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RESEARCH ARTICLE

On Symbol-Triple Distance of a Class of Constacyclic Codes of Length $3p^s$ Over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$

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ABSTRACT Let $p \neq 3$ be any prime. In this paper, we completely determine symbol-triple distance of all γ-constacyclic codes of length $3p^s$ over the finite commutative chain ring $\mathcal{R} = \mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where γ is a unit of $\mathcal{R}$ which is not a cube in $\mathbb{F}_{p^m}$. We give the necessary and sufficient condition for a symbol-triple γ-constacyclic code to be an MDS symbol-triple code. Using that, we establish all MDS symbol-triple γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$. Some examples of the symbol-triple distance of γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are provided. We also list some new MDS symbol-triple γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$.

INDEX TERMS Constacyclic codes, dual codes, chain rings, MDS symbol-triple codes, symbol-triple codes.

I. INTRODUCTION

The class of constacyclic codes is an important class of linear codes in coding theory. Many optimal linear codes are directly derived from constacyclic codes. Constacyclic codes have practical applications as they are effective for encoding and decoding with shift registers.

λ-constacyclic codes of length n over $\mathbb{F}$ are classified as the ideals $\langle g(x)\rangle$ of the ambient ring $\frac{\mathbb{F}[x]}{\langle x^n-\lambda\rangle}$ where $g(x)$ is a divisor of $x^n - \lambda$ and λ is a unit in the finite field $\mathbb{F}_{p^m}$. If $(n, p) = 1$, the code is called a *simple-root code*. Otherwise, it is called *repeated-root code*. Repeat-root codes were studied earlier from the 1960s in some papers (for examples, [1], [2], [24], [25], and [27]). Since the last decade, repeated-roots codes have received much more attention as there have been many more optimal codes obtained from this class of codes. Dinh ([6], [8], [9], [10], [11]) determined the algebraic structures of constacyclic codes in terms of generator polynomials over $\mathbb{F}_{p^m}$ of length mp^s, where $m = 1, 2, 3, 4, 6$.

In 2011, Cassuto and Blaum ([3], [4]) introduced a new metric, called *symbol-pair metric*. Let σ be the code alphabet consisting of q elements. Then each element $v \in \sigma$ is called a *symbol*. In symbol-triple read channels, a codeword $(v_0, v_1, \ldots, v_{n-1})$ is read as $((v_0, v_1, v_2), (v_1, v_2, v_3), \ldots, (v_{n-1}, v_0, v_1))$. A q-ary code of length n is a nonempty subset $C \subseteq \sigma^n$. Assume that $v = (v_0, v_1, \ldots, v_{n-1})$ is a codeword in σ^n. The symbol-triple codeword of v is defined as

$$\gamma(v) = ((v_0, v_1, v_2), (v_1, v_2, v_3), \ldots, (v_{n-1}, v_0, v_1)).$$

Hence, each vector has a unique symbol-triple codeword $\gamma(v) \in (\sigma, \sigma, \sigma)^n$. The symbol-triple distance is an important parameter of symbol-triple codes. Given $v = (v_0, v_1, \ldots, v_{n-1})$, $t = (t_0, t_1, \ldots, t_{n-1})$, the symbol-triple distance between v and t is defined as

$$d_{st}(v, t) = |\{i : (v_i, v_{i+1}, v_{i+2}) \neq (t_i, t_{i+1}, t_{i+2})\}|.$$

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In 2008, the Hamming distance of all cyclic codes of prime power lengths over $\mathbb{F}_{p^m}$ is given by Dinh [6]. In 2010, [7] computed the Hamming distance of all $(\alpha+u\beta)$-constacyclic codes of length p^s over $\mathcal{R}=\mathbb{F}_{p^m}+u\mathbb{F}_{p^m}$. After that, the Hamming distance of all constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m}$ is provided in [14]. In addition, the Hamming distance of all γ-constacyclic codes of prime power lengths over $\mathcal{R}$ is studied in [18]. In 2020, the Hamming distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is established in $\mathcal{R}$ [15], where $\lambda=\alpha+u\beta$ is not a cube. In 2020, the Hamming distances and b-symbol distances of λ-constacyclic codes of length $4p^s$ over $\mathcal{R}$ are determined for $p^m \equiv 1 \pmod 4$ and the non-square unit λ [16]. In this paper, we completely symbol-triple distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathbb{F}_{p^m}$. In addition, we determine all MDS symbol-triple codes. As an application, some new MDS symbol-triple codes are given. Note that the structure of codes of length $3p^s$ is much more complicated than codes of length $4p^s$. Repeated-root constacyclic codes of length $3p^s$ over $\mathcal{R}$ form a very interesting class of constacyclic codes. When λ is not a cube in $\mathbb{F}_{p^m}$, symbol-triple distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ did not study in the past.

Motivated by these, we determine symbol-triple distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is not a cube in $\mathbb{F}_{p^m}$ in this paper. As an application, we identify all the MDS symbol-triple codes among such codes. We also give some new MDS symbol-triple codes.

The rest of this paper is organized as follows. Section II gives some preliminaries. Section III obtains the symbol-triple distance of all γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ (λ is not a cube in $\mathbb{F}_{p^m}$). In Section IV, we give the necessary and sufficient condition for a symbol-triple γ-constacyclic code to be an MDS symbol-triple code and we identify all such codes. Some new MDS symbol-triple codes are provided in Section IV. The conclusion of this paper is given in Section V.

II. PRELIMINARIES

For a unit λ of R, the λ-constacyclic (λ-twisted) shift ρ_λ on R^n is the shift

$$\rho_\lambda(x_0, x_1, \ldots, x_{n-1}) = (\lambda x_{n-1}, x_0, x_1, \ldots, x_{n-2}),$$

and a code C is said to be λ-constacyclic if $\rho_\lambda(C)=C$. If $\lambda=\{1,-1\}$, then C is a cyclic and negacyclic code, respectively.

Proposition 1: [23] *Let C be a linear code. Then C is a λ-constacyclic code of length n over R if and only if C is an ideal of the ring $\frac{R[x]}{\langle x^n-\lambda\rangle}$.*

Proposition 2: [17] The dual of a λ-constacyclic code is a λ^{-1}-constacyclic code.

Let p be a prime and R be a finite chain ring of size p^m.

Proposition 3: [23] *Let C be a linear code C of length n over R. Then $|C|=p^k$, for some integer $k\in\{0,1,\ldots,mn\}$. In addition, $|C|\cdot|C^\perp|=|R|^n$, where $C^\perp$ is the dual code of C.*

Assume that α and β are elements in $\mathbb{F}_{p^m}$. It is easy to see that $\alpha+u\beta$ is an invertible element over $\mathcal{R}$ if and only if $\alpha\neq 0$. Therefore, we divide all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ into the following cases: λ is a cube and $p^m\equiv 1 \pmod 3$, λ is a cube and $p^m\equiv 2 \pmod 3$, $\lambda=\alpha+u\beta$ is not a cube and $0\neq\alpha,\beta\in\mathbb{F}_{p^m}$, λ is not a cube and $0\neq\lambda\in\mathbb{F}_{p^m}$. We give all λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ in the following theorem.

Theorem 1: [15] *Let $p\neq 3$ be any prime. Let C be a λ-constacyclic code of length $3p^s$ over $\mathcal{R}$.*

1) *Assume that λ is a cube in $\mathcal{R}$ and $p^m\equiv 1 \pmod 3$. Let $\lambda_0\in\mathcal{R}$ such that $\lambda_0^3=\lambda$ and $\delta,\theta\in\mathbb{F}_{p^m}$ such that $\delta\theta=1$ and $\delta+\theta=-1$. Then $C=C_1\oplus C_2\oplus C_3$ where C_1 is a λ_0-constacyclic code of length p^s over $\mathcal{R}$, C_2 is a $\delta\lambda_0$-constacyclic code of length p^s over $\mathcal{R}$ and C_3 is a $\theta\lambda_0$-constacyclic code of length p^s over $\mathcal{R}$. In particular, $|C|=|C_1||C_2||C_3|$.*
2) *Assume that λ is a cube in $\mathcal{R}$ and $p^m\equiv 2 \pmod 3$. Let $\lambda_1\in\mathcal{R}$ such that $\lambda=\lambda_1^3$. Then*
   (a) *$C=C_1\oplus C_2$ where C_1 is a λ_1-constacyclic code of length p^s over $\mathcal{R}$ and C_2 is an ideal of $\frac{\mathcal{R}[x]}{\langle x^{2p^s}+\lambda_1x^{p^s}+\lambda_1^2\rangle}$.*
   (b) *$|C|=|C_1||C_2|$, where C_1 is determined as in Theorem 2.2 and all ideals of $\frac{\mathcal{R}[x]}{\langle (x^2+\lambda_1x+\lambda_1^2)^{p^s}\rangle}$ are determined as follows:*
   - *Type 1: (trivial ideals)*

     $$\langle 0\rangle \text{ and } \langle 1\rangle.$$

     Then $n_{C_2}=1$ and $n_{C_2}=p^{4mp^s}$, respectively.
   - *Type 2: (principal ideals with nonmonic polynomial generators)*

     $$\langle u(x^2+\lambda_1x+\lambda_1^2)^j\rangle,$$

     where $0\le j\le p^s-1$. Then $n_{C_2}=p^{2m(p^s-j)}$
   - *Type 3: (principal ideals with monic polynomial generators)*

     $$\langle (\ell(x))^j+u(\ell(x))^t v(x)\rangle,$$

     where $\ell(x)=x^2+\lambda_1x+\lambda_1^2$, $1\le j\le p^s-1$, $0\le t<j$, and either $v(x)$ is 0 or a unit which can be represented as $v(x)=\sum_{i=0}^{j-t-1}(v_{1i}x+v_{0i})(x^2+\lambda_1x+\lambda_1^2)^i$ with $v_{0i},v_{1i}\in\mathbb{F}_{p^m}$ and $v_{10}x+v_{00}\neq 0$. In this case,

     $$n_{C_2}=\begin{cases}\bullet\, p^{4m(p^s-j)}, & \text{if } v(x) \text{ is } 0,\\ 1\le j\le p^s-1 & \\ \text{or } v(x) \text{ is a unit }, & \\ 1\le j\le \frac{p^s+t}{2}, & \\ \bullet\, p^{2m(p^s-t)}, & \text{if } v(x) \text{ is a unit},\\ \text{and } \frac{p^s+t}{2}<j\le p^s-1. & \end{cases}$$

• *Type 4: (non-principal ideals)* $\langle (x^2+\lambda_1 x+\lambda_1^2)^j + u(x^2+\lambda_1 x+\lambda_1^2)^t v(x), u(x^2+\lambda_1 x+\lambda_1^2)^{\omega}\rangle$, *with* $v(x)$ *as in Type 3,* $\deg v(x) \le \omega - r - 1$ *and* $\omega < R$ *and* R *is the smallest integer satisfying* $u(x^2+\lambda_1 x+\lambda_1^2)^R \in \langle (x^2+\lambda_1 x+\lambda_1^2)^j + u(x^2+\lambda_1 x+\lambda_1^2)^t v(x)\rangle$. *In this case,* $n_{C_2} = p^{2m(2p^s-j-\omega)}$

3) Assume that $\lambda = \alpha + u\beta$ *is not a cube in* $\mathcal{R}$. *There is an* $\alpha_1 \in \mathbb{F}_{p^m}$ *satisfying* $\alpha = \alpha_1^{p^s}$. *Then* $(\alpha+u\beta)$*-constacyclic codes of length* $3p^s$ *over* $\mathcal{R}$ *are the ideals* $\langle (x^3-\alpha_1)^i\rangle \subseteq \mathcal{R}_{\alpha,\beta}$, *where* $0 \le i \le 2p^s$ *and each* $(\alpha+u\beta)$*-constacyclic code* $\langle (x^3-\alpha_1)^i\rangle$ *has* $p^{3m(2p^s-1)}$ *codewords.*

4) Assume that $\gamma \in \mathbb{F}_{p^m}\backslash\{0\}$ *is not a cube in* $\mathbb{F}_{p^m}$. *Let* $\gamma_0 \in \mathbb{F}_{p^m}$ *such that* $\gamma_0^{p^s} = \gamma$. *Then* γ*-constacyclic codes of length* $3p^s$ *over* $\mathcal{R}$ *are*

• *Type 1:*

$$\langle 0\rangle \text{ and } \langle 1\rangle.$$

• *Type 2: (principal ideals with nonmonic polynomial generators)*

$$\langle u(x^3-\gamma_0)^i\rangle,$$

where $0 \le i \le p^s - 1$.

• *Type 3: (principal ideals with monic polynomial generators)*

$$\langle ((x^3-\gamma_0)^i + u(x^3-\gamma_0)^t) v(x)\rangle,$$

where $1 \le i \le p^s-1, 0 \le t < i$, *and* $v(x)$ *is 0 or a unit* ($v(x) = \sum_{j=0}^{i-t-1}(h_{2j}x^2 + h_{1j}x + h_{0j}j)(x^3-\gamma_0)^j$ *where* $h_{0j}, h_{1j}, h_{2j} \in \mathbb{F}_{p^m}$ *and* $h_{00} \neq 0$).

• *Type 4: (nonprincipal ideals)*

$$\langle (g(x))^i + u(\sum_{j=0}^{\omega-1}(t_{0j}(x))(g(x))^j), u(g(x))^{\omega}\rangle$$

where $g(x) = x^3 - \gamma_0$, $1 \le i \le p^s - 1$, $a_{0j}, b_{0j}, c_{0j} \in \mathbb{F}_{p^m}$, $t_{0j}(x) = a_{0j}x^2 + b_{0j}x + c_{0j}$, *and* $\omega < T$, *where* T *is the smallest integer satisfying*

$$u(g(x))^T \in \langle (g(x))^i + u\sum_{j=0}^{w-1}(t_{0j}(x))(g(x))^j\rangle$$

or equivalently,

$$\langle (g(x))^i + u(g(x))^t h(x), u(g(x))^{\omega}\rangle$$

with $h(x)$ *as in Type 3 and* $\deg h(x) \le \omega - t - 1$.

In addition, the number of codewords of C, *denoted by* n_C, *is given as follows:*

∘ *If* $C = \langle 0\rangle$ *and* $C = \langle 1\rangle$, *then* $n_C = 1$ *and* $n_C = p^{6mp^s}$, *respectively.*

∘ *If* $C = \langle u(x^3-\gamma_0)^i\rangle$, *where* $0 \le i \le p^s - 1$, *then* $n_C = p^{3m(p^s-i)}$.

∘ *If* $C = \langle (x^3-\gamma_0)^i + u(x^3-\gamma_0)^t h(x)\rangle$ *where* $1 \le i \le p^s-1, 0 \le t < i$, *and* $h(x)$ *is 0 or a unit, then*

$$n_C = \begin{cases} \bullet\, p^{6m(p^s-i)}, & \text{if } h(x) \text{ is } 0, \\ 1 \le i \le p^s-1 \text{ or } h(x) \text{ is a unit}, \\ 1 \le i \le p^{s-1} + \frac{t}{2}, \\ \bullet\, p^{3m(p^s-t)}, & \text{if } h(x) \text{ is a unit}, \\ p^{s-1} + \frac{t}{2} < i \le p^s - 1. \end{cases}$$

∘ *If* $C = \langle (x^3-\gamma_0)^i + u(x^3-\gamma_0)^t h(x), u(x^3-\gamma_0)^{\kappa}\rangle$, *where* $1 \le i \le p^s-1$, $0 \le t \le i$, *either* $h(x)$ *is 0 or a unit, and*

$$\kappa < T = \begin{cases} i, \text{ if } h(x) = 0, \\ \min\{i, p^s - i + t\}, \text{ if } h(x) \neq 0, \end{cases}$$

then $n_C = p^{3m(2p^s-i-\kappa)}$.

Let b be an integer and $b \ge 1$. For a codeword $v = (v_0, v_1, \cdots, v_{n-1}) \in \sigma^n$, we define the b-symbol read codeword of v as

$$\pi_b(v) = ((v_0, \cdots, v_{b-1}), \cdots, (v_{n-1}, v_0, \cdots, v_{b-2})) \in (\sigma^b)^n.$$

Then the b-symbol distance between two codeword v and t in σ^n is denoted by $d_b(v,t)$ and defined as

$$d_b(v,t) = d_H(\pi_b(v), \pi_b(t)).$$

Recently, Yaakobi et al. [26] generalized the coding framework for symbol-pair read channels to that for b-symbol read channels, where the read operation is performed as a consecutive sequence of $b > 2$ symbols. They also generalized some of the known results for symbol-pair read channels to those for b-symbol read channels. In [21], Dinh et al. computed the b-symbol distance for $C = \langle (x^{\eta} - \lambda_0)^j\rangle$ for $0 \le j \le p^s$ and $b \le \eta$ over $\mathbb{F}_{p^m}$, where $(x^n - \lambda_0)$ is irreducible. For symbol-triple distance, we have the following theorem.

Theorem 2: Let $C = \langle (x^3-\lambda_0)^j\rangle \subseteq \frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\lambda\rangle}$ *for* $0 \le j \le p^s$, *the symbol-triple distance* $d_{st}(C)$ *is completely given by*

$$d_{st}(C) = \begin{cases} \bullet 3, \text{ if } j = 0 \\ \bullet 3(\delta+1)p^{\xi}, \\ \text{if } p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \\ \text{and } j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \le \delta \le p-1$, $0 \le \xi \le s-1$.

III. SYMBOL-TRIPLE DISTANCE

In [13], the authors obtained the symbol-pair distances of all constacyclic codes of prime power lengths over $\mathbb{F}_{p^m}$. Later, [18] and [20] gave the symbol-pair distances of all constacyclic codes of length p^s over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$. In this section, when γ is not a cube in $\mathbb{F}_{p^m}$, we determine the symbol-triple distance of all γ-constacyclic code of length $3p^s$ over $\mathcal{R}$, where the structure of γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is given in part 4 of Theorem 1. Denote $\mathrm{d_{st}}(C_F)$ as the symbol-triple distance of $C\big|_{\mathbb{F}_{p^m}}$.

Obviously, if $C = \langle 0\rangle$, then $\mathrm{d_{st}}(C) = 0$. If $C = \langle 1\rangle$, then $\mathrm{d_{st}}(C) = 3$. Then $\mathrm{d_{st}}(C)$ can be determined as follows.

Theorem 3: Let $C = \langle u(x^3 - \gamma_0)^j\rangle$ be a γ-constacyclic code of Type 2 of length $3p^s$ over $\mathcal{R}$, where $0 \le j \le p^s - 1$. Then we have $\mathrm{d_{st}}(C) = \mathrm{d_{st}}(\langle (x^3-\gamma_0)^j\rangle_F)$, and $\mathrm{d_{st}}(C)$ is given by

$$\mathrm{d_{st}}(C) = \begin{cases} \bullet 3, \text{ if } j = 0 \\ \bullet 3(\delta+1)p^{\xi}, \text{ if } p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \\ \text{and } j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \le \delta \le p-1$, $0 \le \xi \le s-1$.

Proof: We divide into two cases, namely, $j = 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$.

Case 1: If $j = 0$, then $\mathrm{d_{st}}(C) = 1$.

Case 2: If $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then $C = \langle u(x^3-\gamma_0)^j\rangle$, $0 \le j \le p^s - 1$. We see that n_C is exactly same as $n_{\langle (x^3-\gamma_0)^j\rangle}$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\gamma\rangle}$ multiplied by u. Therefore, $\mathrm{d_{st}}(C) = \mathrm{d_{st}}(\langle (x^3-\gamma_0)^j\rangle_F)$ and $\mathrm{d_{st}}(C)$ is given by Theorem 2 as follows:

$$\mathrm{d_{st}}(C) = \begin{cases} \bullet 3, \text{ if } j = 0 \\ \bullet 3(\delta+1)p^{\xi}, \text{ if } p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le j \\ \text{and } j \le p^s - p^{s-\xi} + \delta p^{s-\xi-1} \end{cases}$$

where $1 \le \delta \le p-1$, $0 \le \xi \le s-1$. □

The symbol-triple distance of γ-constacyclic codes of Type 3 of length $3p^s$ over $\mathcal{R}$ is provided in the following theorem.

Theorem 4: Let $C = \langle (\alpha(x))^j + u(\alpha(x))^r v(x)\rangle$ be a γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$, where $\alpha(x) = x^3 - \gamma_0$, $1 \le j \le p^s - 1$, $0 \le r < j$ and either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\lambda\rangle}$ or 0. Then we have $\mathrm{d_{st}}(C) = \mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F)$, where T is the smallest integer satisfying $u(x^3-\gamma_0)^T \in \langle (x^3-\gamma_0)^j + u(x^3-\gamma_0)^r v(x)\rangle$, and

$$T = \begin{cases} j, & \text{if } v(x) = 0 \\ \min\{j, p^s - j + r\}, & \text{if } v(x) \ne 0. \end{cases}$$

Hence,

$$\mathrm{d_{st}}(C) = 3(\delta+1)p^{\xi},$$

where $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le T \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, $1 \le \delta \le p-1$ and $0 \le \xi \le s-1$.

Proof: Since T is the smallest integer satisfying $u(\alpha(x))^T \in \langle (\alpha(x))^j + u(\alpha(x))^r v(x)\rangle$, we see that

$$\mathrm{d_{st}}(C) \le \mathrm{d_{st}}(\langle u(\alpha(x))^T\rangle) = \mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F).$$

Let $c(x) \in C$ be an arbitrary polynomial. Then there are two polynomials $f_0(x)$ and $f_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(\alpha(x))^j + u(\alpha(x))^r v(x)] \\ &= f_0(x)(\alpha(x))^j + u[f_0(x)(\alpha(x))^r v(x) + f_u(x)(\alpha(x))^j]. \end{aligned}$$

Now, we consider two cases, namely, $v(x) = 0$ and $v(x) \ne 0$.

Case 1: Assume that $v(x) = 0$. Hence, we have

$$\begin{aligned} \mathrm{wt_{st}}(c(x)) &\ge \max\left\{\mathrm{wt_{st}}(f_0(x)(\alpha(x))^j), \mathrm{wt_{st}}(f_u(x)(\alpha(x))^j)\right\} \\ &\ge \max\left\{\mathrm{wt_{st}}(f_0(x)(\alpha(x))^j), \mathrm{wt_{st}}(f_0(x)(\alpha(x))^j)\right\} \\ &\ge \mathrm{d_{st}}(\langle (\alpha(x))^j\rangle_F), \\ &= \mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F) \end{aligned}$$

This shows that $\mathrm{d_{st}}(C) = \mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F)$.

Case 2: Assume that $v(x) \ne 0$. Then we see that

$$\begin{aligned} \mathrm{wt_{st}}(c(x)) &\ge \max\left\{\mathrm{wt_{st}}(f_0(x)(\alpha(x))^j), \mathrm{wt_{st}}(h(x))\right\} \\ &\ge \max\left\{\mathrm{wt_{st}}(\theta(x))^j), \mathrm{wt_{st}}(\theta(x))^l)\right\} \\ (\text{where } \theta(x) &= f_0(x)(\alpha(x), l = p^s - j + r) \\ &\ge \mathrm{d_{st}}(\langle (\alpha(x))^{\min\{j,\, p^s-j+r\}}\rangle_F), \\ &= \mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F), \end{aligned}$$

where $h(x) = f_0(x)(\alpha(x))^r v(x) + f_u(x)(\alpha(x))^j$. Hence, by combining both the cases, we get $\mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F) \le \mathrm{d_{st}}(C)$. It implies that $\mathrm{d_{st}}(\langle (\alpha(x))^T\rangle_F) = \mathrm{d_{st}}(C)$. □

We compute the symbol-triple distance of γ-constacyclic codes of Type 4 in the following theorem.

Theorem 5: Let $C = \langle (\alpha(x))^j + u(\alpha(x))^r v(x), u(\alpha(x))^{\omega}\rangle$ be a γ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$, where $\alpha(x) = x^3 - \gamma_0$, $v(x)$ is same as given in Type 3, $\deg(v) \le \omega - r - 1$, $\omega < T$, and T is the smallest integer satisfying $u(\alpha(x))^T \in \langle (\alpha(x))^j + u(\alpha(x))^r v(x)\rangle$, i.e., $T = j$, if $v(x) = 0$ and otherwise $T = \min\{j, p^s - j + t\}$. Then we have $\mathrm{d_{st}}(C) = \mathrm{d_{st}}(\langle (\alpha(x))^{\omega}\rangle_F)$, and is given by

$$\mathrm{d_{st}}(C) = 3(\delta+1)p^{\xi},$$

where $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \le \omega \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, $1 \le \delta \le p-1$ and $0 \le \xi \le s-1$.

Proof: From $\omega < T \le j$, we have $C = \langle (\alpha(x))^j + u(\alpha(x))^r v(x), u(\alpha(x))^{\omega}\rangle \supseteq \langle u(\alpha(x))^{\omega}\rangle \supseteq \langle u(\alpha(x))^j\rangle$. Therefore, $\mathrm{d_{st}}(C) \le \mathrm{d_{st}}(\langle u(x-\gamma_0)^{\omega}\rangle) = \mathrm{d_{st}}(\langle (\alpha(x))^{\omega}\rangle_F)$. We need to prove that $\mathrm{d_{st}}(\langle (\alpha(x))^{\omega}\rangle_F) \le \mathrm{d_{st}}(C)$. In order to do this, let $c(x) \in C$ be an arbitrary polynomial and we will prove that $\mathrm{wt_{st}}(c(x)) \ge \mathrm{d_{st}}(\langle (\alpha(x))^{\omega}\rangle_F)$. We see that there exist polynomials $f_0(x), f_u(x), g_0(x)$ and $g_u(x)$ over $\mathbb{F}_{p^m}$ satisfying

$$\begin{aligned} c(x) &= [f_0(x) + uf_u(x)][(\alpha(x))^j + u(\alpha(x))^r v(x)] \\ &\quad + u(\alpha(x))^{\omega}[g_0(x) + ug_u(x)] \\ &= f_0(x)(\alpha(x))^j + u[f_0(x)(\alpha(x))^r v(x) + f_u(x)(\alpha(x))^j] \\ &\quad + ug_0(x)(\alpha(x))^{\omega} \\ &= f_0'(x)(\alpha(x))^{\omega} + u[f_0(x)(\alpha(x))^r v(x) + g_0'(x)(\alpha(x))^{\omega}], \end{aligned}$$

TABLE 1. γ-constacyclic codes over $\mathbb{F}_7 + u\mathbb{F}_7$.

n	s	γ	$\langle g(x)\rangle$	$[n, M, d_{st}]$
21	1	3	$\langle u(x^3-3)\rangle$	$[21, 7^{18}, 6]$
21	1	3	$\langle (x^3-3)^2\rangle$	$[21, 7^{30}, 9]$
21	1	3	$\langle (x^3-3)^2, u(x^3-3)\rangle$	$[21, 7^{33}, 6]$
147	2	2	$\langle u(x^3-2)^{44}\rangle$	$[147, 7^{15}, 63]$
147	2	2	$\langle (x^3-2)^{44}\rangle$	$[147, 7^{30}, 63]$

TABLE 2. γ-constacyclic codes over $\mathbb{F}_{13} + u\mathbb{F}_{13}$.

n	s	γ	$\langle g(x)\rangle$	$[n, M, d_{st}]$
39	1	2	$\langle u(x^3-2)^2\rangle$	$[39, 13^{33}, 9]$
39	1	2	$\langle (x^3-2)^2\rangle$	$[39, 13^{66}, 9]$
39	1	2	$\langle (x^3-2)^{15} + u(x^3-2), u(x^3-2)\rangle$	$[39, 13^{30}, 6]$

where $f_0'(x) = f_0(x)(\alpha(x))^{j-\omega} \in \mathbb{F}_{p^m}[x]$, $g_0'(x) = f_u(x)(\alpha(x))^{j-\omega} + g_0(x) \in \mathbb{F}_{p^m}[x]$. Hence,

$$\begin{aligned} \mathrm{wt_{st}}(c(x)) &\geq \max\left\{\mathrm{wt_{st}}(f_0'(x)(\alpha(x))^{\omega}), \mathrm{wt_{st}}(h'(x))\right\} \\ &\geq \max\left\{\mathrm{wt_{st}}(f_0'(x)(\alpha(x))^{\omega}), \mathrm{wt_{st}}(f_0'(x)(\alpha(x))^{\omega})\right\} \\ &\geq \mathrm{d_{st}}(\langle(\alpha(x))^{\omega}\rangle_F), \end{aligned}$$

where $h'(x) = f_0(x)(\alpha(x))^r v(x) + g_0'(x)(\alpha(x))^{\omega}$. □

If $\lambda = \alpha + u\beta$ is not a cube in $\mathcal{R}$, then there is an $\alpha_1 \in \mathbb{F}_{p^m}$ satisfying $\alpha = \alpha_1^{p^s}$. As in part 3 of Theorem 1, $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ are the ideals $\langle(x^3-\alpha_1)^i\rangle \subseteq \mathcal{R}_{\alpha,\beta}$, where $0 \leq i \leq 2p^s$. When $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$, we determine the symbol-triple distance of all $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ in the following remark.

Remark 1: Let $C \subseteq \mathcal{R}_{\alpha+u\beta} = \frac{\mathcal{R}[x]}{\langle x^{3p^s}-(\alpha+u\beta)\rangle}$, then $C = \langle(x^3-\alpha_1)^j\rangle$, for $j \in \{0, 1, \ldots, 2p^s\}$, and

$$\mathrm{d_{st}}(C) = \begin{cases} \bullet 3, \text{ if } 0 \leq j \leq p^s \\ \bullet 3(\delta+1)p^{\xi}, \\ \text{if } 2p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq j \\ \text{and } j \leq 2p^s - p^{s-\xi} + \delta p^{s-\xi-1} \\ 0, \text{ if } j = 2p^s \end{cases}$$

where $1 \leq \delta \leq p-1$, $0 \leq \xi \leq s-1$.

Proof: We consider three cases.

Case 1: If $j = 0$ and $j = 2p^s$, then $C = \langle 1\rangle$ and $C = \langle 0\rangle$. It is easy to verify that $\mathrm{d_{st}}(C) = 3$ and $\mathrm{d_{st}}(C) = 0$, respectively.

Case 2: If $1 \leq j \leq p^s$. In $\mathcal{R}_{\alpha+u\beta}$, $\mathcal{R}_{\alpha+u\beta} = \langle 1\rangle \supsetneq \langle(x^3-\alpha_1)\rangle \supsetneq \cdots \supsetneq \langle(x^3-\alpha_1)^{p^s}\rangle \supsetneq \cdots \supsetneq \langle(x^3-\alpha_1)^{2p^s}\rangle = \langle 0\rangle$. Thus, we have $u \in \langle(x^3-\alpha_1)^j\rangle$. It implies that $\mathrm{d_{st}}(C) = 3$.

Case 3: If $p^s + 1 \leq j \leq 2p^s - 1$, then we see that $\langle(x^3-\alpha_1)^j\rangle = \langle u(x^3-\alpha_1)^{j-p^s}\rangle$. Hence, $n_{\langle(x^3-\alpha_1)^j\rangle}$ in $\mathcal{R}_{\alpha+u\beta}$ is exactly same as $n_{\langle(x^3-\alpha_1)^{j-p^s}\rangle}$ in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s}-\alpha\rangle}$ multiplied by u. Thus, $\mathrm{wt_{st}}(\langle(x^3-\alpha_1)^j\rangle) = \mathrm{wt_{st}}(\langle(x^3-\alpha_1)^{j-p^s}\rangle)$. By Theorem 2, we can determine the symbol-triple distance of $\langle(x^3-\alpha_1)^{j-p^s}\rangle$. Therefore, $\mathrm{d_{st}}(C) = 3(\delta+1)p^{\xi}$ when $2p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq j \leq 2p^s - p^{s-\xi} + \delta p^{s-\xi-1}$. □

Example 1: We present some examples of symbol-triple distance γ-constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\gamma \in \mathbb{F}_p^*$ and γ is not a cube. In Table 1, we compute the symbol-triple distances for $p = 7, m = 1, s = 1$ and 2 and in Table 2, symbol-triple distances have been computed by taking $p = 13, m = 1, s = 1$ and 2.

IV. MDS SYMBOL-TRIPLE CODES

In 2018, Ding et al. [5] discussed the Singleton bound with respect to $\mathrm{d_b}(C)$. Following them, the Singleton bound with respect to the b-symbol distance is given as $|C| \leq q^{n-\mathrm{d_b}(C)+b}$. For symbol-triple distance, we need to have the following result.

Theorem 6 (Singleton Bound With Respect to Symbol-Triple Distance): Let C be a linear symbol-triple code of length n over $\mathcal{R}$ with symbol-triple distance $\mathrm{d_{st}}(C)$. Then, the Singleton bound with respect to the symbol-triple distance $\mathrm{d_{st}}(C)$ is given by $|C| \leq p^{2m(n-\mathrm{d_{st}}(C)+3)}$.

Proof: Assume that $C = (n, M, \mathrm{d_{st}}(C))$ is a symbol-triple code. After deleting the last $\mathrm{d_{st}}(C)-3$ coordinates from all the codewords in C, we observe that any $\mathrm{d_{st}}(C)-3$ consecutive coordinates contribute at most $\mathrm{d_{st}}(C)-1$ to the symbol-triple distance. Since C has symbol-triple distance $\mathrm{d_{st}}(C)$, it implies that the resulting vectors of length $n - \mathrm{d_{st}}(C) + 3$ are still distinct. The conclusion follows from the fact that the maximum number of distinct vectors of length $n - \mathrm{d_{st}}(C) + 3$ over $\mathcal{R}$ is $p^{2m(n-\mathrm{d_{st}}(C)+3)}$. □

Definition 1: Let C be a symbol-triple linear code of length n over $\mathcal{R}$. Then C is called an MDS symbol-triple code with respect to the symbol-triple distance if $|C| = p^{2m(n-\mathrm{d_{st}}(C)+3)}$.

Next, we give all symbol-triple MDS codes of length $3p^s$ over $\mathcal{R}$ when λ is a unit of the form $\lambda = \gamma \in \mathbb{F}_{p^m}^*$ and λ is not a cube. First, we consider the symbol-triple γ-constacyclic code C of length $3p^s$ over $\mathcal{R}$, where C is a symbol-triple γ-constacyclic code of Type 1 of length $3p^s$ over $\mathcal{R}$, i.e., $C = \langle 0\rangle$ and $C = \langle 1\rangle$.

Theorem 7: Let C be a symbol-triple γ-constacyclic code of Type 1 of length $3p^s$ over $\mathcal{R}$. Then $C = \langle 1\rangle$ is an MDS symbol-triple code.

Proof: Case 1: If $C = \langle 0\rangle$, then the symbol-triple distance is $\mathrm{d_{st}}(C) = 0$. We see that C is an MDS symbol-triple code when $|C| = p^{2m(3p^s-\mathrm{d_{st}}(C)+3)}$, i.e., $1 = p^{2m(3p^s+3)}$, i.e., $3p^s + 3 = 0$. This is a contradiction. Thus, $C = \langle 0\rangle$ is not an MDS symbol-triple code.

Case 2: If $C = \langle 1\rangle$, then $\mathrm{d_{st}}(C) = 3$. Hence, C is an MDS symbol-triple code when $|C| = p^{2m(3p^s-\mathrm{d_{st}}(C)+3)}$, i.e., $p^{6mp^s} = p^{2m(3p^s)}$, which is true for all p and s. Therefore, the code $C = \langle 1\rangle$ is an MDS symbol-triple code. □

We determine the MDS condition for symbol-triple γ-constacyclic codes of Type 2 of length $3p^s$ over $\mathcal{R}$.

Theorem 8: Let $C = \langle u(x^3-\gamma_0)^j\rangle$ be a symbol-triple γ-constacyclic code of Type 2 of length $3p^s$ over $\mathcal{R}$, where $0 \leq j \leq p^s - 1$. Then C is not an MDS symbol-triple γ-constacyclic code.

Proof: Case 1: If $j = 0$, then $d_{st}(C) = 3$. Hence, C is an MDS symbol-triple γ-constacyclic code when $|C| = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $p^{3mp^s} = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $p^{3mp^s} = p^{6mp^s}$, which is not true for any p, m, and s. Therefore, C is not an MDS symbol-triple γ-constacyclic code when $j = 0$.

Case 2: If $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq j \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then we have symbol-triple distance $d_{st}(C) = 3(\delta+1)p^{\xi}$. Thus, C is an MDS symbol-triple γ-constacyclic code if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$, which is equivalent to $p^{3m(p^s-j)} = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $3j = 2\,d_{st}(C) - 3p^s - 6$. We see that

$$\begin{aligned}3j &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{ (equality when } \xi = s-1)\\ &\geq 6(\delta+1)p^{\xi} - 3p^s - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1 = \delta)\\ &\geq 2\,d_{st}(C) - 3p^s - 6 + 3\\ &> 2\,d_{st}(C) - 3p^s - 6.\end{aligned}$$

Thus, C is not an MDS symbol-triple γ-constacyclic code in this case. □

In the following, we consider the MDS condition for symbol-triple γ-constacyclic codes of Type 3 of length $3p^s$ over $\mathcal{R}$.

Theorem 9: Let $C = \langle (x^3 - \gamma_0)^j + u(x^3 - \gamma_0)^r v(x)\rangle$ be a symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$, where $1 \leq j \leq p^s - 1$, $0 \leq r < j$, and either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \gamma\rangle}$ or 0. Then C is an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$ if one of the following conditions holds true:

- If $v(x) = 0$
  - If $s = 1$, then $d_{st}(C) = 3(T+1)$ for $1 \leq T \leq p-1$.
  - If $s \geq 2$, then
    1. $T = 1$, then $d_{st}(C) = 6$,
    2. $T = p^s - 1$, then $d_{st}(C) = 3p^s$.
- If $v(x) \neq 0$
  - If $s = 1$, then $d_{st}(C) = 3(T+1)$ for $1 \leq T \leq p-1$.
  - If $s \geq 2$, then
    1. $T = 1$, then $d_{st}(C) = 6$,
    2. $T = p^s - 1$, $r = p^s - 2$, then $d_{st}(C) = 3p^s$.

Proof: We divide into two cases, namely, $v(x) = 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, and $v(x) \neq 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$.

Case 1: If $v(x) = 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then we have $d_{st}(C) = 3(\delta+1)p^{\xi}$. Thus, C is an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$ if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$ i.e., $p^{6m(p^s-j)} = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $3j = d_{st}(C) - 3$, i.e., $3T = d_{st}(C) - 3$.

Now we see that

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3(p^{\xi+1} - p + (\delta-1) + 1) \text{ (equality when } \xi = s-1)\\ &\geq 3(\delta+1)p^{\xi} - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1 = \delta)\\ &= d_{st}(C) - 3.\end{aligned}$$

It implies that C is an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$ if and only if $s = 1$ (in such case, $j = \delta$, $d_{st}(C) = 3(\delta+1)$), or $\delta = 1$, $\xi = 0$ (in such case, $j = 1$, $d_{st}(C) = 6$), or $\delta = p-1$, $\xi = s-1$ (in such case, $j = p^s - 1$, $d_{st}(C) = 3p^s$).

Case 2: If $v(x) \neq 0$ and $p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1 \leq T \leq p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, we consider the following subcases:

Subcase 1: If $1 \leq j \leq \frac{p^s+r}{2}$, then $T = j$. Also, C is an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$ if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $p^{6m(p^s-j)} = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $3j = d_{st}(C) - 3$, i.e., $3T = d_{st}(C) - 3$. Now we see that

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{(equality when } \xi = s-1)\\ &\geq 3(\delta+1)p^{\xi} - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1 = \delta)\\ &= d_{st}(C) - 3.\end{aligned}$$

Hence, C is an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$ if and only if $s = 1$ (in such case, $j = \delta$, $d_{st}(C) = 3(\delta+1)$), or $\delta = 1$, $\xi = 0$ (in such case, $j = 1$, $d_{st}(C) = 6$), or $\delta = p-1$, $\xi = s-1$ (in such case, $j = p^s - 1$, $r = p^s - 2$, $d_{st}(C) = 3p^s$).

Subcase 2: If $\frac{p^s+r}{2} < j \leq p^s - 1$, then $T = p^s - j + r$. Hence, C is an MDS symbol-triple code of Type 3 of length $3p^s$ over $\mathcal{R}$ if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $p^{3m(p^s-r)} = p^{2m(3p^s - d_{st}(C)+3)}$, which is equivalent to $3r = 2\,d_{st}(C) - 3p^s - 6$, i.e., $3p^s + 3r = 2\,d_{st}(C) - 6$, i.e., $3p^s - 3j + 3r = 2\,d_{st}(C) - 3j - 6$, i.e., $3T = 2\,d_{st}(C) - 3j - 6$. We see that

$$\begin{aligned}3T &\geq 3(p^s - p^{s-\xi} + (\delta-1)p^{s-\xi-1} + 1)\\ &\geq 3p^{\xi+1} - 3p + 3(\delta-1) + 3 \text{(equality when } \xi = s-1)\\ &\geq 6(\delta+1)p^{\xi} - 3p^s - 3(\delta+1) + 3(\delta-1) + 3\\ &\quad \text{(equality when } p-1 = \delta)\\ &\geq 2\,d_{st}(C) - 3p^s - 6 + 3\\ &\geq 2\,d_{st}(C) - 3(j+1) - 6 + 3.\\ &> 2\,d_{st}(C) - 3j - 6.\end{aligned}$$

Therefore, C is not an MDS symbol-triple γ-constacyclic code of Type 3 of length $3p^s$ over $\mathcal{R}$. □

Finally, we determine the MDS condition for symbol-triple γ-constacyclic codes of Type 4 of length $3p^s$ over $\mathcal{R}$.

Theorem 10: Let $C = \langle (x^3 - \gamma_0)^j + u(x^3 - \gamma_0)^r v(x), u(x^3 - \gamma_0)^{\omega}\rangle$ be a symbol-triple γ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$, where $1 \le j \le p^s - 1$, $0 \le r < j$, either $v(x)$ is a unit in $\frac{\mathbb{F}_{p^m}[x]}{\langle x^{3p^s} - \gamma \rangle}$ or 0, $\deg(v) \le \omega - r - 1$, $\omega < T$, and T is the smallest integer satisfying $u(x^3 - \gamma_0)^T \in \langle (x^3 - \gamma_0)^j + u(x^3 - \gamma_0)^r v(x) \rangle$, i.e., $T = j$, if $v(x) = 0$, otherwise $T = \min\{j, p^s - j + r\}$. Then C is not an MDS symbol-triple γ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$.

Proof: If $p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \le \omega \le p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then symbol-triple distance is $d_{st}(C) = 3(\delta + 1)p^{\xi}$. So, C is an MDS symbol-triple γ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$ if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $p^{3m(2p^s - j - \omega)} = p^{2m(3p^s - d_{st}(C)+3)}$, i.e., $3\omega = 2\, d_{st}(C) - 3j - 6$. We see that

$$\begin{aligned} 3\omega &\ge 3(p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1) \\ &\ge 3p^{\xi+1} - 3p + 3(\delta - 1) + 3 \text{ (equality when } \xi = s - 1) \\ &\ge 6(\delta + 1)p^{\xi} - 3p^{\xi+1} - 3(\delta + 1) + 3(\delta - 1) + 3 \\ &\quad \text{(equality when } p - 1 = \delta) \\ &\ge 2\, d_{st}(C) - 3p^s - 6 + 3 \\ &\ge 2\, d_{st}(C) - 3j - 6. \end{aligned}$$

Thus, we see that if $\omega = p^s - 1, j = p^s - 1$, then $d_{st}(C) = d_{st}(\langle (x^3 - \gamma_0)^{\omega} \rangle_F) = 3p^s$. We also have $2\, d_{st}(C) - 3j - 6 = 6p^s - 3(p^s - 1) - 6 = 3(p^s - 1) = 3\omega$. Thus, equality holds when $\omega = p^s - 1$. However, $\omega < T \le p^s - 1$. It implies that $3\omega > 2\, d_{st}(C) - 3j - 6$. Therefore, C is not an MDS symbol-triple γ-constacyclic code of Type 4 of length $3p^s$ over $\mathcal{R}$. □

In Remark 1, we compute the symbol-triple distance of $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$. From this, we have the following remark.

Remark 2: Let $C \subseteq \mathcal{R}_{\alpha+u\beta} = \frac{\mathcal{R}[x]}{\langle x^{3p^s} - (\alpha + u\beta) \rangle}$, then $C = \langle (x^3 - \alpha_1)^j \rangle$ for $j \in \{0, 1, \ldots, 2p^s\}$. Then C is not an MDS symbol-triple $((\alpha + u\beta)$-constacyclic code of length $3p^s$ over $\mathcal{R}$.

Proof: From part 3 of Theorem 1, we have $|C| = p^{3m(2p^s - j)}$.

Case 1: When $0 \le j \le p^s$, by Remark 1, the symbol-triple distance is $d_{st}(C) = 3$. Hence, C is an MDS symbol-triple code if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$ i.e., $p^{3m(2p^s - j)} = p^{2m(3p^s - 3 + 3)}$, i.e., $6p^s - 3j = 6p^s$, i.e., $j = 0$. Thus, $C = \langle 1 \rangle$ is an MDS symbol-triple $(\alpha + u\beta)$-constacyclic code of length $3p^s$ over $\mathcal{R}$.

Case 2: When $2p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1 \le j \le 2p^s - p^{s-\xi} + \delta p^{s-\xi-1}$, then the symbol-triple distance is $d_{st}(C) = 3(\delta + 1)p^{\xi}$. Thus, C is an MDS symbol-triple code if and only if $|C| = p^{2m(3p^s - d_{st}(C)+3)}$ i.e., $p^{3m(2p^s - j)} = p^{2m(3p^s - d_{st}(C)+3)}$ i.e., $6p^s - 3j = 6p^s - d_{st}(C) + 3$ i.e., $3j = d_{st}(C) - 3$. Now, we have

$$\begin{aligned} 3j &\ge 3(2p^s - p^{s-\xi} + (\delta - 1)p^{s-\xi-1} + 1) \\ &\ge 6p^{\xi+1} - 3p + 3(\delta - 1) + 3 \text{ (equality when } \xi = s - 1) \\ &\ge 6(\delta + 1)p^{\xi} - 3(\delta + 1) + 3(\delta - 1) + 3 \\ &\quad \text{(equality when } p - 1 = \delta) \\ &> 3(\delta + 1)p^{\xi} - 3. \end{aligned}$$

Hence, $3j > d_{st}(C) - 3$. Thus, C is not an MDS symbol-triple code. □

Example 2: We provide some new MDS symbol-triple γ-constacyclic codes of length $3p^s$ over $\mathbb{F}_{p^m} + u\mathbb{F}_{p^m}$, where $\gamma \in \mathbb{F}_p^*$ and γ is not a cube as follows.

TABLE 3. New MDS symbol-triple γ-constacyclic codes over $\mathbb{F}_7 + u\mathbb{F}_7$, $\mathbb{F}_{13} + u\mathbb{F}_{13}$ and $\mathbb{F}_{19} + u\mathbb{F}_{19}$.

n	s	γ	$\langle g(x) \rangle$	$[n, M, d_{st}]$
21	1	3	$\langle (x^3 - 3)^2 \rangle$	$[21, 7^{30}, 9]$
39	1	2	$\langle (x^3 - 2)^2 \rangle$	$[39, 13^{66}, 9]$
57	1	2	$\langle (x^3 - 2)^6 \rangle$	$[57, 19^{78}, 21]$
57	1	3	$\langle (x^3 - 3)^8 \rangle$	$[57, 19^{66}, 24]$

V. CONCLUSION

The metrics of constacyclic codes have very significant role in error-correcting coding theory. In [15], we completely determined the Hamming distance of γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$. In this paper, the symbol-triple distances of all γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$ are determined (Theorems 3-5). The symbol-triple distance of $(\alpha + u\beta)$-constacyclic codes of length $3p^s$ over $\mathcal{R}$ is given in Remark 1, where $(\alpha + u\beta)$ is not a cube in $\mathcal{R}$. Example 1 gives us some examples of symbol-triple distance γ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where γ is not a cube in $\mathbb{F}_{p^m}$. We provide the necessary and sufficient conditions for MDS symbol-triple codes of length $3p^s$ over $\mathcal{R}$ (Theorems 7-10 and Remark 2). Some new MDS symbol-triple γ-constacyclic codes of lengths $21, 39, 57$ over $\mathbb{F}_7 + u\mathbb{F}_7$, $\mathbb{F}_{13} + u\mathbb{F}_{13}$ and $\mathbb{F}_{19} + u\mathbb{F}_{19}$ are shown in Example 2.

For future work, it will be very interesting to study symbol-triple distance of λ-constacyclic codes of length $3p^s$ over $\mathcal{R}$, where λ is a cube in $\mathcal{R}$. In a near future, we will discuss the b-symbol metrics for all constacyclic codes of length $3p^s$ over $\mathcal{R}$ and as an application, we will identify all MDS constacyclic codes of length $3p^s$ with respect to b-symbol distances.

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