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RESEARCH ARTICLE

Optimal Constructions of Low-Hit-Zone Frequency-Hopping Sequence Sets via Cyclotomy

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ABSTRACT In complex environments of wireless communication, to improve the communication anti-interference performance, quasi-synchronous frequency-hopping (FH) communication system requires low-hit-zone (LHZ) FH sequence sets with optimal Hamming correlation (HC) properties and flexible parameters. In this paper, based upon cyclotomy theory, two classes of LHZ FH sequence sets with flexible parameters not included in the literatures are designed, and the periodic HC properties of which are analyzed. It turns out that the designed LHZ FH sequence sets are optimal on the maximum periodic HC. And thus the new constructions can be used to provide optimal FH sequence sets for quasi-synchronous FH communication system.

INDEX TERMS Hamming correlation, cyclotomy, frequency-hopping sequence, frequency-hopping communication, quasi-synchronous frequency-hopping.

I. INTRODUCTION

The signals sent and received in complex electromagnetic environments are easy to be intercepted and interfered by the interferers. The main purpose of communication anti-interference is to weaken or destroy the usage performance of the interferers' wireless communication system and to ensure the normal use of ours [1], [2], [3]. Frequency-hopping (FH) communication technology, which is a kind of effective communication anti-interference measure, has been now applied extensively in radio anti-jamming, wireless mobile communication, sonar, modern radar and other electronic systems [4], [5]. In FH communication system, frequency hopper usually consists of frequency synthesizer, FH frequency table and FH sequence. According to FH sequence, frequency hopper takes out the frequency control codes from the FH frequency table to control the frequency synthesizer, and then produces the carrier signal frequencies. Obviously, the carrier signal frequencies hop in a more wide frequency band under the control of FH sequence. In FH network, each user is given an FH sequence, on the basis of which, each sender transmits a message along with the switching frequencies in every time slot, and the corresponding receiver receives the signals under the control of the same FH sequence. There usually exists signal interference measured by the so-called Hamming autocorrelation property of FH sequence if only one FH sequence is employed by all users and there often exists another kind of signal interference measured by the so-called Hamming crosscorrelation property of FH sequences otherwise. In the literatures [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], Hamming correlations (HCs) always are used to measure the performances of FH communication system, such as synchronization, anti-interference, multiple access networking, and so on. The designs of FH sequences with

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optimal HCs are always one of the most active research topics in FH communication research field.

Different from the conventional FH sequence sets, low-hit-zone FH sequence sets exist a low-hit-zone called LHZ in which the HCs are relatively small. The time benchmark is not set strict limits in quasi-synchronous (QS) FH communication system, by using optimal LHZ FH sequence sets, even if there exists relative time delay between different users, the mutual interferences will be reduced to acceptable level provided that the time delay is kept within the LHZ. In addition to QS FH communication system, LHZ FH sequence sets also have great application value in multi-user radar and sonar system. Thus, the studies of LHZ FH sequence sets have significant value. The seminal studies can be retrospect to the work of Peng et al. [18] in which the proverbial Peng-Fan-Lee bounds used to evaluate the maximum periodic HC property of LHZ FH sequence sets are established. After that, according to the Peng-Fan-Lee bounds [18], many optimal LHZ FH sequence sets have been designed. For example, based on m-sequence, the first designs of optimal LHZ FH sequence sets ware presented by Ma and Sun [19] in 2011, and then Han et al. [20] and Zhou et al. [21] gave several designs of optimal LHZ FH sequence sets in 2016 and in 2017, respectively. In addition, by interleaving techniques, Niu et al. [22] presented two designs of optimal LHZ FH sequence sets in 2012 and constructed a kind of optimal LHZ FH sequence sets with large capacity in 2013 [23]. What's more, through cartesian product, Chung and Yang [24] gave some designs of optimal LHZ FH sequence sets in 2013, Zhou et al. [25] presented generalized methods to design several classes of optimal LHZ FH sequence sets in 2017, and Niu et al. [26] presented extension interleaved constructions of optimal LHZ FH sequence sets in 2019.

Cyclotomy is the effective method to construct FH sequences. Many optimal conventional FH sequence sets have been designed based on cyclotomy theory so far [6], [8], [15], [16], [17]. But only two classes of optimal LHZ FH sequence sets based on cyclotomy theory and discrete logarithm function were designed by Wang et al. [27] in 2016. Different from the known designs, we present new methods to design optimal LHZ FH sequence sets with flexible parameters based on cyclotomy theory in this paper. As a result, our designs can offer many optimal LHZ FH sequence sets with new parameters for the QS FH communication system.

The organizations of this paper are as follows: Section II gives the terms and definitions; Section III gives the optimal designs of LHZ FH sequence sets based on cyclotomy theory; and Section IV gives the summary of full paper.

II. PRELIMINARIES

For the convenience of expression, the following notations are used in this paper:

- $(L, N, q, Z, \mathcal{M}(u))$: A LHZ FH sequence set u including N FH sequences with length L, the maximum periodic HC is $\mathcal{M}(u)$ within LHZ of size Z, and the frequency slot set includes q frequencies.
- $\mathcal{M}(X, \tau)$: The maximum periodic HC of X at the time delay τ.
- $\otimes$: The cartesian product.
- $\lceil x \rceil$: The least integer greater than or equal to x.
- $U \cap V$: The intersection set of U and V.
- $U \cup V$: The union set of U and V.
- $\langle x \rangle_L$: The least positive integer of x modulo L.

Let $F = \{f_0, f_1, \ldots, f_{q-1}\}$ be any frequency slot set and $X = \{X_k = (X_k(0), X_k(1), \ldots, X_k(L-1)) | k = 0, 1, \ldots, N-1\}$ be an FH sequence set over F. Assume $X_i = (X_i(0), X_i(1), \ldots, X_i(L-1))$ and $X_j = (X_j(0), X_j(1), \ldots, X_j(L-1))$ are any two FH sequences in X, at the time delay τ where $1 \le \tau < L$ if $X_i = X_j$ and $0 \le \tau < L$ if $X_i \ne X_j$, the periodic HC $H_{X_i,X_j}(\tau)$ of X_i and X_j is defined by

$$H_{X_i,X_j}(\tau) = \sum_{t=0}^{L-1} h(X_i(t), X_j(\langle t+\tau \rangle_L)) \tag{1}$$

where $h(X_i(t), X_j(\langle t+\tau \rangle_L)) = 1$ if $X_i(t) = X_j(\langle t+\tau \rangle_L)$ and $h(X_i(t), X_j(\langle t+\tau \rangle_L)) = 0$ otherwise.

The maximum periodic HC $\mathcal{M}(X)$ of the FH sequence set X is defined as

$$\mathcal{M}(X) = \max\{\max_{1\le\tau<L}\{H_{X_0,X_0}(\tau)|\forall X_0 \in X\}, \max_{0\le\tau<L}\{H_{X_0,X_1}(\tau)|\forall X_0 \ne X_1 \in X\}\}.$$

Let $h_a > 0$ and $h_c > 0$, the LHZ Z of X is defined by

$$Z = \min\{\max_{X_i\in X}\max_{1<\tau\le z_1}\left\{z_1|H_{X_i,X_i}(\tau) \le h_a\right\}, \max_{X_i\ne X_j\in X}\max_{0\le\tau\le z_2}\left\{z_2|H_{X_i,X_j}(\tau) \le h_c\right\}\}.$$

An FH sequence set is called LHZ FH sequence set if $Z > 0$. Assume X is any LHZ FH sequence set, within LHZ, the maximum periodic Hamming autocorrelation (HAC) $\mathcal{M}_a(X)$, the maximum periodic Hamming crosscorrelation (HCC) $\mathcal{M}_c(X)$ and the maximum periodic HC $\mathcal{M}(X)$ are defined by, respectively,

$$\mathcal{M}_a(X) = \max\{H_{X_i,X_i}(\tau), \forall X_i \in X, 1 \le \tau \le Z\},$$
$$\mathcal{M}_c(X) = \max\{H_{X_i,X_j}(\tau), \forall X_i \ne X_j \in X, 0 \le \tau \le Z\},$$
$$\mathcal{M}(X) = \max\{\mathcal{M}_a(X), \mathcal{M}_c(X)\}.$$

In 2006, Peng and Fan [18] obtained the following theoretical bound called Peng-Fan-Lee bound.

Lemma 1 (Peng-Fan-Lee Bound [18]): Assume X is any LHZ FH sequence set $(L, N, q, Z, \mathcal{M}(X))$ and $I = \lfloor \frac{LN}{q} \rfloor$. Then we have

$$\mathcal{M}(X) \ge \left\lceil \frac{(NZ+N-q)L}{(NZ+N-1)q} \right\rceil \tag{2}$$

Definition 1: Let $(L, N, q, Z, \mathcal{M}(X))$ be any LHZ FH sequence set X. For the maximum periodic HC, X is said to be optimal if $\mathcal{M}(X)$ let the equality in (2) hold, and to be almost optimal if $\mathcal{M}(X) - 1$ let the equality in (2) hold.

Definition 2: Let $s_0 = (s_0(0), s_0(1), \ldots, s_0(L-1))$ *and* $s_1 = (s_1(0), s_1(1), \ldots, s_1(L-1))$ *are any two FH sequences,* s_i *is called the shift equivalence sequence of* s_0 *if there exists integer* k *satisfying*

$$s_0(i) = s_1(i+k)$$

for any i *with* $0 \le i < L$. k *is called the equivalence distance.*

Definition 3: Any FH sequence set S *is called shift equivalence FH sequence set if for any sequence* s_i *in* S*, there exists another sequence* s_j *in* S *which is the shift equivalence sequence of* s_i*.*

III. NEW CLASSES OF OPTIMAL LHZ FH SEQUENCE SETS

Let p be any odd prime, F_p be a finite field and p can be rewritten as $p = \theta\beta + 1$ where θ and β are positive integers. Assume α is a primitive element of F_p, all the elements of which are presented by the power of α. Classical cyclotomic class [28] $\mathcal{C}_\vartheta$ is defined as follow:

$$\mathcal{C}_\vartheta = \{\alpha^{\vartheta + r\theta} | 0 \le r \le \beta - 1\}.$$

Assume F_p^* denotes the multiplicative group of F_p, there are few difficulties to verify that

$$\mathcal{C}_i \cap \mathcal{C}_j = \emptyset,\ i \ne j \text{ and } \cup_{\vartheta=0}^{\theta-1} \mathcal{C}_\vartheta = F_p^*.$$

There exist some properties on classical cyclotomic class as follows.

Lemma 2: $(\mathcal{C}_i+\eta)\cap\mathcal{C}_j = \frac{1}{\eta}(\mathcal{C}_i+1)\cap\frac{1}{\eta}\mathcal{C}_j = (\mathcal{C}_{i+h}+1)\cap\mathcal{C}_{j+h}$ *for* $\eta \ne 0$, $\frac{1}{\eta} \in \mathcal{C}_h$.

Lemma 3: $\sum_{i=0}^{\theta-1} |(\mathcal{C}_{i+k}+1)\cap\mathcal{C}_i| = \beta - 1$ *if* $k = 0$ *and* $\sum_{i=0}^{\theta-1} |(\mathcal{C}_{i+k}+1)\cap\mathcal{C}_i| = \beta$ *otherwise.*

We then design two classes of optimal LHZ FH sequence sets with flexible parameters based on classical cyclotomy.

For fixed t' with $0 \le t' \le p-2$, we define

$$\mathcal{D}_i^{t'} = \alpha^{t'}\mathcal{C}_i \bmod p, \quad 0 \le i \le \theta - 1.$$

Obviously, $\mathcal{D}_0^{t'}, \mathcal{D}_1^{t'}, \ldots, \mathcal{D}_{\theta-1}^{t'}$ form a partition of F_p^*, and then we have the following conclusions.

Lemma 4: I. $\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{t+k}+\tau)\cap\mathcal{D}_j^t| = \beta - 1$ *if* $k = 0$ *and* $\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{t+k}+\tau)\cap\mathcal{D}_j^t| = \beta$ *otherwise,* $\tau \ne 0$.

II. $\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{t+k}\cap\mathcal{D}_j^t| = \theta\beta$ *if* $k = 0$ *and* $\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{t+k}\cap\mathcal{D}_j^t| = 0$ *otherwise.*

Proof: Let $\tau = \alpha^w$, $\alpha^{t+k-w} \in \mathcal{C}_h$, $\alpha^{t-w} \in \mathcal{C}_{h'}$, $\alpha^{t+k} \in \mathcal{C}_e$, $\alpha^t \in \mathcal{C}_{e'}$. According to Lemma 2 and Lemma 3, we have

$$\begin{aligned}
&\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{t+k}+\tau)\cap\mathcal{D}_j^t| \\
&\quad= \sum_{j=0}^{\theta-1} |(\alpha^{t+k-w}\mathcal{C}_j+1)\cap\alpha^{t-w}\mathcal{C}_j| \\
&\quad= \sum_{j=0}^{\theta-1} |(\mathcal{C}_{j+h}+1)\cap\mathcal{C}_{j+h'}| \\
&\quad= \begin{cases} \beta - 1 & \text{if } k = 0, \\ \beta & \text{otherwise.} \end{cases}
\end{aligned}$$

and

$$\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{t+k}\cap\mathcal{D}_j^t| = \sum_{j=0}^{\theta-1} |\mathcal{C}_{j+e}\cap\mathcal{C}_{j+e'}| = \begin{cases} \theta\beta & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The results follow. □

Assume integer $q \ge 2$, p is any odd prime satisfying $\gcd(q, p) = 1$, $G = \{0, 1, \ldots, q-1\}$ and $y = (y(0), y(1), \ldots, y(qp-1))$ is a sequence over $U = \{0, 1, \ldots, pq-1\}$. In the sequence y, the support of the element $\rho \in U$ is defined as $Supp_y(\rho)$, such that

$$Supp_y(\rho) = \{t | y(t) = \rho, t = 0, 1, \ldots, qp-1\}.$$

Any element $t \in U$ can be denoted by two dimensional order pair $(\langle t\rangle_q, \langle t\rangle_p) = (t_1, t_2) \in G \otimes F_p$. So, the $Supp_y(\rho)$ can be rewritten as

$$Supp_y(\rho) = \{(t_1, t_2) | y(t) = \rho, t_1 = \langle t\rangle_q, t_2 = \langle t\rangle_p\}.$$

Construction $\mathbb{A}$*:* Design of optimal LHZ FH sequence set $(qp, \theta, \theta+1, p-1, q\beta)$.

Step 1: Let θ, β be two positive integers. And assume $p = \theta\beta + 1$ is an odd prime, α is a primitive element of the finite field F_p. Select a sequence set μ over $Z' = \{1, \alpha, \ldots, \alpha^{\theta-1}\}$, such that

$$\mu = \{\mu_i = (\mu_{i,0}, \mu_{i,1}, \ldots, \mu_{i,q-1}) | i = 0, 1, \ldots, \theta - 1\}.$$

Moreover, μ satisfies $\mathcal{M}(\mu, 0) = 0$ and $\mathcal{M}(\mu) < q$.

Step 2: Let $Z_\theta = \{0, 1, \ldots, \theta-1\}$, design the LHZ FH sequence set $S_a = \{s_i | i = 0, 1, \ldots, \theta-1\}$ where $s_i = (s_i(0), s_i(1), \ldots, s_i(qp-1))$ is defined as follows:

$$\begin{aligned}
Supp_{s_i}(\lambda) &= \{(t_0, t_1) | s_i(t) = \lambda, t_1 = \langle t\rangle_q, t_2 = \langle t\rangle_p\} \\
&= \cup_{\vartheta=0}^{q-1}(\{\vartheta\} \otimes \mathcal{D}_\lambda^{\mu_{i,\vartheta}}),\ \text{any } \lambda \in Z_\theta
\end{aligned}$$

if $t \ne 0 \bmod p$, and

$$Supp_{s_i}(\theta) = \cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\}$$

otherwise.

Theorem 1: With respect to the Peng-Fan-Lee bound (2), S_a *designed by Construction* $\mathbb{A}$ *is an optimal LHZ FH sequence set* $(qp, \theta, \theta+1, p-1, q\beta)$ *if* $q\theta\beta < (\theta p - 1)(q + \theta + 1 - q\beta)$.

Proof: Let $s_{\mu_{i_0}}$ and $s_{\mu_{i_1}}$ be any two FH sequences in S_a. The periodic HC at the time delay τ can be expressed as

$$\begin{aligned}
&H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) \\
&= \sum_{j=0}^{\theta-1} | \cup_{\vartheta=0}^{q-1}(\{\vartheta\} \otimes \mathcal{D}_j^{\mu_{i_1,\vartheta}}) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{\tau\})| \\
&\quad + |(\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\} + \tau) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| \\
&\quad + \sum_{j=0}^{\theta-1} |(\cup_{\vartheta=0}^{q-1}(\{\vartheta\} \otimes \mathcal{D}_j^{\mu_{i_0,\vartheta}}) + \tau) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| \\
&\quad + \sum_{j=0}^{\theta-1} |(\cup_{\vartheta=0}^{q-1}(\{\vartheta\} \otimes \mathcal{D}_j^{\mu_{i_0,\vartheta}}) + \tau) \cap (\cup_{\vartheta=0}^{q-1}(\{\vartheta\} \otimes \mathcal{D}_j^{\mu_{i_1,\vartheta}}))|.
\end{aligned}$$

Furthermore, let $\tau_1 = \langle \tau \rangle_q$, $\tau_2 = \langle \tau \rangle_p$, we can get

$$H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) = |(\cup_{\vartheta=0}^{q-1}\{\vartheta'\} \otimes \{\tau_2\}) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}|$$

where $\vartheta' = \langle \vartheta + \tau_1 \rangle_q$.

Case 1: $i_0 \neq i_1$.

Case 1. 1: $\tau_1 = 0, \tau_2 = 0$. Since $\mathcal{M}(\mu, 0) = 0$, $\mu_{i_0,\vartheta} \neq \mu_{i_1,\vartheta}$. One can verify that

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\}) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| \\ &\quad + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{\mu_{i_0,\vartheta}} \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| \\ &= q. \end{aligned}$$

Case 1. 2: $\tau_1 = 0, \tau_2 \neq 0$. We can get from the Lemma 4,

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) &= \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| \\ &\leq q\beta. \end{aligned}$$

Case 1. 3: $\tau_1 \neq 0, \tau_2 = 0$. We can have

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{q-1}\{\vartheta'\} \otimes \{0\}) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| \\ &\quad + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| \\ &= q + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| \\ &\leq q + \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_1,\vartheta})\theta\beta \\ &= q + H_{\mu_{i_0},\mu_{i_1}}(\tau_1)\theta\beta \\ &\neq qp. \end{aligned}$$

The third-to-last inequality holds because of $|\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| = 0$ when $\mu_{i_0,\vartheta'} \neq \mu_{i_1,\vartheta}$. The last inequality holds because of $\mathcal{M}(\mu) \neq q$.

Case 1. 4: $\tau_1 \neq 0, \tau_2 \neq 0$. One can have

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_1}}}(\tau) &= \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| \\ &= (q - \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_1,\vartheta}))\beta \\ &\quad + \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_1,\vartheta})(\beta - 1) \\ &= (q - H_{\mu_{i_0},\mu_{i_1}}(\tau_1))\beta + H_{\mu_{i_0},\mu_{i_1}}(\tau_1)(\beta - 1) \\ &= q\beta - H_{\mu_{i_0},\mu_{i_1}}(\tau_1). \end{aligned}$$

The third-to-last equality holds because of $\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| = \beta$ if $\mu_{i_0,\vartheta'} \neq \mu_{i_1,\vartheta}$ and $\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_1,\vartheta}}| = \beta - 1$ otherwise.

Thus, the maximum periodic HCC $\mathcal{M}_c(S_a)$ of S_a within LHZ can be given as

$$\mathcal{M}_c(S_a) = q\beta.$$

Case 2: $i_0 = i_1$.

Case 2. 1: $\tau_1 = 0, \tau_2 \neq 0$. We can get from the Lemma 4,

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_0}}}(\tau) &= \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_0,\vartheta}}| \\ &\leq q(\beta - 1). \end{aligned}$$

Case 2. 2: $\tau_1 \neq 0, \tau_2 = 0$. One has

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_0}}}(\tau) &= |(\cup_{\vartheta=0}^{q-1}\{\vartheta'\} \otimes \{0\}) \cap (\cup_{\vartheta=0}^{q-1}\{\vartheta\} \otimes \{0\})| \\ &\quad + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_0,\vartheta}}| \\ &= q + \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_0,\vartheta}}| \\ &\leq q + \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_0,\vartheta})\theta\beta \\ &= q + H_{\mu_{i_0},\mu_{i_0}}(\tau_1)\theta\beta \\ &\neq qp. \end{aligned}$$

The third-to-last inequality holds because of $|\mathcal{D}_j^{\mu_{i_0,\vartheta'}} \cap \mathcal{D}_j^{\mu_{i_0,\vartheta}}| = 0$ when $\mu_{i_0,\vartheta'} \neq \mu_{i_0,\vartheta}$. The last inequality holds because of $\mathcal{M}(\mu) \neq q$.

Case 2. 3: $\tau_1 \neq 0, \tau_2 \neq 0$. One can verify that

$$\begin{aligned} H_{s_{\mu_{i_0}}, s_{\mu_{i_0}}}(\tau) &= \sum_{\vartheta=0}^{q-1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{\mu_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{\mu_{i_0,\vartheta}}| \\ &= (q - \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_0,\vartheta}))\beta \\ &\quad + \sum_{\vartheta=0}^{q-1} h(\mu_{i_0,\vartheta'}, \mu_{i_0,\vartheta})(\beta - 1) \\ &= (q - H_{\mu_{i_0},\mu_{i_0}}(\tau_1))\beta + H_{\mu_{i_0},\mu_{i_0}}(\tau_1)(\beta - 1) \\ &= q\beta - H_{\mu_{i_0},\mu_{i_0}}(\tau_1). \end{aligned}$$

Thus, the maximum periodic HAC $\mathcal{M}_a(S_a)$ of S_a within LHZ can be given by

$$\mathcal{M}_a(S_a) = q\beta.$$

In conclusion, the maximum periodic HC $\mathcal{M}(S_a)$ of S_a within LHZ can be given as

$$\mathcal{M}(S_a) = q\beta.$$

According to the Peng-Fan-Lee bound (2), we use all the parameters of S_a as the input parameters, the value λ_{opt} on the right side of (2) can be given by

$$\begin{aligned}\lambda_{opt} &= \left\lceil \frac{(\theta p-(\theta+1))qp}{(\theta p-1)(\theta+1)} \right\rceil \\ &= \left\lceil \frac{qp}{\theta+1} - \frac{\theta qp}{(\theta p-1)(\theta+1)} \right\rceil \\ &= \left\lceil q\beta - \frac{q(\theta p-1)(\beta-1)+q\theta\beta}{(\theta p-1)(\theta+1)} \right\rceil .\end{aligned}$$

Since $q\theta\beta < (\theta p-1)(q+\theta+1-q\beta)$, one can verify that

$$0 < \frac{q(\theta p-1)(\beta-1)+q\theta\beta}{(\theta p-1)(\theta+1)} < 1$$

which leads to

$$\lambda_{opt} = q\beta.$$

It is obvious that the equality in the Peng-Fan-Lee bound (2) holds, so the conclusion is true. □

Example 1: Let $p = 43$, $q = 4$, $\theta = 14$, $\beta = 3$ and $\alpha = 3$ be a primitive element of the finite field F_{43}. Select a sequence set μ over $Z = \{1, \alpha, \cdots, \alpha^{13}\} = \{1, 3, 9, 27, 13, 19, 14, 7, 21, 20, 10, 5, 4, 2\}$, such that

$$\begin{aligned}\mu = \{&(1, 3, 1, 9), (3, 1, 3, 26), (9, 26, 9, 2), (26, 9, 26, 3),\\ &(13,19,13,14),(19,13,19,7),(14,7,14,13),(7,14,7,19),\\ &(21, 20,21, 10),(20, 21,20, 5),(10, 5,10, 4),(5, 2, 5,20),\\ &(4, 10, 4, 1), (2, 4, 2, 21)\}.\end{aligned}$$

By the construction $\mathbb{A}$, we obtain an LHZ FH sequence set S_a as follows:

$$\begin{aligned}\mathrm{S}_a = \{&\mathcal{S}_{a,0} = (14, 13, 13, 13, 12, 10, 0, 5, 11, 1, 10, 0, 13, 3,\\ &6,10, 10,9, 1,3, 9,7, 1,0, 12,7, 3,1, 5,2, 11,4, 9,2, \ldots),\\ &\ldots\\ &\mathcal{S}_{a,13} = (14, 12,10,0,9, 9,11,6, 8, 0, 7, 1, 10,2, 3,11, 7,\\ &8, 12, 4, 6, 6, 12, 1, 9, 6, 0, 2, 2, 11, 8, 5, 6, 1, 6, \ldots).\}.\end{aligned}$$

As the FIGURE 1 shows, the maximum periodic HC of S_a within LHZ equates to 12. It can check that S_a is an optimal LHZ FH sequence set (172,14,15,42,12) with respect to the Peng-Fan-Lee bound (2). Moreover, the maximum periodic partial HCs of S_a are shown in the FIGURE 2. And on the maximum periodic partial HC, S_b is optimal for some correlation window length L with $158 \le L \le 172$.

Construction $\mathbb{B}$: Construction of optimal LHZ FH sequence set $(2p, \theta, \theta, p-1, 2(\beta+1))$.

Step 1: Assume θ, β are two positive integers. And let $p = \theta\beta+1$ be an odd prime, α be a primitive element of the finite field F_p. Select a sequence set v over $Z' = \{1, \alpha, \ldots, \alpha^{\theta-1}\}$, such that

$$\mathrm{v} = \{v_i = (v_{i,0}, v_{i,1}) | i = 0, 1, \ldots, \theta-1\}$$

which satisfies $\mathcal{M}(v, 0) = 0$ and $\mathcal{M}(v) < 2$.

Step 2: Let $Z_\theta = \{0, 1, \ldots, \theta-1\}$, design the desirable FH sequence set $S_b = \{s_i | i = 0, 1, \ldots, \theta-1\}$ where $s_i = (s_i(0), s_i(1), \ldots, s_i(2p-1))$ is defined by

$$\begin{aligned}Supp_{s_i}(\lambda) &= \{(t_0, t_1) | s_i(t) = \lambda, t_0 = \langle t\rangle_2, t_1 = \langle t\rangle_p\}\\ &= \cup_{\vartheta=0}^{1}(\{\vartheta\} \otimes \mathcal{D}_\lambda^{v_{i,\vartheta}}),\ \text{any } \lambda \in Z_\theta\end{aligned}$$

if $t \not\equiv 0 \bmod p$, and

$$Supp_{s_i}(g) = \cup_{\vartheta=0}^{1}\{\vartheta\} \otimes \{0\},\ \text{any } g \in Z_\theta$$

otherwise.

Theorem 2: For the maximum periodic HC, according to the Peng-Fan-Lee bound (2), S_b designed by Construction $\mathbb{B}$ is an almost optimal LHZ FH sequence set $(2p, \theta, \theta, p-1, 2(\beta+1))$ if β is even.

Proof: We now assume $s_{v_{i_0}}$ and $s_{v_{i_1}}$ are any two FH sequences in S_b. There are the following two cases.

Case 1: $i_0 \ne i_1$. The periodic HCC at the time delay τ can be given by

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{1}\{\vartheta\} \otimes \{0\} + \tau) \cap (\cup_{\vartheta=0}^{1}\{\vartheta\} \otimes \{0\})|\\ &\quad + \sum_{j=0}^{\theta-1} |(\cup_{\vartheta=0}^{1}(\{\vartheta\} \otimes \mathcal{D}_j^{v_{i_0,\vartheta}}) + \tau)\\ &\quad \cap \cup_{\vartheta=0}^{1} (\{\vartheta\} \otimes \mathcal{D}_j^{v_{i_1,\vartheta}})| + \Delta\end{aligned}$$

where

$$\begin{aligned}\Delta = &|\cup_{\vartheta=0}^{1} (\{\vartheta\} \otimes \mathcal{D}_g^{v_{i_1,\vartheta}}) \cap \{\tau\}|\\ &+ |\cup_{\vartheta=0}^{1} (\{\vartheta\} \otimes \mathcal{D}_g^{v_{i_1,\vartheta}}) \cap \{p+\tau\}|\\ &+ |\cup_{\vartheta=0}^{1} (\{\vartheta\} \otimes \mathcal{D}_g^{v_{i_0,\vartheta}}) \cap \{2p-\tau\}|\\ &+ |\cup_{\vartheta=0}^{1} (\{\vartheta\} \otimes \mathcal{D}_g^{v_{i_0,\vartheta}}) \cap \{p-\tau\}|\end{aligned}$$

It can check that $\langle\tau\rangle_2 = \langle 2p-\tau\rangle_2$, $\langle p+\tau\rangle_2 = \langle p-\tau\rangle_2$ and the odd-even property of which is opposite. Let $\tau_2 = \langle\tau\rangle_p$ and $\vartheta' = \langle\vartheta+\tau\rangle_2$. Since τ_2 and $-\tau_2$ belong to the sole cyclotomic class respectively, we obtain

$$\begin{aligned}\Delta &= |(\cup_{\vartheta=0}^{1}\{\vartheta\} \cap \{\langle\tau\rangle_2\}) \otimes \cup_{\vartheta=0}^{1}(\mathcal{D}_g^{v_{i_1,\vartheta}} \cap \{\tau_2\})|\\ &\quad + |(\cup_{\vartheta=0}^{1}\{\vartheta\} \cap \{\langle p+\tau\rangle_2\}) \otimes \cup_{\vartheta=0}^{1}(\mathcal{D}_g^{v_{i_1,\vartheta}} \cap \{\tau_2\})|\\ &\quad + |(\cup_{\vartheta=0}^{1}\{\vartheta\} \cap \{\langle 2p-\tau\rangle_2\}) \otimes \cup_{\vartheta=0}^{1}(\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\})|\\ &\quad + |(\cup_{\vartheta=0}^{1}\{\vartheta\} \cap \{\langle p-\tau\rangle_2\}) \otimes \cup_{\vartheta=0}^{1}(\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\})|\\ &= |\cup_{\vartheta=0}^{1} (\mathcal{D}_g^{v_{i_1,\vartheta}} \cap \{\tau_2\})| + |\cup_{\vartheta=0}^{1} (\mathcal{D}_g^{v_{i_1,\vartheta'}} \cap \{\tau_2\})|\\ &\quad + |\cup_{\vartheta=0}^{1} (\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\})| + |\cup_{\vartheta=0}^{1} (\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{-\tau_2\})|\\ &= \sum_{\vartheta=0}^{1}(|\mathcal{D}_g^{v_{i_1,\vartheta}} \cap \{\tau_2\}| + |\mathcal{D}_g^{v_{i_1,\vartheta'}} \cap \{\tau_2\}|)\\ &\quad + \sum_{\vartheta=0}^{1}(|\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{-\tau_2\}|)\\ &= |\mathcal{D}_g^{v_{i_1,\vartheta}} \cap \{\tau_2\}| + |\mathcal{D}_g^{v_{i_1,\vartheta'}} \cap \{\tau_2\}|\\ &\quad + |\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{-\tau_2\}|.\end{aligned}$$

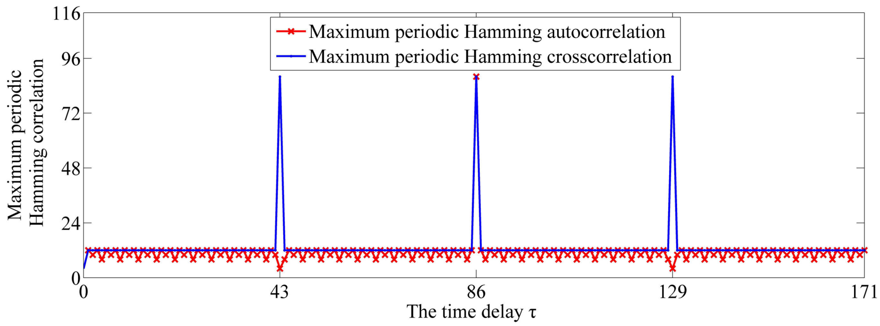

FIGURE 1. The maximum periodic HCs of S_a in example 1.

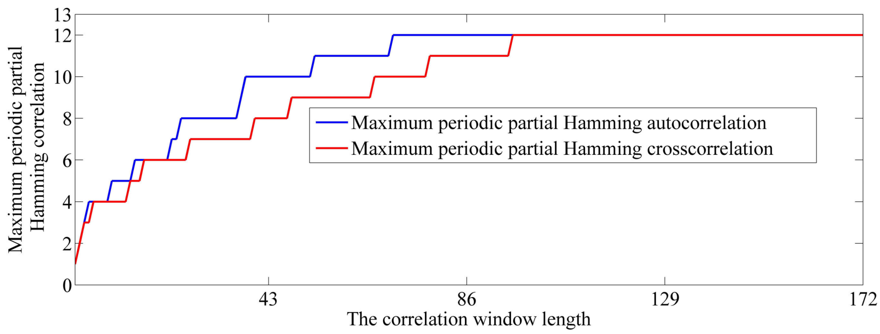

FIGURE 2. The maximum periodic partial HCs of S_a in example 1.

Therefore, we have

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{1}\{\vartheta'\}\otimes\{\tau_2\})\cap(\cup_{\vartheta=0}^{1}\{\vartheta\}\otimes\{0\})| \\ &+\sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1}|(\mathcal{D}_j^{v_{i_0,\vartheta'}}+\tau_2)\cap\mathcal{D}_j^{v_{i_1,\vartheta}}| \\ &+|\mathcal{D}_g^{v_{i_1,\vartheta}}\cap\{\tau_2\}|+|\mathcal{D}_g^{v_{i_1,\vartheta'}}\cap\{\tau_2\}| \\ &+|\mathcal{D}_g^{v_{i_0,\vartheta}}\cap\{-\tau_2\}|+|\mathcal{D}_g^{v_{i_0,\vartheta'}}\cap\{-\tau_2\}|. \qquad (3)\end{aligned}$$

According to (3), it can be divided into the following four cases to discuss.

Case 1. 1: $\langle\tau\rangle_2=0$, $\tau_2=0$. One can have

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{1}\{\vartheta\}\otimes\{0\})\cap(\cup_{\vartheta=0}^{1}\{\vartheta\}\otimes\{0\})| \\ &= 2.\end{aligned}$$

Case 1. 2: $\langle\tau\rangle_2=0$, $\tau_2\neq0$. We have

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1}|(\mathcal{D}_j^{v_{i_0,\vartheta}}+\tau_2)\cap\mathcal{D}_j^{v_{i_1,\vartheta}}| \\ &+2|\mathcal{D}_g^{v_{i_1,\vartheta}}\cap\{\tau_2\}|+2|\mathcal{D}_g^{v_{i_0,\vartheta}}\cap\{-\tau_2\}|.\end{aligned}$$

Let $\tau_2=\alpha^{\sigma}$, $0\leq\sigma<p-1$. Since $\alpha^{\frac{p-1}{2}}=-1$, one can check that $-\tau_2=\alpha^{\frac{p-1}{2}+\sigma}$ and $\frac{p-1}{2}+\sigma\equiv\frac{\theta\beta}{2}+\sigma\equiv\sigma \bmod \theta$ for β is even. So, τ_2 and $-\tau_2$ belong to the same cyclotomic class. For $\vartheta=0,1$, $v_{i_1,\vartheta}\neq v_{i_0,\vartheta}$ for $\mathcal{M}(v,0)=0$. Thus

$$H_{s_{v_{i_0}},s_{v_{i_1}}}(t)\leq2\beta+2=2(\beta+1).$$

Case 1. 3: $\langle\tau\rangle_2\neq0$, $\tau_2=0$. According to the proof of the Case 1. 3 in Theorem 1, one can verify that

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= |(\cup_{\vartheta=0}^{1}\{\vartheta'\}\otimes\{0\})\cap(\cup_{\vartheta=0}^{1}\{\vartheta\}\otimes\{0\})| \\ &+\sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1}|\mathcal{D}_j^{v_{i_0,\vartheta'}}\cap\mathcal{D}_j^{v_{i_1,\vartheta}}| \\ &\leq 2+H_{v_{i_0},v_{i_1}}(\tau)\theta\beta \\ &\neq 2p.\end{aligned}$$

Case 1. 4: $\langle\tau\rangle_2\neq0$, $\tau_2\neq0$. We have

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &= \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1}|(\mathcal{D}_j^{v_{i_0,\vartheta'}}+\tau_2)\cap\mathcal{D}_j^{v_{i_1,\vartheta}}| \\ &+|\mathcal{D}_g^{v_{i_1,\vartheta}}\cap\{\tau_2\}|+|\mathcal{D}_g^{v_{i_1,\vartheta'}}\cap\{\tau_2\}| \\ &+|\mathcal{D}_g^{v_{i_0,\vartheta}}\cap\{-\tau_2\}|+|\mathcal{D}_g^{v_{i_0,\vartheta'}}\cap\{-\tau_2\}|.\end{aligned}$$

According to the proof of Case 1. 2, there exist the following three cases:

Case 1. 4. 1: $v_{i_1,\vartheta}\neq v_{i_0,\vartheta'}$, $\vartheta=0,1$. We can get

$$H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau)\leq2\beta+2=2(\beta+1).$$

Case 1. 4. 2: $v_{i_1,\vartheta}=v_{i_0,\vartheta'}$, $\vartheta=0,1$. One can have

$$\begin{aligned}H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) &\leq \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1}|(\mathcal{D}_j^{v_{i_1,\vartheta}}+\tau_2)\cap\mathcal{D}_j^{v_{i_1,\vartheta}}|+4 \\ &= 2(\beta+1).\end{aligned}$$

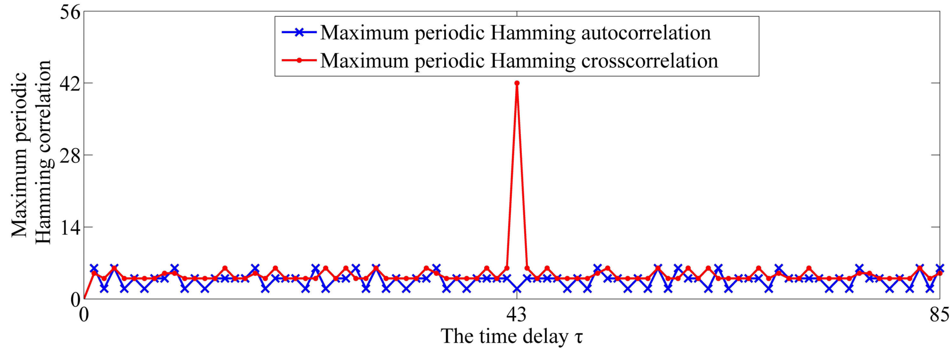

FIGURE 3. The maximum periodic HCs of S_b in example 2.

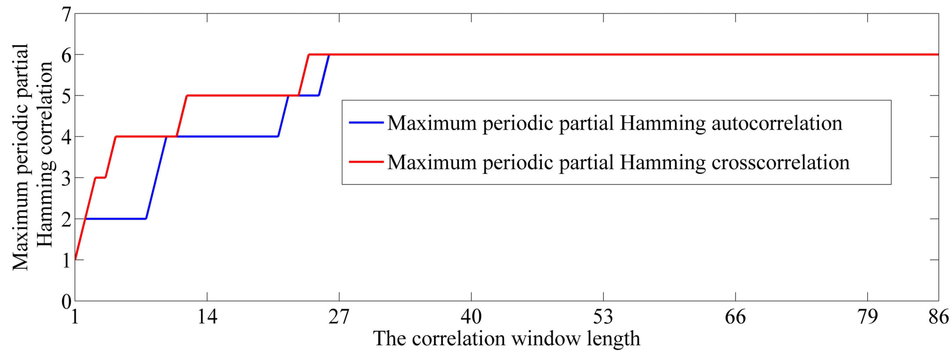

FIGURE 4. The maximum periodic partial HCs of S_a in example 2.

Case 1. 4. 3: Without losing generality, assume $v_{i_0,1} \neq v_{i_1,0}$ and $v_{i_0,0} = v_{i_1,1}$. One can get

$$\begin{aligned} H_{s_{v_{i_0}},s_{v_{i_1}}}(\tau) \leq & \sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{v_{i_0,1}} + \tau_2) \cap \mathcal{D}_j^{v_{i_1,0}}| \\ & + \sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{v_{i_0,0}} + \tau_2) \cap \mathcal{D}_j^{v_{i_0,0}}| + 3 \\ = & \ 2(\beta+1). \end{aligned}$$

The maximum periodic HCC $\mathcal{M}_c(S_b)$ of S_b within LHZ can be given as

$$\mathcal{M}_c(S_b) = 2(\beta+1).$$

Then we discuss the periodic HAC of S_b.

Case 2: $i_0 = i_1$. The periodic HAC at the time delay τ can be given by

$$\begin{aligned} H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) = & \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{v_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{v_{i_0,\vartheta}}| \\ & + |\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{\tau_2\}| \\ & + |\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{-\tau_2\}| \\ & + |(\cup_{\vartheta=0}^{1}\{\vartheta'\} \otimes \{\tau_2\}) \cap (\cup_{\vartheta=0}^{1}\{\vartheta\} \otimes \{0\})| \end{aligned}$$

There exist the following three cases:

Case 2. 1: $\langle\tau\rangle_2 = 0$, $\tau_2 \neq 0$. We have

$$\begin{aligned} H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) = & \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{v_{i_0,\vartheta}} + \tau_2) \cap \mathcal{D}_j^{v_{i_0,\vartheta}}| \\ & + 2|\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{\tau_2\}| + 2|\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\}|. \end{aligned}$$

It is easy to check that

$$H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) \leq 2(\beta-1)+4 = 2(\beta+1).$$

Case 2. 2: $\langle\tau\rangle_2 \neq 0$, $\tau_2 = 0$. One can verify that

$$\begin{aligned} H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) = & \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1} |\mathcal{D}_j^{v_{i_0,\vartheta'}} \cap \mathcal{D}_j^{v_{i_0,\vartheta}}| \\ \leq & \ 2 + H_{v_{i_0},v_{i_0}}(\tau)\theta\beta \\ \neq & \ 2p. \end{aligned}$$

Case 2. 3: $\langle\tau\rangle_2 \neq 0$, $\tau_2 \neq 0$. We can obtain

$$\begin{aligned} H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) = & \sum_{\vartheta=0}^{1}\sum_{j=0}^{\theta-1} |(\mathcal{D}_j^{v_{i_0,\vartheta'}} + \tau_2) \cap \mathcal{D}_j^{v_{i_0,\vartheta}}| \\ & + |\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{\tau_2\}| \\ & + |\mathcal{D}_g^{v_{i_0,\vartheta}} \cap \{-\tau_2\}| + |\mathcal{D}_g^{v_{i_0,\vartheta'}} \cap \{-\tau_2\}|. \end{aligned}$$

It can be divided into the following two cases to discuss.

Case 2. 3. 1: $v_{i_0,\vartheta} \neq v_{i_0,\vartheta'}$. We get

$$H_{s_{v_{i_0}},s_{v_{i_0}}}(\tau) \leq 2\beta+2 = 2(\beta+1).$$

TABLE 1. Comparisons between some known optimal LHZ FH sequence sets and new ones.

Ref.	Parameters	Constraints	Shift Equivalence (Y or N)
[19]	$(q(q^n-1), q\|I\|, q, Z, q^n)$	$q^n-1=M(Z+1), M\neq 1$	Y
	$(q(q^n-1), q^n-1, q, q-1, q^n)$	-	Y
	$(s(q^n-1), M, q, Z, s(q^{n-1}-1))$	$q^n-1=M(Z+1), \gcd(s, q^n-1)=1, s<M$	Y
[20]	$(q^n-1, q^k(q-1), q^k, \frac{q^n-1}{q-1}-1, q^{n-k})$	$1\leq k\leq n$	Y
	$(\frac{q^n-1}{t}, q-1, q^k, \frac{q^n-1}{q-1}-1, \frac{q^{n-k}-1}{t})$	$1\leq k\leq n, t\|(q-1), \gcd(t,n)=1$	Y
[22]	$(T\frac{q^n-1}{l}, Ml, q, w-1, T\frac{q^{n-1}-1}{l})$	$l\|(q^n-1), \frac{q^n-1}{l}=Mw, \gcd(l,n)=1$ $T=\lambda w+1, \lambda\geq 1, T<\frac{lq(q^n-2)}{q-1}$	N
	$(T\frac{r(q^n-1)}{l}, M\lfloor\frac{l}{r}\rfloor, q, w-1, T\frac{r(q^{n-1}-1)}{l})$	$l\|(q^n-1), 2\leq r\leq l, \frac{r(q^n-1)}{l}=Mw, \gcd(l,n)=1$ $T=\lambda w+1, \lambda\geq 1, T<\frac{lq(q^n-2)}{q-1}$	N
	$(T(q^n-1), Mq^k, q^k, w-1, T(q^{n-k}-1))$	$1\leq k\leq n, q^n-1=Mw$ $T=\lambda w+1, \lambda\geq 1, T<\frac{q^k(q^n-2)}{q^k-1}$	N
[23]	$(Tn, Ml, Q, w-1, TH_m)$	$T\geq 2, Mw=n, T=\lambda w+1$	N
[24]	$(p^2(q_1-1), pq_1, pq_1, \min\{p^2-1, q_1-2\}, p)$	$\gcd(p^2, q_1-1)=1, 2p\leq q_1-1$	N
	$(p^2(q_1-1)(q_2-1), pq_1q_2, pq_1q_2,$ $\min\{p^2-1, q_1-2, q_2-2\}, p)$	$\gcd(p, q_1-1, q_2-1)=1, 3p<\min\{q_1-2, q_2-2\}$	N
	$(\frac{(q_1-1)\cdots(q_k-1)}{d}, q_1\cdots q_k, q_1\cdots q_k, q_1-1, 1)$	$2<q_1<\cdots<q_k, d=\gcd(q_1, q_2, ..., q_k)$	N
	$(k_1k_2(q_1-1)(q_2-1), \frac{(q_1-1)(q_2-1)}{k_1k_2},$ $q_1q_2, \min\{q_1-1, q_2-1\}, k_1k_2)$	$k_1\|q_1-1, k_2\|q_2-1, \gcd(k_1(q_1-1), k_2(q_2-1))=1$ $k_1(q_1-1)<k_2(q_2-1)$ and $q_1>k_1k_2^2+2k_1k_2$ or $k_1(q_1-1)>k_2(q_2-1)$ and $q_2>k_2k_1^2+2k_1k_2$	N
[25]	$((q_1-1)(q_2^n-1), q_1, q_1q_2, q_2^n-2, q_2^{n-1}-1)$	$q_1>q_2^n, \gcd(q_1-1, q_2^n-1)=1$	N
	$(p^2(p^2-1), p, p^2, p^2-2, p(p-1))$	$\gcd(p^2, p^2-1)=1$	N
[27]	$(l(p^n-1), e, e+1, p^n-2, lf)$	$p^n-1=ef, lfe^2<(fe^2-1)(e+1-lf)$	N
	$(l(p^n-1), e, e+1, \frac{p^n-1-m}{m}, lf)$	$p^n-1=ef, lfe^2m<(fe^2-m)(e+1-lf)$	N
Th.2	$(lp, \theta, \theta+1, p-1, l\beta)$	$p-1=\theta\beta, l\theta p<(\theta p-1)(l+\theta+1-l\beta)$	N
Th.3	$(2p, \theta, \theta, p-1, 2(\beta+1))$	$p-1=\theta\beta$, β is even	N

• q is prime power. $p, q_1, q_2, ..., q_k$ are prime numbers. $Q, k, k_1, n, m, w, t, r, l, M, T, Z, \theta, \beta$ are positive integers.

Case 2. 3. 2: $v_{i_0,\vartheta}=v_{i_0,\vartheta'}$. One can have

$$H_{s_{v_{i_0}}, s_{v_{i_0}}}(\tau)\leq 2(\beta-1)+4=2(\beta+1).$$

The maximum periodic HAC $\mathcal{M}_a(S_b)$ of S_b within LHZ can be given as

$$\mathcal{M}_a(S_b)=2(\beta+1).$$

Thus, the maximum periodic HC $\mathcal{M}(S_b)$ of S_b within LHZ can be given by

$$\mathcal{M}(S_b)=2(\beta+1).$$

The optimality of S_b is checked according to Peng-Fan-Lee bound (2) as follow:

$$\left\lceil\frac{2(p-1)p}{\theta p-1}\right\rceil=\left\lceil 2\beta+\frac{2\beta}{\theta p-1}\right\rceil=2\beta+1.$$

Obviously, $\mathcal{M}(S_b)-1$ let the equality in (2) hold. So, the conclusion follows. □

Example 2: Let $p=43$, $\theta=21$, $\beta=2$ and $\alpha=3$ be a primitive element of the finite field F_{43}. And select a sequence set v over $Z=\{1,\alpha,\cdots,\alpha^{20}\}=\{1, 3, 9, 16, 5, 15, 2, 6, 18, 11, 10, 13, 4, 12, 7, 21, 20, 17, 8, 19, 14\}$, such that

$$\begin{aligned}v=\{&(1,3), (3,9), (9,16), (16,5), (5,15), (15,2), (2,6),\\ &(6,18), (18,11), (11,10), (10,13), (13,4), (4,12), (12,7),\\ &(7,21), (21,20), (20,17), (17,8), (8,19), (19,1)\}.\end{aligned}$$

By the construction $\mathbb{B}$, we can obtain an LHZ FH sequence set S_b as follows:

$$\begin{aligned}S_b\\ =\{&\mathcal{S}_{b,0}=(0, 20, 6, 0, 12, 3, 7, 13, 18, 1, 10, 8, 13, 10, 20,\\ &4, 3, 16, 8, 18, 16, 14, 15, 15, 19, 7, 17, 2, 5, 19, 11, \ldots),\\ &\mathcal{S}_{b,1}=(0, 19, 5, 20, 11, 2, 6, 12, 17, 0, 9, 7, 12, 9, 19,\\ &3, 2, 15, 7, 17, 15, 13, 14, 14, 18, 6, 16, 1, 4, 18, 10, \ldots),\\ &\cdots\\ &\mathcal{S}_{b,19}=(0, 2, 9, 3, 15, 6, 10, 16, 0, 4, 13, 11, 16, 13, 2,\\ &7, 6, 19, 11, 0, 19, 17, 18, 18, 1, 10, 20, 5, 8, 1, 14, 15, \ldots),\\ &\mathcal{S}_{b,20}=(0, 0, 8, 1, 14, 4, 9, 14, 20, 2, 12, 9, 15, 11, 1, 5,\\ &5, 17, 10, 19, 18, 15, 17, 16, 0, 8, 19, 3, 7, 20, 13, 13, \ldots).\}.\end{aligned}$$

As shown in the FIGURE 3, it is clear that the maximum periodic HC of S_b equates to 6 when $0\leq\tau\leq 42$. It is easy to verify that S_b is an almost optimal LHZ FH sequence set (86, 21, 21, 42, 6) according to Peng-Fan-Lee bound (2). Besides, the maximum periodic partial HCs of S_b are shown in the FIGURE 4. For some correlation window length L with $72\leq L\leq 86$, S_b is optimal on the maximum periodic partial HC.

IV. CONCLUSION

In this paper, we designed two new classes of LHZ FH sequence sets with optimal maximum periodic HC property based on cyclotomic theory. The new LHZ FH sequence sets have new parameters not included in the literatures (comparisons between some known optimal LHZ FH sequence sets and new ones are listed in Table 1), and which may be used to provide the selection sequences for QS FH communications.

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