced stability/instability phenomena, and consequently, a large number of papers have been published on the stability analysis of systems affected by

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time delays (see [21], [22], [23], [24], [25], [26], [27], [28], [29], [30] and the references therein).

As it can be seen from the literature [31], [32], [33], [34], [35], [36], [37], [38], many techniques developed for the stability robustness analysis of two time-delay systems against the delay uncertainties starts by considering the systems with the characteristic quasi-polynomial of the form

$$Q(s) = A(s) + B(s)e^{-\tau_1 s} + C(s)e^{-\tau_2 s} + D(s)e^{-(\tau_1+\tau_2)s} \quad (1)$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are time delays, and $A(s)$, $B(s)$, $C(s)$, $D(s)$ are polynomials in s. There are three reasons for choosing (1) as a model system to develop a robustness analysis technique. First, characteristic equation (1) contains rich dynamics, i.e., the effects two independent delays and delay cross-talk. Second, the visualization of delay map on the delay parameter plane facilitates the explanation of the underlying concepts of the techniques. Moreover, the class of systems (1) often appear in practical applications [39], [40], [41]. A necessary and sufficient condition for system (1) to be exponentially stable is that all its characteristic roots lie in the open left-half plane (LHP) of the complex plane. If there exists any root in the right-half plane (RHP), the system is absolute unstable. Hence, according to the continuity argument [5], i.e., the roots of the parametric quasi-polynomial (1) are continuous functions of the delays, the delay parameter plane can be partitioned into regions within each of which the number of RHP characteristic roots of the corresponding quasi-polynomial (1) remains invariant. In [42], such a partition of the delay parameter space is referred to as a τ-decomposition, which is a special case of D-partition [43], [44], [45].

Although the characteristic equation (1) has an infinite number of characteristic roots, the effort for exhaustive and complete construction of the τ-decomposition boundaries on the delay parameter plane can be greatly reduced by utilizing the cluster treatment of characteristic roots (CTCR) paradigm [46], [47]. According to the CTCR paradigm, the τ-decomposition boundaries consist of kernel and offspring curves. The set of kernel curves is the set of smallest positive values of τ_1 and τ_2 which are the representative part of the boundaries while all the offspring curves can be generated by a nonlinear transformation. Under the assumption of $Q(0) \neq 0$, the boundaries that separate the τ-decomposition regions are those points which render at least a pair of complex-conjugate characteristic roots $s = \pm i\omega$, $i = \sqrt{-1}$, of (1). The set of frequencies $\omega > 0$ satisfying the condition $Q(\pm i\omega) = 0$ for some positive τ_1 and τ_2 is called the crossing frequency set. Mathematically, it is defined by

$$\Omega = \{\omega > 0 : Q(\pm i\omega; \tau_1, \tau_2) = 0\} \quad (2)$$

for some $(\tau_1, \tau_2) \in \mathbb{R}^2_+$.

It has been shown that the crossing frequency set Ω consists of a finite number of intervals of finite length [33], [47]. Hence, the exhaustive and complete determination of the crossing frequency set Ω play a crucial role in applying a frequency-sweeping technique to construct delay map in the delay parameter space. In literature, there are two main approaches for determining the crossing frequency set Ω for the frequency sweeping. The first approach, which was presented by Gu et al. [33], is based on using the unity magnitude property $\left|e^{\pm i\tau_k\omega}\right| = 1, \forall\omega > 0$, and a graphic insight to derive a set of equalities and inequalities for characterizing the crossing frequency set for two-delay systems without the delay cross-talk term, i.e. $D(s)$=0 in (1). This approach is in fact an extension of the direct method [2], [48] for single delay systems. Due to the fact that this approach is geometric in nature, it is not easy to be implemented or algorithmized. The second approach is an algebraic approach [31], [49], [50], [51], which consists of two steps: First, the Rekasius substitutions [52], [53]

$$e^{-\tau_k s} = \frac{1 - T_k s}{1 + T_k s}, k = 1, 2 \quad (3)$$

are used to convert the quasi-polynomial (1) into the following equivalent parametric algebraic polynomial

$$P(s, T_1, T_2) = (1 + T_1 s)(1 + T_2 s)\, Q(s)\Big|_{e^{-\tau_1 s}=\frac{1-T_1 s}{1+T_1 s}, e^{-\tau_2 s}=\frac{1-T_2 s}{1+T_2 s}} \quad (4)$$

Next, the resultant theory is applied to multivariable polynomials $P_R(\omega, T_1, T_2)$ and $P_I(\omega, T_1, T_2)$, which are the real and imaginary parts of (4) to determine the lower and upper bounds of the set Ω. Except for the Rekasius substitution, the half-angle transformations [31], [54]

$$e^{-i\tau_k\omega} = \frac{1 - z_k^2}{1 + z_k^2} - i\frac{2z_k}{1 + z_k^2}, z_k = \tan(\frac{\tau_k\omega}{2}), \ k = 1, 2 \quad (5)$$

are also used to convert quasi-polynomial (1) into an equivalent characteristic polynomial. In spite of the fact that the resultant-based approaches have been extended to dynamic systems with two or multiple time delays, including neutral systems [55], distributed delay systems [38], [56], and fractional-order systems [57], an exact determination of crossing frequency set is still lack. Hence, it may waste time in sweeping frequencies that lie between the lower and upper bounds but not actually belong to the crossing frequency set.

The aim of this paper is to present a simple frequency-sweeping procedure for robustness analysis of LTI systems having the characteristic quasi-polynomial of (1). Such a task is particularly indispensable for the design of robust delay-based controllers, such as Smith predictors [41], PID-deadtime controllers in [39] and [40], and tuned model controllers [58], for feedback control of time-delayed plants. The presented approach is based on the observation that the multivariable polynomials $P_R(\omega, T_1, T_2)$ and $P_I(\omega, T_1, T_2)$ are both first-degree polynomials in pseudo delay T_1 or T_2. Hence, a direct variable elimination allows one to obtain explicit ω-parametrized expressions for pseudo delays $T_1(\omega)$ and $T_2(\omega)$. Since all the mathematical derivations of the presented procedure is based on the explicit expression for the solution of a quadratic polynomial in the variable T_1 or T_2, the procedure is very simple and easy to be understood. The main contribution of the paper lies in deploying the discriminant of a quadratic polynomial to determine exactly the crossing frequency set, without invoking the polynomial resultant calculation and wasting time in sweeping frequencies that do not belong to admissible crossing frequency intervals. Moreover, as a by-product of the presented procedure, the non-zero finite frequencies corresponding to infinite pseudo-delays are also revealed. These frequencies divide the crossing frequency intervals into subintervals over each of which the frequency-sweeping technique provides continuous stability crossing curves in the domain of pseudo delays.

The rest of the article is organized as follows. Section II gives some preliminaries, including the CTCR paradigm [59], for characterizing delay maps. Our main contributions are presented in Section III. In Section IV, two case-study examples are provided to illustrate the proposed procedure and its effectiveness for depicting delay map. Finally, in Section V, conclusions are given.

II. SOME PRELIMINARY FACTS

Consider an LTI two-delay system having the characteristic function (1) with the degree of $A(s)$ being larger than those of $B(s)$, $C(s)$ and $D(s)$. Such a delay system is of retarded type and hence it is asymptotically or exponentially stable if and only if the stability abscissa of the quasi-polynomial in (1), which is defined by

$$\alpha(\tau_1, \tau_2) = \sup\{Re(s) | Q(s) = 0\} \tag{6}$$

is negative, where $Re(s)$ denotes the real part of the complex number s. As indicated by Hale [5], the abscissa $\alpha(\tau_1, \tau_2)$ is a continuous function of the delay parameters τ_1 and τ_2. This property implies that the first quadrant of the (τ_1, τ_2) plane can be partitioned into regions within each of which the corresponding characteristic function (1) has the same number of roots in the open right-half plane $\mathbb{C}_+$. This is the underlying idea of the D-partition theory [33], [34], [35] and its derivative τ-decomposition technique [32]. Hence, the set of stability crossing curves, i.e., τ-decomposition boundaries in the delay parameter plane consists of all $(\tau_1, \tau_2) \in \mathbb{R}_+^2$ such that the corresponding characteristic function (1) has at least a pair of purely imaginary roots $s = \pm i\omega$. In order to depict all τ-decomposition boundaries, it is necessary to determine the crossing frequency set Ω defined in (2).

Due to the $i2\pi$ periodicity of the exponential function, it can be seen that a frequency point $\tilde{\omega} \in \Omega$ maps to the following infinitely many delay grid point solutions of (1):

$$\begin{aligned}&\left(\tau_{1,j}(\tilde{\omega}), \tau_{2,k}(\tilde{\omega})\right)\\ &\quad= (\tilde{\tau}_1, \tilde{\tau}_2) + \left(\frac{j2\pi}{\tilde{\omega}}, \frac{k2\pi}{\tilde{\omega}}\right), \ j, k = 0, 1, \ldots\end{aligned} \tag{7}$$

where $0 < \tilde{\tau}_k\tilde{\omega} < 2\pi, k = 1, 2$. For all $\omega \in \Omega$, the solutions in (7) constitute the set of stability crossing curves (SCCs) in the parameter plane:

$$\Gamma = \{(\tau_1, \tau_2) \in \mathbb{R}_+^2 : Q(\pm i\omega) = 0 \forall \omega \in \Omega\} \tag{8}$$

The subset

$$\begin{aligned}\Gamma_{0,0} = \{&(\tau_1, \tau_2) \in \mathbb{R}_+^2 : Q(\pm i\omega) = 0,\\ &0 < \tau_k\omega < 2\pi (k = 1, 2) \forall \omega \in \Omega\}\end{aligned} \tag{9}$$

of the stability crossing curve set Γ is referred to as the kernel SCC and the stability crossing curves in the set $\Gamma \backslash \Gamma_{0,0}$ are called offspring SCCs [47].

Since the set of SSCs contains an infinite number of curve branches, it is not practical to find exhaustively and completely the entire SSC set from the solutions of the quasi-polynomial in (1). Instead, it has been shown that when $s = \pm i\omega$, the Rekasius substitution of (3) are exact if the following relations hold:

$$\tau_k = \frac{2}{\omega}\left(\tan^{-1}(T_k\omega) + l\pi\right), l = 0, 1, 2, \ldots, k = 1, 2 \tag{10}$$

where the function $\tan^{-1}(x)$ takes values in the range $[0, \pi)$ and $\omega \in \Omega$. As a result, the $s = \pm i\omega$ roots can be conveniently solved from the converted polynomial $P(s, T_1, T_2)$ in (4). Let the crossing frequency set Ω_T of the converted algebraic polynomial $P(s, T_1, T_2)$ be defined similarly as Ω in (2):

$$\Omega_T = \{\omega > 0 : P(\pm i\omega, T_1, T_2) = 0 \forall (T_1, T_2) \in R^2\} \tag{11}$$

It is noted that the identity $\Omega = \Omega_T$ holds. Moreover, the algebraic nature of the polynomial $P(s, T_1, T_2)$ implies that $\omega < \infty$. Hence, the crossing frequency set Ω consists of finite number of intervals with finite length [33], [47].

Once the crossing frequency set Ω_T has been exactly and exhaustively determined, the stability crossing set in the pseudo-delay parameter plane can be constructed as follows

$$\Upsilon_T = \left\{(T_1(\omega), T_2(\omega)) \in \mathbb{R}^2 \,\middle|\, \begin{array}{c} P(\pm i\omega, T_1, T_2) = 0 \\ \forall \omega \in \Omega_T \end{array}\right\} \tag{12}$$

From this stability crossing set, we can obtain from the relations (9) and (10) the kernel stability crossing curves in the delay plane:

$$\Gamma_{0,0} = \left\{(\tau_1(\omega), \tau_2(\omega)) \in \mathbb{R}_+^2 \,\middle|\, \begin{array}{c} \tau_k(\omega) = \frac{2\tan^{-1}(\omega T_k(\omega))}{\omega} \\ \forall (T_1(\omega), T_2(\omega)) \in \Upsilon_T, \\ 0 < \omega\tau_k(\omega) < 2\pi, k = 1, 2 \end{array}\right\} \tag{13}$$

Besides, the sets of offspring crossing curves in the delay plane are given by.

$$\begin{aligned}\Gamma_{j,k} = \Big\{&\left(\tau_1(\omega) + \frac{j2\pi}{\omega}, \tau_2(\omega) + \frac{k2\pi}{\omega}\right)\\ &\in \mathbb{R}_+^2 \,\big|\, \forall (\tau_1(\omega), \tau_2(\omega) \in \Gamma_{0,0}\Big\}\end{aligned}$$

III. MAIN RESULTS

A. CONSTRUCTION OF CROSSING FREQUENCY Set Ω

The main objective of this subsection is to present a simple procedure for revealing the exact and complete crossing frequency set Ω. As begin, we note that with the expression

(3) for Q, the converted algebraic polynomial $P(s, T_1, T_2)$ defined in (4) can be explicitly written as

$$\begin{aligned} P(s) &= (1+T_1 s)(1+T_2 s)A(s) + (1-T_1 s)(1+T_2 s)B(s) \\ &\quad + (1+T_1 s)(1-T_2 s)C(s) + (1-T_1 s)(1-T_2 s)D(s) \\ &= F_{00}(s) + F_{10}(s)T_1 + F_{01}(s)T_2 + F_{11}(s)T_1T_2 \end{aligned} \tag{14}$$

where

$$\begin{aligned} F_{00}(s) &= A(s) + B(s) + C(s) + D(s) \\ F_{10}(s) &= s\,(A(s) - B(s) + C(s) - D(s)) \\ F_{01}(s) &= s\,(A(s) + B(s) - C(s) - D(s)) \\ F_{11}(s) &= s^2\,(A(s) - B(s) - C(s) + D(s)) \end{aligned} \tag{15}$$

Let $s = i\omega$, we can express the equation $P(i\omega) = 0$ as follows:

$$\begin{aligned} P(i\omega) &= F_{00}(i\omega) + F_{10}(i\omega)T_1 + F_{01}(i\omega)T_2 \\ &\quad + F_{11}(i\omega)T_1T_2 = 0 \end{aligned} \tag{16}$$

Further, let

$$F_{jk}(i\omega) = R_{jk}(\nu) + i\omega I_{jk}(\nu),\ \nu = \omega^2, j, k = 1, 2 \tag{17}$$

and note $\omega > 0$, we can obtain from the real and imaginary parts of equation (16) the following two equations:

$$R_{00}(\nu) + R_{10}(\nu)T_1 + R_{01}(\nu)T_2 + R_{11}(\nu)T_1T_2 = 0 \tag{18a}$$
$$I_{00}(\nu) + I_{10}(\nu)T_1 + I_{01}(\nu)T_2 + I_{11}(\nu)T_1T_2 = 0 \tag{18b}$$

It is noted that, for a given $\nu = \omega^2$ the above two equations are both linear with respective to the unknowns T_1 or T_2. Hence, explicit expressions for the solutions T_1 and T_2 can be derived. To do this, we first solve (18a) for T_2:

$$T_2 = -\frac{R_{00}(\nu) + R_{10}(\nu)T_1}{R_{01}(\nu) + R_{11}(\nu)T_1},\ R_{01}(\nu) + R_{11}(\nu)T_1 \neq 0 \tag{19}$$

Substitution of this T_2 into (18b) yields the following quadratic equation for T_1:

$$a_1(\nu)T_1^2 + b_1(\nu)T_1 + c_1(\nu) = 0 \tag{20}$$

where

$$\begin{aligned} a_1(\nu) &= R_{11}(\nu)I_{10}(\nu) - R_{10}(\nu)I_{11}(\nu) \\ b_1(\nu) &= R_{11}(\nu)I_{00}(\nu) + R_{01}(\nu)I_{10}(\nu) \\ &\quad - R_{00}(\nu)I_{11}(\nu) - R_{10}(\nu)I_{01}(\nu) \\ c_1(\nu) &= R_{01}(\nu)I_{00}(\nu) - R_{00}(\nu)I_{01}(\nu) \end{aligned} \tag{21}$$

The solutions of the quadratic equation (20) can be readily written down:

$$T_{1\pm}(\nu) = \frac{-b_1(\nu) \pm \sqrt{\Delta_1(\nu)}}{2a_1(\nu)} \tag{22}$$

where the discriminant $\Delta_1(\nu)$ is given by

$$\Delta_1(\nu) = b_1^2(\nu) - 4a_1(\nu)c_1(\nu) \tag{23}$$

Similarly, if we solve (18a) for

$$T_1 = -\frac{R_{00}(\nu) + R_{01}(\nu)T_2}{R_{10}(\nu) + R_{11}(\nu)T_2},\ R_{10}(\nu) + R_{11}(\nu)T_2 \neq 0 \tag{24}$$

and substitute this solution into (18b), we obtain the following quadratic equation for T_2:

$$a_2(\nu)T_2^2 + b_2(\nu)T_2 + c_2(\nu) = 0 \tag{25}$$

where

$$\begin{aligned} a_2(\nu) &= R_{11}(\nu)I_{01}(\nu) - R_{01}(\nu)I_{11}(\nu) \\ b_2(\nu) &= R_{11}(\nu)I_{00}(\nu) + R_{10}(\nu)I_{01}(\nu) \\ &\quad - R_{01}(\nu)I_{10}(\nu) - R_{00}(\nu)I_{11}(\nu) \\ c_2(\nu) &= R_{10}(\nu)I_{00}(\nu) - R_{00}(\nu)I_{10}(\nu) \end{aligned} \tag{26}$$

The solutions of the quadratic equation (25) can be readily written down:

$$T_{2\pm} = \frac{-b_2(\nu) \pm \sqrt{\Delta_2(\nu)}}{2a_2(\nu)} \tag{27}$$

where the discriminant $\Delta_2(\nu)$ is given by

$$\Delta_2(\nu) = b_2^2(\nu) - 4a_2(\nu)c_2(\nu) \tag{28}$$

It can be verified that the following identity holds:

$$\Delta(\nu) = \Delta_2(\nu) = \Delta_1(\nu) \tag{29}$$

In the sequel, the notation $\Delta(\nu)$ without a subscript will be used to represent such a discriminant.

Now, we are at the position to state our main contributions. First, we note from (22) or (27) that no real solution exists for T_1 and T_2 when the discriminant $\Delta(\nu) < 0$ and two real solutions (counting multiplicity) of T_1 and T_2 exist if $\Delta(\nu) \geq 0$. In other words, a $\nu > 0$ with $\Delta(\nu) \geq 0$ belongs to crossing frequency set. Hence, the distinct positive real zeros of equation

$$\Delta(\nu) = 0 \tag{30}$$

which are denoted by $\tilde{\nu}_1 < \tilde{\nu}_2 < \cdots < \tilde{\nu}_m$ provide a set of critical squared frequencies N_c for the existence of real (T_1, T_2) solutions to the simultaneous equations of (18). Deleting the $\tilde{\nu}_l' s$ that satisfy the condition

$$\left.\frac{d\Delta(\nu)}{d\nu}\right|_{\nu=\tilde{\nu}_l} = 0,\ \left.\frac{d^2\Delta(\nu)}{d\nu^2}\right|_{\nu=\tilde{\nu}_l} \neq 0 \tag{31}$$

from the set N_c and relabeling the remaining elements as $\tilde{\nu}_1 < \tilde{\nu}_2 < \cdots < \tilde{\nu}_n$, we obtain a set of n frequency intervals:

$$\Omega_k = \left(\sqrt{\tilde{\nu}_{k-1}}, \sqrt{\tilde{\nu}_k}\right) := (\tilde{\omega}_{k-1}, \tilde{\omega}_k),\ k = 1, 2, \ldots, n \tag{32}$$

where $\tilde{\nu}_0 = 0$. Moreover, when ν increases continuously from $\tilde{\nu}_k - \varepsilon$ to $\tilde{\nu}_k + \varepsilon$, where ε is a sufficiently small positive number, the sign of the discriminant $\Delta(\nu)$ changes, i.e., $\Delta(\tilde{\nu}_k - \varepsilon)\Delta(\tilde{\nu}_k + \varepsilon) < 0$. With the above preparation, we have the following theorem:

Theorem 1: Let $\{\tilde{\nu}_k\}_{k=1}^n$ be a set of squared frequencies that satisfy the conditions (30) and (31), then the frequency intervals $\omega \in \Omega_k$ defined by (32) own the following properties: (i)$\Omega_n \subset \Omega$; (ii) if $\Omega_k \subset \Omega$, then $\Omega_{k-1} \cap \Omega = \emptyset$ and

$\Omega_{k+1} \cap \Omega = \emptyset$. *In other words, the crossing frequency set can be explicitly constructed as follows:*

$$\Omega = \begin{cases} \bar{\Omega}_0 \cup \bar{\Omega}_2 \cdots \cup \bar{\Omega}_n \text{ if } \exists \omega \in \Omega_0 \ni \Delta(\omega^2) > 0 \text{ if } n \text{ is even} \\ \bar{\Omega}_1 \cup \bar{\Omega}_3 \cdots \cup \bar{\Omega}_n \text{ if } \exists \omega \in \Omega_0 \ni \Delta(\omega^2) < 0 \text{ if } n \text{ is odd} \end{cases} \quad (33)$$

where $\bar{\Omega}_k$ *denotes the closure of* Ω_k.

Proof: Let ε be a sufficiently small number such that $\tilde{\nu}_n - \varepsilon \in (\tilde{\nu}_{n-1}, \tilde{\nu}_n) \triangleq \Omega_n^2$ and $\tilde{\nu}_n + \varepsilon \in (\tilde{\nu}_n, \infty) \triangleq \Omega_{n+1}^2$. Since $\tilde{\nu}_n$ is the largest ν that satisfies the conditions posed in (30) and (31), therefore, there exists no any real solution (T_1, T_2) to the equations in (18) for $\nu > \tilde{\nu}_n$. Hence, we have that $\Omega_{n+1} \cap \Omega = \emptyset$. It is noted that the condition implies that a sign change of the discriminant $\Delta(\nu)$ occurs as ν moves across a critical squared frequency point $\tilde{\nu}_k$. This property of the critical $\tilde{\nu}_k$ and the fact that the sign of the discriminant $\Delta(\omega^2)$ remains invariant for all $\omega \in \Omega_k$ allow one to conclude that if $\Omega_k \subset \Omega$ then $\Omega_{k-1} \cap \Omega = \emptyset$ and $\Omega_{k+1} \cap \Omega = \emptyset$, or conversely, if $\Omega_k \cap \Omega = \emptyset$ then $\Omega_{k-1} \subset \Omega$ and $\Omega_{k+1} \subset \Omega$. Hence, starting from the property that $\Omega_{n+1} \cap \Omega = \emptyset$ we obtain the result of (33). ∎

B. IDENTIFICATION OF CROSSING FREQUENCIES for $T_1 = \pm\infty$ and $T_2 = \pm\infty$

As shown in (19) an infinite T_2 will be obtained if the following condition holds:

$$R_{01}(\nu) + R_{11}(\nu)T_1 = 0 \quad (34)$$

Similarly, it follows from (24) that $T_1 = -\infty$ or∞ if the following condition holds:

$$R_{10}(\nu) + R_{11}(\nu)T_2 = 0 \quad (35)$$

Here it is curious to ask: what finite ν and T_1 (resp. ν and T_2) that make condition (34) (resp. (35)) hold? To the best of authors' knowledge, such a question has not previously been raised. To answer this question, we take a look at the expression in (22) for the solutions $T_{1\pm}$. From this equation it becomes quite obvious that any of the positive real roots, which are denoted by $\tilde{\nu}_l^{1\infty}, l = 1, 2, \ldots, n_{1\infty}$, of the polynomial $a_1(\nu)$ renders an infinite T_1. Since the condition (35) also implies the infinity of T_1, the value of T_2 corresponding to the squared frequency $\nu = \tilde{\nu}_l^{1\infty}$ can be obtained as

$$T_{2,l}^{1\infty} = -\frac{R_{10}(\tilde{\nu}_l^{1\infty})}{R_{11}(\tilde{\nu}_l^{1\infty})} \quad (36)$$

which comes from (35). It is noted that when the squared frequency ν moves across the point $\tilde{\nu}_l^{1\infty}$ the corresponding stability crossing curve exhibits a discontinuity which jumps from $(T_1, T_2) = (\infty, T_{2,l}^{1\infty})$ to $(-\infty, T_{2,l}^{1\infty})$ or from $(-\infty, T_{2,l}^{1\infty})$ to $(\infty, T_{2,l}^{1\infty})$.

By a similar argument, one can show that the set finite-value pairs (ν, T_1) that make T_2 positive or negative infinite are given by

$$(\tilde{\nu}_l^{2\infty}, T_{1,l}^{2\infty}) = \left(\tilde{\nu}_l^{2\infty}, -\frac{R_{01}(\tilde{\nu}_l^{2\infty})}{R_{11}(\tilde{\nu}_l^{2\infty})}\right), l = 1, 2, \ldots, n_{2\infty} \quad (37)$$

where $\tilde{\nu}_l^{2\infty}, l = 1, 2, \ldots, n_{2\infty}$, are all positive real roots of the polynomial $a_2(\nu)$ in (26).

Finally, we note that since $\{\tilde{\nu}_l^{1\infty}\}_{l=1}^{n_{1\infty}} \subset \Omega$ and $\{\nu_l^{2\infty}\}_{l=1}^{n_{2\infty}} \subset \Omega$ the points of the set $\{\tilde{\nu}_l^{1\infty}\}_{l=1}^{n_{1\infty}} \cup \{\tilde{\nu}_l^{2\infty}\}_{l=1}^{n_{2\infty}}$ subdivide a crossing frequency interval Ω_k into subintervals $\Omega_{k,j}, j = 1, 2, \ldots, n_k$. The stability crossing curve corresponding to a crossing frequency subinterval $\Omega_{k,j}$ is continuous.

IV. ILLUSTRATIVE EXAMPLES

Example 1. Consider a two-delay system having the characteristic equation (1) with [37]:

$$A(s) = s^2 + 7.1s + 21.1425$$
$$B(s) = 6s + 14.8$$
$$C(s) = 2s + 7.3$$
$$D(s) = 8$$

The converted algebraic characteristic polynomial $P(s, T_1, T_2)$ defined in (14) is

$$\begin{aligned} P(s, T_1, T_2) = {} & T_1T_2s^4 + (T_1 + T_2 - 0.9T_1T_2)s^3 \\ & + (1 + 3.1T_1 + 11.1T_2 + 7.0425T_1T_2)s^2 \\ & + (15.1 + 5.6425T_1 + 20.6425T2)s + 51.2425 \end{aligned}$$

The polynomials $R_{j,k}(\nu), I_{j,k}(\nu), j = 1, 2$ associated with the even and odd parts of $P(i\omega, T_1, T_2)$, which are defined in (16), are obtained as follows:

$$\{R_{00}, R_{01}, R_{10}, R_{11}\} = \{-\nu + 51.2425, -11.1\nu, -3.1\nu, \nu^2 - 7.042\nu\}$$
$$\{I_{00}, I_{01}, I_{10}, I_{11}\} = \{15.1, -\nu + 20.6425, -\nu + 5.6425, 0.9\nu\}$$

Following (21) and (26), we form the following two sets of polynomials

$$\begin{aligned} a_1(\nu) &= \nu(\nu^2 - 15.475\nu + 39.73731) \\ b_1(\nu) &= \nu(-24\nu + 151.1) \\ c_1(\nu) &= \nu^2 + 95.725\nu + 1057.773, \\ a_2(\nu) &= \nu(\nu^2 - 37.675\nu + 145.3748) \\ b_2(\nu) &= \nu(153.82 - 8\nu) \\ c_2(\nu) &= \nu^2 - 10.075\nu + 289.1358. \end{aligned}$$

The discriminant $\Delta(\nu)$ is then evaluated to be

$$\begin{aligned} \Delta(\nu) &= b_k^2(\nu) - 4a_k(\nu)c_k(\nu) \\ &= \nu(-4\nu^4 + 255\nu^3 - 5717.465\nu^2 \\ &\quad + 73091.96\nu - 168132.3) \end{aligned}$$

where $k = 1$ or 2. The positive real roots of this polynomial are given by

$$\tilde{\nu}_1 = 2.863417, \tilde{\nu}_2 = 38.03817$$

It can be verified that the derivative $d\Delta(\nu)/d\nu$ does not vanish when $\nu = \tilde{\nu}_1 \text{or} \tilde{\nu}_2$.

Following ***Theorem*** 1, we obtain readily the crossing frequency set as

$$\Omega = (\sqrt{\tilde{\nu}_1}, \sqrt{\tilde{\nu}_2}) = (1.69216, 6.16751)$$

The positive real roots of polynomials $a_1(\nu)$ and $a_2(\nu)$ and the corresponding (T_1, T_2) are, respectively, found to be

$$\begin{aligned}\{\tilde{\nu}_1^{1\infty}, \tilde{\omega}_1^{1\infty}, T_{2,1}^{1\infty}\} &= \{3.250675, 1.802963, \ -0.817548\}\\ \{\tilde{\nu}_2^{1\infty}, \tilde{\omega}_2^{1\infty}, T_{2,2}^{1\infty}\} &= \{12.22433, 3.496330, \ 0.598245\},\\ \{u_1^{2\infty}, \omega_1^{2\infty}, T_{1,1}^{2\infty}\} &= \{4.364194, 2.08907, -4.144410\}\\ \{u_2^{2\infty}, \omega_2^{2\infty}, T_{1,2}^{2\infty}\} &= \{33.31081, 5.771551, 0.422562\}.\end{aligned}$$

It can be seen that $\omega_1^{1\infty}, \omega_2^{1\infty}, \omega_1^{2\infty}$ and $\omega_2^{2\infty}$ all lie in the interior of the crossing frequency interval Ω, and that they divide the interval into following five subintervals:

$$\begin{aligned}\Omega_1 &= (\tilde{\omega}_1, \tilde{\omega}_1^{1\infty}) = (1.6921633, 1.802963)\\ \Omega_2 &= (\tilde{\omega}_1^{1\infty}, \tilde{\omega}_1^{2\infty}) = (1.802963, 2.08907)\\ \Omega_2 &= (\tilde{\omega}_1^{2\infty}, \tilde{\omega}_2^{1\infty}) = (2.08907, 3.496330)\\ \Omega_4 &= (\tilde{\omega}_2^{1\infty}, \tilde{\omega}_2^{2\infty}) = (3.496330, 5.771551)\\ \Omega_5 &= (\tilde{\omega}_2^{2\infty}, \tilde{\omega}_2) = (5.771551, 6.167509)\end{aligned}$$

Since for each fixed ω, the simultaneous equations in (18) have two solutions (T_{1+}, T_{2+}) and (T_{1-}, T_{2-}), performing frequency sweeping over above five frequency intervals for the stability crossing curves gives ten segments of continuous curves in the pseudo-delay plane, which are shown in Figure 1. In this figure, (T_{1+}, T_{2+}) curves are red-colored while (T_{1-}, T_{2-}) curves are blue-colored. The mapping of the stability crossing curves in Figure 1 to the kernel stability crossing curves in $(\tau_1\omega, \tau_2\omega)$-plane is shown in Figure 2. The kernel stability crossing curves in (τ_1, τ_2)-plane are shown in Figure 3. Figure 4 shows the complete delay map of the system. The numbers shown in regions of this figure represent the number of unstable characteristic roots of the quasi-polynomial $Q(s, \tau_1, \tau_2)$ when the delay points (τ_1, τ_2) locate in the corresponding region. It can be seen that the results obtained in this example are the same as those given by Sipahi & Olgac [47].

Example 2. Consider a two-delay system having the characteristic equation (1) with [33]:

$$\begin{aligned}A(s) &= (s^2 + 2s + 1)(16s^2 + 8s + 1)\\ B(s) &= 2(16s^2 + 8s + 1)\\ C(s) &= 1.5(s^2 + 2s + 1)\\ D(s) &= 0\end{aligned}$$

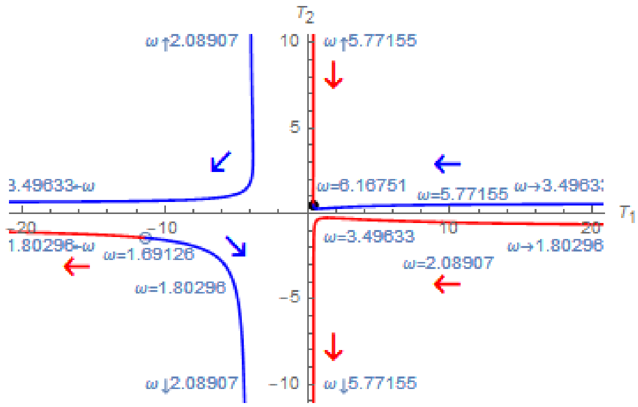

FIGURE 1. Stability crossing curves corresponding to $\Omega = (1.69216, 6.16751)$ in the pseudo-delay plane of Example 1.

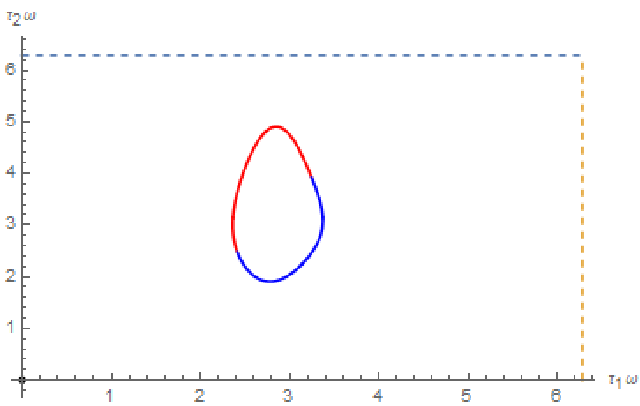

FIGURE 2. The stability crossing curves in $(\tau_1\omega, \tau_2\omega)$-plane of Example 1.

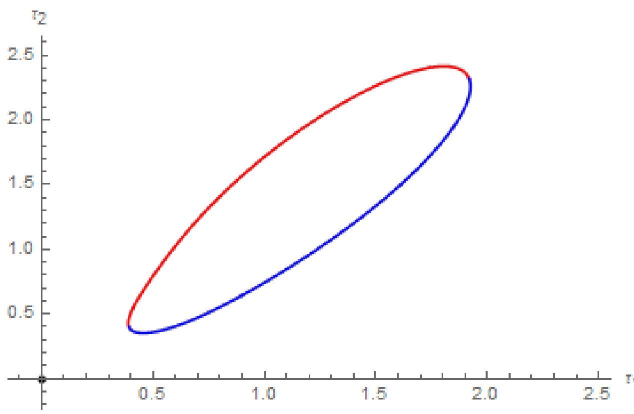

FIGURE 3. The kernel stability crossing curves in (τ_1, τ_2)-plane of Example 1.

With these polynomials, we form the following two sets of polynomials

$$\begin{cases} a_1(\nu) = 256\nu^5 + 1568\nu^4 + 446.75\nu^3 + 33.5\nu^2 - 1.25\nu) \\ b_1(\nu) = -4096\nu^3 - 512\nu^2 - 16\nu \\ c_1(\nu) = 256\nu^4 - 480\nu^3 + 2238.75\nu^2 + 281.5\nu + 6.75, \end{cases}$$

$$\begin{cases} a_2(\nu) = 256\nu^5 + 592\nu^4 - 607.75\nu^3 - 47.5\nu^2 - 3.75\nu \\ b_2(\nu) = -48\nu^3 - 96\nu^2 - 48\nu \\ c_2(\nu) = 256\nu^4 + 496\nu^3 - 793.75\nu^2 - 131.5\nu + 2.25. \end{cases}$$

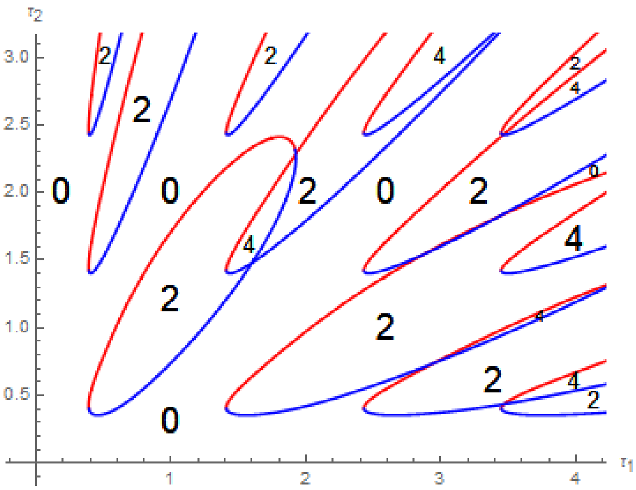

FIGURE 4. The complete stability crossing curves of Example 1.

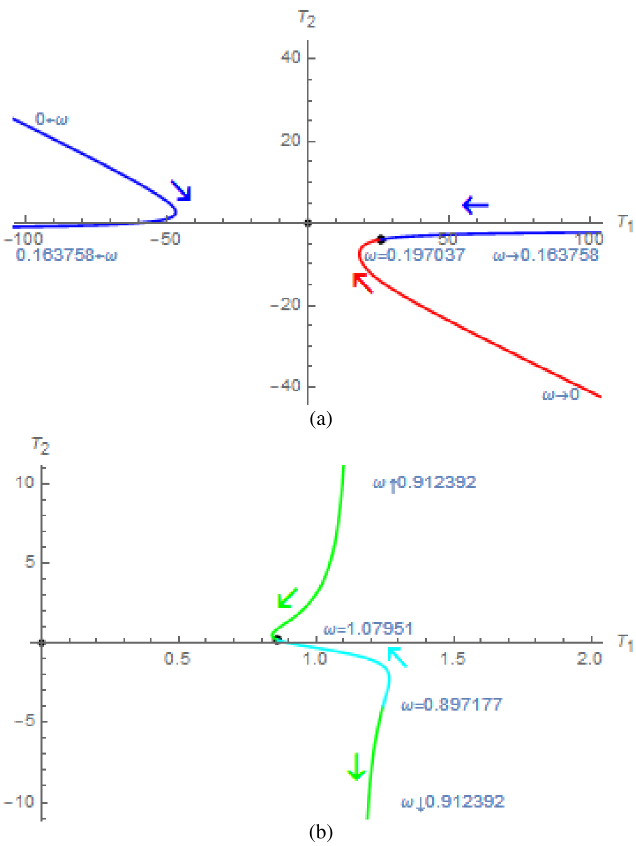

FIGURE 5. (a). The stability crossing curves in the pseudo-delay plane corresponding to the crossing frequency interval $\Omega_1 = (0, 0.197037)$ of Example 2. Figure 5(b). The stability crossing curves corresponding to $\Omega_2 = (0.897177, 1.07951)$ in the pseudo-delay plane of Example 2.

The discriminant is then evaluated to be

$$\begin{aligned}\Delta(\nu) &= b_k^2(\nu) - 4a_k(\nu)c_k(\nu)\\ &= -262144\nu^9 - 1114112\nu^8 + 260608\nu^7\\ &\quad + 3270976\nu^6 - 1514178.2\nu^5 - 456584.4\nu^4\\ &\quad - 23391.65\nu^3 + 633.6541\nu^2 + 30.092\nu\end{aligned}$$

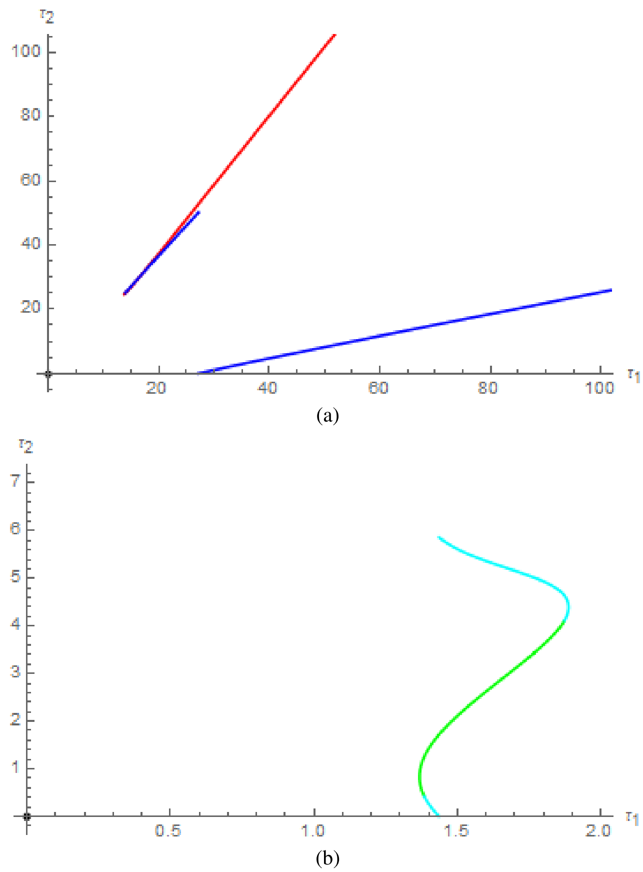

FIGURE 6. (a). The kernel stability crossing curves corresponding to $\Omega_1 = (0, 0.197037)$ in the (τ_1, τ_2)-plane of Example 2. Figure 6(b). The kernel stability crossing curves corresponding to $\Omega_2 = (0.897177, 1.07951)$ in the (τ_1, τ_2)-plane of Example 2.

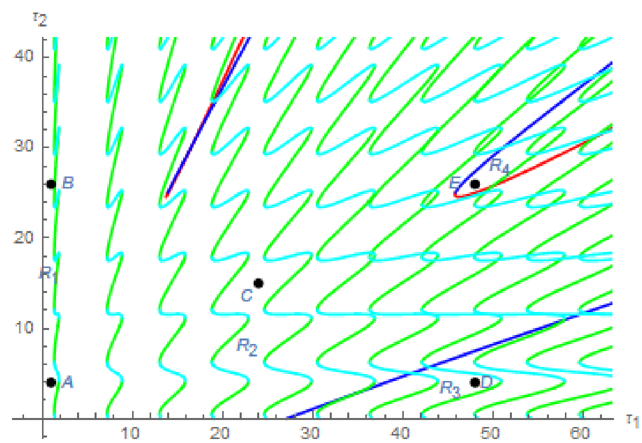

FIGURE 7. The complete stability crossing curves corresponding to $\Omega_1 = (0, 0.197037)$ and $\Omega_2 = (0.897177, 1.07951)$ of Example 2.

The positive real roots of this discriminant polynomial are given by

$$\begin{aligned}\tilde{\nu}_1 &= 0.0388234\\ \tilde{\nu}_2 &= 0.804927\\ \tilde{\nu}_3 &= 1.16533\end{aligned}$$

It can be verified that the derivative $d\Delta(\nu)/d\nu$ does not vanish when $\nu = \tilde{\nu}_1, \tilde{\nu}_2,$ or$\tilde{\nu}_3$. Following ***Theorem*** 1, we readily

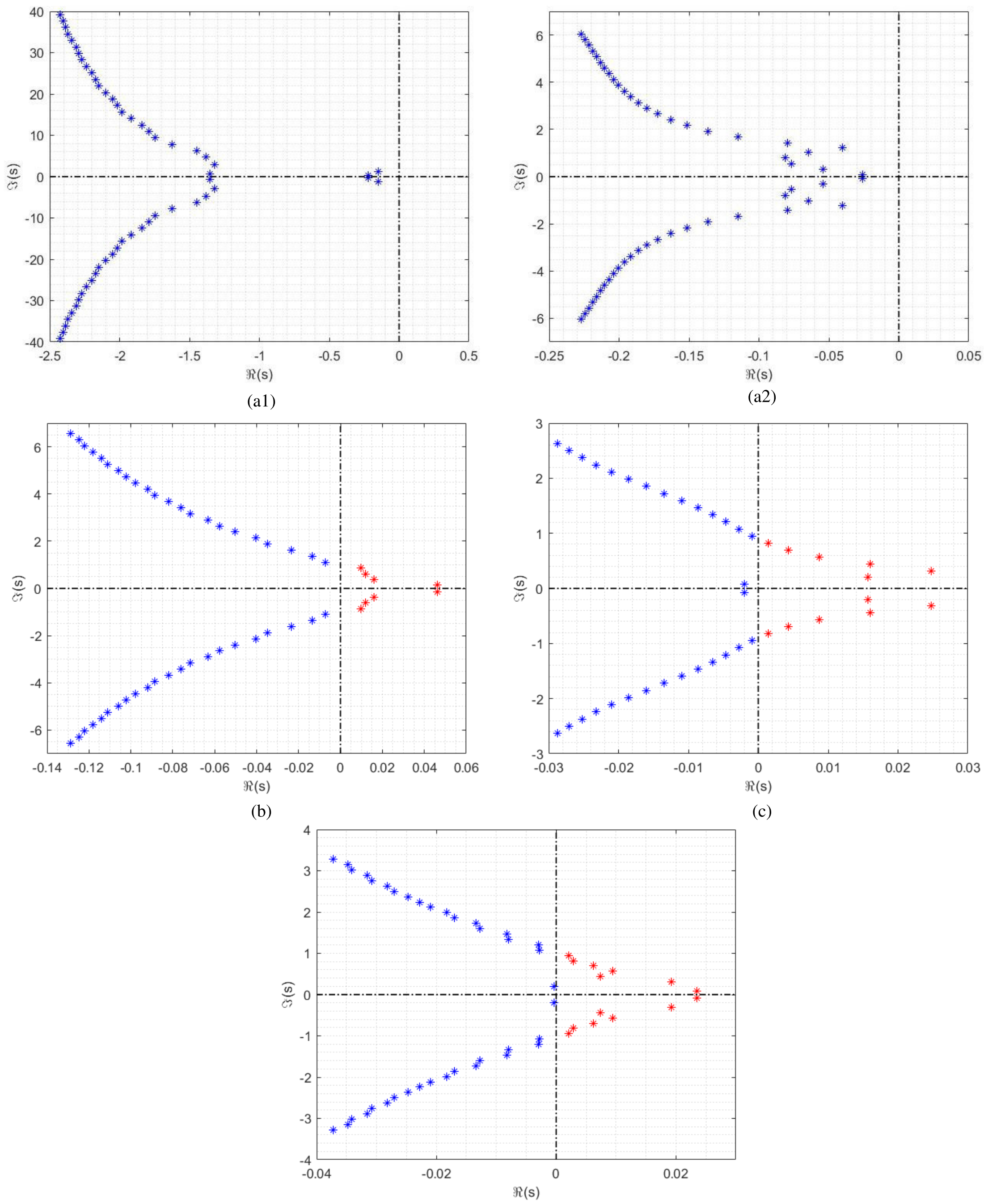

FIGURE 8. (a-1). The root distribution of $Q(s, \tau_1, \tau_2)$ with delay point A($\tau_1 = 1, \tau_2 = 4$) in region R_1 of Figure 7. Figure 8(a-2). The root distribution of $Q(s, \tau_1, \tau_2)$ with delay point B ($\tau_1 = 1, \tau_2 = 26$) in region R_1 of Figure 7. Figure 8(b). The root distribution of $Q(s, \tau_1, \tau_2)$ with delay point C ($\tau_1 = 24, \tau_2 = 15$) in region R_2 of Figure 7. Figure 8(c). The root distribution of $Q(s, \tau_1, \tau_2)$ for delay point D ($\tau_1 = 48, \tau_2 = 4$) in region R_3 of Figure 7. Figure 8(d). The root distribution of $Q(s, \tau_1, \tau_2)$ for delay point E ($\tau_1 = 48, \tau_2 = 26$) in region R_4 of Figure 7.

obtain the crossing frequency set

$$\begin{aligned} \bar{\Omega} &= \bar{\Omega}_1 \cup \bar{\Omega}_2 \\ \Omega_1 &= (0, \sqrt{\tilde{\nu}_1}) = (0, 0.197037) \\ \Omega_2 &= (\sqrt{\tilde{\nu}_2}, \sqrt{\tilde{\nu}_3}) = (0.897177, 1.07951) \end{aligned}$$

The positive real roots and the associated frequency ω and pseudo-delay T of polynomials $a_1(\nu)$ and $a_2(\nu)$ are, respectively, found to be

$$\tilde{\nu}^{1\infty} = 0.0268166, \tilde{\omega}^{1\infty} = 0.163758, T_2^{1\infty} = -1.64546$$

and

$$\tilde{\nu}^{2\infty} = 0.83246, \tilde{\omega}^{2\infty} = 0.912392, T_1^{2\infty} = 1.141375$$

Since $\tilde{\omega}^{1\infty} \in \Omega_1$ and $\tilde{\omega}^{2\infty} \in \Omega_2$, the frequency interval Ω_1 and Ω_2 are divided into subintervals:

$$\begin{aligned}
\bar{\Omega}_1 &= \bar{\Omega}_{1,1} \cup \bar{\Omega}_{1,2},\\
\Omega_{1,1} &= (\tilde{\omega}_0, \tilde{\omega}^{1\infty}) = (0, 0.163758),\\
\Omega_{1,2} &= (\tilde{\omega}^{1\infty}, \tilde{\omega}_1) = (0.163758, 0.197037).\\
\bar{\Omega}_2 &= \bar{\Omega}_{2,1} \cup \bar{\Omega}_{2,2},\\
\Omega_{2,1} &= (\tilde{\omega}_1, \tilde{\omega}^{2\infty}) = (0.897177, 0.912392),\\
\Omega_{2,2} &= (\tilde{\omega}^{2\infty}, \tilde{\omega}_3) = (0.912392, 1.07951).
\end{aligned}$$

By sweeping the frequencies over the intervals $\Omega_{1,1}$, $\Omega_{1,2}$, $\Omega_{2,1}$ and $\Omega_{2,2}$, we obtain the stability crossing curves in the pseudo-delay plane, which are shown in Figures 5(a) and 5(b) for the frequency intervals Ω_1 and Ω_2 respectively. In Figure 5(a), red-colored curves are corresponding to the solution set (T_{1+}, T_{2+}) while blue-colored ones are corresponding to the solutions sets (T_{1-}, T_{2-}). In Figure 5(b), green-colored curves are corresponding to the solution set (T_{1+}, T_{2+}) while cyan-colored ones are corresponding to the solutions sets (T_{1-}, T_{2-}). The kernel stability crossing curves in the (τ_1, τ_2)-plane are shown in Figures (6a) and (6b). Using the nonlinear transformation (9), we construct the complete stability chart in Figure 7. To verify the correctness of the constructed stability chart, we have applied the DDE-BIFTOOL to compute the actual root distributions of the characteristic quasi-polynomial with five the delay points taken form regions R_k, $k = 1, 2, 3, 4$, in Figures 8(a)-(d), respectively. The number of unstable characteristic roots corresponding to delay points A, B, C, D, and E are 0, 0, 8, 12, and 14, respectively. Let N_k denotes the number of unstable characteristic roots for delay points in region R_k. It can be verified that the number of unstable characteristic roots is increased by two the delay point moves across one stability crossing curve.

V. CONCLUSION

This paper dealt with the problem of stability robustness analysis for LTI two-delay systems with/without a delay cross-talk. More precisely, we present a procedure for constructing stability chart in the delay plane. The procedure offers the following advantages: (i) All the mathematical derivations are based on using the explicit expression for a quadratic polynomial in the pseudo-delay T_1 or T_2 with coefficients being polynomial in the frequency ω, the end points of all admissible crossing frequency intervals can be exactly identified without resorting to the use of resultant theory; (ii) It allows one to determine exactly the crossing frequency set rather than just the lower and upper bounds of the set obtained by the resultant method; (iii) The revelation of set of the non-zero finite frequencies corresponding to infinite pseudo-delays, as a by-product of the procedure, allows one to identify continuous curve segments in the pseudo-delay plane; (iv) It is extremely simple and easy to follow. These advantages have been demonstrated by two case studies.

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