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RESEARCH ARTICLE

Semi-Analytical Scheme for Solving Intuitionistic Fuzzy System of Differential Equations

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ABSTRACT The aim of this article is to implement the Generalized Modified Adomian Decomposition Method to compute the semi-numerical solution of the linear system of intuitionistic fuzzy initial value problems. Here, we consider the initial values as generalized trapezoidal intuitionistic fuzzy numbers. The technique is applied to brine tanks problem and coupled mass spring systems.Theoretically, different approaches to solving a system of generalized trapezoidal intuitionistic fuzzy differential equations are discussed in this study under the presumption that the coefficients of the system of the differential equations are associated to generalized trapezoidal intuitionistic fuzzy numbers. The approximate results are compared with exact solutions which shows good efficiency. The corresponding graphs at different levels of uncertainty show the example's numerical outcomes. The graphical representations further demonstrate the effectiveness and accuracy of the proposed method in comparison to existing semi-numerical methods in the literature.

INDEX TERMS Fuzzy set, fuzzy number, generalized trapezoidal intuitionistic fuzzy number, system of fuzzy differential equation, analytical technique, engineering applications.

I. INTRODUCTION

System of differential equation plays a significant role in modeling and studying many naturally occurring phenomenon such as population models, economic models, friction model, bacteria culture model, predator-prey model, weight loss and oil production model, bank account and drug concentration problem, human immunodeficiency virus (HIV) model. In classical set theory, the variables or parameters are taken as crisp numbers. But in actual case, these variables or parameters are usually uncertain or vague. So, these variables may be considered as a fuzzy numbers. In other words, to overcome uncertainty we use fuzzy numbers. So the system of differential equations are converted to system of fuzzy differential equations (SFDE).

The concept of fuzzy set theory was firstly introduced by Zadeh in 1965, as the extension of classical set theory [1]. The concept of fuzzy set theory has been applied to various fields of science and engineering to handle vagueness and uncertainty. In 1987, Kandel and Byatt [2] introduced the fuzzy differential equations. The fuzzy differential equations have been applied in numerous daily life problems [3], [4], [5]. Vasavi et al. [6] discussed fuzzy differential for cooling problems. Devi and Ganesan used fuzzy differential equations in modelling electric circuit problem [7]. Ahmad et al. [8] studied a mathematical method to find the solution of fuzzy integro differential equations. Sadeghi et al. [9] studied the system of fuzzy differential equation. Buckley et al find the solution of system of first

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order linear fuzzy differential equations by extension principle [10]. Hashemi et al. find the series solution of SFDE [11]. In 1986, Atanassov [12] introduced an extension of fuzzy set theory known as intuitionistic fuzzy set. The intuitionistic fuzzy set [13] not only provides the information about membership values but also the non-membership values respectively, and so that the sum of both values is less than one. Intuitionistic fuzzy differential equations are being studied widely and being used in various fields of Physics, Chemistry, Biology as well as among other fields of science and engineering. Melliani and Chadli obtained the approximate and numerical solutions of intuitionistic fuzzy differential equations with linear differential operators [14], [15]. Gulzar et al. worked on fuzzy algebra [16], [17], [18]. Akin and Bayeg [19], [20] studied a method to find general solution of second order intuitionistic fuzzy differential equation and to solve the system of intuitionistic fuzzy differential equations with intuitionistic fuzzy initial values. Mondal and Roy [21], [22], [23] studied the generalized intuitionistic fuzzy Laplace transform method and to solve the system of differential equations with initial value as triangular intuitionistic fuzzy number. Saw et al introduced a method for solving system of linear intuitionistic fuzzy equations [24].

The Adomian Decomposition Method (ADM) which is a semi analytical method was first presented by Adomian in 1980's [25], [26]. This method is very efficient in finding the solutions of differential equations, algebraic equations as well as integral equations. In this article, we will propose the Generalized Modified Adomian Decomposition Method (GMADM) to find the solutions of the system of linear intuitionistic fuzzy differential equations with initial values as generalized trapezoidal intuitionistic fuzzy number. This modification was proposed by Wazwaz [27]. He presented a reliable modification to the ADM. In this modification Wazwaz divides the original function into two parts, one part assigned to the initial term of the series and the other to the second term. This modification results in a different series being generated. The efficiency of this method depends only on the choice of the parts into which the original function is to be divided.

First order system of fuzzy differential equations is important among all the fuzzy differential equations. There are many approaches to solve the SFDEs. Buckley and Feuring [28] solving the linear system of first order ordinary differential equations with fuzzy initial conditions by extension principle using triangular fuzzy number. The geometric approach is developed by Gasilova et al. [29] and series solution is developed by Hashemi et al. [30]. Mondal and Roy [31] studied strong and weak solution of first order homogeneous intuitionistic fuzzy differential equation, subsequently and studied system of differential equation in literature. Melliani, et al. [32] discussed the existence and uniqueness of the solution of the intuitionistic fuzzy differential equation and its system using the analytical technique. Therefore, finding an efficient and accurate algorithm for investigating FIE has been one the hot areas of research in recent time. To achieve these goals, various methods and procedures were used to handle differentia equations, using triangular fuzzy number, for details, see [9], [33].

In this study, motivated by the aforementioned work, we solve the system of differential equations using a GMADM and a more generalized fuzzy system of differential equations, namely a trapezoidal intuitionistic fuzzy system of intuitionistic differential equations.

The main contributions of this research work are summarized below.

- GMADM is used to solve a system of differential equations using initial conditions as a Generalized trapezoidal intuitionistic fuzzy number.
- In order to solve a system of fuzzy intuitionistic differential equations that have not before been explored, the computational complexity of the suggested GMADM is discussed.
- Applications of system of Generalized trapezoidal intuitionistic fuzzy differential equations in mechanical engineering are taken into consideration in a Generalized trapezoidal intuitionistic fuzzy environment.
- Computational tools are used to evaluate the effectiveness and applicability of the suggested analytical scheme.

This paper is organized as follows: In section II, we recall some basic definitions which we will use in further sections. In sections III, we introduced our proposed method. In section IV, the efficiency of this method has been illustrated by applications. In the last section, we give conclusions.

II. PRELIMINARIES

In this section, the fundamental definitions of fuzzy set and intuitionistic fuzzy set are presented.

Definition 1 [34]: Let $\overset{\circ}{U}$ be the largest set under consideration then $\overset{\star}{F}$ be a subset of $\overset{\circ}{U}$ is said to be a fuzzy set if it is defined as:

$$\mu_{\overset{\star}{F}}(\hat{u}) = \overset{\circ}{U} \longrightarrow [0, 1],$$

defines the degree of membership of an element $\hat{u} \epsilon \overset{\circ}{U}$ to the set $\overset{\star}{F}$ which is a subset of $\overset{\circ}{U}$.

Definition 2 [34]: α-cut of a fuzzy set $\overset{\star}{F}$ is a crisp set $\overset{\star}{F}_\alpha$ which is defined as:

$$\overset{\star}{F}_\alpha = \{(\hat{u} \mid \mu_{\overset{\star}{F}}(\hat{u}) \geq \alpha : \hat{u} \epsilon \overset{\circ}{U}\}.$$

Definition 3 [23]: If $\overset{\star}{F}$ is a fuzzy set then height of a fuzzy set is denoted by $h(\overset{\star}{F})$ and is defined as the largest membership function obtained by any element in that set i.e.,

$$h(\overset{\star}{F}) = \sup \mu\overset{\star}{F}(\hat{u}).$$

Definition 4 [19]: Let $\mathring{U}$ be a nonempty finite set of real numbers, then an intuitionistic fuzzy set $\overset{\star}{I}$ on $\mathring{U}$ is:

$$\overset{\star}{I} = \{(\hat{u}, \mu_{\overset{\star}{I}}(\hat{u}), \upsilon_{\overset{\star}{I}}(\hat{u})) : \hat{u} \,\epsilon\, \mathring{U}\},$$

where the functions,

$$\mu_{\overset{\star}{I}}(\hat{u}) = \mathring{U} \longrightarrow [0, 1],$$
$$\upsilon_{\overset{\star}{I}}(\hat{u}) = \mathring{U} \longrightarrow [0, 1],$$

define the degree of membership and degree of non-membership respectively, of an element $\hat{u} \,\epsilon\, \mathring{U}$ to the set $\overset{\star}{I}$ which is a subset of $\mathring{U}$, and for every $\hat{u} \,\epsilon\, \mathring{U}$, the

$$0 \leq \mu_{\overset{\star}{I}}(\hat{u}) + \upsilon_{\overset{\star}{I}}(\hat{u}) \leq 1,$$

condition must be satisfied.

Definition 5 [23]: An intuitionistic fuzzy set $\overset{\star}{I}$ is said to be normal if there exists an $\hat{u}_0 \,\epsilon\, \mathring{U}$, such that $\mu_{\overset{\star}{I}}(\hat{u}_0) = 1$ so $\upsilon_{\overset{\star}{I}}(\hat{u}_0) = 0$.

Definition 6 [23]: An intuitionistic fuzzy set $\overset{\star}{I}$ is said to be convex set for the membership function if it satisfy the following condition:

$$\mu_{\overset{\star}{I}}(\hat{u})(\eta\hat{u} + (1-\eta)\hat{s}) \geq min(\mu_{\overset{\star}{I}}(\hat{u}), \mu_{\overset{\star}{I}}(\hat{s}));$$
$$\forall \hat{u}, \hat{s} \,\epsilon\, \mathring{U}, \eta \,\epsilon [0, 1].$$

Definition 7 [23]: An intuitionistic fuzzy set $\overset{\star}{I}$ is said to be concave set for the non-membership function if it satisfy the following condition:

$$\upsilon_{\overset{\star}{I}}(\hat{u})(\eta\hat{u} + (1-\eta)\hat{s}) \geq max(\upsilon_{\overset{\star}{I}}(\hat{u}), \upsilon_{\overset{\star}{I}}(\hat{s}));$$
$$\forall \hat{u}, \hat{s} \,\epsilon\, \mathring{U}, \eta \,\epsilon [0, 1].$$

A. GENERALIZED INTUITIONISTIC FUZZY NUMBER

Definition 8 [35]: A generalized intuitionistic fuzzy number

$$\overset{\star}{T} = <(\overset{\star}{s}_1, \overset{\star}{s}_2, \overset{\star}{s}_3, \overset{\star}{s}_4; \nu_A); (\overset{\star}{t}_1, \overset{\star}{t}_2, \overset{\star}{t}_3, \overset{\star}{t}_4; \nu_B) >$$

is said to be generalized trapezoidal intuitionistic fuzzy number (GTIFN) (as shown in Figure 1) if its membership and non-membership functions are defined as follows:

$$\mu_{\overset{\star}{T}}(\hat{u}) = \begin{cases} \nu_A(\frac{\hat{u}-\overset{\star}{s}_1}{\overset{\star}{s}_2-\overset{\star}{s}_1}), \overset{\star}{s}_1 \leq \hat{u} \leq \overset{\star}{s}_2, \\ \nu_A, \overset{\star}{s}_2 \leq \hat{u} \leq \overset{\star}{s}_3, \\ \nu_A(\frac{\overset{\star}{s}_4-\hat{u}}{\overset{\star}{s}_4-\overset{\star}{s}_3}), \overset{\star}{s}_3 \leq \hat{u} \leq \overset{\star}{s}_4, \\ 0, otherwise, \end{cases} \quad \ldots$$

$$\upsilon_{\overset{\star}{T}}(\hat{u}) = \begin{cases} \frac{(\overset{\star}{t}_2-\hat{u})+\nu_B(\hat{u}-\overset{\star}{t}_1)}{\overset{\star}{t}_2-\overset{\star}{t}_1}, \overset{\star}{t}_1 \leq \hat{u} \leq \overset{\star}{t}_2, \\ \nu_B, \overset{\star}{t}_2 \leq \hat{u} \leq \overset{\star}{t}_3, \\ \frac{(\hat{u}-\overset{\star}{t}_3)+\nu_B(\overset{\star}{t}_4-\hat{u})}{\overset{\star}{t}_4-\overset{\star}{t}_3}, \overset{\star}{t}_3 \leq \hat{u} \leq \overset{\star}{t}_4, \\ 1, otherwise, \end{cases}$$

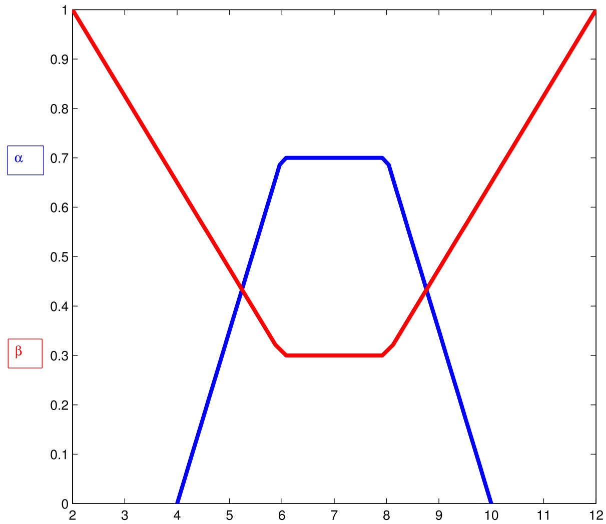

FIGURE 1. Generalized trapezoidal intuitionistic fuzzy number.

where $\overset{\star}{t}_1 \leq \overset{\star}{s}_1 \leq \overset{\star}{t}_2 \leq \overset{\star}{s}_2 \leq \overset{\star}{s}_3 \leq \overset{\star}{t}_3 \leq \overset{\star}{s}_4 \leq \overset{\star}{t}_4, 0 \leq \nu_A, \nu_B \leq 1$ and $0 < \nu_A + \nu_B \leq 1$.

Definition 9 [36], [37]: (α, β)—cut set of a generalized trapezoidal intuitionistic fuzzy number $\overset{\star}{T} = <(\overset{\star}{s}_1, \overset{\star}{s}_2, \overset{\star}{s}_3, \overset{\star}{s}_4; \nu_A);(\overset{\star}{t}_1, \overset{\star}{t}_2, \overset{\star}{t}_3, \overset{\star}{t}_4; \nu_B)>$ is a crisp subset of $\mathring{U}$ which is defined as:

$$\begin{aligned} &\overset{\star}{T}(\alpha, \beta) \\ &= \{(\hat{u}, \mu_{\overset{\star}{T}}(\hat{u}), \upsilon_{\overset{\star}{T}}(\hat{u})) : \hat{u} \,\epsilon U, \mu_{\overset{\star}{T}}(\hat{u}) \geq \alpha, \upsilon_{\overset{\star}{T}}(\hat{u}) \geq \beta\} \\ &= \{[\overset{\star}{T}_1(\alpha), \overset{\star}{T}_2(\alpha)]; [\overset{\star}{T}_1(\beta), \overset{\star}{T}_2(\beta)]\}, \end{aligned}$$

where $\alpha + \beta < 1, \alpha \,\epsilon [0, \nu_A]$ and $\beta \,\epsilon [\nu_B, 1]$.

Figure 1: Shows membership and non-membership function of generalized trapezoidal intuitionistic fuzzy number.

B. ARITHMETIC OPERATIONS ON GTIFNs

Definition 10 [37], [38]: Let $\overset{\star}{T}_1 = <(\overset{\star}{s}_1, \overset{\star}{s}_2, \overset{\star}{s}_3, \overset{\star}{s}_4; \nu_{A_1}); (\overset{\star}{t}_1, \overset{\star}{t}_2, \overset{\star}{t}_3, \overset{\star}{t}_4; \nu_{A_2}) >$ and $\overset{\star}{T}_2 = <(\overset{\star}{u}_1, \overset{\star}{u}_2, \overset{\star}{u}_3, \overset{\star}{u}_4; \nu_{B_1}); (\overset{\star}{v}_1, \overset{\star}{v}_2, \overset{\star}{v}_3, \overset{\star}{v}_4; \nu_{\overset{\star}{t}_2}) >$ be two GTIFNs and ϖ be a real number. Then

- $$\begin{aligned} \overset{\star}{T}_1 + \overset{\star}{T}_2 = <(&\overset{\star}{s}_1 + \overset{\star}{u}_1, \overset{\star}{s}_2 + \overset{\star}{u}_2, \overset{\star}{s}_3 + \overset{\star}{u}_3, \overset{\star}{s}_4 \\ &+ \overset{\star}{u}_4; min\{\nu_{A_1}, \nu_{B_1}\}); \\ &(\overset{\star}{t}_1 + \overset{\star}{v}_1, \overset{\star}{t}_2 + \overset{\star}{v}_2, \overset{\star}{t}_3 + \overset{\star}{v}_3, \overset{\star}{t}_4 \\ &+ \overset{\star}{v}_4; max\{\nu_{A_2}, \nu_{B_2}\})>. \end{aligned}$$

- $$\begin{aligned} \overset{\star}{T}_1 - \overset{\star}{T}_2 = <(&\overset{\star}{s}_1 - \overset{\star}{u}_4, \overset{\star}{s}_2 - \overset{\star}{u}_3, \overset{\star}{s}_3 - \overset{\star}{u}_2, \overset{\star}{s}_4 \\ &- \overset{\star}{u}_1; min\{\nu_{A_1}, \nu_{B_1}\}); \\ &(\overset{\star}{t}_1 - \overset{\star}{v}_4, \overset{\star}{t}_2 - \overset{\star}{v}_3, \overset{\star}{t}_3 - \overset{\star}{v}_2, \overset{\star}{t}_4 \\ &- \overset{\star}{v}_1; max\{\nu_{A_2}, \nu_{B_2}\})>. \end{aligned}$$

- $$\overset{\star}{T}_1 \times \overset{\star}{T}_2 = <(\overset{\star}{s}_1\overset{\star}{u}_1, \overset{\star}{s}_2\overset{\star}{u}_2, \overset{\star}{s}_3\overset{\star}{u}_3, \overset{\star}{s}_4\overset{\star}{u}_4; min\{v_{A_1}, v_{B_1}\}); \\ (\overset{\star}{t}_1\overset{\star}{v}_1, \overset{\star}{t}_2\overset{\star}{v}_2, \overset{\star}{t}_3\overset{\star}{v}_3, \overset{\star}{t}_4\overset{\star}{v}_4; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\overset{\star}{T}_1 > 0, \overset{\star}{T}_2 > 0$.
- $$\overset{\star}{T}_1 \times \overset{\star}{T}_2 = <(\overset{\star}{s}_1\overset{\star}{u}_4, \overset{\star}{s}_2\overset{\star}{u}_3, \overset{\star}{s}_3\overset{\star}{u}_2, \overset{\star}{s}_4\overset{\star}{u}_1; min\{v_{A_1}, v_{B_1}\}); \\ (\overset{\star}{t}_1\overset{\star}{v}_4, \overset{\star}{t}_2\overset{\star}{v}_3, \overset{\star}{t}_3\overset{\star}{v}_2, \overset{\star}{t}_4\overset{\star}{v}_1; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\overset{\star}{T}_1 < 0, \overset{\star}{T}_2 > 0$.
- $$\overset{\star}{T}_1 \times \overset{\star}{T}_2 = <(\overset{\star}{s}_4\overset{\star}{u}_4, \overset{\star}{s}_3\overset{\star}{u}_3, \overset{\star}{s}_2\overset{\star}{u}_2, a_1\overset{\star}{u}_1; min\{v_{A_1}, v_{B_1}\}); \\ (\overset{\star}{t}_4\overset{\star}{v}_4, \overset{\star}{t}_3\overset{\star}{v}_3, \overset{\star}{t}_2\overset{\star}{v}_2, \overset{\star}{t}_1\overset{\star}{v}_1; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\overset{\star}{T}_1 < 0, \overset{\star}{T}_2 < 0$.
- $$\overset{\star}{T}_1 \div \overset{\star}{T}_2 = <(\frac{\overset{\star}{s}_1}{\overset{\star}{u}_4}, \frac{\overset{\star}{s}_2}{\overset{\star}{u}_3}, \frac{\overset{\star}{s}_3}{\overset{\star}{u}_2}, \frac{\overset{\star}{s}_4}{\overset{\star}{u}_1}; min\{v_{A_1}, v_{B_1}\}); \\ (\frac{\overset{\star}{t}_1}{\overset{\star}{v}_4}, \frac{\overset{\star}{t}_2}{\overset{\star}{v}_3}, \frac{\overset{\star}{t}_3}{\overset{\star}{v}_2}, \frac{\overset{\star}{t}_4}{\overset{\star}{v}_4}; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\overset{\star}{T}_2 > 0$.
- $$\varpi\overset{\star}{T}_1 = <(\varpi\overset{\star}{s}_1, \varpi\overset{\star}{s}_2, \varpi\overset{\star}{s}_3, \varpi\overset{\star}{s}_4; min\{v_{A_1}, v_{B_1}\}); \\ (\varpi\overset{\star}{t}_1, \varpi\overset{\star}{t}_2, \varpi\overset{\star}{t}_3, \varpi\overset{\star}{t}_4; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\varpi > 0$.
- $$\varpi\overset{\star}{T}_1 = <(\varpi\overset{\star}{s}_4, \varpi\overset{\star}{s}_3, \varpi\overset{\star}{s}_2, \varpi\overset{\star}{s}_1; min\{v_{A_1}, v_{B_1}\}); \\ (\varpi\overset{\star}{t}_4, \varpi\overset{\star}{t}_3, \varpi\overset{\star}{t}_4, \varpi b_5; max\{v_{A_2}, v_{B_2}\})>,$$
  where $\varpi < 0$.

III. THE GENERALIZED MODIFIED ADOMIAN DECOMPOSITION METHOD

Let us consider the system of intuitionistic fuzzy differential equations with linear differential operator as follows:

$$\left.\begin{aligned} \overset{\maltese}{L}\overset{\star}{x}(t) + \overset{\maltese}{R}\overset{\star}{x}(t) + \overset{\maltese}{R}\overset{\star}{y}(t) + N(t, \overset{\star}{x}(t), \overset{\star}{y}(t)) &= \overset{\star}{g}(t), \\ \overset{\maltese}{L}\overset{\star}{y}(t) + \overset{\maltese}{R}\overset{\star}{x}(t) + \overset{\maltese}{R}\overset{\star}{y}(t) + N(t, \overset{\star}{x}(t), \overset{\star}{y}(t)) &= \overset{\star}{h}(t), \end{aligned}\right\} \tag{1}$$

where $\overset{\maltese}{L}$ is the highest order linear differential operator, $\overset{\maltese}{R}$ is the remaining part of the linear differential operator, N may be linear or nonlinear function of t, $\overset{\star}{x}(t)$ and $\overset{\star}{y}(t)$, $\overset{\star}{g}(t)$ and $\overset{\star}{h}(t)$ are non-homogeneous terms. Here, in this case we take N as a linear function of $\overset{\star}{x}(t), \overset{\star}{y}(t)$ and t.Taking (α, β)-cut of (1), we get, (2), as shown at the bottom of the page.

From (2), we obtain the following equations:

$$\left.\begin{aligned} &\overset{\maltese}{L}\overset{\star}{x}_1(t,\alpha) + \overset{\maltese}{R}\overset{\star}{x}_1(t,\alpha) + \overset{\maltese}{R}\overset{\star}{y}_1(t,\alpha) + \\ &N_1(t, \overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)) = \overset{\star}{g}_1(t,\alpha), \\ &\overset{\maltese}{L}\overset{\star}{y}_1(t,\alpha) + \overset{\maltese}{R}\overset{\star}{x}_1(t,\alpha) + \overset{\maltese}{R}\overset{\star}{y}_1(t,\alpha) + \\ &N_1(t, \overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)) = \overset{\star}{h}_1(t,\alpha). \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} &\overset{\maltese}{L}\overset{\star}{x}_2(t,\alpha) + \overset{\maltese}{R}\overset{\star}{x}_2(t,\alpha) + \overset{\maltese}{R}\overset{\star}{y}_2(t,\alpha) + \\ &N_2(t, \overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha)) = \overset{\star}{g}_2(t,\alpha), \\ &\overset{\maltese}{L}\overset{\star}{y}_2(t,\alpha) + \overset{\maltese}{R}\overset{\star}{x}_2(t,\alpha) + \overset{\maltese}{R}\overset{\star}{y}_2(t,\alpha) + \\ &N_2(t, \overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha)) = \overset{\star}{h}_2(t,\alpha). \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} &\overset{\maltese}{L}\overset{\star}{x}_1(t,\beta) + \overset{\maltese}{R}\overset{\star}{x}_1(t,\beta) + \overset{\maltese}{R}\overset{\star}{y}_1(t,\beta) + \\ &N_1(t, \overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)) = \overset{\star}{g}_1(t,\beta), \\ &\overset{\maltese}{L}\overset{\star}{y}_1(t,\beta) + \overset{\maltese}{R}\overset{\star}{x}_1(t,\beta) + \overset{\maltese}{R}\overset{\star}{y}_1(t,\beta) + \\ &N_1(t, \overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)) = \overset{\star}{h}_1(t,\beta). \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} &\overset{\maltese}{L}([\overset{\star}{x}_1(t,\alpha), \overset{\star}{x}_2(t,\alpha)]; [\overset{\star}{x}_1(t,\beta), \overset{\star}{x}_2(t,\beta)]) + \overset{\maltese}{R}([\overset{\star}{x}_1(t,\alpha), \overset{\star}{x}_2(t,\alpha)]; \\ &[\overset{\star}{x}_1(t,\beta), \overset{\star}{x}_2(t,\beta)]) + \overset{\maltese}{R}([\overset{\star}{y}_1(t,\alpha), \overset{\star}{y}_2(t,\alpha)]; [\overset{\star}{y}_1(t,\beta), \overset{\star}{y}_2(t,\beta)]) \\ &+([N_1(t, \overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)), N_2(t, \overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha))]; \\ &[N_1(t, \overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)), N_2(t, \overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta))]) \\ &= ([\overset{\star}{g}_1(t,\alpha), \overset{\star}{g}_2(t,\alpha)]; [\overset{\star}{g}_1(t,\beta), \overset{\star}{g}_2(t,\beta)]), \\ &\overset{\maltese}{L}([\overset{\star}{y}_1(t,\alpha), \overset{\star}{y}_2(t,\alpha)]; [\overset{\star}{y}_1(t,\beta), \overset{\star}{y}_2(t,\beta)]) + \overset{\maltese}{R}([\overset{\star}{x}_1(t,\alpha), \overset{\star}{x}_2(t,\alpha)]; \\ &[\overset{\star}{x}_1(t,\beta), \overset{\star}{x}_2(t,\beta)]) + \overset{\maltese}{R}([\overset{\star}{y}_1(t,\alpha), \overset{\star}{y}_2(t,\alpha)]; [\overset{\star}{y}_1(t,\beta), \overset{\star}{y}_2(t,\beta)]) \\ &+([N_1(t, \overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)), N_2(t, \overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha))]; \\ &[N_1(t, \overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)), N_2(t, \overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta))]) \\ &= ([\overset{\star}{h}_1(t,\alpha), \overset{\star}{h}_2(t,\alpha)]; [\overset{\star}{h}_1(t,\beta), \overset{\star}{h}_2(t,\beta)]). \end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
&\overset{\maltese}{L}\overset{\star}{x}_2(t,\beta)+\overset{\maltese}{R}\overset{\star}{x}_2(t,\beta)+\overset{\maltese}{R}\overset{\star}{y}_2(t,\beta)+\\
&N_1(t,\overset{\star}{x}_2(t,\beta),\overset{\star}{y}_2(t,\beta))=\overset{\star}{g}_2(t,\beta),\\
&\overset{\maltese}{L}\overset{\star}{y}_2(t,\beta)+\overset{\maltese}{R}\overset{\star}{x}_2(t,\beta)+\overset{\maltese}{R}\overset{\star}{y}_2(t,\beta)+\\
&N_1(t,\overset{\star}{x}_2(t,\beta),\overset{\star}{y}_2(t,\beta))=\overset{\star}{h}_2(t,\beta).
\end{aligned}\right\}\tag{6}$$

Applying the $\overset{\maltese}{L}^{-1}$ operator on both sides of (3), (4), (5) and (6), we get;

$$\left.\begin{aligned}
\overset{\star}{x}_1(t,\alpha)&=\Psi_1(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_1(t,\alpha))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_1(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(N_1(t,\overset{\star}{x}_1(t,\alpha),\overset{\star}{y}_1(t,\alpha))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_1(t,\alpha)),\\
\overset{\star}{y}_1(t,\alpha)&=\Phi_1(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_1(t,\alpha))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_1(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(N_1(t,\overset{\star}{y}_1(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_1(t,\alpha)).
\end{aligned}\right\}\tag{7}$$

$$\left.\begin{aligned}
\overset{\star}{x}_2(t,\alpha)&=\Psi_2(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_2(t,\alpha))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_2(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(N_2(t,\overset{\maltese}{L}^{-1}(\overset{\star}{x}_2(t,\alpha)),\overset{\star}{y}_2(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_2(t,\alpha)),\\
\overset{\star}{y}_2(t,\alpha)&=\Phi_2(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_2(t,\alpha))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_2(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(N_2(t,\overset{\maltese}{L}^{-1}(\overset{\star}{x}_2(t,\alpha)),\overset{\star}{y}_2(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_2(t,\alpha)),
\end{aligned}\right\}\tag{8}$$

$$\left.\begin{aligned}
\overset{\star}{x}_1(t,\beta)&=\Psi_1(t,\beta)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_1(t,\beta))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_1(t,\beta))\\
&-\overset{\maltese}{L}^{-1}(N_1(t,\overset{\star}{x}_1(t,\beta),\overset{\star}{y}_1(t,\beta)))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_1(t,\beta)),\\
\overset{\star}{y}_1(t,\beta)&=\Phi_1(t,\beta)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_1(t,\beta))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_1(t,\beta))\\
&-\overset{\maltese}{L}^{-1}(N_1(t,\overset{\star}{x}_1(t,\beta),\overset{\star}{y}_1(t,\beta)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_1(t,\beta)),
\end{aligned}\right\}\tag{9}$$

$$\left.\begin{aligned}
\overset{\star}{x}_2(t,\beta)&=\Psi_2(t,\beta)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_2(t,\beta))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_2(t,\beta))\\
&-\overset{\maltese}{L}^{-1}(N_2(t,\overset{\star}{x}_2(t,\beta),\overset{\star}{y}_2(t,\beta)))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_2(t,\beta)),\\
\overset{\star}{y}_2(t,\beta)&=\Phi_2(t,\beta)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_2(t,\beta))-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_2(t,\beta))\\
&-\overset{\maltese}{L}^{-1}(N_2(t,\overset{\star}{x}_2(t,\beta),\overset{\star}{y}_2(t,\beta)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_2(t,\beta)),
\end{aligned}\right\}\tag{10}$$

where,

$$\left.\begin{aligned}
\Psi_i(t,\alpha)&=\overset{\maltese}{L}\Psi_i(t,\alpha)=0,\ i=1,2\\
\Phi_i(t,\alpha)&=\overset{\maltese}{L}\Phi_i(t,\alpha)=0,\ i=1,2
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\Psi_i(t,\beta)&=\overset{\maltese}{L}\Psi_i(t,\beta)=0,\ i=1,2\\
\Phi_i(t,\beta)&=\overset{\maltese}{L}\Phi_i(t,\beta)=0,\ i=1,2
\end{aligned}\right\}$$

the above functions are found by using the initial conditions. Now by using the GMADM the solutions of the (7), (8), (9) and (10), can be expressed in the form of an infinite series for the unknown functions as follows:

$$\left.\begin{aligned}
\overset{\star}{x}_1(t,\alpha)&=\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha),\\
\overset{\star}{y}_1(t,\alpha)&=\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha),
\end{aligned}\right\}\tag{11}$$

$$\left.\begin{aligned}
\overset{\star}{x}_2(t,\alpha)&=\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha),\\
\overset{\star}{y}_2(t,\alpha)&=\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha),
\end{aligned}\right\}\tag{12}$$

$$\left.\begin{aligned}
\overset{\star}{x}_1(t,\beta)&=\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta),\\
\overset{\star}{y}_1(t,\beta)&=\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta),
\end{aligned}\right\}\tag{13}$$

$$\left.\begin{aligned}
\overset{\star}{x}_2(t,\beta)&=\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta),\\
\overset{\star}{y}_2(t,\beta)&=\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta),
\end{aligned}\right\}\tag{14}$$

Using (11), (12), (13) and (14), into (7), (8), (9) and (10), we have:

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha)=\Psi_1(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha))-\overset{\maltese}{L}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha),\\
&\qquad\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_1(t,\alpha)),\\
&\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha)=\Phi_1(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha))-\overset{\maltese}{L}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\alpha),\\
&\qquad\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_1(t,\alpha)),
\end{aligned}\right\}\tag{15}$$

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha)=\Psi_2(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha))-\overset{\maltese}{L}^{-1}(N_2(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha),\\
&\qquad\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{g}_2(t,\alpha)),\\
&\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha)=\Phi_2(t,\alpha)-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha))\\
&-\overset{\maltese}{L}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha))-\overset{\maltese}{L}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\alpha),\\
&\qquad\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\alpha)))+\overset{\maltese}{L}^{-1}(\overset{\star}{h}_2(t,\alpha)),
\end{aligned}\right\}\tag{16}$$

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta)=\Psi_1(t,\beta)-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta),\\
&\quad\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta)))+\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_1(t,\beta)),\\
&\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta)=\Phi_1(t,\beta)-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{1_n}(t,\beta),\\
&\quad\sum_{n=0}^{\infty}\overset{\star}{y}_{1_n}(t,\beta)))+\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_1(t,\beta)),
\end{aligned}\right\}\tag{17}$$

$$\left.\begin{aligned}
&\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta)=\Psi_2(t,\beta)-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_2(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta),\\
&\quad\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta)))+\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_2(t,\beta)),\\
&\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta)=\Phi_2(t,\beta)-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_1(t,\sum_{n=0}^{\infty}\overset{\star}{x}_{2_n}(t,\beta),\\
&\quad\sum_{n=0}^{\infty}\overset{\star}{y}_{2_n}(t,\beta)))+\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_2(t,\beta)).
\end{aligned}\right\}\tag{18}$$

According to the GMADM the recursive relation for the (15), (16),(17) and (18), is as follows:

$$\left.\begin{aligned}
&\overset{\star}{x}_{1_0}(t,\alpha)=\Psi_1(t,\alpha),\\
&\overset{\star}{y}_{1_0}(t,\alpha)=\Phi_1(t,\alpha),\\
&\overset{\star}{x}_{1_1}(t,\alpha)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_1(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_0}(t,\alpha))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_0}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_0}(t,\alpha),\overset{\star}{y}_{1_0}(t,\alpha))),\\
&\overset{\star}{y}_{1_1}(t,\alpha)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_1(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_0}(t,\alpha))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_0}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_0}(t,\alpha),\overset{\star}{y}_{1_0}(t,\alpha))),\\
&\overset{\star}{x}_{1_{k+1}}(t,\alpha)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_k}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_k}(t,\alpha))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_k}(t,\alpha),\overset{\star}{y}_{1_k}(t,\alpha))),\ k\geq 1,\\
&\overset{\star}{y}_{1_{k+1}}(t,\alpha)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_k}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_k}(t,\alpha))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_k}(t,\alpha),\overset{\star}{y}_{1_k}(t,\alpha))),\ k\geq 1,
\end{aligned}\right\}\tag{19}$$

$$\left.\begin{aligned}
&\overset{\star}{x}_{2_0}(t,\alpha)=\Psi_2(t,\alpha),\\
&\overset{\star}{y}_{2_0}(t,\alpha)=\Phi_2(t,\alpha),\\
&\overset{\star}{x}_{2_1}(t,\alpha)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_2(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_0}(t,\alpha))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_0}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_0}(t,\alpha),\overset{\star}{y}_{2_0}(t,\alpha))),\\
&\overset{\star}{y}_{2_1}(t,\alpha)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_2(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_0}(t,\alpha))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_0}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_0}(t,\alpha),\overset{\star}{y}_{2_0}(t,\alpha))),\\
&\overset{\star}{x}_{2_{k+1}}(t,\alpha)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_k}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_k}(t,\alpha))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_k}(t,\alpha),\overset{\star}{y}_{2_k}(t,\alpha))),\ k\geq 1,\\
&\overset{\star}{y}_{2_{k+1}}(t,\alpha)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_k}(t,\alpha))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_k}(t,\alpha))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_k}(t,\alpha),\overset{\star}{y}_{2_k}(t,\alpha))),\ k\geq 1,
\end{aligned}\right\}\tag{20}$$

$$\left.\begin{aligned}
&\overset{\star}{x}_{1_0}(t,\beta)=\Psi_1(t,\beta),\\
&\overset{\star}{y}_{1_0}(t,\beta)=\Phi_1(t,\beta),\\
&\overset{\star}{x}_{1_1}(t,\beta)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_1(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_0}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_0}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_0}(t,\beta),\overset{\star}{y}_{1_0}(t,\beta))),\\
&\overset{\star}{y}_{1_1}(t,\beta)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_1(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_0}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_0}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_0}(t,\beta),\overset{\star}{y}_{1_0}(t,\beta))),\\
&\overset{\star}{x}_{1_{k+1}}(t,\beta)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_k}(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_k}(t,\beta))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_k}(t,\beta),\overset{\star}{y}_{1_k}(t,\beta))),\ k\geq 1,\\
&\overset{\star}{y}_{1_{k+1}}(t,\beta)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{1_k}(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{1_k}(t,\beta))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_1(t,\overset{\star}{x}_{1_k}(t,\beta),\overset{\star}{y}_{1_k}(t,\beta))),\ k\geq 1,
\end{aligned}\right\}\tag{21}$$

$$\left.\begin{aligned}
&\overset{\star}{x}_{2_0}(t,\beta)=\Psi_2(t,\beta),\\
&\overset{\star}{y}_{2_0}(t,\beta)=\Phi_2(t,\beta),\\
&\overset{\star}{x}_{2_1}(t,\beta)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{g}_2(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_0}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_0}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_0}(t,\beta),\overset{\star}{y}_{2_0}(t,\beta))),\\
&\overset{\star}{y}_{2_1}(t,\beta)=\overset{\maltese}{L}{}^{-1}(\overset{\star}{h}_2(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_0}(t,\beta))\\
&-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_0}(t,\beta))-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_0}(t,\beta),\overset{\star}{y}_{2_0}(t,\beta))),\\
&\overset{\star}{x}_{2_{k+1}}(t,\beta)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_k}(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_k}(t,\beta))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_k}(t,\beta),\overset{\star}{y}_{2_k}(t,\beta))),\ k\geq 1,\\
&\overset{\star}{y}_{2_{k+1}}(t,\beta)=-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{x}_{2_k}(t,\beta))-\overset{\maltese}{L}{}^{-1}(\overset{\maltese}{R}\overset{\star}{y}_{2_k}(t,\beta))\\
&\quad-\overset{\maltese}{L}{}^{-1}(N_2(t,\overset{\star}{x}_{2_k}(t,\beta),\overset{\star}{y}_{2_k}(t,\beta))),\ k\geq 1.
\end{aligned}\right\}\tag{22}$$

The nth term approximation to the solution is defined as follows:

$$\left.\begin{aligned}\varphi_{1n}(t,\alpha) &= \sum_{i=0}^{n-1} \overset{\star}{x}_{1_i}(t,\alpha),\\ \phi_{1n}(t,\alpha) &= \sum_{i=0}^{n-1} \overset{\star}{y}_{1_i}(t,\alpha),\end{aligned}\right\}$$

$$\left.\begin{aligned}\varphi_{2n}(t,\alpha) &= \sum_{i=0}^{n-1} \overset{\star}{x}_{2_i}(t,\alpha),\\ \phi_{2n}(t,\alpha) &= \sum_{i=0}^{n-1} \overset{\star}{y}_{2_i}(t,\alpha),\end{aligned}\right\}$$

$$\left.\begin{aligned}\varphi_{1n}(t,\beta) &= \sum_{i=0}^{n-1} \overset{\star}{x}_{1_i}(t,\beta),\\ \phi_{1n}(t,\beta) &= \sum_{i=0}^{n-1} \overset{\star}{y}_{1_i}(t,\beta),\end{aligned}\right\}$$

$$\left.\begin{aligned}\varphi_{2n}(t,\beta) &= \sum_{i=0}^{n-1} \overset{\star}{x}_{2_i}(t,\beta),\\ \phi_{1n}(t,\beta) &= \sum_{i=0}^{n-1} \overset{\star}{y}_{1_i}(t,\beta).\end{aligned}\right\}$$

Hence,

$$\begin{cases}\{\lim\limits_{n\to\infty}(\phi_{1n}(t,\alpha))\,,\ \lim\limits_{n\to\infty}\phi_{1n}(t,\alpha)\} = \{\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)\},\\ \{\lim\limits_{n\to\infty}(\phi_{2n}(t,\alpha))\,,\ \lim\limits_{n\to\infty}\phi_{2n}(t,\alpha)\} = \{\overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha)\},\\ \{\lim\limits_{n\to\infty}(\phi_{1n}(t,\beta))\,,\ \lim\limits_{n\to\infty}\phi_{1n}(t,\beta)\} = \{\overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)\},\\ \{\lim\limits_{n\to\infty}(\phi_{2n}(t,\beta))\,,\ \lim\limits_{n\to\infty}\phi_{2n}(t,\beta)\} = \{\overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta)\}.\end{cases}$$

Raza et al. [39], Ray [40], Zo'bi et al. [41], and many others discuss the convergence of the ADM or Decomposition method and GMADM.

A. STABILITY OF GMADM

When the solution produced by a technique is unaffected by small changes in the inputs and parameters and when it is expected that changes in the parameters carried on by impacts in equations and conditions, the method is said to be stable. By giving examples and analyzing the stability of the GMADM in this study, we suggested contrasting the GMADM with other existing methods i.e., ADM and Taylor series method (TSM).

B. APPLICATIONS

Example 1 Brine Tanks Problem [42], [43]: Two tanks A and B are connected by pipes. Tank A contains 100 gal of brine and tank B contains 200 gal of brine. Through one pipe solution is pumped from first tank to the second at 30 gal/min. Through the other solution is pumped at the rate of 10 gal/min from the second tank to the first, and the brine in tank B flows out at 20 gal/min. If $\overset{\star}{x}(t)$ and $\overset{\star}{y}(t)$ denotes the amount of salt in tanks A and B respectively, then what will be the salt content in each tank at any time t. It is to be noted that at time $t = 0$ there is <(98, 99, 100, 101, 0.7); (97, 99, 100, 102; 0.2)> lb salt in tank A and <(50, 51, 52, 53; 0.7); (49, 51, 52, 54; 0.2)> lb in tank B.

The system of intuitionistic fuzzy differential equation related to above problem is as follows:

$$\begin{aligned}\frac{d\overset{*}{\overset{\star}{x}}(t,\alpha)}{dt} &= \frac{\overset{\star}{y}}{20} - \frac{3\overset{\star}{x}}{10},\\ \frac{d\overset{\star}{y}(t,\alpha)}{dt} &= \frac{3\overset{\star}{x}}{10} - \frac{3\overset{\star}{y}}{20}.\end{aligned} \tag{23}$$

with initial conditions,

$$\begin{cases}\overset{\star}{x}(0) = <(98, 99, 100, 101, 0.7); (97, 99, 100, 102; 0.2)>,\\ \overset{\star}{y}(0) = <(50, 51, 52, 53; 0.7); (49, 51, 52, 54; 0.2)>.\end{cases} \tag{24}$$

By taking (α, β)-cut of (24) and (24), we obtain the following equations:

$$\begin{cases}\frac{d\overset{\star}{x}(t,\alpha)}{dt} = \frac{\overset{\star}{y}}{20} - \frac{3\overset{\star}{x}}{10}, \overset{\star}{x}_1(0,\alpha) = 1.429\alpha + 98,\\ \frac{d\overset{\star}{y}(t,\alpha)}{dt} = \frac{3\overset{\star}{x}}{10} - \frac{3\overset{\star}{y}}{20}, \overset{\star}{y}_1(0,\alpha) = 1.429\alpha + 50,\end{cases} \tag{25}$$

$$\begin{cases}\frac{d\overset{\star}{x}(t,\alpha)}{dt} = \frac{\overset{\star}{y}}{20} - \frac{3\overset{\star}{x}}{10}, \overset{\star}{x}_2(0,\alpha) = -1.429\alpha + 101,\\ \frac{d\overset{\star}{y}(t,\alpha)}{dt} = \frac{3\overset{\star}{x}}{10} - \frac{3\overset{\star}{y}}{20}, \overset{\star}{y}_2(0,\alpha) = -1.429\alpha + 53,\end{cases} \tag{26}$$

$$\begin{cases}\frac{d\overset{\star}{x}(t,\beta)}{dt} = \frac{\overset{\star}{y}}{20} - \frac{3\overset{\star}{x}}{10}, \overset{\star}{x}_1(0,\beta) = -2.5\beta + 99,\\ \frac{d\overset{\star}{y}(t,\beta)}{dt} = \frac{3\overset{\star}{x}}{10} - \frac{3\overset{\star}{y}}{20}, \overset{\star}{y}_1(0,\beta) = -2.5\beta + 51,\end{cases} \tag{27}$$

$$\begin{cases}\frac{d\overset{\star}{x}(t,\beta)}{dt} = \frac{\overset{\star}{y}}{20} - \frac{\overset{\star}{x}}{20}, \overset{\star}{x}_2(0,\beta) = -2.5\beta + 99.5,\\ \frac{d\overset{\star}{y}(t,\beta)}{dt} = \frac{\overset{\star}{x}}{20} - \frac{\overset{\star}{y}}{20}, \overset{\star}{y}_2(0,\beta) = 2.5\beta - 51.5.\end{cases} \tag{28}$$

Here $\overset{\maltese}{L} = \frac{d}{dt}$ and by taking $\overset{\maltese}{L}^{-1}(.) = \int\limits_0^t (.)\,dt$ on both sides of (25), (26), (27) and (28), and using the initial conditions we obtain;

$$\begin{cases}\overset{\star}{x}_1(t,\alpha) = \int_0^t(-0.30\overset{\star}{x}_1(u,\alpha) + 0.05\overset{\star}{y}_1(u,\alpha))du\\ \qquad\qquad +1.429\alpha + 98,\\ \overset{\star}{y}_1(t,\alpha) = \int_0^t(-0.150\overset{\star}{y}_1(u,\alpha) + 0.30\overset{\star}{x}_1(u,\alpha))du\\ \qquad\qquad +1.429\alpha + 50,\end{cases} \tag{29}$$

$$\begin{cases}\overset{\star}{x}_2(t,\alpha) = \int_0^t(-0.30\overset{\star}{x}_2(u,\alpha) + 0.05\overset{\star}{y}_2(u,\alpha))du\\ \qquad\qquad -1.429\alpha + 101,\\ \overset{\star}{y}_2(t,\alpha) = \int_0^t(-0.15\overset{\star}{y}_2(u,\alpha) + 0.30\overset{\star}{x}_2(u,\alpha))du\\ \qquad\qquad -1.429\alpha + 53,\end{cases} \tag{30}$$

$$\begin{cases}\overset{\star}{x}_1(t,\beta) = \int_0^t(-0.30\overset{\star}{x}_1(u,\beta) + 0.05\overset{\star}{y}_1(u,\beta))du\\ \qquad\qquad +99 - 2.5\beta,\\ \overset{\star}{y}_1(t,\beta) = \int_0^t(-0.15\overset{\star}{y}_1(u,\beta) + 0.30\overset{\star}{x}_1(u,\beta))du\\ \qquad\qquad +51 - 2.5\beta,\end{cases} \tag{31}$$

$$\begin{cases}\overset{\star}{x}_2(t,\beta) = \int_0^t(-0.05\overset{\star}{x}_2(u,\beta) + 0.05\overset{\star}{y}_2(u,\beta))du\\ \qquad\qquad +2.5\beta + 99.5,\\ \overset{\star}{y}_2(t,\beta) = \int_0^t(-0.05\overset{\star}{y}_2(u,\beta) + 0.05\overset{\star}{x}_2(u,\beta))du\\ \qquad\qquad +0.25\beta + 0.65.\end{cases} \tag{32}$$

Now by using GMADM the solution of (29), (30), (31) and (32), can be expressed as;

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\alpha) = 1.429\alpha + 98, \\ \overset{\star}{y}_{1_0}(t,\alpha) = 1.429\alpha + 50, \\ \overset{\star}{x}_{1_1}(t,\alpha) = -0.357250\alpha t - 26.90t, \\ \overset{\star}{y}_{1_1}(t,\alpha) = 0.214350\alpha t + 21.90t, \\ \overset{\star}{x}_{1_{k+1}}(t,\alpha) = \int_0^t(-0.30\overset{\star}{x}_{1_k}(u,\alpha) \\ \quad +0.05\overset{\star}{y}_{1_k}(u,\alpha))du,\ k\geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\alpha) = \int_0^t(-0.15\overset{\star}{y}_{1k}(u,\alpha) \\ \quad +0.30\overset{\star}{x}_{1_k}(u,\alpha))du,\ k\geq 1. \end{cases} \tag{33}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\alpha) = -1.429\alpha + 101, \\ \overset{\star}{y}_{2_0}(t,\alpha) = -1.429\alpha + 50, \\ \overset{\star}{x}_{2_1}(t,\alpha) = 0.357250\alpha t - 27.650t, \\ \overset{\star}{y}_{2_1}(t,\alpha) = -0.214350\alpha t + 22.350t, \\ \overset{\star}{x}_{2_{k+1}}(t,\alpha) = \int_0^t(-0.30\overset{\star}{x}_{2_k}(u,\alpha) \\ \quad +0.05\overset{\star}{y}_{2_k}(u,\alpha))du,\ k\geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\alpha) = \int_0^t(-0.15\overset{\star}{y}_{2k}(u,\alpha) \\ \quad +0.30\overset{\star}{x}_{2_k}(u,\alpha))du,\ k\geq 1. \end{cases} \tag{34}$$

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\beta) = 99 - 2.5\beta, \\ \overset{\star}{y}_{1_0}(t,\beta) = 51 - 2.5\beta, \\ \overset{\star}{x}_{1_1}(t,\beta) = -27.150t + 0.6250\beta t, \\ \overset{\star}{y}_{1_1}(t,\beta) = 22.050t - 0.3750\beta t, \\ \overset{\star}{x}_{1_{k+1}}(t,\beta) = \int_0^t(-0.30\overset{\star}{x}_{1_k}(u,\beta) \\ \quad +0.05\overset{\star}{y}_{1_k}(u,\beta))du,\ k\geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\beta) = \int_0^t(-0.15\overset{\star}{y}_{1_k}(u,\beta) \\ \quad +0.30\overset{\star}{x}_{1_k}(u,\beta))du,\ k\geq 1. \end{cases} \tag{35}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\beta) = 2.5\beta + 99.5, \\ \overset{\star}{y}_{2_0}(t,\beta) = 2.5\beta + 51.5, \\ \overset{\star}{x}_{2_1}(t,\beta) = -0.6250\beta t - 27.2750t, \\ \overset{\star}{y}_{2_1}(t,\beta) = 0.3750\beta t + 22.1250t, \\ \overset{\star}{x}_{2_{k+1}}(t,\beta) = \int_0^t(-0.30\overset{\star}{x}_{2_k}(u,\beta) \\ \quad +0.05\overset{\star}{y}_{2_k}(u,\beta))du,\ k\geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\beta) = \int_0^t(-0.15\overset{\star}{y}_{2_k}(u,\beta) \\ \quad +0.30\overset{\star}{x}_{2_k}(u,\beta))du,\ k\geq 1. \end{cases} \tag{36}$$

By solving (33), (34), (35) and (36), we get the approximate solution after four iterations as follows:

$$\begin{aligned} &(\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)) \\ &= (1.429\alpha + 98 - 0.357250\alpha t - 26.90t \\ &\quad + 0.0589462500t^2\alpha + 4.582500000t^2 \\ &\quad - 0.007055687500\alpha t^3 - 0.5528750000t^3 \\ &\quad + 0.0006463992188\alpha t^4 + 0.05074218750t^4, \\ &\quad 1.429\alpha + 50 + 0.2143500000\alpha t + 21.90000000t \\ &\quad - 0.06966375000\alpha t^2 - 5.677500000t^2 \\ &\quad + 0.009377812500\alpha t^3 + 0.7421250000t^3 \\ &\quad - 0.0008808445312\alpha t^4 - 0.06929531250t^4), \end{aligned}$$

$$\begin{aligned} &(\overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha)) \\ &= (-1.429\alpha + 101 + 0.357250\alpha t - 27.650t \\ &\quad - 0.05894625000\alpha t^2 + 4.706250000t^2 \\ &\quad + 0.007055687499\alpha t^3 - 0.5676874999t^3 \\ &\quad - 0.0006463992188\alpha t^4 + 0.05209921875t^4, \\ &\quad - 1.429\alpha + 53 - 0.2143500000\alpha t + 22.35000000t \\ &\quad + 0.06966375000\alpha t^2 - 5.823750000t^2 \\ &\quad - 0.009377812500\alpha t^3 + 0.7618125000t^3 \\ &\quad + 0.0008808445312\alpha t^4 - 0.07114453125t^4), \end{aligned}$$

$$\begin{aligned} &(\overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)) \\ &= (99 - 2.5\beta - 27.150t + 0.6250\beta t \\ &\quad + 4.623750t^2 - 0.1031250000\beta t^2 \\ &\quad - 0.5578125000t^3 + 0.01234375000\beta t^3 \\ &\quad + 0.05119453125t^4 - 0.001130859375\beta t^4 \\ &\quad , 51 - 2.5\beta + 22.050t - 0.3750\beta t \\ &\quad - 5.726250000t^2 + 0.1218750000\beta t^2 \\ &\quad + 0.7486875000t^3 - 0.01640625000\beta t^3 \\ &\quad - 0.06991171875t^4 + 0.001541015625\beta t^4), \end{aligned}$$

$$\begin{aligned} &(\overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta)) \\ &= (2.5\beta + 99.5 - 0.6250\beta t - 27.2750t \\ &\quad + 0.1031250\beta t^2 + 4.644375000t^2 \\ &\quad - 0.01234375000\beta t^3 - 0.5602812500t^3 \\ &\quad + 0.001130859375\beta t^4 + 0.05142070312t^4, \\ &\quad 2.5\beta + 51.5 + 0.3750\beta t + 22.1250t \\ &\quad - 0.1218750000\beta t^2 - 5.750625000t^2 \\ &\quad + 0.01640625000\beta t^3 + 0.7519687500t^3 \\ &\quad - 0.001541015625\beta t^4 - 0.07021992188t^4). \end{aligned}$$

In Table 1, $\overset{\star}{x}_1(t,\alpha)$,$\overset{\star}{x}_2(t,\alpha)$,$\overset{\star}{y}_1(t,\alpha)$ and $\overset{\star}{y}_2(t,\alpha)$ represents approximate solution of the membership functions of the Example1 for $\alpha \in [0, 0.7]$ and $\overset{\star}{x}_1(t,\beta)$, $\overset{\star}{x}_2(t,\beta)$,$\overset{\star}{y}_1(t,\beta)$ and $\overset{\star}{y}_2(t,\beta)$ represents approximate solution of non-membership function of the Example1 for $\beta \in [0.2, 1.0]$.

The following is the mathematical and exact solution to Example 1 using the classical method, as shown in Figure 2(a,b) and Table 2:

$$\begin{aligned} &(\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)) \\ &= (-\frac{1}{34000}(-17+\sqrt{17})(92000+1429\alpha \\ &\quad - 6000\sqrt{17})e^{\frac{1}{40}(-9+\sqrt{17})t} \\ &\quad + \frac{1}{34000}(17+\sqrt{17})(92000+1429\alpha \\ &\quad + 6000\sqrt{17})e^{-\frac{1}{40}(9+\sqrt{17})t}, \end{aligned}$$

TABLE 1. Illustrates the approximation to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 1 for t=1.

Table 1: Approximate solution of example1 at $t = 1$.									
α	$\overset{*}{\overset{\star}{x}}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	75.18	57.90	77.54	61.06	0.2	75.57	58.42	76.75	60.00
0.1	75.29	58.05	77.43	60.90	0.3	75.38	58.16	76.95	60.27
0.2	75.41	58.20	77.32	60.75	0.4	75.18	57.90	77.15	60.53
0.3	75.52	58.35	77.20	60.60	0.5	74.98	57.63	77.34	60.79
0.4	75.63	58.50	77.09	60.45	0.6	74.79	57.37	77.54	61.06
0.5	75.74	58.65	76.98	60.30	0.7	74.59	57.11	77.37	61.32
0.6	75.85	58.80	76.87	60.15	0.8	74.39	56.84	77.93	61.58
0.7	75.97	58.95	76.75	60.00	0.9	74.20	56.58	78.13	61.84
					1.0	74.00	56.32	78.33	62.11

TABLE 2. Illustrates the exact solution to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 3 for t=1.

Table 2: Exact solution of example1 at $t = 1$.									
α	$\overset{\star}{x}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	74.78	51.00	77.52	70.35	0.2	75.57	67.39	76.75	69.03
0.1	74.89	51.15	77.41	70.13	0.3	75.38	67.13	76.95	69.38
0.2	75.00	51.21	77.32	69.84	0.4	75.19	66.87	77.14	69.64
0.3	75.14	51.43	77.19	69.79	0.5	74.99	66.62	77.34	69.90
0.4	75.22	51.54	77.09	69.68	0.6	74.79	66.36	77.74	70.47
0.5	75.32	51.69	76.95	69.35	0.7	74.59	66.06	77.93	70.73
0.6	75.45	51.80	76.87	69.36	0.8	74.39	65.79	78.13	70.99
0.7	75.56	51.96	76.72	69.10	0.9	74.20	65.44	78.13	70.99
					1.0	73.99	65.12	78.32	71.24

TABLE 3. Illustrates the approximate solution iterations in column 2, residual error in column 3, and CPU time required by the numerical technique GMADM to determine the approximate solution of the system of generalized fuzzy intuitionistic differential equations used in Example 1 in column 4. Whenever the error of all methods is taken into account, we can conclude that the GMADM has a better convergence behavior and is more stable than the ADM and TSM, respectively.

Table 3: Error comparsion			
Paramater	Iterations	Error	CPU-time
GMADM	4	0.1E-15	1.12451
ADM	8	5.8E-08	2.14510
TSM	7	3.1E-05	2.10325

$$
\begin{aligned}
&-\frac{1}{68000}(-17+\sqrt{17})(92000+1429\alpha \\
&-6000\sqrt{17})e^{\frac{1}{40}(-9+\sqrt{17})t} \\
&\times\sqrt{17}-\frac{1}{68000}(17+\sqrt{17})(92000 \\
&+1429\alpha+6000\sqrt{17})e^{-\frac{1}{40}(9+\sqrt{17})t} \\
&\times\sqrt{17}-\frac{3}{68000}(-17+\sqrt{17})(92000+1429\alpha \\
&-6000\sqrt{17})e^{\frac{1}{40}(-9+\sqrt{17})t} \\
&+\frac{3}{68000}(17+\sqrt{17})(92000+1429\alpha \\
&+6000\sqrt{17})e^{-\frac{1}{40}(9+\sqrt{17})t}),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_2(t,\alpha),\overset{\star}{y}_2(t,\alpha)) \\
&=(\frac{1}{66000}(-33+\sqrt{33})(-98000+1429\alpha \\
&\quad+3000\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t} \\
&\quad-\frac{1}{66000}(33+\sqrt{33})(-98000+1429\alpha \\
&\quad-3000\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}, \\
&\quad\frac{1}{132000}(-33+\sqrt{33})(-98000+1429\alpha \\
&\quad+3000\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t}\sqrt{33} \\
&\quad+\frac{1}{132000}(33+\sqrt{33})(-98000+1429\alpha \\
&\quad-3000\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}\sqrt{33} \\
&\quad+\frac{1}{44000}(-33+\sqrt{33})(-98000+1429\alpha \\
&\quad+3000\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t} \\
&\quad-\frac{1}{44000}(33+\sqrt{33})(-98000+1429\alpha \\
&\quad-3000\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_1(t,\beta),\overset{\star}{y}_1(t,\beta)) \\
&=(\frac{1}{132}(-33+\sqrt{33})(-192+5\beta+6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t} \\
&\quad-\frac{1}{132}(33+\sqrt{33})(-192+5\beta-6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}, \\
&\quad\frac{1}{264}(-33+\sqrt{33})(-192+5\beta+6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t}\sqrt{33} \\
&\quad+\frac{1}{264}(33+\sqrt{33})(-192+5\beta-6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}\sqrt{33} \\
&\quad+\frac{1}{88}(-33+\sqrt{33})(-192+5\beta+6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t}\sqrt{33} \\
&\quad-\frac{1}{88}(33+\sqrt{33})(-192+5\beta-6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_2(t,\beta),\overset{\star}{y}_2(t,\beta)) \\
&=(-\frac{1}{132}(-33+\sqrt{33})(193+5\beta-6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t} \\
&\quad+\frac{1}{132}(33+\sqrt{33})(193+5\beta+6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}, \\
&\quad-\frac{1}{264}(-33+\sqrt{33})(193+5\beta-6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t}\sqrt{33} \\
&\quad-\frac{1}{264}(33+\sqrt{33})(193+5\beta+6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}\sqrt{33} \\
&\quad-\frac{1}{88}(-33+\sqrt{33})(193+5\beta-6\sqrt{33})e^{\frac{1}{40}(-9+\sqrt{33})t} \\
&\quad+\frac{1}{88}(33+\sqrt{33})(193+5\beta+6\sqrt{33})e^{-\frac{1}{40}(9+\sqrt{33})t}).
\end{aligned}
$$

Table 2, $\overset{\star}{x}_1(t,\alpha)$, $\overset{\star}{x}_2(t,\alpha)$, $\overset{\star}{y}_1(t,\alpha)$ and $\overset{\star}{y}_2(t,\alpha)$ represents exact solution of the membership functions of the Example1 for $\alpha \epsilon [0, 0.7]$ and $\overset{\star}{x}_1(t,\beta)$, $\overset{\star}{x}_2(t,\beta)$, $\overset{\star}{y}_1(t,\beta)$ and $\overset{\star}{y}_2(t,\beta)$ represents exact solution of non-membership function of the Example1 for $\beta \epsilon [0.2, 1.0]$.

Example 2 Coupled Oscillators: Consider a mechanical system [44], [45], [46], [47] constituting of two masses $m_1 = 1Kg$ and $m_2 = 1Kg$ that are free to slide over a friction less horizontal surface. The masses are attached to one another, and to two rigid walls, with the help of three springs. The spring constants for this system are $k_1 = 1Nm^{-1}$, $k_2 = 2Nm^{-1}$ and $k_3 = 1Nm^{-1}$. The instantaneous state of the system is conveniently specified by the $\overset{\star}{x}(t)$ and $\overset{\star}{y}(t)$ respectively. Thus, the equations of motions of two masses are as follows:

$$\begin{cases} m\dfrac{d^2\overset{\star}{x}(t)}{dt^2} = -k_1\overset{\star}{x} - k_2(\overset{\star}{x} - \overset{\star}{y}), \\ m\dfrac{d^2\overset{\star}{y}(t)}{dt^2} = -k_3\overset{\star}{y} - k_2(\overset{\star}{y} - \overset{\star}{x}). \end{cases} \tag{37}$$

Using the given data, we get;

$$\begin{cases} \dfrac{d^2\overset{\star}{x}(t)}{dt^2} = 2\overset{\star}{y} - 3\overset{\star}{x}, \\ \dfrac{d^2\overset{\star}{y}(t)}{dt^2} = 2\overset{\star}{x} - 3\overset{\star}{y}, \end{cases} \tag{38}$$

with initial conditions,

$$\begin{cases} \overset{\star}{x}(0) = <(2,4,8,15;0.6);(1,4,8,18;0.3)>, \\ \overset{\star}{x}'(0) = <(2,5,8,10;0.6)(1,5,8,12;0.3)>, \\ \overset{\star}{y}(0) = <(8,9,10,11;0.6)(7,9,10,12;0.3)>, \\ \overset{\star}{y}'(0) = <(11,12,13,14;0.6);(10,12,13,15;0.3)>. \end{cases} \tag{39}$$

By taking (α,β)-cut of (38), (39), we get the following equations:

$$\begin{cases} \dfrac{d^2\overset{\star}{x}(t,\alpha)}{dt^2} = 2\overset{\star}{y} - 3\overset{\star}{x}, \\ \quad \overset{\star}{x}_1(0,\alpha) = 3.33\alpha + 2, \overset{\star}{x}_1'(0,\alpha) = 5\alpha + 2, \\ \dfrac{d^2\overset{\star}{y}(t,\alpha)}{dt^2} = 2\overset{\star}{x} - 3\overset{\star}{y}, \\ \quad \overset{\star}{y}_1(0,\alpha) = 1.67\alpha + 8, \overset{\star}{y}_1'(0,\alpha) = 1.67\alpha + 11, \end{cases} \tag{40}$$

$$\begin{cases} \dfrac{d^2\overset{\star}{x}(t,\alpha)}{dt^2} = 2\overset{\star}{y} - 3\overset{\star}{x}, \\ \quad \overset{\star}{x}_2(0,\alpha) = 15 - 11.67\alpha, \overset{\star}{x}_2'(0,\alpha) = 10 - 3.33\alpha, \\ \dfrac{d^2\overset{\star}{y}(t,\alpha)}{dt^2} = 2\overset{\star}{x} - 3\overset{\star}{y}, \\ \quad \overset{\star}{y}_2(0,\alpha) = -1.67\alpha + 11, \overset{\star}{y}_2'(0,\alpha) = -1.67\alpha + 14, \end{cases} \tag{41}$$

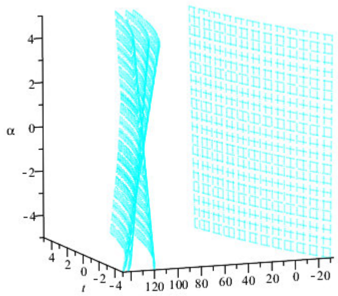

FIGURE 2. Exact solution of $(\overset{\star}{x}_{1_k}, \overset{\star}{y}_{1_k})$ of generalized trapezoidal intuitionistic fuzzy number.

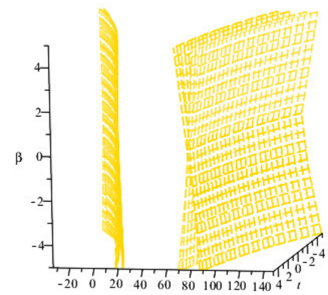

FIGURE 3. Exact solution of $(\overset{\star}{x}_{2_k}, \overset{\star}{y}_{2_k})$ of generalized trapezoidal intuitionistic fuzzy number.

$$\begin{cases} \dfrac{d^2\overset{\star}{x}(t,\beta)}{dt^2} = 2\overset{\star}{y} - 3\overset{\star}{x}, \\ \quad \overset{\star}{x}_1(0,\beta) = -4.29\beta + 5.29, \overset{\star}{x}_1'(0,\beta) = 6.71 - 0.85\beta, \\ \dfrac{d^2\overset{\star}{y}(t,\beta)}{dt^2} = 2\overset{\star}{x} - 3\overset{\star}{y}, \\ \quad \overset{\star}{y}_1(0,\beta) = 9 - 2.86\beta, \overset{\star}{y}_1'(0,\beta) = 12 - 2.86\beta, \end{cases} \tag{42}$$

$$\begin{cases} \dfrac{d^2\overset{\star}{x}(t,\beta)}{dt^2} = 2\overset{\star}{y} - 3\overset{\star}{x}, \\ \quad \overset{\star}{x}_2(0,\beta) = 3.71 + 14.29\beta, \overset{\star}{x}_1'(0,\beta) = 6.29 + 5.71\beta, \\ \dfrac{d^2\overset{\star}{y}(t,\beta)}{dt^2} = 2\overset{\star}{x} - 3\overset{\star}{y}, \\ \quad \overset{\star}{y}_2(0,\beta) == 9.14 + 2.86\beta, \overset{\star}{y}_1'(0,\beta) = 12.14 + 2.86\beta. \end{cases} \tag{43}$$

Here $\overset{\maltese}{L} = \frac{d^2}{dt^2}$ and by taking $\overset{\maltese}{L}^{-1}(.) = \int\limits_0^t\int\limits_0^t (.)\,dtdt$ on both sides of (40), (41), (42) and (40), and using the initial conditions we obtain;

$$\begin{cases} \overset{\star}{x}_1(t,\alpha) = \int_0^t (3\overset{\star}{x}_1(u,\alpha)(-t+u) + 2\overset{\star}{y}_1(u,\alpha)(t-u))du \\ \qquad +2 + 5\alpha t + 3.33\alpha + 2t, \\ \overset{\star}{y}_1(t,\alpha) = \int_0^t (2\overset{\star}{x}_1(u,\alpha)(t-u) + 3\overset{\star}{y}_1(u,\alpha)(-t+u))du \\ \qquad +8 + 1.67\alpha t + 1.67\alpha + 11t, \end{cases} \tag{44}$$

Figure 2-3: Shows exact solution of the system of generalized trapezoidal intuitionistic fuzzy initial value problem used

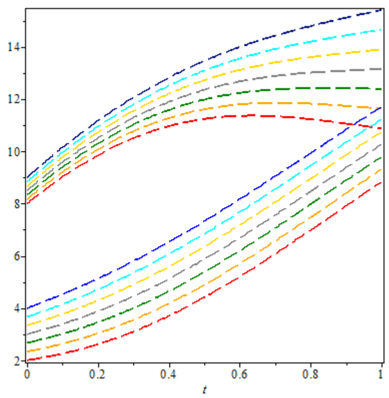

FIGURE 4. Exact solution of $(\overset{\star}{x}_{1_k}(t,\alpha),\overset{\star}{y}_{1_k}(t,\alpha))$ of generalized trapezoidal intuitionistic fuzzy number.

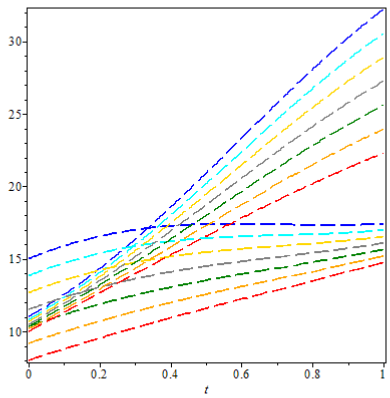

FIGURE 5. Exact solution of $(\overset{\star}{x}_{2_k}(t,\alpha),\overset{\star}{y}_{2_k}(t,\alpha))$ of generalized trapezoidal intuitionistic fuzzy number.

in example 1.

$$\begin{cases} \overset{\star}{x}_2(t,\alpha) = \int_0^t (3\overset{\star}{x}_2(u,\alpha)(-t+u) + 2\overset{\star}{y}_2(u,\alpha)(t-u))du \\ \qquad +15 - 3.33\alpha t - 11.67\alpha + 10t, \\ \overset{\star}{y}_2(t,\alpha) = \int_0^t (2\overset{\star}{x}_2(u,\alpha)(t-u) + 3\overset{\star}{y}_2(u,\alpha)(-t+u))du \\ \qquad +11 - 1.67\alpha t - 1.67\alpha + 14t, \end{cases} \tag{45}$$

$$\begin{cases} \overset{\star}{x}_1(t,\beta) = \int_0^t (3\overset{\star}{x}_1(u,\beta)(-t+u) + 2\overset{\star}{y}_1(u,\beta)(t-u))du \\ \qquad +5.29 - 0.85\beta t + 6.71t - 4.29\beta, \\ \overset{\star}{y}_1(t,\beta) = \int_0^t (2\overset{\star}{x}_1(u,\beta)(t-u) + 3\overset{\star}{y}_1(u,\beta)(-t+u))du \\ \qquad +9 - 2.86\beta t - 2.86\beta + 12t, \end{cases} \tag{46}$$

$$\begin{cases} \overset{\star}{x}_2(t,\beta) = \int_0^t (3\overset{\star}{x}_2(u,\beta)(-t+u) + 2\overset{\star}{y}_2(u,\beta)(t-u))du \\ \qquad +3.71 + 5.71\beta t + 6.29t + 14.29\beta, \\ \overset{\star}{y}_2(t,\beta) = \int_0^t (2\overset{\star}{x}_2(u,\beta)(t-u) + 3\overset{\star}{y}_2(u,\beta)(-t+u))du \\ \qquad +9.14 + 2.86\beta t + 12.14t + 2.86\beta. \end{cases} \tag{47}$$

Figures 3-7: Displays the membership and non-membership functions, as well as the exact and approximate

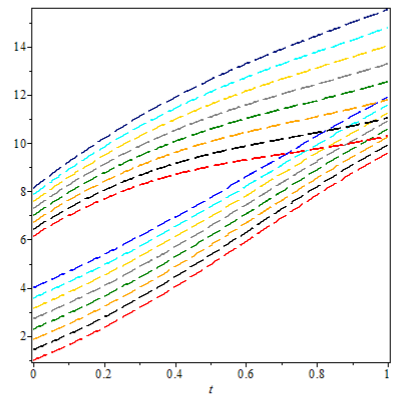

FIGURE 6. Exact solution of $(\overset{\star}{x}_{1_k}(t,\beta),\overset{\star}{y}_{1_k}(t,\beta))$ of generalized trapezoidal intuitionistic fuzzy number.

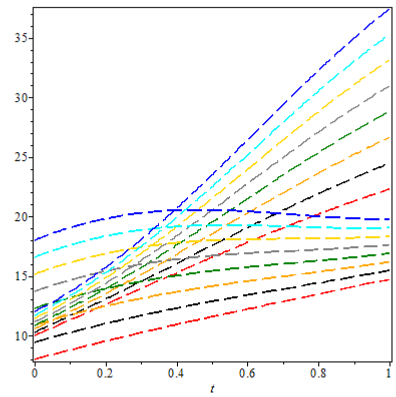

FIGURE 7. Exact solution of $(\overset{\star}{x}_{2_k}(t,\beta),\overset{\star}{y}_{2_k}(t,\beta))$ of generalized trapezoidal intuitionistic fuzzy number.

solutions, for the generalized trapezoidal intuitionistic fuzzy initial value problem used in example 2.

Now by using GMADM the solution of (44), (45), (46) and (47), can be expressed as;

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\alpha) = 2 + t(5\alpha+2) + 3.33\alpha, \\ \overset{\star}{y}_{1_0}(t,\alpha) = 8 + t(1.67\alpha+11) + 1.67\alpha, \\ \overset{\star}{x}_{1_1}(t,\alpha) = -1.943333334\alpha t^3 + 2.666666667t^3 \\ \qquad + 5.00000000t^2 - 3.325000000\alpha t^2, \\ \overset{\star}{y}_{1_1}(t,\alpha) = 2.498333334\alpha t^3 - 4.166666668t^3 \\ \qquad - 8.00000000t^2 + 4.155000000\alpha t^2, \\ \overset{\star}{x}_{1_{k+1}}(t,\alpha) = \displaystyle\int_0^t (3\overset{\star}{x}_{1_k}(u,\alpha)(-t+u) \\ \qquad + 2\overset{\star}{y}_{1_k}(u,\alpha)(t-u))du, k \geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\alpha) = \displaystyle\int_0^t (2\overset{\star}{x}_{1_k}(u,\alpha)(t-u) \\ \qquad + 3\overset{\star}{y}_{1_k}(u,\alpha)(-t+u))du, k \geq 1. \end{cases} \tag{48}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\alpha) = 15 + t(-3.33\alpha + 10) - 11.67\alpha, \\ \overset{\star}{y}_{2_0}(t,\alpha) = 11 + t(-1.67\alpha + 14) - 1.67\alpha, \\ \overset{\star}{x}_{2_1}(t,\alpha) = 1.108333334\alpha t^3 - 0.3333333334t^3 \\ \qquad - 11.50000000t^2 + 15.83500000\alpha t^2, \\ \overset{\star}{y}_{2_1}(t,\alpha) = -1.385000000\alpha t^3 - 0.3333333334t^3 \\ \qquad + 13.50000000t^2 - 20.83500000\alpha t^2, \\ \overset{\star}{x}_{2_{k+1}}(t,\alpha) = \int_0^t (3\overset{\star}{x}_{2_k}(u,\alpha)(-t+u) \\ \qquad + 2\overset{\star}{y}_{2_k}(u,\alpha)(t-u))du, k \geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\alpha) = \int_0^t (2\overset{\star}{x}_{2_k}(u,\alpha)(t-u) \\ \qquad + 3\overset{\star}{y}_{2_k}(u,\alpha)(-t+u))du, k \geq 1. \end{cases} \tag{49}$$

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\beta) = 5.29 + t(-0.85\beta + 6.71) - 4.29\beta, \\ \overset{\star}{y}_{1_0}(t,\beta) = 9 + t(-2.86\beta + 12) - 2.86\beta, \\ \overset{\star}{x}_{1_1}(t,\beta) = -0.528333333\beta t^3 + 0.645000000t^3 \\ \qquad + 1.06500000t^2 + 3.575000000\beta t^2, \\ \overset{\star}{y}_{1_1}(t,\beta) = 0.863333334\beta t^3 - 1.526666667t^3 \\ \qquad - 2.92000000t^2 - 4.290000000\beta t^2, \\ \overset{\star}{x}_{1_{k+1}}(t,\beta) = \int_0^t (2\overset{\star}{x}_{1_k}(u,\beta)(t-u) \\ \qquad + 3\overset{\star}{y}_{1_k}(u,\beta)(-t+u))du, k \geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\beta) = \int_0^t (2\overset{\star}{x}_{1_k}(u,\beta)(t-u) \\ \qquad + 3\overset{\star}{y}_{1_k}(u,\beta)(-t+u))du, k \geq 1. \end{cases} \tag{50}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\beta) = 3.71 + t(5.71\beta + 6.29) + 14.29\beta, \\ \overset{\star}{y}_{2_0}(t,\beta) = 9.14 + t(2.86\beta + 12.14) + 2.86\beta, \\ \overset{\star}{x}_{2_1}(t,\beta) = -1.901666667\beta t^3 + 0.901666667t^3 \\ \qquad + 3.57500000t^2 - 18.57500000\beta t^2, \\ \overset{\star}{y}_{2_1}(t,\beta) = 2.376666667\beta t^3 - 1.876666667t^3 \\ \qquad - 6.29000000t^2 + 24.29000000\beta t^2, \\ \overset{\star}{x}_{2_{k+1}}(t,\beta) = \int_0^t (3\overset{\star}{x}_{2_k}(u,\beta)(-t+u) \\ \qquad + 2\overset{\star}{y}_{2_k}(u,\beta)(t-u))du, k \geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\beta) = \int_0^t (2\overset{\star}{x}_{2_k}(u,\beta)(t-u) \\ \qquad + 3\overset{\star}{y}_{2_k}(u,\beta)(-t+u))du, k \geq 1. \end{cases} \tag{51}$$

By solving (48), (49), (50) and (51), we get the approximate solution after four iterations as follows:

$$\begin{aligned}
&(\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha)) \\
&= (2 + 5\alpha t + 2t + 3.33\alpha - 1.943333334\alpha t^3 \\
&\quad + 2.666666667t^3 + 5.00000000t^2 - 3.325000000\alpha t^2 \\
&\quad + 0.541333334\alpha t^5 - 0.816666667t^5 - 2.583333330t^4 \\
&\quad + 1.523749999\alpha t^4 - 0.0750198413\alpha t^7 \\
&\quad + 0.1134920635t^7 \\
&\quad + 0.502777777t^6 - 0.295513888\alpha t^6 \\
&\quad + 0.00607264110\alpha t^9 \\
&\quad - 0.00918761024t^9 - 0.0523313492t^8 \\
&\quad + 0.0307552083\alpha t^8, \\
&\quad 8 + 1.67\alpha t + 11t + 1.67\alpha + 2.498333334\alpha t^3 \\
&\quad - 4.166666668t^3 \\
&\quad - 8.00000000t^2 + 4.155000000\alpha t^2 - 0.763416667\alpha t^5 \\
&\quad + 1.158333334t^5 + 3.66666667t^4 - 2.147083332\alpha t^4 \\
&\quad + 0.1060853175\alpha t^7 - 0.1605158731t^7 - 0.711111110t^6 \\
&\quad + 0.417874999\alpha t^6 - 0.00858799053\alpha t^9 \\
&\quad + 0.0129932761t^9 \\
&\quad + 0.0740079365t^8 - 0.0434942956\alpha t^8),
\end{aligned}$$

$$\begin{aligned}
&(\overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha)) \\
&= (15 - 3.33\alpha t + 10t - 11.67\alpha + 1.108333334\alpha t^3 \\
&\quad - 0.3333333334t^3 - 11.50000000t^2 + 15.83500000\alpha t^2 \\
&\quad - 0.304750000\alpha t^5 + 0.01666666662t^5 + 5.12500000t^4 \\
&\quad - 7.43125000\alpha t^4 + 0.0422162699\alpha t^7 \\
&\quad - 0.00198412699t^7 \\
&\quad - 0.993055554t^7 + 1.442263887\alpha t^6 \\
&\quad - 0.00341724537\alpha t^9 \\
&\quad + 0.000159832452t^9 + 0.1033482144t^8 \\
&\quad - 0.150104911\alpha t^8, \\
&\quad 11 - 1.67\alpha t + 14t - 1.67\alpha - 1.385000000\alpha t^3 \\
&\quad - 0.3333333334t^3 + 13.50000000t^2 - 20.83500000\alpha t^2 \\
&\quad + 0.429416667\alpha t^5 - 0.01666666682t^5 - 7.20833333t^4 \\
&\quad + 10.48708333\alpha t^4 - 0.0596964286\alpha t^7 \\
&\quad + 0.00277777778t^7 \\
&\quad + 1.404166665t^6 - 2.03954167\alpha t^6 + 0.00483269953\alpha t^9 \\
&\quad - 0.000225970018t^9 - 0.146155754t^8 \\
&\quad + 0.212280011\alpha t^8),
\end{aligned}$$

$$\begin{aligned}
&(\overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta)) \\
&= (5.29 - 85\beta t + 6.71t - 4.29\beta - 0.528333333\beta t^3 \\
&\quad + 0.645000000t^3 + 1.06500000t^2 + 3.575000000\beta t^2 \\
&\quad + 0.1655833334\beta t^5 - 0.2494166672t^5 - 0.752916666t^4 \\
&\quad - 1.608749999\beta t^4 - 0.0230257937\beta t^7 \\
&\quad + 0.0348630952t^7 \\
&\quad + 0.1476249998t^6 + 0.311819444\beta t^6 \\
&\quad + 0.00186405974\beta t^9 \\
&\quad - 0.00282277888t^9 - 0.0153687996t^8 \\
&\quad - 0.0324516369\beta t^8, \\
&\quad 9 - 2.86\beta t + 12t - 2.86\beta + 0.863333334\beta t^3 \\
&\quad - 1.526666667t^3
\end{aligned}$$

$$
\begin{aligned}
&- 2.92000000t^2 - 4.290000000\beta t^2 - 0.2351666672\beta t^5\\
&+ 0.358000000t^5 + 1.085000000t^4 + 2.264166666\beta t^4\\
&+ 0.0325674603\beta t^7 - 0.0493253968t^7 - 0.208888889t^6\\
&- 0.440916666\beta t^6 - 0.00263618827\beta t^9\\
&+ 0.00399206349t^9\\
&+ 0.0217351191t^8 + 0.0458933532\beta t^8),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta))\\
&= (3.71 + 5.71\beta t + 6.29t + 14.29\beta - 1.901666667\beta t^3\\
&\quad + 0.901666667t^3 + 3.57500000t^2 - 18.57500000\beta t^2\\
&\quad + 0.522916667\beta t^5 - 0.322916667t^5 - 1.942083333t^4\\
&\quad + 8.69208333\beta t^4 - 0.0724384921\beta t^7 + 0.0450575397t^7\\
&\quad + 0.378486111t^6 - 1.686819440\beta t^6\\
&\quad + 0.00586361883\beta t^9\\
&\quad - 0.00364801036t^9 - 0.0393960814t^8\\
&\quad + 0.175556795\beta t^8,\\
&\quad 9.14 + 2.86\beta t + 12.14t + 2.86\beta + 2.376666667\beta t^3\\
&\quad - 1.876666667t^3 - 6.29000000t^2 + 24.29000000\beta t^2\\
&\quad - 0.736833334\beta t^5 + 0.461833333t^5 + 2.764166670t^4\\
&\quad - 12.26416666\beta t^4 + 0.1024325397\beta t^7\\
&\quad - 0.0637420635t^7\\
&\quad - 0.535361111t^6 + 2.38536111\beta t^6\\
&\quad - 0.00829238317\beta t^9\\
&\quad + 0.00515911596t^9 + 0.0557147819t^8\\
&\quad - 0.248274306\beta t^8),
\end{aligned}
$$

In Table 3, $\overset{\star}{x}_1(t,\alpha)$,$\overset{\star}{x}_2(t,\alpha)$, $\overset{\star}{y}_1(t,\alpha)$ and $\overset{\star}{y}_2(t,\alpha)$ represents analytical solution of the membership functions of the Example1 for $\alpha \epsilon [0, 0.6]$ and $\overset{\star}{x}_1(t,\beta)$, $\overset{\star}{x}_2(t,\beta)$, $\overset{\star}{y}_1(t,\beta)$ and $\overset{\star}{y}_2(t,\beta)$ represents analytical solution of non-membership function of the Example2 for $\beta \epsilon [0.3, 1.0]$.

The following is the mathematical and exact solution to Example 2 using the classical method, as shown in Figure 4(a-d) and Table 4:

$$
\begin{aligned}
&(\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha))\\
&= ((\frac{667}{200}\alpha + \frac{13}{2})sin(t) + (5 + \frac{5}{2}\alpha)cos(t)\\
&\quad + \frac{9}{1000}\sqrt{5}(-100 + 37\alpha)\\
&\quad \times sin(\sqrt{5}t) + (-3 + \frac{83}{100}\alpha)cos(\sqrt{5}t), (\frac{667}{200}\alpha + \frac{13}{2})sin(t)\\
&\quad + (5 + \frac{5}{2}\alpha)cos(t) - \frac{9}{1000}\sqrt{5}(-100 + 37\alpha)sin(\sqrt{5}t)\\
&\quad - (-3 + \frac{83}{100}\alpha)cos(\sqrt{5}t)),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha))\\
&= (-\frac{1}{500}\sqrt{5}(200 + 83\alpha)sin(\sqrt{5}t) + (2 - 5\alpha)cos(\sqrt{5}t)\\
&\quad + (-\frac{5}{2}\alpha + 12)sin(t) + (13 - \frac{667}{100}\alpha)cos(t),\\
&\quad \frac{1}{500}\sqrt{5}(200 + 83\alpha)\\
&\quad \times sin(\sqrt{5}t) - (2 - 5\alpha)cos(\sqrt{5}t) + (-\frac{5}{2}\alpha + 12)sin(t)\\
&\quad + (13 - \frac{667}{100}\alpha)cos(t)),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta))\\
&= (\frac{1}{1000}\sqrt{5}(-529 + 201\beta)sin(\sqrt{5}t)\\
&\quad + (-\frac{371}{200} - \frac{143}{200}\beta)cos(\sqrt{5}t)\\
&\quad + (-\frac{371}{200}\beta + \frac{1871}{200})sin(t) + (\frac{1429}{200} - \frac{143}{40}\beta)cos(t),\\
&\quad -\frac{1}{1000}\sqrt{5}\\
&\quad \times (-529 + 201\beta)sin(\sqrt{5}t) - (-\frac{371}{200} - \frac{143}{200}\beta)cos(\sqrt{5}t)\\
&\quad + (-\frac{371}{200}\beta + \frac{1871}{200})sin(t) + (\frac{1429}{200} - \frac{143}{40}\beta)cos(t)),
\end{aligned}
$$

$$
\begin{aligned}
&(\overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta))\\
&= ((\frac{857}{200}\beta + \frac{1843}{200})sin(t) + (\frac{257}{40} + \frac{343}{40}\beta)cos(t) + \frac{3}{200}\sqrt{5}\\
&\quad \times (-39 + 19\beta)sin(\sqrt{5}t) + (-\frac{543}{200} + \frac{1143}{200}\beta)cos(\sqrt{5}t),\\
&\quad (\frac{857}{200}\beta + \frac{1843}{200})sin(t) + (\frac{257}{40} + \frac{343}{40}\beta)cos(t)\\
&\quad - \frac{3}{200}\sqrt{5}(-39 + 19\beta)sin(\sqrt{5}t)\\
&\quad - (-\frac{543}{200} + \frac{1143}{200}\beta)cos(\sqrt{5}t)).
\end{aligned}
$$

In Table 4, $\overset{\star}{x}_1(t,\alpha)$,$\overset{\star}{x}_2(t,\alpha)$, $\overset{\star}{y}_1(t,\alpha)$ and $\overset{\star}{y}_2(t,\alpha)$ represents exact solution of the membership functions of the Example2 for $\alpha \epsilon [0, 0.6]$ and $\overset{\star}{x}_1(t,\alpha)$,$\overset{\star}{x}_2(t,\alpha)$, $\overset{\star}{y}_1(t,\alpha)$ and $\overset{\star}{y}_2(t,\alpha)$ represents exact solution of non-membership function of the Example2 for $\beta \epsilon [0.3, 1.0]$.

Example 3: Consider the first order non-homogenous system of intuitionistic fuzzy differential equation as follows:

$$
\left.\begin{aligned}
\frac{d\overset{\star}{x}}{dt} &= 2\overset{\star}{x}(t) + 3\overset{\star}{y}(t) - 7,\\
\frac{d\overset{\star}{y}}{dt} &= -\overset{\star}{x}(t) - 2\overset{\star}{y}(t) + 5.
\end{aligned}\right\} \tag{52}
$$

TABLE 4. Illustrates the approximation to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 2 for t=1.

Table 4: Approximate solution of example2 at $t = 0.5$.									
α	$\overset{\star}{x}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	4.408	10.87	17.38	32.20	0.3	7.746	15.51	11.57	22.28
0.1	4.891	11.62	16.42	30.54	0.4	7.348	14.77	12.85	24.44
0.2	5.377	12.38	15.45	28.89	0.5	6.950	14.02	14.13	26.60
0.3	5.863	13.14	14.48	27.24	0.6	6.552	13.27	15.41	28.77
0.4	6.350	13.89	13.51	25.59	0.7	6.154	12.52	16.69	30.93
0.5	6.836	14.65	12.54	23.93	0.8	5.757	11.77	17.97	33.09
0.6	7.322	15.40	11.57	22.28	0.9	5.359	11.02	19.25	35.25
					1.0	4.961	10.288	20.53	37.41

TABLE 5. Illustrates the exact solution to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 2 for t=1.

Table 5: Exact solution of example2 at $t = 0.5$.									
α	$\overset{\star}{x}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	4.382	10.63	17.23	17.09	0.3	7.699	11.40	11.49	14.37
0.1	4.864	10.90	16.27	16.64	0.4	7.305	10.99	12.75	15.02
0.2	5.347	11.18	15.32	16.19	0.5	6.911	10.57	14.02	15.67
0.3	5.829	11.46	14.36	15.73	0.6	6.519	10.16	15.28	16.32
0.4	6.313	11.73	13.40	15.28	0.7	6.126	9.750	16.56	16.98
0.5	6.795	12.01	12.45	14.83	0.8	5.732	9.336	17.80	17.62
0.6	7.276	12.28	11.49	14.37	0.9	5.337	8.932	19.08	18.27
					1.0	4.945	8.513	20.34	18.92

TABLE 6. Illustrates the approximate solution iterations in column 2, residual error in column 3, and CPU time required by the numerical technique GMADM to determine the approximate solution of the system of generalized fuzzy intuitionistic differential equations used in Example 2 in column 4. Whenever the error of all methods is taken into account, we can conclude that the GMADM has a better convergence behavior and is more stable than the ADM and TSM, respectively.

Table 6: Error comparsion			
Paramater	Iterations	Error	CPU-time
GMADM	4	0.2E-19	1.00145
ADM	7	3.1E-07	1.94512
TSM	6	4.2E-09	1.14532

with initial conditions

$$\left.\begin{aligned} \overset{\star}{x}(0) &= <(2, 4, 8, 15; 0.6); (1, 4, 8, 18; 0.3)>, \\ \overset{\star}{y}(0) &= <(2, 5, 8, 10; 0.6)(1, 5, 8, 12; 0.3)>. \end{aligned}\right\} \tag{53}$$

By taking (α, β)-cuts of (52) and (53), we get,

$$\left.\begin{aligned} \frac{d\overset{\star}{x}_1(t,\alpha)}{dt} &= 2\overset{\star}{x}_1(t,\alpha) + 3\overset{\star}{y}_1(t,\alpha) - 7, \\ \overset{\star}{x}_1(0,\alpha) &= 3.33\alpha + 2, \\ \frac{d\overset{\star}{y}_1(t,\alpha)}{dt} &= -\overset{\star}{x}_1(t,\alpha) - 2\overset{\star}{y}_1(t,\alpha) + 5, \\ \overset{\star}{y}_1(0,\alpha) &= 5\alpha + 2, \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} \frac{d\overset{\star}{x}_2(t,\alpha)}{dt} &= 2\overset{\star}{x}_2(t,\alpha) + 3\overset{\star}{y}_2(t,\alpha) - 7, \\ \overset{\star}{x}_2(0,\alpha) &= -11.67\alpha + 15, \\ \frac{d\overset{\star}{y}_2(t,\alpha)}{dt} &= -\overset{\star}{x}_2(t,\alpha) - 2\overset{\star}{y}_2(t,\alpha) + 5, \\ \overset{\star}{y}_2(0,\alpha) &= -3.33\alpha + 10, \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} \frac{d\overset{\star}{x}_1(t,\beta)}{dt} &= 2\overset{\star}{x}_1(t,\beta) + 3\overset{\star}{y}_1(t,\beta) - 7, \\ \overset{\star}{x}_1(0,\beta) &= -4.29\beta + 5.29, \\ \frac{d\overset{\star}{y}_1(t,\beta)}{dt} &= -\overset{\star}{x}_1(t,\beta) - 2\overset{\star}{y}_1(t,\beta) + 5, \\ \overset{\star}{y}_1(0,\beta) &= -0.85\beta + 6.71, \end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned} \frac{d\overset{\star}{x}_2(t,\beta)}{dt} &= 2\overset{\star}{x}_2(t,\beta) + 3\overset{\star}{y}_2(t,\beta) - 7, \\ \overset{\star}{x}_2(0,\beta) &= 14.29\beta + 3.71, \\ \frac{d\overset{\star}{y}_2(t,\beta)}{dt} &= -\overset{\star}{x}_2(t,\beta) - 2\overset{\star}{y}_2(t,\beta) + 5, \\ \overset{\star}{y}_2(0,\beta) &= 5.71\beta + 6.29. \end{aligned}\right\} \tag{57}$$

Here $\overset{\maltese}{L} = \frac{d}{dt}$ and by taking $\overset{\maltese}{L}^{-1}(.) = \int_0^t (.)\,dt$ on both sides of (54), (55), (56) and (57), and using the initial conditions we obtain;

$$\begin{cases} \overset{\star}{x}_1(t,\alpha) = \displaystyle\int_0^t (2\overset{\star}{x}_1(u,\alpha) + 3\overset{\star}{y}_1(u,\alpha))du \\ \qquad +3.33\alpha + 2 - 7t, \\ \overset{\star}{y}_1(t,\alpha) = \displaystyle\int_0^t (-\overset{\star}{x}_1(u,\alpha) - 2\overset{\star}{y}_1(u,\alpha)) \\ \qquad +5\alpha + 2 + 5t, \end{cases} \tag{58}$$

$$\begin{cases} \overset{\star}{x}_2(t,\alpha) = \displaystyle\int_0^t (2\overset{\star}{x}_2(u,\alpha) + 3\overset{\star}{y}_2(u,\alpha))du \\ \qquad -11.67\alpha + 15 - 7t, \\ \overset{\star}{y}_2(t,\alpha) = \displaystyle\int_0^t (-\overset{\star}{x}_2(u,\alpha) - 2\overset{\star}{y}_2(u,\alpha)) \\ \qquad -3.33\alpha + 10 + 5t, \end{cases} \tag{59}$$

$$\begin{cases} \overset{\star}{x}_1(t,\beta) = \displaystyle\int_0^t (2\overset{\star}{x}_1(u,\beta) + 3\overset{\star}{y}_1(u,\beta))du \\ \qquad +5.29 - 4.29\beta - 7t, \\ \overset{\star}{y}_1(t,\beta) = \displaystyle\int_0^t (-\overset{\star}{x}_1(u,\beta) - 2\overset{\star}{y}_1(u,\beta)) \\ \qquad +6.71 - 0.85\beta + 5t, \end{cases} \tag{60}$$

$$\begin{cases} \overset{\star}{x}_2(t,\beta) = \displaystyle\int_0^t (2\overset{\star}{x}_2(u,\beta) + 3\overset{\star}{y}_2(u,\beta))du \\ \qquad +14.29\beta + 3.71 - 7t, \\ \overset{\star}{y}_2(t,\beta) = \displaystyle\int_0^t (-\overset{\star}{x}_2(u,\beta) - 2\overset{\star}{y}_2(u,\beta)) + 5.71\beta \\ \qquad +6.29 + 5t. \end{cases} \tag{61}$$

Now by using GMADM we get;

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\alpha) = 3.33\alpha + 2, \\ \overset{\star}{y}_{1_0}(t,\alpha) = 5\alpha + 2, \\ \overset{\star}{x}_{1_1}(t,\alpha) = 21.66000000\alpha t + 3t, \\ \overset{\star}{y}_{1_1}(t,\alpha) = -13.33000000\alpha t - t, \\ \overset{\star}{x}_{1_{k+1}}(t,\alpha) = \displaystyle\int_0^t (2\overset{\star}{x}_{1k}(u,\alpha) + 3\overset{\star}{y}_{1k}(u,\alpha))du,\ k \geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\alpha) = \displaystyle\int_0^t (-\overset{\star}{x}_{1k}(u,\alpha) - 2\overset{\star}{y}_{1_k}(u,\alpha))du,\ k \geq 1. \end{cases} \tag{62}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\alpha) = -11.67\alpha + 15, \\ \overset{\star}{y}_{2_0}(t,\alpha) = -3.33\alpha + 10, \\ \overset{\star}{x}_{2_1}(t,\alpha) = -33.33000000\alpha t + 53t, \\ \overset{\star}{y}_{2_1}(t,\alpha) = 18.33000000\alpha t - 30t, \\ \overset{\star}{x}_{2_{k+1}}(t,\alpha) = \displaystyle\int_0^t (2\overset{\star}{x}_{2k}(u,\alpha) + 3\overset{\star}{y}_{2k}(u,\alpha))du,\ k \geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\alpha) = \displaystyle\int_0^t (-\overset{\star}{x}_{2k}(u,\alpha) - 2\overset{\star}{y}_{2_k}(u,\alpha))du,\ k \geq 1, \end{cases} \tag{63}$$

$$\begin{cases} \overset{\star}{x}_{1_0}(t,\beta) = 5.29 - 4.29\beta, \\ \overset{\star}{y}_{1_0}(t,\beta) = 6.71 - 0.85\beta, \\ \overset{\star}{x}_{1_1}(t,\beta) = 23.71000000t - 11.13000000\beta t, \\ \overset{\star}{y}_{1_1}(t,\beta) = -13.71000000t + 5.990000000\beta t, \\ \overset{\star}{x}_{1_{k+1}}(t,\beta) = \displaystyle\int_0^t (2\overset{\star}{x}_{1k}(u,\alpha) + 3\overset{\star}{y}_{1k}(u,\alpha))du,\ k \geq 1, \\ \overset{\star}{y}_{1_{k+1}}(t,\beta) = \displaystyle\int_0^t (-\overset{\star}{x}_{1k}(u,\alpha) - 2\overset{\star}{y}_{1_k}(u,\alpha))du,\ k \geq 1. \end{cases} \tag{64}$$

$$\begin{cases} \overset{\star}{x}_{2_0}(t,\beta) = 3.71 + 14.29\beta, \\ \overset{\star}{y}_{2_0}(t,\beta) = 6.29 + 5.71\beta, \\ \overset{\star}{x}_{2_1}(t,\beta) = 19.29000000t + 45.71000000\beta t, \\ \overset{\star}{y}_{2_1}(t,\beta) = -11.29000000t - 25.71000000\beta t, \\ \overset{\star}{x}_{2_{k+1}}(t,\beta) = \displaystyle\int_0^t (2\overset{\star}{x}_{2k}(u,\alpha) + 3\overset{\star}{y}_{2k}(u,\alpha))du,\ k \geq 1, \\ \overset{\star}{y}_{2_{k+1}}(t,\beta) = \displaystyle\int_0^t (-\overset{\star}{x}_{2k}(u,\alpha) - 2\overset{\star}{y}_{2_k}(u,\alpha))du,\ k \geq 1. \end{cases} \tag{65}$$

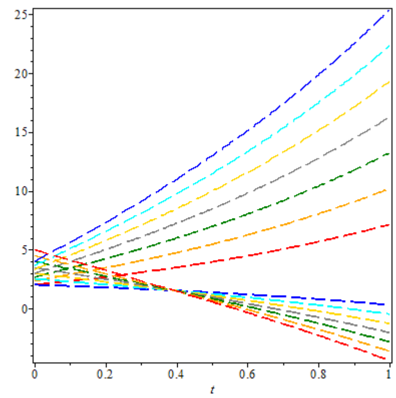

FIGURE 8. Exact solution of $(\overset{\star}{x}_{1_k}(t,\alpha), \overset{\star}{y}_{1_k}(t,\alpha))$ of generalized trapezoidal intuitionistic fuzzy number.

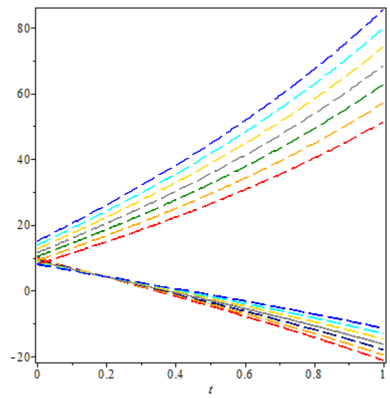

FIGURE 9. Exact solution of $(\overset{\star}{x}_{2_k}(t,\alpha), \overset{\star}{y}_{2_k}(t,\alpha))$ of generalized trapezoidal intuitionistic fuzzy number.

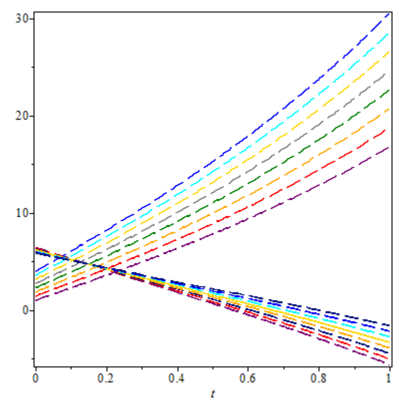

FIGURE 10. Exact solution of $(\overset{\star}{x}_{1_k}(t,\beta), \overset{\star}{y}_{1_k}(t,\beta))$ of generalized trapezoidal intuitionistic fuzzy number.

Figures 8-11: Displays the membership and non-membership functions, as well as the exact and approximative solutions, for the generalised trapezoidal intuitionistic fuzzy initial value problem used in example 3.

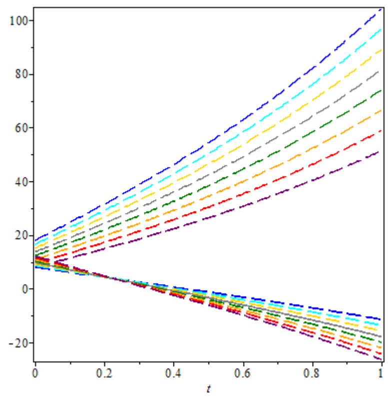

FIGURE 11. Exact solution of ($\overset{\star}{x}_{2_k}(t,\beta)$, $\overset{\star}{y}_{2_k}(t,\beta)$) of generalized trapezoidal intuitionistic fuzzy number.

By solving the (62), (63), (64) and (65), we get the approximate solution after three iteration as follows:

$$
\begin{aligned}
&(\overset{\star}{x}_1(t,\alpha), \overset{\star}{y}_1(t,\alpha))\\
&= (3.33\alpha + 2 + 3t + 21.66000000\alpha t + 1.500000000t^2\\
&\quad + 1.665000000\alpha t^2 + 0.5000000000t^3 + 3.610000000\alpha t^3\\
&\quad + 0.1250000000t^4 + 0.1387500007\alpha t^4, 5\alpha + 2 - t\\
&\quad - 13.33000000\alpha t + 2.500000000\alpha t^2 - 0.5000000000t^2\\
&\quad - 0.1666666666t^3 - 2.221666666\alpha t^3\\
&\quad - 0.04166666670t^4\\
&\quad + 0.2083333329\alpha t^4),\\
&(\overset{\star}{x}_2(t,\alpha), \overset{\star}{y}_2(t,\alpha))\\
&= (-11.67\alpha + 15 + 53t - 33.33000000\alpha t + 8.000000000t^2\\
&\quad - 5.835000000\alpha t^2 + 8.833333330t^3 - 5.554999998\alpha t^3\\
&\quad + 0.6666666650t^4 - 0.4862499990\alpha t^4,\\
&\quad - 3.33\alpha + 10 - 30t\\
&\quad + 18.33000000\alpha t - 1.665000000\alpha t^2 + 3.50000000t^2\\
&\quad - 5.000000000t^3 + 3.055000000\alpha t^3 + 0.2916666675t^4\\
&\quad - .1387500006\alpha t^4),\\
&(\overset{\star}{x}_1(t,\beta), \overset{\star}{y}_1(t,\beta))\\
&= (-4.29\beta + 5.29 + 23.71000000t - 11.13000000\beta t\\
&\quad + 3.145000000t^2 - 2.145000000\beta t^2 + 3.951666665t^3\\
&\quad - 1.854999999\beta t^3 + 0.2620833332t^4\\
&\quad - 0.1787499999\beta t^4,\\
&\quad - 0.85\beta + 6.71 - 13.71000000t + 5.990000000\beta t\\
&\quad - 0.4250000000\beta t^2 + 1.855000000t^2 - 2.284999999t^3\\
&\quad + 0.9983333329\beta t^3 + 0.1545833333t^4\\
&\quad - 0.03541666665\beta t^4),\\
&(\overset{\star}{x}_2(t,\beta), \overset{\star}{y}_2(t,\beta))\\
&= (14.29\beta + 3.71 + 19.29000000t + 45.71000000\beta t\\
&\quad + 2.355000000t^2 + 7.145000000\beta t^2 + 3.214999999t^3\\
&\quad + 7.618333330\beta t^3 + 0.1962499999t^4\\
&\quad + 0.5954166665\beta t^4,\\
&5.71\beta + 6.29 - 11.29000000t - 25.71000000\beta t\\
&\quad + 2.855000000\beta t^2 + 1.645000000t^2 - 1.881666666t^3\\
&\quad - 4.284999998\beta t^3 + 0.1370833333t^4\\
&\quad + 0.2379166666\beta t^4).
\end{aligned}
$$

The following is the mathematical and exact solution to Example 3 using the classical method, as shown in Figure 4(a-d) and Table 6:

$$
\begin{aligned}
\{\overset{\star}{x}_1(t,\alpha) &= -\frac{1833}{200}\alpha e^{-t} + (3 + \frac{2499}{200}\alpha)e^{t} - 1,\\
\overset{\star}{y}_1(t,\alpha) &= \frac{1833}{200}\alpha e^{-t} - \frac{1}{3}(3 + \frac{2499}{200}\alpha)e^{t} + 3\},\\
\{\overset{\star}{x}_2(t,\alpha) &= (\frac{69}{2} - \frac{45}{2}\alpha)e^{t} + (-\frac{37}{2} + \frac{1083}{100}\alpha)e^{-t} - 1,\\
\overset{\star}{y}_2(t,\alpha) &= -(-\frac{37}{2} + \frac{1083}{100}\alpha)e^{-t}\\
&\quad - \frac{1}{3}(\frac{69}{2} - \frac{45}{2}\alpha)e^{t} + 3\},\\
\{\overset{\star}{x}_1(t,\beta) &= (-\frac{871}{100} + \frac{171}{50}\beta)e^{-t} + (15 - \frac{771}{100}\beta)e^{t} - 1,\\
\overset{\star}{y}_1(t,\beta) &= -(-\frac{871}{100} + \frac{171}{50}\beta)e^{-t}\\
&\quad - \frac{1}{3}(15 - \frac{771}{100}\beta)e^{t}\} + 3,\\
\{\overset{\star}{x}_2(t,\beta) &= (12 + 30\beta)e^{t} + (-\frac{729}{100} - \frac{1571}{100}\beta)e^{-t} - 1,\\
\overset{\star}{y}_2(t,\beta) &= -(-\frac{729}{100} - \frac{1571}{100}\beta)e^{-t}\\
&\quad - \frac{1}{3}(12 + 30\beta)e^{t} + 3\},
\end{aligned}
$$

IV. RESULTS AND DISCUSSION

We discuss how the computing efficiency, stability, and residual error robustness of the proposed modified technique, GMADM, outperforms the ADM and TSM approaches.

- Tables 1 to 9 illustrate how GMADM, a recently created approach, is more reliable and consistent than ADM and TSM. While solving a system of fuzzy intuitionistic differential equations, it is observed that GMADM converges more quickly and accurately than ADM and TSM.
- Tables 3, 6, and 9 clearly demonstrate that GMADM is superior to ADM and TSM in terms of iterations, residual error, and CPU time.
- Figures 2-11 compare the numerical simulation of our recently modified family GMADM to a precise solution of the generalized trapezoidal intuitionistic fuzzy initial value problem used in Example 1-3 respectively.
- Figures 2-3, 4-11 illustrate the precise and approximate solutions for the membership and non-membership functions of the generalized trapezoidal intuitionistic

TABLE 7. **Illustrates the approximation to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 3 for t=1.**

Table 7: Approximate solution of numerical example at t=1.

α	$\overset{\star}{x}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	7.125	0.292	85.50	-21.21	0.3	30.48	-5.572	51.37	-11.46
0.1	10.17	-0.493	79.81	-19.58	0.4	28.52	-5.004	58.91	-13.58
0.2	13.21	-1.277	74.13	-17.96	0.5	26.56	-4.436	66.45	-15.70
0.3	16.25	-2.061	68.44	-16.33	0.6	24.60	-3.867	73.98	-17.82
0.4	19.29	-2.846	62.75	-14.71	0.7	22.64	-3.301	81.52	-19.93
0.5	22.33	-3.630	57.06	-13.08	0.8	20.68	-2.733	89.05	-22.05
0.6	25.37	-4.414	51.37	-11.46	0.9	18.72	-2.165	96.59	-24.17
					1.0	16.76	-1.597	104.13	-26.29

TABLE 8. **Illustrates the exact solution to the generalized trapezoidal intuitionistic fuzzy initial value problem from Example 3 for t=1.**

Table 8: Exact solution of numerical example at t=1.

α	$\overset{\star}{x}_1(t,\alpha)$	$\overset{\star}{y}_1(t,\alpha)$	$\overset{\star}{x}_2(t,\alpha)$	$\overset{\star}{y}_2(t,\alpha)$	β	$\overset{\star}{x}_1(t,\beta)$	$\overset{\star}{y}_1(t,\beta)$	$\overset{\star}{x}_2(t,\beta)$	$\overset{\star}{y}_2(t,\beta)$
0	7.155	0.282	85.98	-21.45	0.3	30.66	-5.669	51.67	-11.61
0.1	10.21	-0.513	80.26	-19.81	0.4	28.69	-5.096	59.25	-13.75
0.2	13.27	-1.308	74.54	-18.17	0.5	26.72	-4.523	66.82	-15.89
0.3	16.33	-2.103	68.82	-16.53	0.6	24.75	-3.950	74.40	-18.03
0.4	19.39	-2.899	63.10	-14.89	0.7	22.78	-3.378	81.97	-20.17
0.5	22.45	-3.693	57.39	-13.25	0.8	20.81	-2.805	89.55	-22.31
0.6	25.51	-4.488	51.67	-11.61	0.9	18.84	-2.232	97.13	-24.45
					1.0	16.87	-1.659	104.7	-26.59

TABLE 9. **Illustrates the approximate solution iterations in column 2, residual error in column 3, and CPU time required by the numerical technique GMADM to determine the approximate solution of the system of generalized fuzzy intuitionistic differential equations used in Example 3 in column 4. Whenever the error of all methods is taken into account, we can conclude that the GMADM has a better convergence behavior and is more stable than the ADM and TSM, respectively.**

Table 9: Error comparsion

Paramater	Iterations	Error	CPU-time
GMADM	3	0.6E-18	0.0145
ADM	9	4.7E-06	0.1646
TSM	8	0.9E-05	0.1531

fuzzy system of initial value problems used in example 1-3, respectively.

- The numerical results obtained in Tables 1-2,4-6,7-8, and Figures 1-11 clearly demonstrate that the exact and approximate solutions are matched up to 30 decimal places using GMADM, 7 decimal places using ADM, and 9 decimal places using TSM. The numerical simulation of our methods demonstrates unequivocally how much superior our method is to ADM and TSM.

V. CONCLUSION

In this work, Generalized Modified Adomian Decomposition Method have been utilized for computing the approximate solution of the linear system of generalized trapezoidal intuitionistic fuzzy initial value problems. We used the initial conditions as generalized trapezoidal intuitionistic fuzzy numbers. We have applied this procedure to brine tank problems and coupled oscillators. Moreover, by comparing the approximate results with exact solution, we have shown that this method is more reliable. Future studies will therefore focus on the solution of systems of higher order generalized tripezodial intuitionistic fuzzy system of differential equations as well as a system of nonlinear first order differential equations and their application [48], [49], [50], [51], [52], [53] in a more generalized fuzzy environment utilizing GMADM.

COMPETING INTERESTS

The authors confirm that there is no competing interests between them.

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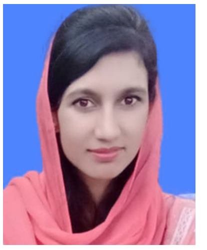

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