IEEE Access
Multidisciplinary ⁝ Rapid Review ⁝ Open Access Journal

Received 9 December 2022, accepted 16 January 2023, date of publication 2 February 2023, date of current version 7 February 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3241804

RESEARCH ARTICLE

Consensus Disturbance Rejection of Linear Discrete-Time Multi-Agent Systems With Communication Noises

YUANYUAN JIA, JINGJING WANG, AND JIAMAN BO
School of Finance and Mathematics, Huainan Normal University, Huainan, Anhui 232001, China

Corresponding author: Yuanyuan Jia (46311182@qq.com)

This work was supported in part by the Key Scientific Research Projects of Huainan Normal University under Grant 2021XJZD022 and Grant 2021XJZD004, in part by the Key Projects of Scientific Research Preparation Plan in Anhui Province under Grant 2022AH051585, in part by the Natural Science Fund of Colleges and Universities in Anhui Province under Grant KJ2021A0966, and in part by the Natural Science Fund of Colleges and Universities in Anhui Province under Grant KJ2021A0967.

ABSTRACT This paper studies consensus disturbance rejection problem of discrete-time multi-agent systems with communication noises. The involved agents are modeled by linear dynamics with disturbance, and are assumed to receive information corrupted by additive communication noises receiving from their neighbours. The control aim, not only is to guarantee the average state of the agents that is subject to deterministic disturbance, but also is to exclude the communication noises. In order to achieve this control aim, some disturbance observers are designed under two kinds of stochastic-approximation type control gains. By utilizing the martingale convergence theorem and some mathematical techniques, two sufficient conditions are obtained to ensure consensus and disturbance rejection in mean square. Finally, a numerical example is given to show the applicability of considered methods.

INDEX TERMS Consensus control, disturbance rejection, multi-agent systems, communication noises.

I. INTRODUCTION

Consensus is a typical problem in cooperation control of multi-agent systems. The so-called consensus usually stands for all the agents reaching a common agreement as time evolves [1]. In consensus control, the important progress is to design an appropriate control strategy according to local information of an agent from its neighbours. This control strategy is a kind of distributed control that has a wide range of applications, such as distributed filtering, distributed estimation, formation control and distributed optimization [2], [3], [4], [5], [6], [7]. Therefore, the consensus control problem has naturally attracted increasing interests in relative research fields [8], [9], [10], [11], [12], [13], [14].

When considering the consensus of multi-agent systems, disturbance rejection is one of basic requirements [15], [16], [17]. It not only expands the applicability of the control theory, but also has been developed by tremendous applications, including navigation, robotics and process control [18], [19], [20]. The disturbance rejection in these applications, appeared as disturbance suppression, external disturbance, output regulation or antidisturbance, is essentially an approach for estimating the disturbance [21], [22], [23], [24], [25]. This approach presents a requirement that has a capacity to cancel the disturbance according to remodeling the output or state measurements as a new control input [13], [20]. From this perspective, the disturbance rejection can be regraded as a kind of observer involving the internal model of disturbance. In distributed control, the disturbance rejection naturally relates to distributed observer that is formed by both the disturbance and measurement of relative state. For example, in [16] and [17], some kinds of disturbed disturbance rejection control have been applied to studying networked systems by utilizing event-triggered and adaptive control strategies, respectively. In [21], mismatched disturbance has been introduced into output consensus problem of higher-order multi-agent systems. Then, some works have considered the disturbance rejection into some practical problems, such as formation control [25], [26].

In addition to disturbance rejection, there is an other requirement to implement distributed control strategy in

The associate editor coordinating the review of this manuscript and approving it for publication was Laura Celentano.

many existing works [9], [11], [12], which is assumed that all the agents can send and receive the signals through an ideal communication channel. It is to say that all the agents can get accurate state information from their neighbouring agents. However, signal transmission among the agents may be subject to a noisy process in digital implementations. The reason caused this phenomenon is some random influences that spontaneously exist in electric devices or other complex environments. In networked control systems, these random influences can be modeled by communication noises. For sake of excluding such communication noises, the primary aim is to design an appropriate control gain that is called the stochastic-approximation type gain in many existing works [27], [28], [29], [30]. The main idea of this control gain is to construct some sequences of martingale difference such that the communication noises can be transformed into a certain stationary process. It further means that the effect of the communication noises is converged under the action of such control gain. Recently, many works have focused on this problem [31], [32], [33], [34], [35]. For example, in [27], consensus problems have been studied for multi-agent systems with communication noises, in which several kinds of communication topologies have been considered, respectively. In [30], leader-following consensus convergence rate has been discussed for multi-agent systems with communication noises. In [33], a kind of communication noise, named as networked attack, has been introduced into distributed resilient fault-tolerant control via an observer-based strategy.

As mentioned above, the disturbance and communication noises are unavoidable phenomena in the distributed control problem. Although the existing works related to such topic have been obtained many remarkable results [15], [28], [29], they can not be applied to the disturbance and communication noises at the same time. In this case, how to design an appropriate distributed disturbance observer with the stochastic-approximation type gain that, not only can estimate and cancel the disturbance, but also can be utilized to deal with the communication noises, has still some challenges. On the other hand, compared with the continuous-time systems, discrete-time behaviors of agents, especially general linear dynamics, play an important factor to understand the cooperation control mechanism, and also have many advantages in digitized signal of communication process [6]. However, how to analyze such general linear discrete-time dynamics in the case of the disturbance and communication noises are still some difficulties. The primary difficulties are to derive the relationship among the general linear discrete-time dynamics, the disturbance and communication noises, such that this relationship can be utilized to represent the final control aim.

Motivated by above considerations, disturbance and communication noise of general linear discrete-time multi-agent systems have been considered in this paper, where the disturbance and communication noises are molded by an exosystem and a sequence of additive noises, respectively. Then, an observer-based consensus control has been introduced to estimate the disturbance and the state of each agent. Meanwhile, two kinds of stochastic-approximation type control gains have been applied to excluding the influence of communication noises, respectively. Under the estimated results and martingale convergence theorem, two sufficient conditions have been derived to ensure consensus and disturbance rejection in mean square, which means that consensus among the agents can be guaranteed. Finally, the applicability of obtained results is shown by an example. The mainly contributions of this paper are listed as follows: (1) The additive communication noises have been introduced into the consensus disturbance rejection problem of linear discrete-time multi-agent systems. (2) Two kinds of stochastic-approximation type control gains have been utilized to deal with the consensus disturbance rejection problem with communication noises.

Notations: Let $\mathbb{R}^{N\times M}$ and $\mathbb{R}^n$ be the set of $N \times M$ real matrix and n-dimensional Euclidean space, respectively. $\mathbb{N}^+$ stands for the set of positive natural numbers. I_N means the $N\times N$ identity matrix. O_N presents the $N\times N$ null matrix. $\mathbf{1}_n$ and $\mathbf{0}_n$ represent the n-dimensional column identity and null vectors, respectively. For a vector $x \in \mathbb{R}^n$ and a matrix $X \in \mathbb{R}^{N\times N}$, $\|x\| = x^Tx$ and $\|X\| = \sqrt{\lambda_N(X^TX)}$, where the superscript T stands for the transposition of a vector or matrix, and $\lambda_i(\cdot)$ is the i-th smallest eigenvalue. For any matrix A with appropriate dimensions, $\mathrm{Sym}(A) = A + A^T$. The symbol $\otimes$ stands for the Kronecker product of matrix. For a given random variable ξ, the corresponding mathematical expectation is given by $E[\xi]$.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. PRELIMINARIES

Denote a probability space as $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the set of events, $\mathcal{F}$ means σ-algebra on the set Ω, and $\mathcal{P}$ is the corresponding probability. For a sequence of random variables $\{\xi(k), k = 1, 2, \ldots\}$, the σ-algebra $\sigma\{\{\xi \in \Delta\}, \Delta \in \mathcal{B}, k = 1, 2, \ldots\}$ is denoted by $\sigma\{\xi(k), k = 1, 2, \ldots\}$, where $\mathcal{B}$ presents a Borel set. If a random variable is $\mathcal{F}$-measurable, ξ is adapted to a σ-algebra $\mathcal{F}$. Define $\{\zeta(k), k = 1, 2, \ldots\}$ as an adapted sequence of random variables, then, $\{\zeta(k), k = 1, 2, \ldots\}$ is said to be a *martingale* relative to the filtration $\{\xi(k), k = 1, 2, \ldots\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$, if for each k, $E[\zeta(k+1)|\xi(k)] = \zeta(k)$ [36]. Correspondingly, the adapted sequence $\{\gamma(k), k = 1, 2, \ldots\}$ is said to be a *martingale difference* relative to the filtration $\{\xi(k), k = 1, 2, \ldots\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$, if for each k, $E[\gamma(k+1)|\xi(k)] = 0$, a.s., and $E|\gamma(k+1)| < \infty$. By simple construction, one can imply $\gamma(k+1) = \zeta(k+1) - \zeta(k)$ for a martingale $\{\zeta(k), k = 1, 2, \ldots\}$.

The communication topology is given by a weighted undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, \ldots, N\}$ is the set of N nodes, $\mathcal{E} = e(i,j)$ means the edge set, and $\mathcal{A} = [a_{ij}]$ presents the corresponding adjacency matrix. For the adjacency matrix $\mathcal{A} = [a_{ij}]$, $a_{ij} = a_{ji} > 0$ if $e(i,j) \in \mathcal{E}$

(implies $e(j, i) \in \mathcal{E}$), otherwise $a_{ij} = 0$. The associated Laplacian matrix $\mathcal{L}$ of the graph $\mathcal{G}$ denotes as follows: for any $i \neq j$, $\ell_{ij} = \ell_{ji} = -a_{ij}$ if $e(i, j) \in \mathcal{E}$, otherwise $\ell_{ij} = \ell_{ji} = 0$, and $\ell_{ii} = \sum_{j=1,j\neq i}^{N} \ell_{ij}$. Through all this paper, the undirected graph $\mathcal{G}$ is assumed to be connected and simple, which means that all the eigenvalues of matrix $\mathcal{L}$ satisfy: $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \ldots \leq \lambda_N(\mathcal{L})$, where $\lambda_i(\mathcal{L})$ is the i-th smallest eigenvalue of matrix $\mathcal{L}$ [37].

B. PROBLEM STATEMENT

Now, consider a linear discrete-time agent with disturbance in an n-dimensional Euclidean space, the dynamics of agent i is given below:

$$x_i(k+1) = Ax_i(k) + Bu_i(k) + Dw_i(k), \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$ and $D \in \mathbb{N}^{n\times m}$ are constant matrices, $x_i(k) \in \mathbb{R}^n$ is the state vector at time instant $k \in \mathbb{N}^+$, $u_i(k) \in \mathbb{R}^p$ is the control input at time instant $k \in \mathbb{N}^+$, and $w_i(k) \in \mathbb{R}^m$ stands for a disturbance at time instant $k \in \mathbb{N}^+$ that is attributed to an exosystem with the form

$$w_i(k+1) = Sw_i(k), \tag{2}$$

with $S \in \mathbb{R}^{m\times m}$ being a known constant matrix.

The information available to agent i at each time instant $k \in \mathbb{N}^+$ is the relative measurements among its neighbours, denoted by $\zeta_i(k)$. Then, this relative information $\zeta_i(k)$ is given by

$$\zeta_i(k) = \sum_{j=i,j\neq i}^{N} a_{ij}(x_i(k) - y_{ji}(k)), \tag{3}$$

where $y_{ji}(k) \in \mathbb{R}^n$ stands for the state information from agent j to agent i, this information is corrupted by communication noises $\xi_{ji}(k) \in \mathbb{R}^n$ with following form

$$y_{ji}(k) = x_j(k) + \xi_{ji}(k). \tag{4}$$

Similar to [15], assume that the disturbance (1) is matched, which means that there exists a matrix $F \in \mathbb{R}^{p\times m}$ such that $D = BF$ holds. Then, the control input $u_i(k)$ for disturbance rejection is designed by

$$u_i(k) = -K\chi_i(k) - F\varpi_i(k), \tag{5}$$

where K is a control gain matrix with a proper dimension such that $(A - BK)$ is Hurwitz, $\chi_i(k) \in \mathbb{R}^n$ and $\varpi_i(k) \in \mathbb{R}^m$ are the relative estimations of states $x_i(k)$ and $w_i(k)$, respectively. These estimations $\chi_i(k)$ and $\varpi_i(k)$ have the following form

$$\chi_i(k+1) = (A - BK)\chi_i(k) + c_i(k)\Big(\zeta_i(k) - \sum_{j=i,j\neq i}^{N} a_{ij}(\chi_i(k) - \chi_j(k))\Big), \tag{6}$$

$$\varpi_i(k+1) = S\varpi_i(k) + c_i(k)\Big(\zeta_i(k) - \sum_{j=i,j\neq i}^{N} a_{ij}(\chi_i(k) - \chi_j(k))\Big), \tag{7}$$

where $c_i(k)$ ($i \in \mathcal{V}$) is a stochastic-approximation type control gain designed later. Note that the relative estimations $\chi_i(k)$ and $\varpi_i(k)$ (6) and (7) can be regarded as a kind of consensus-based disturbance observers of the systems (1) and (2).

Denote $\varepsilon_i(k) = x_i(k) - \chi_i(k)$ and $\delta_i(k) = w_i(k) - \varpi_i(k)$. Then, from (1) to (7), the following systems can be obtained

$$\varepsilon_i(k+1) = A\varepsilon_i(k) + D\delta_i(k) - c_i(k)\Big(\sum_{j=i}^{N} \ell_{ij}\varepsilon_j(k) - \sum_{j=i,j\neq i}^{N} a_{ij}\xi_{ji}(k)\Big), \tag{8}$$

$$\delta_i(k+1) = S\delta_i(k) - c_i(k)\Big(\sum_{j=i}^{N} \ell_{ij}\varepsilon_j(k) - \sum_{j=i,j\neq i}^{N} a_{ij}\xi_{ji}(k)\Big). \tag{9}$$

Denote $e(k) = [\varepsilon(k), \delta(k)]^T$, where $\varepsilon(k) = [\varepsilon_1^T(k), \ldots, \varepsilon_N^T(k)]^T$ and $\delta(k) = [\delta_1^T(k), \ldots, \delta_N^T(k)]^T$. Thus, the systems (8) and (9) can be rewritten in the following Kronecker-product from

$$e(k+1) = \Big((I_N \otimes \overline{A}) - c(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)e(k) + c(k)(\Lambda \otimes I_{2n})\xi(k), \tag{10}$$

where $\xi(k) = [\xi_{11}^T(k), \ldots, \xi_{N1}^T(k), \xi_{12}^T(k), \ldots, \xi_{N2}^T(k), \ldots, \xi_{1N}^T(k), \ldots, \xi_{NN}^T(k),]^T \otimes I_2$, $c(k) = diag\{c_1(t), \ldots, c_N(t), c_1(t), \ldots, c_N(t)\} \otimes I_n$, $\Lambda = diag\{\mathcal{A}_1^T, \ldots, \mathcal{A}_N^T\} \in \mathbb{R}^{N\times N^2}$ with $\mathcal{A}_i$ being the i-th row of the adjacency matrix $\mathcal{A}$, and the matrices $\overline{A}$ and $\overline{\mathcal{I}}$ have the following form

$$\overline{A} = \begin{bmatrix} A & D \\ O_n & S \end{bmatrix}, \quad \overline{\mathcal{I}} = \begin{bmatrix} \widehat{\mathcal{I}} \\ \widehat{\mathcal{I}} \end{bmatrix} \quad \widehat{\mathcal{I}} = \begin{bmatrix} I_n & O_n \end{bmatrix}.$$

For sake of obtaining the main results, the following definition, lemmas and assumptions are required.

Definition 1: The multi-agent system (1) is said to achieve consensus disturbance rejection in mean square, if there exist two random vectors x^* and w^* such that, for $i \in \mathcal{V}$

$$\lim_{k\to\infty} E[\|x_i(k) - \chi_i(k) - x^*\|^2] = 0,$$
$$\lim_{k\to\infty} E[\|w_i(k) - \varpi_i(k) - w^*\|^2] = 0.$$

Lemma 1: If the pair (S, D) is observable, the pair $(\overline{\mathcal{A}}, \widehat{\mathcal{I}})$ is observable, with $\overline{\mathcal{A}}$ and $\widehat{\mathcal{I}}$ in (10).

Proof: This lemma can be regarded as a special case of Lemma 1 in [15]. Therefore, the proof is omitted here. ■

Note that the consensus-based disturbance observers (6) and (7) exist if Lemma 1 holds. It is to say that the states $x_i(k)$ and $w_i(k)$ can be estimated by using the observers (6) and (7).

Lemma 2: [27] Let $\{u(k), k = 1, 2, \ldots\}$, $\{\alpha(k), k = 1, 2, \ldots\}$ and $\{q(k), k = 1, 2, \ldots\}$ be real sequences, satisfying $0 \leq q(k) \leq 1$, $\alpha(k) \geq 0$, $k = 1, 2, \ldots$, $\sum_{k=1}^{\infty} q(k) = \infty$, $\alpha(k)/q(k) \to 0$, $k \to \infty$, and

$$u(k+1) \leq (1 - q(k))u(k) + \alpha(k).$$

Then, $\lim_{k\to\infty} u(k) \leq 0$. In particular, if $u(k) \geq 0$, $k = 1, 2, \ldots$, then $u(k) \to 0$, $k \to \infty$.

Assumption 1: $\{\xi_i(k), k \in \mathbb{N}^+, i \in \mathcal{V}\} \subset \mathcal{S}$, where $\mathcal{S} = \{\zeta | (\zeta \in \mathbb{R}^n, \mathcal{F}^{\zeta}(k)$ is a martingale difference, $\sigma_\zeta = \sup_{k>0} E[\|\zeta(k)\|^2] < \infty\}$.

Assumption 2: For any $i, j \in \mathcal{V}$, $c_i(k) = c_j(k) = \overline{c}(k) \leq 1$, $\sum_{k=1}^{\infty} \overline{c}(k) = \infty$, and $\lim_{k\to\infty} \overline{c}(k) = 0$.

Assumption 3: $\sum_{k=1}^{\infty} c_i(k) = \infty$, $\lim_{k\to\infty} c_i(k) = 0$, $\max_{i\leq i,j\leq N} |c_i(k) - c_j(k)| = o(\sum_{j=1}^{N} c_j(k))$ as $k \to \infty$, and $c_i(k) \leq 1$ for all k and i, where the operator $o(\cdot)$ means the infinitesimal of higher order.

It is worth pointing out that the above assumptions are primarily used to describe the parameters of the systems (8) and (9). Assumption 1 shows that communication noises $\xi_{ji}(k)$ are a bounded and adaptive process, which are more general than the case in [33]. Assumptions 2 and 3 will be utilized to design the control gain $c_i(k)$ in the rest of this paper.

Remark 1: When considering the cooperation problems of multi-agent systems, the consensus disturbance rejection is a basic requirement. Recently, there are many works that have been discussed about this topic [15], [16], [17], [22], [23]. However, some of these works have mainly concerned with the design of disturbance estimator rather than the communication noises. Compare with the results in [15], [16], and [17], the communication noises have been introduced into the disturbance rejection problem of linear discrete-time multi-agent systems in this paper. The communication noises are molded by a sequence of additive noises, which can reflect many potential properties of distributed control.

III. MAIN RESULTS

In this section, the consensus disturbance rejection problem with communication noises (10) is studied in mean square under the control input (5) and the relative estimations (6) and (7). In order to study this problem, two kinds of stochastic-approximation type control gains have been introduced. First, consider the case that for any $i, j \in \mathcal{V}$ and $k \in \mathbb{N}^+$, $c_i(k) = c_j(k)$, i.e., $c_i(k) = \overline{c}(k)$ for all $i \in \mathcal{V}$.

Theorem 1: Suppose that Assumptions 1 and 2 hold. The linear discrete-time multi-agent system (10) can realize the consensus disturbance rejection in mean square, if there exist a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ and a scalar $k \in \mathbb{N}^+$, such that the following conditions hold

$$\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P > 0, \tag{11}$$

$$\eta\overline{c}(k) \leq \overline{a} < 1, \tag{12}$$

$$2\overline{c}(k)\|\mathcal{L}\|^2 > \eta, \tag{13}$$

where $\overline{a} = \|\overline{A}\|^2 + \lambda_{2n}(P)$, and $\eta = \lambda_1(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)\lambda_2(\mathcal{L})$.

Proof: Denote $J = (1/N)\mathbf{1}_N\mathbf{1}_N^T$, and $\rho(k) = ((I_N - J)\otimes I_{2n})e(k)$. Consider the Lyapunov function $V(k) = \rho^T(k)\rho(k)$ for the consensus error of the system (10). Note that the undirected graph $\mathcal{G}$ is connected and simple, then $\mathcal{L} = \mathcal{L}^T$, $(I_N - J)(I_N - J) = (I_N - J)$ and $\mathbf{1}_N^T\mathcal{L} = 0_N$. One has

$$\begin{aligned}
\rho(k+1) &= \Big((I_N - J)\otimes I_{2n}\Big)e(k+1)\\
&= \Big((I_N - J)\otimes I_{2n}\Big)\Big[\Big((I_N\otimes\overline{A}) - \overline{c}(k)(\mathcal{L}\otimes\overline{\mathcal{I}})\Big)\\
&\quad\times e(k) + \overline{c}(k)(\Lambda\otimes I_{2n})\xi(k)\Big]\\
&= \Big[\Big((I_N - J)\otimes\overline{A}\Big) - \overline{c}(k)\Big(((I_N - J)\mathcal{L})\otimes\overline{\mathcal{I}}\Big)\Big]\\
&\quad\times e(k) + \overline{c}(k)\Big[\Big(((I_N - J)\Lambda)\otimes I_{2n}\Big)\Big]\xi(k)\\
&= \Big((I_N\otimes\overline{A}) - \overline{c}(k)(\mathcal{L}\otimes\overline{\mathcal{I}})\Big)\rho(k)\\
&\quad+ \overline{c}(k)\Big(((I_N - J)\Lambda)\otimes I_{2n}\Big)\xi(k)
\end{aligned}\tag{14}$$

and therefore, one can write

$$\begin{aligned}
V(k+1) &= \Big[\Big((I_N\otimes\overline{A}) - \overline{c}(k)(\mathcal{L}\otimes\overline{\mathcal{I}})\Big)\rho(k)\\
&\quad+ \overline{c}(k)\Big(((I_N - J)\Lambda)\otimes I_{2n})\Big)\xi(k)\Big]^T\\
&\quad\times\Big[\Big((I_N\otimes\overline{A}) - \overline{c}(k)(\mathcal{L}\otimes\overline{\mathcal{I}})\Big)\rho(k)\\
&\quad+ \overline{c}(k)\Big(((I_N - J)\Lambda)\otimes I_{2n})\Big)\xi(k)\Big]\\
&= \rho^T(k)\Big((I_N\otimes(\overline{A}^T\overline{A})) + \overline{c}^2(k)(\mathcal{L}^2\otimes\overline{\mathcal{I}}^T\overline{\mathcal{I}})\\
&\quad- \overline{c}(k)(\mathcal{L}\otimes\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}))\Big)\rho(k) + \rho^T(k)\\
&\quad\times\Big[\overline{c}(k)\Big(((I_N - J)\Lambda)\otimes\overline{A}^T\Big)\\
&\quad- \overline{c}^2(k)\Big((\mathcal{L}(I_N - J)\Lambda)\otimes\overline{\mathcal{I}}^T\Big)\Big]\xi(k)\\
&\quad+ \xi^T(k)\Big[\overline{c}(k)\Big((\Lambda(I_N - J))\otimes\overline{A}\Big)\\
&\quad- \overline{c}^2(k)\Big((\Lambda(I_N - J)\mathcal{L})\otimes\overline{\mathcal{I}}\Big)\Big]\rho(k)\\
&\quad+ \overline{c}^2(k)\xi^T(k)\Big((\Lambda(I_N - J)\Lambda)\otimes I_{2n}\Big)\xi(k).
\end{aligned}\tag{15}$$

From the properties of Laplacian matrix [37], there exists a unitary matrix $U = [u_1, u_2, \ldots, u_N] \in \mathbb{R}^{N\times N}$ such that $U^T\mathcal{L}U = diag\{0, \lambda_2(\mathcal{L}), \ldots, \lambda_N(\mathcal{L})\} \triangleq \Phi$, where $u_1 = 1/\sqrt{N}[1, 1, \ldots, 1]^T \in \mathbb{R}^N$ and $u_i = [u_{1i}, u_{2i}, \ldots, u_{Ni}]^T \in \mathbb{R}^N$ for $i = 2, \ldots, N$. Therefore, one constructs $z(k) = (U\otimes I_{2n})\rho(k)$, and gets the following equation

$$\begin{aligned}
&-\rho^T(k)\big(\mathcal{L}\otimes\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}})\big)\rho(k)\\
&\quad= -z^T(k)(U^T\otimes I_{2n})\big(\mathcal{L}\otimes\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}})\big)(U\otimes I_{2n})z(k)\\
&\quad= -z^T(k)\Big((U^T\mathcal{L}U)\otimes\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}})\Big)z(k)\\
&\quad= -z^T(k)(\Phi\otimes\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}))z(k).
\end{aligned}\tag{16}$$

According to (11), there exists a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$, such that $\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P$ is positive semidefinite. Thus, there also exists a unitary matrix $V = [v_1, v_2, \ldots, v_{2n}] \in \mathbb{R}^{2n\times 2n}$ such that $V^T(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)V = diag\{\lambda_1(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P), \ldots, \lambda_{2n}(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)\} \in$

$\mathbb{R}^{2n\times 2n} \triangleq \Psi$, where $v_i = [v_{1i}, v_{2i}, \ldots, v_{(2n)i}]^T \in \mathbb{R}^{2n}$ for $i = 1, \ldots, 2n$. Let $y(k) = (I_N \otimes V)z(k)$, it follows from (16)

$$\begin{aligned}&-z^T(k)(\Phi \otimes (\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P))z(k)\\ &\quad= -y^T(k)(I_N \otimes V^T)\big(\Phi \otimes (\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)\big)\\ &\qquad\times (I_N \otimes V)y(k)\\ &\quad= -y^T(k)\Big(\Phi \otimes (V^T(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)V)\Big)y(k)\\ &\quad= -y^T(k)(\Phi \otimes \Psi)y(k).\end{aligned} \tag{17}$$

Note that $\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}})+P$ is a positive definite matrix, and $\mathcal{L}$ is a positive semidefinite matrix. It further derives from (16) and (17)

$$\begin{aligned}&-E[\rho^T(k)(\mathcal{L} \otimes (\overline{A}^T\overline{\mathcal{I}} + \overline{\mathcal{I}}^T\overline{A} + P)\rho(k)]\\ &\quad= -E[y^T(k)(\Phi \otimes \Psi)y(k)]\\ &\quad\leq -\lambda_1(\mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}}) + P)\lambda_2(\mathcal{L})\\ &\qquad\times E[y^T(k)(U^TU \otimes V^TV)y(k)]\\ &\quad= -\eta E[V(k)].\end{aligned} \tag{18}$$

Based on the discussions from (16)-(18), taking mathematical expectation on both sides of (15), one has

$$\begin{aligned}E[V(k+1)] &\leq E[\rho^T(k)(\|\overline{A}\|^2 + \lambda_{2n}(P) - \eta\overline{c}(k)\\ &\quad + 2\overline{c}^2(k)\|\mathcal{L}\|^2)\rho(k) + \xi^T(k)\overline{c}^2(k)\|\Lambda\|^2\\ &\quad\times \|I_N - J\|^2\xi(k)]\\ &\leq (\overline{a} - \eta\overline{c}(k) + 2\overline{c}^2(k)\|\mathcal{L}\|^2)E[V(k)]\\ &\quad + \overline{c}^2(k)\|\Lambda\|^2\|I_N - J\|^2\sigma_\xi,\end{aligned} \tag{19}$$

where $\sigma_\xi = \sup_{k\in\mathbb{N}^+} E[\|\xi(k)\|^2]$ from Assumption 1.

Due to Assumptions 1 and 2, $\overline{c}(k) \to 0$ as $k \to \infty$, thus, one knows that there is a $k_0 > 0$, such that, for any $k \geq k_0$,

$$\overline{c}(k)\|\mathcal{L}\|^2 < \eta/2,$$

and from (11) and (12), one gets

$$\eta\overline{c}(k) \leq \overline{a} < 1.$$

In this case, one gets

$$0 \leq \overline{a} - \eta\overline{c}(k) + 2\overline{c}^2(k)\|\mathcal{L}\|^2 < 1, \tag{20}$$

which means that the following inequalities hold

$$\sum_{k=k_0}^{\infty}\left(\eta\overline{c}(k) - \overline{c}^2(k)\|\mathcal{L}\|^2\right) \geq \tfrac{\eta}{2}\textstyle\sum_{k=k_0}^{\infty}\overline{c}(k) = \infty, \tag{21}$$

and for $k \to \infty$,

$$\frac{\overline{c}^2(k)}{\eta} \to 0. \tag{22}$$

Together with (19), (21), (22) and Lemma 2, one gets $\lim_{k\to\infty} E[V(k)] = 0$. It completes the proof. ■

The above theorem studies the case of the control gain $\overline{c}(k)$ for all the agents. This condition may be not applied to some practical cases. For instance, there may be a small change between the control gain $\overline{c}(k)$ for some parts of the agents and the control gain $c_i(k)$ of the i-th agent. Thus, one has the following theorem to investigate this case.

Theorem 2: Suppose that Assumptions 1 and 3 hold. The linear discrete-time multi-agent system (10) can realize the consensus disturbance rejection in mean square, if there exists a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ and a scalar $k \in \mathbb{N}^+$, such that (11), (12), and the following condition hold

$$\mu(k) \leq 1, \tag{23}$$

where $\mu(k) = \overline{a} - \eta\widetilde{c}(k) + 2\widetilde{c}^2(k)\|\mathcal{L}\|^2 + (2\|\mathcal{A}\| + \varsigma\|\mathcal{L}\|\|\overline{\mathcal{I}}\|)\|\mathcal{L}\|\|\overline{\mathcal{I}}\|\|\Xi(k)\|$, $\varsigma = \max_i \sup_{k>0} c_i(k)$, $\widetilde{c}(k) = \sum_{i=1}^{N} c_i(k)/N$, and $\eta, \overline{a}$ are given in Theorem 1.

Proof: Denote $\Xi(k) = diag\{\Xi_1(k), \ldots, \Xi_N(k), \Xi_1(k), \ldots, \Xi_N(k)\} \otimes I_n$, where $\Xi_i(k) = \widetilde{c}(k) - c_i(k)$, and $c(k) = \widetilde{c}(k) - \Xi(k)$. Similar to (14), one has

$$\begin{aligned}\rho(k+1) &= \Big(I_N \otimes \overline{A} - \widetilde{c}(k)(\mathcal{L} \otimes \overline{\mathcal{I}}) + (J \otimes I_{2n})\\ &\quad\times \Xi(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)\rho(k) + ((I_N - J) \otimes I_{2n})\\ &\quad\times (\widetilde{c}(k)I_{2nN} - \Xi(k))(\Lambda \otimes I_{2n})\xi(k).\end{aligned} \tag{24}$$

Consider the same Lyapunov function in Theorem 1, one gets

$$\begin{aligned}V(k+1) &= \Big[\Big(I_N \otimes \overline{A} - \widetilde{c}(k)(\mathcal{L} \otimes \overline{\mathcal{I}}) + (J \otimes I_{2n})\\ &\quad\times \Xi(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)\rho(k) + ((I_N - J) \otimes I_{2n})\\ &\quad\times (\widetilde{c}(k)I_{2nN} - \Xi(k))(\Lambda \otimes I_{2n})\xi(k)\Big]^T\\ &\quad\times \Big[\Big(I_N \otimes \overline{A} - \widetilde{c}(k)(\mathcal{L} \otimes \overline{\mathcal{I}}) + (J \otimes I_{2n})\\ &\quad\times \Xi(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)\rho(k) + ((I_N - J) \otimes I_{2n})\\ &\quad\times (\widetilde{c}(k)I_{2nN} - \Xi(k))(\Lambda \otimes I_{2n})\xi(k)\Big]\\ &= \rho^T(k)\Big((I_N \otimes (\overline{A}^T\overline{A})) + \widetilde{c}^2(k)(\mathcal{L}^2 \otimes \overline{\mathcal{I}}^T\overline{\mathcal{I}})\\ &\quad - \widetilde{c}(k)(\mathcal{L} \otimes \mathrm{Sym}(\overline{A}^T\overline{\mathcal{I}})) + \mathrm{Sym}((\mathcal{L} \otimes \overline{\mathcal{I}}^T)\\ &\quad\times \Xi(k)(J \otimes \overline{A})) + (\mathcal{L} \otimes \overline{\mathcal{I}}^T)\Xi(k)(J \otimes I_{2n})\\ &\quad\times \Xi(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)\rho(k) + \rho^T(k)\Big(((I_N - J)\\ &\quad\otimes \overline{A}^T) - \widetilde{c}(k)(\mathcal{L} \otimes \overline{\mathcal{I}}^T)\Big)(\widetilde{c}(k)I_{2nN} - \Xi(k))\\ &\quad\times (\Lambda \otimes I_{2n})\xi(k) + \xi^T(k)(\Lambda \otimes I_{2n})(\widetilde{c}(k)I_{2nN}\\ &\quad - \Xi(k))\Big(((I_N - J) \otimes \overline{A}) - \widetilde{c}(k)(\mathcal{L} \otimes \overline{\mathcal{I}})\Big)\\ &\quad\times \rho(k) + \xi^T(k)(\Lambda \otimes I_{2n})(\widetilde{c}(k)I_{2nN} - \Xi(k))\\ &\quad\times ((I_N - J) \otimes I_{2n})(\widetilde{c}(k)I_{2nN} - \Xi(k))\\ &\quad\times (\Lambda \otimes I_{2n})\xi(k).\end{aligned} \tag{25}$$

Taking mathematical expectation of the last term of (25), one has

$$\begin{aligned}&E[\xi^T(k)(\Lambda \otimes I_{2n})(\widetilde{c}(k)I_{2nN} - \Xi(k))((I_N - J) \otimes I_{2n})\\ &\qquad\times (\widetilde{c}(k)I_{2nN} - \Xi(k))(\Lambda \otimes I_{2n})\xi(k)]\\ &\quad\leq E[\xi^T(k)\|\widetilde{c}(k)I_{2nN} - \Xi(k)\|^2\|\Lambda\|^2\|I_N - J\|^2\xi(k)]\\ &\quad\leq \|\widetilde{c}(k)I_{2nN} - \Xi(k)\|^2\|\Lambda\|^2\|I_N - J\|^2\sigma_\xi\\ &\quad\leq 2(nN)^2\widetilde{c}^2(k)\|\Lambda\|^2\|I_N - J\|^2\sigma_\xi,\end{aligned}$$

where σ_ξ is given by the proof of Theorem 1.

Similar to the discussions (15)-(19), and taking mathematical expectation on both sides of (25), one has

$$\begin{aligned} E[V(k+1)] \leq\ & E[\rho^T(k)(\bar{a} - \eta\tilde{c}(k) + 2\tilde{c}^2(k)\|\mathcal{L}\|^2 + (2\|\mathcal{A}\| \\ & + \varsigma\|\mathcal{L}\|\|\bar{\mathcal{I}}\|)\|\mathcal{L}\|\|\bar{\mathcal{I}}\|\|\Xi(k)\|)\rho(k)] \\ & + 2(nN)^2\tilde{c}^2(k)\|\Lambda\|^2\|I_N - J\|^2\sigma_\xi \\ \leq\ & \mu(k)E[V(k)] + 2(nN)^2\tilde{c}^2(k)\|\Lambda\|^2 \\ & \times \|I_N - J\|^2\sigma_\xi, \end{aligned} \tag{26}$$

where $\mu(k) = \bar{a} - \eta\tilde{c}(k) + 2\tilde{c}^2(k)\|\mathcal{L}\|^2 + (2\|\mathcal{A}\| + \varsigma\|\mathcal{L}\|\|\bar{\mathcal{I}}\|)\|\mathcal{L}\|\|\bar{\mathcal{I}}\|\|\Xi(k)\|$, $\varsigma = \max_i \sup_{k>0} c_i(k)$, and η and $\bar{a}$ are given by the proof of Theorem 1.

Due to $\|\Delta(k)\| \leq \max_{1\leq i,j\leq N} \|c_i(k) - c_j(k)\|$, $\|\Delta(k)\| = o(\tilde{c}(k))$ holds as $k \to \infty$. Then, from Assumption 3, one has $\lim_{k\to\infty} \tilde{c}(k) = 0$ and $\sum_{k=1}^{\infty} \tilde{c}(k) = \infty$. Based on (25), (26) and the above discussions, one gets that there exists a $k_1 > 0$, such that, for all $k \geq k_1$, $0 < \mu(k) \leq 1$,

$$\sum_{k=k_1}^{\infty} \mu(k) = \infty,$$

and as $k \to \infty$,

$$\tilde{c}^2(k)/\mu(k) = 0,$$

which further implies that $\lim_{k\to\infty} E[V(k)] = 0$. It completes the proof. ■

Remark 2: Theorems 1 and 2 consider two different kinds of the stochastic-approximation type gain $c_i(k)$. For each pair of agents $i, j \in \mathcal{V}$, the stochastic-approximation type gain $c_i(k)$ satisfies $\bar{c}(k) = c_i(k) = c_j(k)$ in Theorem 1, and for all agents $i \in \mathcal{V}$, each control gain $c_i(k)$ can be distinction in Theorem 2. Different from the disturbance rejection problems in [22] and [23], the reason introduced these control gains into Theorems 1 and 2 is that the additive communication noises can be handled under the consensus-based disturbance observers (6) and (7). These control gains result in more complicated mathematical analysis.

Remark 3: Actually, there are some restricted conditions during the proofs of Theorems 1 and 2. Specifically, these restrictions are mainly expressed as the condition (11). In view of the matrix theory, the condition (11) means that both the dynamics and disturbances of the system must be stable. The reason for these restrictions is to ensure the Lyapunov function $V(k)$ is a monotone nonincreasing function. However, from the perspective of the system analysis, these restrictions are conservative condition. Therefore, our future works will try to reduce the conservatism of these restrictions.

Remark 4: Note that Lemma 2 depends on the martingale convergence theorem [36], which plays an important role to analyze of the consensus disturbance rejection problem (10). The main advantage of this method is to reduce the requirements of the additive noise $\xi_{ji}(k)$. Actually, there are several works considering the additive noises that are assumed to be the Gaussian noises, or the independent and identically noises [28]. Different from these works, the method in this paper are only assumed that the additive noise $\xi_{ji}(k)$ is a martingale difference sequence with bounded second order moments.

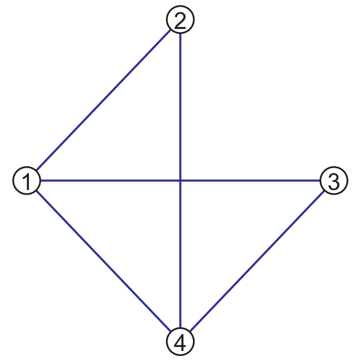

FIGURE 1. The structure of network.

Remark 5: It should be pointed out that there have some recent works related to the disturbance [15], [17], [18], [21], [22] or communication noise [14], [27], [27], [28], [30], [31], respectively. Specifically, the disturbance $w_i(k)$ in these works needs to be estimated by utilizing the disturbance observer (7) that depends on the state observer (6) for estimating $x_i(k)$. The existence of these observers (6) and (7) can be guaranteed by Lemma 1. Different from the disturbance, the communication noise is usually assumed that the relative measurements $\zeta_i(k)$ in (3) satisfy some kinds of random sequences. It inevitably makes the convergence analysis more difficult. However, in practice, the systems may not only involve the disturbance, but also may be subject to the communication noise. In order to fill this gap, this paper explores the relationship between the disturbance and communication noise, which leads to a more general case.

IV. SIMULATION EXAMPLE

An example is given to explain the effectiveness of the main results in this section.

In what follows, consider the linear discrete-time multi-agent system with disturbances and communication noises, where the topology is given in Fig. 1 with weighted communication links all being 0.1.

The matrices A, B, and S in the system (10) are given as:

$$A = \begin{bmatrix} -0.02 & -0.02 \\ 0.15 & -0.03; \end{bmatrix}, B = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix},$$
$$S = \begin{bmatrix} 0.01 & 0.16 \\ -0.07 & -0.13 \end{bmatrix}, D = \begin{bmatrix} -0.02 & 0.01 \\ 0.04 & -0.02; \end{bmatrix}.$$

For all $i, j \in \mathcal{V}$, Fig. 2 draws the additive noise $\xi_{ji}(k)$ that is formed by a normal distribution with mean 2 and variance 12.

Note that $A - BK$ is Hurwitz and the disturbance (1) is matched. In this case, it is easy to check that $\mathfrak{Re}(\lambda_i(A - BK)) = -0.04 < 0$ for all $i = 1, 2$, and $D = BF$ if one chooses $K = [-0.1, 0.1]$ and $F = [-0.2, 0.1]$. Moreover, one selects the stochastic-approximation type gain $c_i(k) = \bar{c}(k) = 1/(1+k)^{0.3}$ for all $i = \{1, 2, 3, 4\}$. Then, one calculates $\lambda_1(\mathrm{Sym}(\bar{A}^T\bar{\mathcal{I}}) + 0.5I_4) = 0.1661 > 0$, $\bar{a} =$

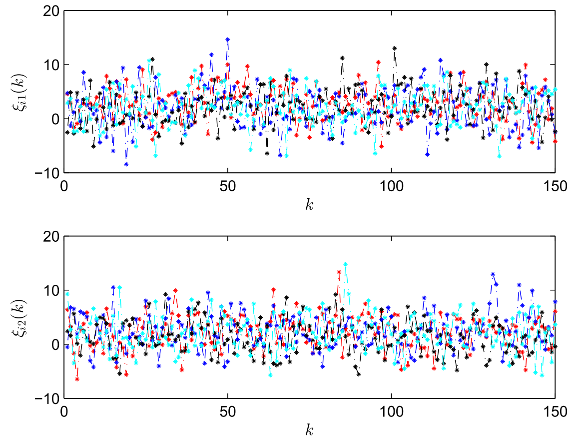

FIGURE 2. Evolution of the states ξ_{ji}.

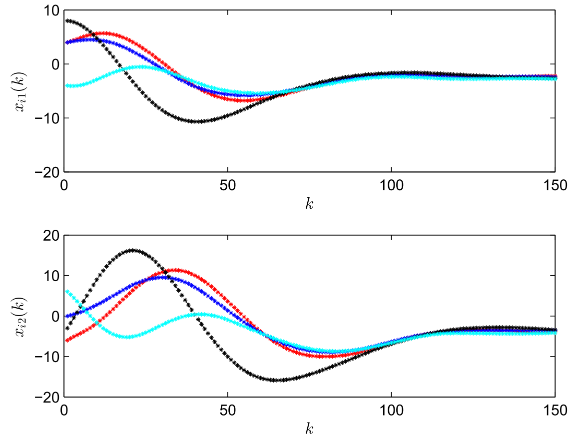

FIGURE 3. Evolution of the states x_i.

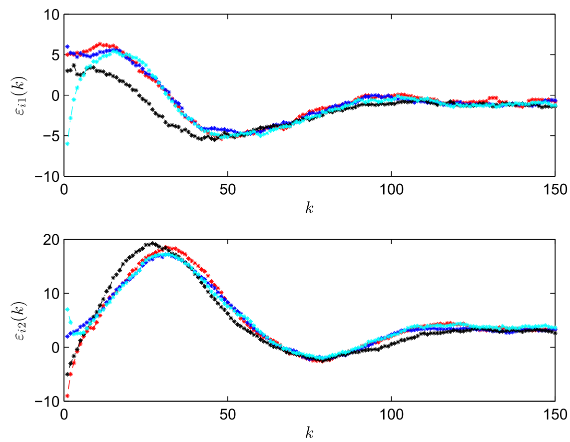

FIGURE 4. Evolution of the states ε_i.

$0.1396 < 1$, and $\sup_k\{\overline{a}-\eta\overline{c}(k)+2\overline{c}^2(k)\|\mathcal{L}\|^2\} = 0.734 < 1$. Thus, all conditions in Theorem 1 are satisfied, which means that the consensus disturbance rejection with communication noises is guaranteed in mean square. Fig. 2 draws the state trajectories x_i of the linear discrete-time multi-agent system (1) with the above parameters. In Figs. 4 and 5, the relative

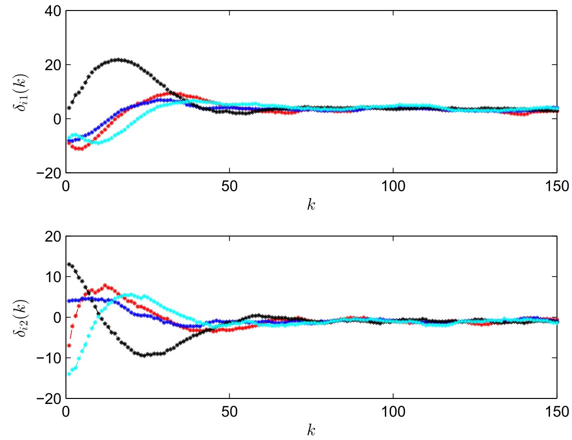

FIGURE 5. Evolution of the states δ_i.

estimation errors $\varepsilon_i(k), \delta_i(k)$ are illustrated, respectively. It should be pointed out that all states of x_i reach consensus as times goes on in Fig. 3, and meanwhile, according to Definition 1, the relative errors $\varepsilon_i(k) = x_i(k) - \chi_i(k)$ and $\delta_i(k) = w_i(k) - \varpi_i(k)$ approach to two random vectors x^* and w^* in Figs. 4 and 5.

V. CONCLUSION

This paper considers the consensus disturbance rejection problem of the linear discrete-time multi-agent systems with communication noises. Each agent, not only is influenced by the disturbances, but also is subject to the communication noises when receiving the state information among its neighbours. The consensus disturbance rejection guarantees the state of each agent that converges to a consensus state in mean square. In order to do this, two disturbance observers are designed to estimate the states of agents and disturbances. Then, two sufficient conditions are derived to ensure consensus disturbance rejection in mean square. Finally, a simulation is provided to show the theoretical results. For the further research, more complex cases may be considered in consensus disturbance rejection problem.

REFERENCES

[1] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[2] R. Olfati-Saber, "Distributed Kalman filter with embedded consensus filters," in *Proc. 44th IEEE Conf. Decis. Control*, Dec. 2005, pp. 8179–8184.

[3] N. Lynch, *Distributed Algorithms*. San Mateo, CA, USA: Morgan Kaufmann, 1996.

[4] Y. Liu, Z. Wang, Y. Yuan, and F. E. Alsaadi, "Partial-nodes-based state estimation for complex networks with unbounded distributed delays," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 8, pp. 3906–3912, Aug. 2018.

[5] Y. Liu, Z. Wang, Y. Yuan, and W. Liu, "Event-triggered partial-nodes-based state estimation for delayed complex networks with bounded distributed delays," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 6, pp. 1088–1098, Jun. 2019.

[6] Z. Li, Y. Tang, T. Huang, and J. Kurths, "Formation control with mismatched orientation in multi-agent systems," *IEEE Trans. Netw. Sci. Eng.*, vol. 6, no. 3, pp. 314–325, Jul. 2019.

[7] X.-H. Wang and H.-B. Ji, "Leader-follower consensus for a class of nonlinear multi-agent systems," *Int. J. Control, Autom. Syst.*, vol. 10, no. 1, pp. 27–35, 2012.
[8] Y. Tang, D. Zhang, X. Jin, D. Yao, and F. Qian, "A resilient attitude tracking algorithm for mechanical systems," *IEEE/ASME Trans. Mechatronics*, vol. 24, no. 6, pp. 2550–2561, Dec. 2019.
[9] Y. Cui, Y. Liu, W. Zhang, and F. E. Alsaadi, "Event-based consensus for a class of nonlinear multi-agent systems with sequentially connected topology," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 10, pp. 3506–3518, Oct. 2018.
[10] W. Zhang, Y. Tang, Q. Miao, and W. Du, "Exponential synchronization of coupled switched neural networks with mode-dependent impulsive effects," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 24, no. 8, pp. 1316–1326, Aug. 2013.
[11] W. Zhang, Y. Tang, Y. Liu, and J. Kurths, "Event-triggering containment control for a class of multi-agent networks with fixed and switching topologies," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 64, no. 3, pp. 619–629, Mar. 2017.
[12] W. Zhang, Y. Tang, T. Huang, and J. Kurths, "Sampled-data consensus of linear multi-agent systems with packet losses," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 11, pp. 2516–2527, Nov. 2017.
[13] J. Xu, X. Wu, and Y. Tang, "Optimal linear exponential quadratic Gaussian estimation with intermittent observations," *IEEE Control Syst. Lett.*, vol. 3, no. 4, pp. 936–941, Oct. 2019.
[14] Y. Cui, Y. Liu, W. Zhang, and F. E. Alsaadi, "Stochastic stability for a class of discrete-time switched neural networks with stochastic noise and time-varying mixed delays," *Int. J. Control, Autom. Syst.*, vol. 16, no. 1, pp. 158–167, Feb. 2018.
[15] Z. Ding, "Consensus disturbance rejection with disturbance observers," *IEEE Trans. Ind. Electron.*, vol. 62, no. 9, pp. 5829–5837, Sep. 2015.
[16] J. Sun, J. Yang, S. Li, and W. X. Zheng, "Sampled-data-based event-triggered active disturbance rejection control for disturbed systems in networked environment," *IEEE Trans. Cybern.*, vol. 49, no. 2, pp. 556–566, Feb. 2019.
[17] J. Sun, Z. Geng, Y. Lv, Z. Li, and Z. Ding, "Distributed adaptive consensus disturbance rejection for multi-agent systems on directed graphs," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 1, pp. 629–639, Mar. 2018.
[18] Y. Yuan, P. Zhang, Z. Wang, L. Guo, and H. Yang, "Active disturbance rejection control for the ranger neutral buoyancy vehicle: A delta operator approach," *IEEE Trans. Ind. Electron.*, vol. 64, no. 12, pp. 9410–9420, Dec. 2017.
[19] S. Li, J. Yang, W.-H. Chen, and X. Chen, *Disturbance Observer-Based Control: Methods and Applications*. Boca Raton, FL, USA: CRC Press, 2014.
[20] B. A. Francis, "Optimal linear exponential quadratic Gaussian estimation with intermittent observations," *SIAM J. Control Optim.*, vol. 15, no. 3, pp. 486–505, 1975.
[21] X. Wang, S. Li, and J. Lam, "Distributed active anti-disturbance output consensus algorithms for higher-order multi-agent systems with mismatched disturbances," *Automatica*, vol. 74, pp. 30–37, Dec. 2016.
[22] Z.-G. Wu, Y. Xu, Y.-J. Pan, H. Su, and Y. Tang, "Event-triggered control for consensus problem in multi-agent systems with quantized relative state measurements and external disturbance," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 65, no. 7, pp. 2232–2242, Jul. 2018.
[23] D. Meng and Y. Jia, "Robust consensus algorithms for multiscale coordination control of multivehicle systems with disturbances," *IEEE Trans. Ind. Electron.*, vol. 63, no. 2, pp. 1107–1119, Feb. 2016.
[24] N. H. Jo, C. Jeon, and H. Shim, "Noise reduction disturbance observer for disturbance attenuation and noise suppression," *IEEE Trans. Ind. Electron.*, vol. 64, no. 2, pp. 1381–1391, Feb. 2017.
[25] C. Dong, W. Zhang, Q. Wang, and Y. Liu, "Time-varying anti-disturbance formation control for high-order non-linear multi-agent systems with switching directed topologies," *IET Control Theory Appl.*, vol. 14, no. 2, pp. 271–282, 2020.
[26] J. Toji and H. Ichihara, "Disturbance rejection for interconnected positive systems and its application to formation control of quadrotors," *IET Control Theory Appl.*, vol. 13, no. 7, pp. 988–995, Apr. 2019.
[27] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Trans. Autom. Control*, vol. 55, no. 9, pp. 2043–2057, Sep. 2010.
[28] T. Li and J. Wang, "Mean square average-consensus under measurement noises and fixed topologies: Necessary and sufficient conditions," *Automatica*, vol. 45, no. 8, pp. 1929–1936, 2009.
[29] F. Li and Y. Tang, "False data injection attack for cyber-physical systems with resource constraint," *IEEE Trans. Cybern.*, vol. 50, no. 2, pp. 729–738, Feb. 2020.
[30] L. Cheng, Y. Wang, W. Ren, Z.-G. Hou, and M. Tan, "On convergence rate of leader-following consensus of linear multi-agent systems with communication noises," *IEEE Trans. Autom. Control*, vol. 61, no. 11, pp. 3586–3592, Nov. 2016.
[31] X. Zong, T. Li, G. Yin, L. Y. Wang, and J.-F. Zhang, "Stochastic consentability of linear systems with time delays and multiplicative noises," *IEEE Trans. Autom. Control*, vol. 63, no. 4, pp. 1059–1074, Apr. 2018.
[32] S. Zhu, Y. C. Soh, and L. Xie, "Distributed inference for relay-assisted sensor networks with intermittent measurements over fading channels," *IEEE Trans. Signal Process.*, vol. 64, no. 3, pp. 742–756, Feb. 2016.
[33] C. Deng and C. Wen, "Distributed resilient observer-based fault-tolerant control for heterogeneous multiagent systems under actuator faults and DoS attacks," *IEEE Trans. Control Netw. Syst.*, vol. 7, no. 3, pp. 1308–1318, Sep. 2020.
[34] S. Liu, L. Xie, and H. Zhang, "Distributed consensus for multi-agent systems with delays and noises in transmission channels," *Automatica*, vol. 47, no. 5, pp. 920–934, May 2011.
[35] Y. Wang, L. Cheng, W. Ren, Z.-G. Hou, and M. Tan, "Seeking consensus in networks of linear agents: Communication noises and Markovian switching topologies," *IEEE Trans. Autom. Control*, vol. 60, no. 5, pp. 1374–1379, May 2015.
[36] J. Davidson, *Stochastic Limit Theory*. Oxford, U.K.: Oxford Univ. Press, 1994.
[37] R. A. Horn and C. R. Johnson, *Martix Analysis*. New York: NY, USA: Springer-Verlag, 2001.
[38] M. Huang and J. H. Manton, "Coordination and consensus of networked agents with noisy measurements: Stochastic algorithms and asymptotic behavior," *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 134–161, 2009.

YUANYUAN JIA received the bachelor's degree in applied mathematics from Fuyang Normal University, Anhui, China, in 2006, and the master's degree in basic mathematics from Qinghai Normal University, Qinghai, China, in 2009. She is currently an Assistant Professor at Huainan Normal University. Her current research interests include synchronization/consensus, network control systems, and genetic regulatory networks.

JINGJING WANG received the bachelor's degree in applied mathematics from the Anhui University of Science and Technology, Anhui, China, and the master's degree in finance from the Nanjing University of Science and Technology, Jiangsu, China. She is currently an Assistant Professor at Huainan Normal University. Her current research interest includes actuarial and financial mathematics.

JIAMAN BO is currently pursuing the degree with Huainan Normal University. She is also a Senior Researcher with the Department of Mathematics and Applied Mathematics.