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RESEARCH ARTICLE

Functional Observers for Descriptor Systems With Unknown Inputs

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ABSTRACT This paper addresses the problem of estimating a functional vector for a class of linear time-invariant descriptor systems with unknown inputs. The unknown inputs are considered in both the system dynamics and measurement equations. The proposed observer has an order less than or equal to the dimension of the functional vector to be estimated. The existence of functional ODE observers is proved under simple rank conditions on the system coefficient matrices. A few numerical examples are included to illustrate the proposed theory and designed algorithm.

INDEX TERMS Descriptor systems (differential-algebraic equations), linear systems observers, functional (partial state) observers, unknown inputs.

I. INTRODUCTION

In this paper, we study a class of linear time-invariant (LTI) descriptor systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t) + Fv(t), \tag{1a}$$
$$y(t) = Cx(t) + Gv(t), \tag{1b}$$
$$z(t) = Kx(t), \tag{1c}$$

where $E \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, $C \in \mathbb{R}^{p\times n}$, $F \in \mathbb{R}^{m\times q}$, $G \in \mathbb{R}^{p\times q}$, and $K \in \mathbb{R}^{r\times n}$ are known constant matrices. The vectors $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^k$, $v(t) \in \mathbb{R}^q$, and $y(t) \in \mathbb{R}^p$ are the semistate vector, the known input (control) vector, the unknown (immeasurable) input vector, and the (measured) output vector, respectively. Notably, unlike the standard state space theory, $x(t)$ does not satisfy the semigroup property and cannot be initialized with an arbitrary initial condition; therefore, we call $x(t)$ the semistate vector instead of the state vector. The functional vector $z(t) \in \mathbb{R}^r$ contains unmeasured (output) variables, and therefore observers are needed to estimate them. An observer that estimates z without estimating the whole semistate vector x is said to be a functional (or partial state) observer. Moreover, any functional observer becomes a full-state observer if K is the identity matrix of order n. The first order matrix polynomial $(\lambda E - A)$, in the indeterminate λ, is called matrix pencil for (1). We call the system (1) regular if $m = n$ and $\det(\lambda E - A)$ is not an identically zero polynomial in λ. In the present paper, we do not necessarily assume that the system is regular or square. Moreover, we also allow considering the cases when matrices E and A may be under or over-determined.

Descriptor systems result from mathematical modeling of physical processes where the dynamics are subject to algebraic constraints. To name but a few, descriptor systems have found extensive applications in electrical circuits, mechanical systems, and chemical engineering; for more motivation and references, we refer the readers to the books [1], [2], [3]. Due to separate independent investigations by researchers from different disciplines, there are several names synonymous with descriptor systems, viz. singular systems [1], [2], [4], generalized state space systems [5], or systems described by differential-algebraic equations (DAEs) [6], [7]. From the seminal work of El-Tohami et al. [5], the theory and design of observers for descriptor systems have been well developed. Recently, a relatively complete discussion on necessary and sufficient conditions for the existence of full-state observers for linear descriptor systems without unknown inputs has been given by Jaiswal et al. [8].

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On the other hand, many applications need only a part or combination of semistates instead of having information on the whole semistate vector. Some particular examples are feedback control, and fault or disturbance detection [9]. Functional observers eliminate the redundancy because the full-state observers may estimate even those states which are either directly measurable or are of no use. Besides the applications' viewpoint, theoretically, functional observers are essential because these can be designed under much weaker conditions than those which are necessary for the existence of full-state observers. Therefore, functional observer design is an active area of research, even in the case of standard state space systems, e.g., see [10]. Moreover, as far as the linear descriptor systems are concerned, considerable attention has also been paid to the design of functional observers. The contributions made toward such observers can be broadly classified into two approaches. In the first approach, the observers are made in DAE form itself by adding a linear correction term to the dynamics of the original descriptor system [2], [11], [12]. On the other hand, the second approach provides observers that are described only by ordinary differential equations (ODEs) [13], [14], [15], [16], [17]. Such observers are often called ODE (or Luenberger-type) observers. An ODE observer is always preferred because it can be arbitrarily initialized and is easy to implement using standard software, such as the ode45 module in MATLAB.

It is notable that, due to the appearance of input derivatives in the solutions, descriptor systems are susceptible to small changes in input variables. However, there are many practical situations where control systems arise either with noise/disturbances or inputs are not entirely accessible. Therefore, the observer design problem for descriptor systems with unknown inputs is of great significance for its theoretical and practical importance. Many results have been obtained on full-state observers for descriptor systems with unknown inputs in the last three decades; see [18], [19], [20], [21], [22], [23], [24] and references therein. On the other hand, the literature on functional observers for descriptor systems with unknown input is not very rich [12], [25], [26], [27], [28]. Berger [12] has studied descriptor systems (1) in the context of disturbance decoupled estimation and established a necessary and sufficient condition for the existence of functional DAE observers. The initial contributions on functional ODE observers [25], [26], [27] are for regular or square linear descriptor systems, and the assumptions under which the observers are designed are too restrictive. More recently, Zhang et al. [28] have proved the existence of prescribed-time functional ODE observers for the systems of the form (1) under some rank conditions, which are weaker than those in the previous works [25], [26], [27].

The main aim of the current paper is to generalize the existing results on full-state observers for linear descriptor systems with unknown inputs to the case of functional observers. There are two main reasons for rendering such generalizations. First, the functional observers can be designed with considerably weaker assumptions. Second, functional observers can have significantly lower order than full-state observers. The observer design approach used in the current paper is purely algebraic and hence, easily implementable. The main features of the approach are that (i) it is not restricted to the square descriptor systems; (ii) the formulated functional observer is governed by ODEs only; (iii) it handles the presence of unknown inputs in both the dynamics of semistates and the outputs; (iv) the used existence conditions are milder than the previous existing works on functional ODE observers; and (v) the proposed observer may have reduced order less than the dimension of the functional vector to be estimated. The above fourth and fifth features of the approach make this paper unique to the earlier works [12], [25], [26], [27], [28]. Notably, the order of the realized observers is equal to the dimension of the functional vector in [15], [16], [17], [25], [26], [27], and [28]. However, the functional vector z, in general, may contain information available in the output vector y. It is further to note that the article [28] describes an observer design method under the intended necessary and sufficient conditions. However, in the current paper, we show that the conditions established in [28] are still restrictive and need not be necessary in the case of asymptotic observers.

The paper is organized as follows. Section II starts with the problem statement and collects some preliminary results used in this article. Section III contains the main contribution and provides a less restrictive set of conditions for the existence of functional ODE observers. Here, the proposed observer has an order less or equal to the dimension of the functional vector to be estimated. Two numerical examples are given to illustrate the observer design procedure in Section IV. Finally, Section V concludes the article.

We use the following notations: 0 and I, respectively, stand for appropriate dimensional zero and identity matrices. For more clarity, the identity matrix of size $n \times n$ is sometimes denoted by I_n. All missing blocks are zero matrices of appropriate dimensions in a block-partitioned matrix. The symbols Row(A), $A^\top$, and A^+ represent the row space, the transpose, and the Moore-Penrose inverse (MP-inverse) of any matrix A, respectively. $\mathbb{C}$ denotes the set of complex numbers and $\bar{\mathbb{C}}^+ := \{\lambda \in \mathbb{C} \mid Re(\lambda) \geq 0\}$. A matrix pencil $(\lambda E - A)$ is said to have *normal-rank* s if $\text{rank}(\lambda E - A) = s$, for all finite $\lambda \in \mathbb{C}$ except some individual values. A block diagonal matrix having matrices X, Y, and Z on its main diagonal is represented by blk-diag(X, Y, Z). The notion $\hat{z}(t) \to z(t)$ as $t \to \infty$ means that $\lim_{t\to\infty} \text{ess sup}_{[t,\infty)} ||\hat{z}(t) - z(t)|| = 0$.

II. PROBLEM STATEMENT AND PRELIMINARIES

In this paper, we propose a method to design an observer of the following form:

$$\dot{w}(t) = Nw(t) + \begin{bmatrix} H & L \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \quad w(t) \in \mathbb{R}^l \tag{2a}$$

$$\hat{z}(t) = Rw(t) + \begin{bmatrix} M_1 & M_2 \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \tag{2b}$$

Any observer of the above form is called an ODE observer because (2a) is a system of ODEs only. Throughout the study, it is assumed that the system designer has already defined the system (1) in such a way that we have an input $u(t)$ and a certain (unknown) initial condition $Ex(0-)$ such that the solution set of (1) is nonempty. Such a pair of $Ex(0-)$ and $u(t)$ is called an admissible pair for (1), see also [8, Definition 4]. The tuple $(x, u, v, y, z) \in \mathscr{L}^1_{loc}(\mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^r)$ is said to be a solution of (1) if it satisfies (1) for almost all $t \in \mathbb{R}$ and $Ex \in \mathcal{AC}(\mathbb{R} \to \mathbb{R}^n)$, where $\mathscr{L}^1_{loc}$ and $\mathcal{AC}$ represent the set of locally Lebesgue integrable functions and the set of absolutely continuous functions, respectively. We denote the solution set of (1) by $\mathscr{B}$, which is also called the behavior of (1) in [6] and [29], where the authors have used $\mathscr{B}$ to define various solution and observability concepts for (1). We now exploit $\mathscr{B}$ to define functional ODE observer for (1)

Definition 1: System (2) is said to be a functional ODE observer for (1), if for every $(x, u, v, y, z) \in \mathscr{B}$, we have a solution $(w, u, y, \hat{z})$ of (2) such that $\lim_{t\to\infty} ess\ sup_{[t,\infty)} ||\hat{z}(t) - z(t)|| = 0$.

The non-negative integer l in (2a) is called the order of the observer. If $l < r$, the observer is called a reduced-order observer. Moreover, If $l = 0$, the observer is expressed only by (2b) and is called a static observer.

The main problem that we have considered in the current article is to design a functional ODE observer (2) for a given system (1). Mathematically, the problem is to design matrices N, H, L, R, M_1, and M_2 of appropriate dimensions such that $\hat{z}(t) \to z(t)$ as $t \to \infty$.

We now recall the following fundamental results of matrix theory from our recent work [15]. All these results will be used in the sequel of the current paper.

Lemma 1: System $XA = B$ is solvable for the unknown X if and only if $\operatorname{rank}\begin{bmatrix} A \\ B \end{bmatrix} = \operatorname{rank} A$, *equivalently, $BA^+A = B$. Moreover,*

$$X = BA^+ - Z(I - AA^+),$$

where Z is an arbitrary matrix of appropriate dimension.

Lemma 2: Let $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{p\times n}$, and $\mathbb{F}$ be any field. Then for a consistent system in unknowns x and z: $Ax = b$ and $z = Bx$, the vector $z \in \mathbb{F}^p$ can be uniquely determined if and only if $\operatorname{rank}\begin{bmatrix} A \\ B \end{bmatrix} = \operatorname{rank} A$.

Lemma 3: Let X, Y, and Z be any matrices of compatible dimensions. If X has full row rank and/or Z has full column rank, then

$$\operatorname{rank}\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} = \operatorname{rank} X + \operatorname{rank} Z.$$

Lemma 4: Let X and Y be any two matrices of compatible dimensions. Then $\operatorname{rank}(XY) = \operatorname{rank} Y$ *if and only if the matrix* $\begin{bmatrix} X \\ I - YY^+ \end{bmatrix}$ *has full column rank.*

Lemma 5: Let A, B, C, and D be any matrices of compatible dimensions such that $\operatorname{rank}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A & B \end{bmatrix}$, *then* $\operatorname{rank}\begin{bmatrix} A \\ C \end{bmatrix} = \operatorname{rank} A$ *and* $\operatorname{rank}\begin{bmatrix} B \\ D \end{bmatrix} = \operatorname{rank} B$.

The following lemma is motivated by the work of Jaiswal et al. [17].

Lemma 6: Let $E \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times k}$, and $F \in \mathbb{R}^{m\times q}$. Then there exist two orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$UEV = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \quad UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ 0 & A_{32} \end{bmatrix}, \tag{3a}$$

$$UB = \begin{bmatrix} B_{11} \\ B_{21} \\ 0 \end{bmatrix}, \ and UF = \begin{bmatrix} F_{11} \\ F_{21} \\ 0 \end{bmatrix}, \tag{3b}$$

where

1) *E_{11} has full row rank,*
2) $\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix}$ *has full row rank,*
3) *A_{32} has full column rank.*

Proof: Take the matrix quadruple (E, A, B, F) and compute an orthogonal row compression matrix $U_1 \in \mathbb{R}^{m\times m}$ of $\begin{bmatrix} E & B & F \end{bmatrix}$ such that

$$U_1 \begin{bmatrix} E & B & F \end{bmatrix} = \begin{bmatrix} E_1 & B_1 & F_1 \\ 0 & 0 & 0 \end{bmatrix},$$

where matrix $\begin{bmatrix} E_1 & B_1 & F_1 \end{bmatrix}$ has full row rank, say r_1. Denote

$$U_1 A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{r_1\times n}$, and compute an orthogonal column compression matrix $V_1 \in \mathbb{R}^{n\times n}$ of A_2 such that

$$A_2 V_1 = \begin{bmatrix} 0 & A_{32} \end{bmatrix},$$

where $A_{32} \in \mathbb{R}^{m-r_1\times c_1}$ has full column rank. Denote $E_1 V_1 = \begin{bmatrix} \tilde{E}_1 & \bar{E}_1 \end{bmatrix}$, where $\tilde{E}_1 \in \mathbb{R}^{r_1\times n-c_1}$ and further compute an orthogonal row compression $U_2 \in \mathbb{R}^{r_1\times r_1}$ of $\tilde{E}_1$ such that

$$U_2 \tilde{E}_1 = \begin{bmatrix} E_{11} \\ 0 \end{bmatrix},$$

where E_{11} has full row rank. Thus, the desired decomposition is obtained by taking the orthogonal matrices

$$U = \begin{bmatrix} U_2 & 0 \\ 0 & I_{m-r_1} \end{bmatrix} U_1 \text{ and } V = V_1.$$

□

We end this section by recalling the following fundamental result from standard state space control systems theory.

Definition 2 [30]: The matrix pair (A, C) is called detectable if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \quad \text{for all } \lambda \in \bar{\mathbb{C}}^+.$$

Equivalently, a matrix Z of compatible dimension exists such that the matrix $A - ZC$ is Hurwitz (stable).

III. OBSERVER DESIGN

By setting the notations

$$\Gamma := \begin{bmatrix} E & A & B & 0 & 0 & F & 0 & 0 \\ 0 & E & 0 & A & 0 & 0 & F & 0 \\ 0 & 0 & 0 & E & A & 0 & 0 & F \\ 0 & 0 & 0 & C & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & G \\ 0 & 0 & 0 & 0 & K & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi := \begin{bmatrix} E & A & B & 0 & 0 & F & 0 & 0 \\ 0 & E & 0 & A & 0 & 0 & F & 0 \\ 0 & 0 & 0 & E & A & 0 & 0 & F \\ 0 & 0 & 0 & C & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & G \\ 0 & 0 & 0 & 0 & K & 0 & 0 & 0 \\ 0 & 0 & 0 & K & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$\Omega := \begin{bmatrix} E & A & B & 0 & 0 & F & 0 & 0 \\ 0 & E & 0 & A & 0 & 0 & F & 0 \\ 0 & 0 & 0 & E & A & 0 & 0 & F \\ 0 & 0 & 0 & C & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & C & 0 & 0 & G \\ 0 & 0 & 0 & K & \lambda K & 0 & 0 & 0 \end{bmatrix},$$

we assume that system (1) satisfies the rank conditions:

$$\operatorname{rank}\Gamma = \operatorname{rank}\Psi, \tag{4}$$

and

$$\forall\,\lambda \in \bar{\mathbb{C}}^{+}:\ \operatorname{rank}\Gamma = \operatorname{rank}\Omega. \tag{5}$$

Remark 1: The assumptions (4) - (5) are milder than the conditions used in earlier works on functional ODE observers for linear descriptor systems with unknown inputs [25], [26], [27], [28]. Thus, the current paper covers a larger class of descriptor systems, and this fact will be explained again in Section IV by considering a numerical example.

In order to design observers, we first transform system (1) into a new coordinate system by using the matrices U and V in Lemma 6. In view of the decomposition (3), system (1) can be written as

$$\begin{aligned} E_{11}\dot{x}_1 + E_{12}\dot{x}_2 &= A_{11}x_1 + A_{12}x_2 + B_{11}u + F_{11}v, && (6a)\\ E_{22}\dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_{21}u + F_{21}v, && (6b)\\ 0 &= A_{32}x_2, && (6c)\\ y &= C_1x_1 + C_2x_2 + Gv, && (6d)\\ z &= K_{11}x_1 + K_{12}x_2, && (6e) \end{aligned}$$

where $x = V\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, $KV = \begin{bmatrix} K_{11} & K_{12} \end{bmatrix}$, and the number of columns in C_1 and K_{11} are the same as in E_{11}. The fact that the matrix A_{32} has full column rank implies $x_2 = 0$ and hence system (6) reduces to

$$\begin{aligned} E_{11}\dot{x}_1 &= A_{11}x_1 + B_{11}u + F_{11}v, && (7a)\\ \bar{y} &= \bar{C}_1x_1 + G_1v, && (7b)\\ z &= K_{11}x_1, && (7c) \end{aligned}$$

where $\bar{y} = \begin{bmatrix} -B_{21}u \\ y \end{bmatrix}$, $\bar{C}_1 = \begin{bmatrix} A_{21} \\ C_1 \end{bmatrix}$, and $G_1 = \begin{bmatrix} F_{21} \\ G \end{bmatrix}$. Let $\operatorname{rank}(G_1) = q_1$. Then, by using the singular value decomposition (SVD), there exist two orthogonal matrices U_3 and V_3 such that

$$U_3G_1V_3 = \begin{bmatrix} I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

Therefore, premultiplying (7b) by U_3 and assuming that $v = V_3\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, we obtain

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} x_1 + \begin{bmatrix} I_{q_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix},$$

where $U_3\bar{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ and $U_3\bar{C}_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$. Thus, system (7) can be rewritten as

$$\begin{aligned} E_{11}\dot{x}_1 &= \Phi x_1 + B_{11}u + F_{12}y_1 + F_{13}v_2, && (9a)\\ y_2 &= C_{21}x_1, && (9b)\\ z &= K_{11}x_1, && (9c) \end{aligned}$$

where $F_{11}V_3 = \begin{bmatrix} F_{12} & F_{13} \end{bmatrix}$ and $\Phi = A_{11} - F_{12}C_{11}$. It is notable that the functional vector z in (9) is precisely the same as in (1).

If $\operatorname{rank}\begin{bmatrix} C_{21} \\ K_{11} \end{bmatrix} \neq \operatorname{rank} C_{21} + \operatorname{rank} K_{11}$, then some part of z is already known form the output y_2. Consequently, there exist a permutation matrix P and two matrices S_{11} and $\bar{P}$ of appropriate dimensions such that $\operatorname{Row}(S_{11}) \cap \operatorname{Row}(C_{21}) = \{0\}$, $\operatorname{Row}(S_{11}) \subset \operatorname{Row}(K_{11})$, S_{11} has full row rank (say l), and

$$K_{11} = P\begin{bmatrix} S_{11} \\ \bar{P}C_{21} \end{bmatrix}.$$

Therefore, (9) reduces to

$$\begin{aligned} E_{11}\dot{x}_1(t) &= \Phi x_1(t) + B_{11}u(t) + F_{12}y_1(t) + F_{13}v_2(t), && (10a)\\ y_2(t) &= C_{21}x_1(t), && (10b)\\ z(t) &= P\begin{bmatrix} S_{11} \\ \bar{P}C_{21} \end{bmatrix} x_1(t) = P\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}, && (10c) \end{aligned}$$

where $z_1(t) = S_{11}x_1(t)$ and $z_2(t) = \bar{P}y_2(t)$. Since z_2 is exactly known from the output y_2, it is sufficient to design a functional observer for z_1 to estimate the functional vector z. It is notable that if

$$\operatorname{rank}\begin{bmatrix} C_{21} \\ K_{11} \end{bmatrix} = \operatorname{rank} C_{21} + \operatorname{rank} K_{11},$$

then $S_{11} = K_{11}$, $P = I$, and $\bar{P}$ is an empty matrix.

Now, we propose the following system as a functional ODE observer for (10):

$$\begin{aligned} \dot{w}(t) &= Nw(t) + TB_{11}u(t) + TF_{12}y_1(t) + Ly_2(t), && (11a)\\ \hat{z}(t) &= Rw(t) + My_2(t), && (11b) \end{aligned}$$

where $w(t) \in \mathbb{R}^l$, $R = P\begin{bmatrix} I_l \\ 0 \end{bmatrix}$, $M = P\begin{bmatrix} \bar{M} \\ \bar{P} \end{bmatrix}$, and $\bar{M}$ is a matrix of appropriate dimension to be determined.

It is notable that the above observer for z in (10) is also a functional ODE observer for z in (1). In the following theorem, we prove that there exist matrices N, T, L, and $\bar{M}$ such that (11) is a functional ODE observer for (1) if some linear matrix equations are consistent and the matrix N is Hurwitz.

Theorem 1: The observer (11) estimates the functional $z(t)$ in (1) if the following conditions hold:

$$\begin{bmatrix} T & \bar{M} & Q & N \end{bmatrix} \Sigma = \Theta, \tag{12a}$$

$$\text{and } N \text{ is Hurwitz}, \tag{12b}$$

where $Q = N\bar{M} - L$, $\Theta = \begin{bmatrix} S_{11} & 0 & 0 \end{bmatrix}$, *and* $\Sigma = \begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & -S_{11} & 0 \end{bmatrix}$.

Proof: Let $e = \hat{z} - z$ and $e_1 = w - TE_{11}x_1$. Then

$$\begin{aligned} e &= Rw + My_2 - K_{11}x_1 \\ &= Rw + P\begin{bmatrix} \bar{M} \\ \bar{P} \end{bmatrix} y_2 - P\begin{bmatrix} S_{11} \\ \bar{P}C_{21} \end{bmatrix} x_1 \\ &= Re_1 + R(TE_{11} + \bar{M}C_{21} - S_{11})x_1 \end{aligned} \tag{13}$$

and

$$\begin{aligned} \dot{e}_1 &= \dot{w} - TE_{11}\dot{x}_1 \\ &= Ne_1 + (NTE_{11} + LC_{21} - T\Phi)x_1 - TF_{13}v_2. \end{aligned} \tag{14}$$

Thus from (13) and (14), we obtain that $e \to 0$ as $t \to \infty$ if the following conditions hold

$$TE_{11} + \bar{M}C_{21} = S_{11}, \tag{15a}$$

$$NTE_{11} + LC_{21} - T\Phi = 0, \tag{15b}$$

$$TF_{13} = 0, \tag{15c}$$

$$\text{and } N \text{ is Hurwitz}. \tag{15d}$$

Here, it is notable that Eq. (15b) is nonlinear in the unknowns. To make it linear, we substitute (15a) in (15b) and obtain

$$T\Phi + QC_{21} - NS_{11} = 0, \tag{16}$$

where $Q = N\bar{M} - L$. Clearly, Eqs. (15a), (15c), and (16) can be rewritten as (12a). This completes the proof. □

Now, we transform the assumptions (4) and (5) from system (1) to system (10). Define

$$\begin{aligned} \tilde{U}_1 &= \text{blk-diag}(U, U, U, I_p, I_p, I_r), \\ \tilde{V}_1 &= \text{blk-diag}(V, V, I_k, V, V, I_q, I_q, I_q), \\ \tilde{U}_2 &= \text{blk-diag}(I_{m_1}, U_3, U_3, I_r), \\ \tilde{V}_2 &= \text{blk-diag}(I_{n_1}, I_{n_1}, V_3, V_3), \end{aligned}$$

where $E_{11} \in \mathbb{R}^{m_1 \times n_1}$. Since the rank of a matrix does not change by pre- and post-multiplication of invertible matrices. Therefore,

$$\text{rank}\,\Gamma = \text{rank}\,\tilde{U}_1\Gamma\tilde{V}_1. \tag{17}$$

We now write Γ in terms of the system coefficients E, A, B, F, C, and G in the right hand side of (17). Then Lemma 3 and Lemma 6 infer that

$$\text{rank}\,\Gamma = \text{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\,\text{rank}\,A_{32} + \text{rank}\,E_{11} + \text{rank}\,\Gamma_0, \tag{18}$$

where $\Gamma_0 = \begin{bmatrix} E_{11} & A_{11} & 0 & F_{11} \\ \bar{C}_1 & 0 & G_1 & 0 \\ 0 & \bar{C}_1 & 0 & G_1 \\ 0 & K_{11} & 0 & 0 \end{bmatrix}$.

Again, to simplify the rank of the matrix Γ_0, we use the fact that $\text{rank}\,\Gamma_0 = \text{rank}(\tilde{U}_2\Gamma_0\tilde{V}_2)$ and perform the following operations:

1) Substitute decomposition (8), $F_{11}V_3 = \begin{bmatrix} F_{12} & F_{13} \end{bmatrix}$, $U_3\bar{C}_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$ in $\tilde{U}_2\Gamma_0\tilde{V}_2$ and obtain

$$\tilde{U}_2\Gamma_0\tilde{V}_2 = \begin{bmatrix} E_{11} & A_{11} & 0 & 0 & F_{12} & F_{13} \\ C_{11} & 0 & I_{q_1} & 0 & 0 & 0 \\ C_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{11} & 0 & 0 & I_{q_1} & 0 \\ 0 & C_{21} & 0 & 0 & 0 & 0 \\ 0 & K_{11} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2) Perform elementary row operations due to pre-multiply the matrix $\tilde{U}_2\Gamma_0\tilde{V}_2$ by

$$\begin{bmatrix} I & & & -F_{12} & & \\ & I & & & & \\ & & I & & & \\ & & & I & & \\ & & & & I & \\ & & & & & I \end{bmatrix}.$$

3) Use Lemma 3 twice for full rank matrices I_{q_1} and obtain

$$\begin{aligned} \text{rank}\,\Gamma_0 &= \text{rank}(\tilde{U}_2\Gamma_0\tilde{V}_2) \\ &= 2q_1 + \text{rank}\begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & K_{11} & 0 \end{bmatrix}. \end{aligned}$$

4) Substitute $K_{11} = P\begin{bmatrix} S_{11} \\ \bar{P}C_{21} \end{bmatrix}$, use the facts P is a permutation matrix and

$$\text{rank}\,\Gamma_0 = 2q_1 + \text{rank}\begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & S_{11} & 0 \\ 0 & \bar{P}C_{21} & 0 \end{bmatrix}.$$

5) Perform elementary row operations due to pre-multiply the right most matrix in the above expression by

$$\begin{bmatrix} I & & & & \\ & I & & & \\ & & I & & \\ & & & I & \\ & & -\bar{P} & & I \end{bmatrix}.$$

Therefore, we obtain

$$\operatorname{rank}\Gamma = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\operatorname{rank}A_{32} + \operatorname{rank}E_{11} + 2q_1 + \operatorname{rank}\Gamma_1, \quad (19)$$

where $\Gamma_1 = \begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & S_{11} & 0 \end{bmatrix}$.

By the similar arguments which are used to obtain the rank identity (19), it is easy to obtain that

$$\operatorname{rank}\Psi = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\operatorname{rank}A_{32} + \operatorname{rank}E_{11} + 2q_1 + \operatorname{rank}\Psi_1 \quad (20)$$

and

$$\operatorname{rank}\Omega = \operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} & F_{11} \\ 0 & E_{22} & B_{21} & F_{21} \end{bmatrix} + 3\operatorname{rank}A_{32} + \operatorname{rank}E_{11} + 2q_1 + \operatorname{rank}\Omega_1, \quad (21)$$

where

$$\Psi_1 = \begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & S_{11} & 0 \\ S_{11} & 0 & 0 \end{bmatrix} \text{ and } \Omega_1 = \begin{bmatrix} E_{11} & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ S_{11} & \lambda S_{11} & 0 \end{bmatrix}.$$

Thus, from (19), (20), and (21), it follows that system (1) satisfies (4) and (5) if and only if the system (10) satisfy the conditions

$$\operatorname{rank}\Psi_1 = \operatorname{rank}\Gamma_1 \quad (22)$$

and

$$\forall\lambda \in \bar{\mathbb{C}}^+ : \ \operatorname{rank}\Omega_1 = \operatorname{rank}\Gamma_1. \quad (23)$$

Thus we have proved the follow result.

Lemma 7: System (1) satisfies conditions (4) and (5) if and only if conditions (22) and (23) hold for system (10).

The rank identities (22) and (23) will be used to prove the following theorem. The proof of the following theorem is motivated by recent works [16], [17], where the authors have studied systems of type (1) without unknown input and established a sufficient condition for the existence of functional ODE observers.

Theorem 2: The system (12) is solvable for the unknowns if and only if (22) and (23) hold.

Proof: Lemma 1 infers that Eq. (12a) is solvable for the unknowns if and only if (22) holds. Moreover,

$$\begin{bmatrix} T & \bar{M} & Q & N \end{bmatrix} = \Theta\Sigma^+ - Z(I - \Sigma\Sigma^+), \quad (24)$$

where Z is an arbitrary matrix of appropriate dimension. Thus,

$$\begin{bmatrix} T & \bar{M} & Q & N \end{bmatrix} = \begin{bmatrix} T_1 & \bar{M}_1 & Q_1 & N_1 \end{bmatrix} - Z\begin{bmatrix} T_2 & \bar{M}_2 & Q_2 & N_2 \end{bmatrix}, \quad (25)$$

where

$$T_1 = \Theta\Sigma^+\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad T_2 = (I - \Sigma\Sigma^+)\begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{M}_1 = \Theta\Sigma^+\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{M}_2 = (I - \Sigma\Sigma^+)\begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}, \quad Q_1 = \Theta\Sigma^+\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix},$$

$$N_1 = \Theta\Sigma^+\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix},$$

$$Q_2 = (I - \Sigma\Sigma^+)\begin{bmatrix} 0 \\ 0 \\ I \\ 0 \end{bmatrix}, \quad \text{and } N_2 = (I - \Sigma\Sigma^+)\begin{bmatrix} 0 \\ 0 \\ 0 \\ I \end{bmatrix}.$$

Now, it remains to prove that the matrix N obtained in (25) is Hurwitz. By Definition 2, there esixts a matrix Z such that N obtained in (25) is Hurwitz if and only if the matrix pair (N_1, N_2) is detectable, *i.e.*, for all $\lambda \in \bar{\mathbb{C}}^+$,

$$\text{matrix } \begin{bmatrix} N_1 - \lambda I \\ N_2 \end{bmatrix} \text{ has full column rank.} \quad (26)$$

In view of Lemma 1, (22) is equivalent to the fact that $\Theta = \Theta\Sigma^+\Sigma$. Hence, (23) can be rewritten as, for all $\lambda \in \bar{\mathbb{C}}^+$,

$$\operatorname{rank}\left[\Theta\Sigma^+ + \lambda \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} \right]\Sigma = \operatorname{rank}\Sigma. \quad (27)$$

Then Lemma 4 implies that (27) holds if and only if, for all $\lambda \in \bar{\mathbb{C}}^+$,

$$\begin{bmatrix} \Theta\Sigma^+ + \lambda \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{bmatrix} \\ I - \Sigma\Sigma^+ \end{bmatrix} \text{ has full column rank.} \quad (28)$$

By substituting the values of $\Theta\Sigma^+$ and $I - \Sigma\Sigma^+$ in (28), we obtain, for all $\lambda \in \bar{\mathbb{C}}^+$,

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ T_1 & \bar{M}_1 & Q_1 & N_1 - \lambda I \\ T_2 & \bar{M}_2 & Q_2 & N_2 \end{bmatrix} \text{ has full column rank,}$$

which, by a direct consequence of Lemma 3, completes the proof. □

The following remark is warranted on Theorems 1 - 2 and Lemma 7.

Remark 2: Lemma 7 proves that conditions (4) and (5) hold for system (1) if and only if conditions (22) and (23) hold for system (10). Thus, both of Theorems 1 - 2 infer that the functional ODE observer (11) estimates the functional $z(t)$ in (1) if the assumptions (4) and (5) hold true. However, it is not always recommended to verify assumption (5) initially because it requires the determination of rank for polynomial matrix Ω. It is a simple observation from Theorem 2 that under the rank assumption (22), which is equivalent to (4), the condition (23) and hence (5) holds if and only if the matrix pair (N_1, N_2) is detectable, cf. Definition 2.

Based on Theorems 1 and 2, we now summarize the observer design procedure in the form of Algorithm 1 below.

Algorithm 1 Computational steps to construct functional ODE observer (11) for system (1)

1) Compute orthogonal matrices U and V as in Lemma 6.
2) Convert system (1) into the form (7) as explained in Section III.
3) Use the SVD of G_1 to compute orthogonal matrices U_3 and V_3 as in (8).
4) Convert system (7) into the form (9) as explained in Section III.
5) If rank $\begin{bmatrix} C_{21} \\ K_{11} \end{bmatrix} \neq$ rank $C_{21}+$rank K_{11}, then compute the matrices P, S_{11}, and $\bar{P}$ such that
$$K_{11} = P\begin{bmatrix} S_{11} \\ \bar{P}C_{21} \end{bmatrix}.$$
Otherwise, take $S_{11} = K_{11}$, $P = I$, and $\bar{P}$ an empty matrix.
6) Calculate $\Theta\Sigma^+$ and $I - \Sigma\Sigma^+$ by using the expressions just below Eq. (12).
7) Extract N_1 and N_2 from $\Theta\Sigma^+$ and $I - \Sigma\Sigma^+$, respectively by using (25).
8) Compute Z such that the matrix $N = N_1 - ZN_2$ is Hurwitz, also see Remark (3) below.
9) Compute T, $\bar{M}$, Q, and N by using (25).
10) Compute $L = N\bar{M} - Q$ and $M = P\begin{bmatrix} \bar{M} \\ \bar{P} \end{bmatrix}$ (see the expressions just below Eq. (11)).

Remark 3: The existence of Z in step 8) of Algorithm 1 is guaranteed due to the detectability of the matrix pair (N_1, N_2), cf. Definition 2. Moreover, we can compute Z by applying the concepts of Lyapunov stability theory, which infers that Z can be obtained by solving the following linear matrix inequality (LMI) for $\bar{P}_2 = \bar{P}_2^\top > 0$ (positive definite) and $\bar{Z}$:
$$N_1^\top \bar{P}_2 + \bar{P}_2 N_1 - N_2^\top \bar{Z}^\top - \bar{Z}N_2 < 0,$$
where $\bar{Z} = \bar{P}_2 Z$, for more details, we refer to Chapter 5 in [31].

It is a well-known fact that if normal-rank$(\lambda E - A) < n$, then there exists more than one solution to (1a) [6]. Since we estimate the functional vector via the observer design approach, for any admissible pair of initial condition and control input, z has to be unique in (1c) even if the uniqueness of the semistate vector x is not recognized from (1a) alone. In the following theorem, we show that conditions (22) and (23) ensure the uniqueness of z_1 in (10), which is equivalent to saying that under assumptions (4) and (5), the functional vector z in (1) can be determined uniquely for any admissible pair $(Ex(0-), u(t))$.

Theorem 3: Suppose conditions (22) and (23) hold for system (10). Then, for each admissible pair $(E_{11}x_1(0-), u(t))$ and output $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, there exists a unique vector $z_1(t)$ satisfying (10).

Proof: Using the Laplace transform technique on (10) and assuming no new notations for the Laplace transforms of system variables, we obtain
$$\bar{E}_{11}(s)\bar{x}_1(s) = \begin{bmatrix} E_{11} \\ 0 \end{bmatrix} x_1(0-) + \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} u(s) + \bar{F}\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix},$$
$$z(s) = P\begin{bmatrix} S_{11} & 0 \\ \bar{P}C_{21} & 0 \end{bmatrix}\bar{x}_1(s),$$
where $\bar{E}_{11}(s) = \begin{bmatrix} sE_{11} - \Phi & -F_{13} \\ C_{21} & 0 \end{bmatrix}$, $\bar{F} = \begin{bmatrix} F_{12} & 0 \\ 0 & I_{p_2} \end{bmatrix}$, and $\bar{x}_1(s) = \begin{bmatrix} x_1(s) \\ v_2(s) \end{bmatrix}$.

On the other hand, from (22) and (23), we obtain
$$\text{rank}\,\Psi_1 = \text{rank}\,\Omega_1, \quad \text{for each } \lambda \in \bar{\mathbb{C}}^+.$$
Let p_2 be the number of rows in matrix C_{21}. Setting, for $\lambda \in \bar{\mathbb{C}}^+ \setminus \{0\}$,
$$\tilde{U}_3 = \begin{bmatrix} I_{m_1} & & & \\ & \frac{1}{\lambda}I_{p_2} & & \\ & \frac{1}{\lambda}I_{p_2} & I_{p_2} & \\ & & & I_l \end{bmatrix},$$
$$\tilde{V}_3 = \begin{bmatrix} \lambda I_{n_1} & & \\ -I_{n_1} & I_{n_1} & \\ & & I_{n_1} \end{bmatrix},$$
$$\text{and } \tilde{U}_4 = \begin{bmatrix} I_{m_1} & & & & \\ & \frac{1}{\lambda}I_{p_2} & & & \\ & \frac{1}{\lambda}I_{p_2} & I_{p_2} & & \\ & & & \lambda I_l & I_l \\ & & & & \frac{1}{\lambda}I_l \end{bmatrix},$$
we obtain
$$\text{rank}(\tilde{U}_4\Psi_1\tilde{V}_3) = \text{rank}(\tilde{U}_3\Omega_1\tilde{V}_3).$$

This is equivalent to the fact that, for $\lambda \in \bar{\mathbb{C}}^+ \setminus \{0\}$,

$$\operatorname{rank}\begin{bmatrix} \lambda E_{11}-\Phi & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & \lambda S_{11} & 0 \\ S_{11} & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda E_{11}-\Phi & \Phi & F_{13} \\ C_{21} & 0 & 0 \\ 0 & C_{21} & 0 \\ 0 & \lambda S_{11} & 0 \end{bmatrix}.$$

Hence, Lemma 5 infers that

$$normal-rank\begin{bmatrix} sE_{11}-\Phi & -F_{13} \\ C_{21} & 0 \\ S_{11} & 0 \end{bmatrix} = normal-rank\ \bar{E}_{11}(s),$$

i.e., z_1 can be uniquely determined (see Lemma 2). □

IV. NUMERICAL ILLUSTRATION

Example 1: In this example, we illustrate Algorithm 1 for designing functional ODE observer of a descriptor system of the form (1). Consider (1) described by the coefficient matrices:

$$E=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad A=\begin{bmatrix}-1&1&0&0\\-1&0&0&1\\0&-1&1&1\\0&0&1&0\end{bmatrix},\quad F=\begin{bmatrix}-1\\0\\0\\0\end{bmatrix},$$

$$B=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix},\quad C=\begin{bmatrix}1&0\\0&0\\0&1\\0&1\end{bmatrix}^\top,\quad G=\begin{bmatrix}0\\1\end{bmatrix},$$

$$K=\begin{bmatrix}1&0\\1&1\\0&-1\\-1&0\end{bmatrix}^\top.$$

This system does not satisfy the conditions proposed in [28]. Therefore, we cannot design a functional ODE observer for this system by using the methods available in the articles [25], [26], [27], [28]. However, the system coefficient matrices satisfy conditions (4) and (5), and therefore, a functional ODE observer exists for this system.

First, by using the step 1) *in Algorithm 1, we obtain*

$$U=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\ \text{and}\ V=\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}.$$

Then step 2) *of Algorithm 1 infers the following system coefficient matrices for (7):*

$$E_{11}=\begin{bmatrix}0&1&0\\-1&0&0\end{bmatrix},\quad A_{11}=\begin{bmatrix}-1&-1&0\\0&-1&-1\end{bmatrix},$$

$$F_{11}=\begin{bmatrix}1\\0\end{bmatrix},$$

$$B_{11}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},\quad \bar{C}_1=\begin{bmatrix}-1&0&1\\0&-1&0\\0&0&1\end{bmatrix},$$

$$G_1=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$\text{and } K_{11}=\begin{bmatrix}1&-1&-1\\1&0&0\end{bmatrix}.$$

Now, by following the step 3)*, we calculate*

$$U_3=\begin{bmatrix}0&0&1\\0&1&0\\-1&0&0\end{bmatrix}\ \text{and}\ V_3=1,$$

and thus, step 4) *of Algorithm 1 provides the coefficient matrices of (9):*

$$E_{11}=\begin{bmatrix}0&1&0\\-1&0&0\end{bmatrix},\quad \Phi=\begin{bmatrix}-1&-1&-1\\0&-1&-1\end{bmatrix},$$

$$F_{12}=\begin{bmatrix}1\\0\end{bmatrix},\quad F_{13}=[empty]_{2\times 0},\ B_{11}=\begin{bmatrix}-1&0\\0&-1\end{bmatrix},$$

$$C_{21}=\begin{bmatrix}0&-1&0\\1&0&-1\end{bmatrix},\ \text{and } K_{11}=\begin{bmatrix}1&-1&-1\\1&0&0\end{bmatrix}.$$

Here,

$$\operatorname{rank}\begin{bmatrix}C_{21}\\K_{11}\end{bmatrix}=3\neq 4=\operatorname{rank}C_{21}+\operatorname{rank}K_{11}.$$

Thus, step 5) *ensures that some part of the functional vector can be determined from the output* y_2*, and thus, we obtain*

$$S_{11}=\begin{bmatrix}1&0&0\end{bmatrix},\quad \bar{P}=\begin{bmatrix}1&1\end{bmatrix},\ \text{and } P=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Therefore, (10c) reduces to

$$z=P\begin{bmatrix}S_{11}x_1\\\bar{P}y_2\end{bmatrix}=\begin{bmatrix}\bar{P}y_2\\S_{11}x_1\end{bmatrix}=:\begin{bmatrix}z_1\\z_2\end{bmatrix}. \tag{29}$$

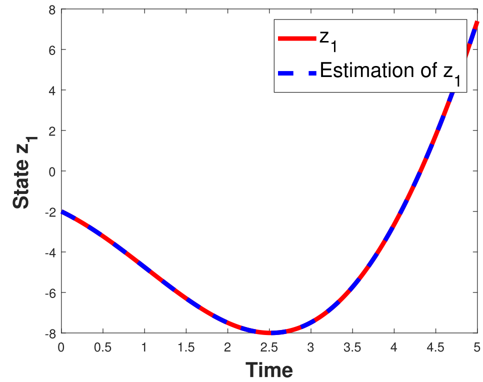
Clearly, we can determine z_1 *from the new output* y_2*, and observer is required only for* z_2*. Thus, by following the remaining steps of Algorithm 1, we obtain the first order functional ODE observer (11) with the coefficient matrices:*

$$N=\begin{bmatrix}-10\end{bmatrix},\ T=\begin{bmatrix}5.5&-1\end{bmatrix},\ L=\begin{bmatrix}-50.5&4.5\end{bmatrix},$$

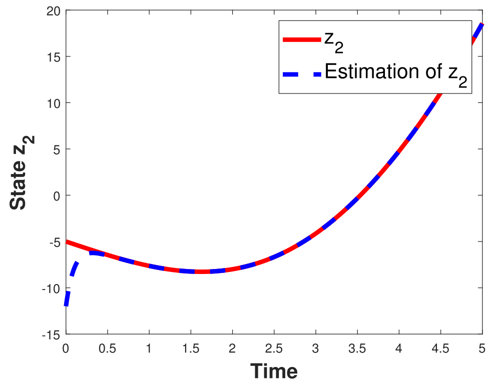
$$R=\begin{bmatrix}0\\1\end{bmatrix},\ \text{and } M=\begin{bmatrix}1&1\\5.5&0\end{bmatrix}.$$

Figure 1 shows the time responses of actual $z(t)$ *and estimated functional* $\hat{z}(t)$*. For the sake of numerical simulation, we take* $u=\begin{bmatrix}\cos(t)\\t\end{bmatrix}$*,* $v=\sin(t)$*,* $x_1(0)=\begin{bmatrix}-5&2&1\end{bmatrix}^\top$*, and* $w(0)=-1$*. Figure 1a shows that* $\hat{z}_1$ *is exactly the same as* z_1 *because* z_1 *is available from the output* y_2 *in (29).*

Example 2: In this example, we implement the proposed functional ODE observer design method to estimate the voltage v_2 *in an electronic circuit shown in Figure 2 [28]. Here,* C_1 *and* C_2 *denote capacitors,* R_1 *and* R_2 *stand for resistors, and L is an inductor. Moreover,* v_1 *and* v_2 *represent the voltages of* C_1 *and* C_2*, whereas* i_1 *and* i_L *represent the currents*



(a)



(b)

FIGURE 1. Time responses of actual and estimated functional in Example 1.

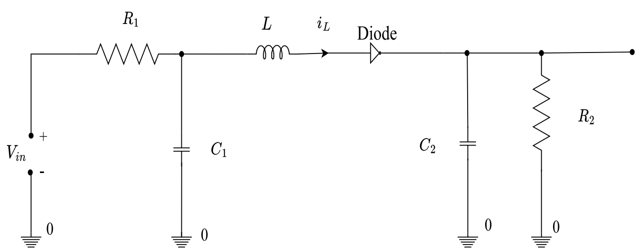


FIGURE 2. An electronic circuit.

flowing across capacitor C_1 and inductor L, respectively. We assume that the voltage source input $v_{in} = v_s$ is unknown. After applying the Kirchhoff law, we obtain that the circuit admits the following mathematical description [28]:

$$\frac{dv_1}{dt} = \frac{1}{C_1} i_1(t), \tag{30a}$$
$$\frac{dv_2}{dt} = \frac{1}{C_2} i_L(t) - \frac{1}{C_2 R_2} v_2(t), \tag{30b}$$
$$\frac{di_L}{dt} = \frac{1}{L} v_1(t) - \frac{1}{L} v_2(t), \tag{30c}$$
$$v_s(t) = v_1(t) + R_1 i_L(t) + R_1 i_1(t). \tag{30d}$$

By taking the semistate vector $x = \begin{bmatrix} v_1 & v_2 & i_L & i_1 \end{bmatrix}^\top$, the measurable output $y = \begin{bmatrix} v_1 & i_L & i_1 \end{bmatrix}^\top$, and the unmeasured output $z = v_2$, the above system can be easily written in the form (1) with coefficient matrices:

$$E = \begin{bmatrix} C_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & -L & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{R_2} & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & R_1 & R_1 \end{bmatrix},$$
$$B = 0_{4\times 1}, \quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
$$G = 0_{3\times 1}, \text{ and } K = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}.$$

For simulation purpose, we take the circuit parameters $C_1 = 100\ mF$, $C_2 = 100\ mF$, $R_1 = 4\ \Omega$, $R_2 = 4\ \Omega$, $L = 0.1\ H$ [28]. It can be checked easily that the system satisfies assumptions (4) and (5), therefore we can design a functional ODE observer (11) by using Algorithm 1 to estimate v_2:

$$N = \begin{bmatrix} -10 \end{bmatrix}, \quad T = \begin{bmatrix} -0.0100 & 0 & -9.9975 \end{bmatrix}, \; R = \begin{bmatrix} 1 \end{bmatrix},$$
$$L = \begin{bmatrix} 9.9975 & 10.0075 & 0 \end{bmatrix}, \text{ and } M = \begin{bmatrix} 0 & -0.9998 & 0 \end{bmatrix}.$$

In Figure 3, the true and estimated values of $z = v_2$ have been plotted by taking $x(0) = \begin{bmatrix} -1 & 2 & -2 & -4 \end{bmatrix}^\top$, $w(0) = 4$, and the unknown input $v_s(t) = 4\sin(1.5t)$. Clearly, Figure 3 reveals that the designed functional ODE observer converges to the true v_2 of the system (30).

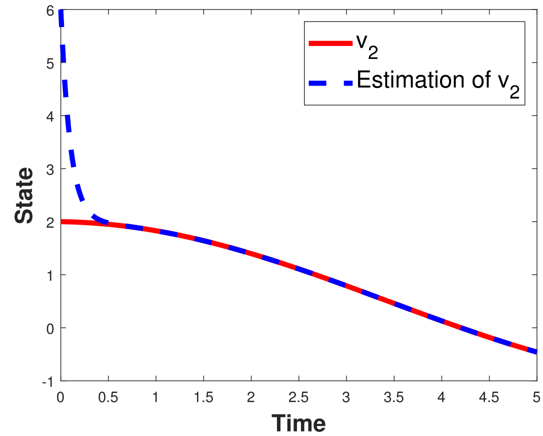


FIGURE 3. Time responses of the actual and estimated v_2.

V. CONCLUSION

This paper has established a new set of the existence conditions for functional ODE observers for a general class of linear time-invariant descriptor systems with unknown inputs. These conditions are much milder than those obtained in the earlier works on functional ODE observers. A numerically stable and easily implantable algorithm has been proposed for the observer design. If there is any redundancy between the measured output and the functional vector to be estimated, the designed algorithm provides an observer of an order less than the dimension of the functional vector. However, the existence conditions established in this paper are less restrictive, but they are still sufficient. Hence, there is still room for additional works to fill this gap and to work on the necessary and sufficient conditions for the existence of functional observers for linear descriptor systems with unknown inputs.

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