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APPLIED RESEARCH

On the Output Regulation for an Underactuated Inverse Pendulum When the Exosystem Is a High-Gain Observer

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ABSTRACT This paper is devoted to solving the output regulation problem on the basis of the new Francis equations for arbitrary reference/disturbance signals, whose model are obtained by High-Gain observers, providing in this way, the regulation of unmodeled but measurable reference signals. The design is given by fixing a steady state globally attractive by means of LMIs which allows controlling the decay rate by considering input bounds; while the regulation problem is solved by computing the steady-state input based on a modified set of the regulation equations, when the exosystem is constructed upon High-Gain observers, extending in this way, the classical output regulation theory for unmodeled reference/disturbance signal. The Furuta pendulum is used to illustrate the viability of the proposed approach.

INDEX TERMS Output regulation, high-gain observer, linear matrix inequality.

I. INTRODUCTION

The problem regarding controlling the output of a system that asymptotically tracks a desired reference and rejects undesired disturbances, retaining stability in closed-loop systems is frequently addressed due to its wide applicability in mechanical systems, aeronautics, robotics and different areas of science.

A common scenario appears when the reference signal is fully known, making it easy to find a controller. However, in practical cases, the signal can diverge from the considered, inheriting control problems and accuracy loss. Similar problems arise when the tracking signal comes directly from the sensor measurement. In those cases, the controller may not ensure the stability of the closed-loop system.

On the other hand, when the signal has a significant amplitude or high frequency, fulfilling global stability is not a simple task, due to the relative degree (defined and constant over the region of interest) with a smooth and bounded signal [1].

To achieve global stability, controllers based on dynamic models of reference/disturbances have been developed as the works of Francis [2] and Francis and Wonham [3]. They have shown that the solvability of a multivariable linear regulator problem corresponds to the solvability of a system of two linear matrix equations, called Francis Equations. Later, Isidori and Byrnes [4] showed that the result established by Francis is a particular case of the nonlinear problem. Additionally, the solvability for the nonlinear case requires an error feedback regulator, depending on a set of nonlinear partial differential equations called Francis-Isidori-Byrnes (FIB). Unfortunately, these equations, present, in many cases, a considerable complexity during their solving process. Furthermore, when variables of state are unknown, the internal model principle is

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used to generate the appropriate signal control trending the error to zero for the close-loop system when the exosystem is partially or fully known.

In the classical Isidori's regulation theory, a problem to consider is that the exosystem must be known a priori, making it impractical for real situations because the dynamical model that describes the path to track is not always available, although, in many cases, measurable. This implies that the controller must be designed with partial information. At the same time tracking control with time-varying external disturbances has been addressed by feedback linearization for a three-rotor UAV and proof stability via Lyapunov analysis [5]. The extension of the output regulation to random references/disturbances in a quadrotor for a hybrid exosystem [6] where the Kalman filter is used.

There has been a development of approaches to solving the output regulation, considering the different conditions of the problem. In [7] is introduced an optimal average cost learning framework for output regulation controller design when linear systems have unknown dynamics. In [8] is used a fuzzy adaptive output feedback control scheme to approximate unknown functions for a class of nonlinear uncertain strict-feedback systems under the action of nonlinear exosystems. Moreover, in the case of PDE's introducing an adaptive internal model that estimates unknown frequencies, output regulation, and disturbance rejection are achieved, even if the disturbance is generated by an unknown finite-dimensional exosystem [9], [10], or the exosystem coefficients are time-varying [11]. Likewise, [12] proposes an event-triggered control based on an adaptive internal model for a class of uncertain linear systems, where the system matrix of the exosystem contains unknown parameters. References [13], [14] develops an internal model designed by the circle criterion introducing the Nussbaum gain to deduce adaptive laws for a backstepping controller.

In literature, in order to overcome the unmodeled references/disturbances, High-Gain observers are used in linearizable plants and observers of lower dimension (Cascade High-Gain observer) to estimate the states without facing the peaking phenomenon [15]. In [16] High-Gain observers allow the controller to achieve the stabilization of the hybrid internal model for linear systems with a relative degree greater than one and dealing with periodic jumps in the plant/exosystem. For the neutrally stable exosystem whose frequencies are not known a priori, but in a compact set, a robust stabilizer and an internal model (adaptively tuned) produce a controller for the output-zeroing condition [17]. In this context, for output regulation of uncertain nonlinear systems in output feedback under disturbances generated by a class of nonlinear exosystems, Ding [18] proposed the design of a High-Gain internal model, or in the case of an unknown function in the system dynamics Nazrulla and Khalil [19] demonstrated that regulation can be achieved through the incorporation of a High-Gain observer in sliding mode control. Also, in [20] is proposed the combination of a nonlinear internal model and an extended High-Gain observer, where the observer is used to estimate the unmatched or unmeasured terms of the redesigned outputs of the plant.

High-Gain observers are capable of estimating the states of nonlinear systems even when the dynamics are fully unknown, but the system must be Lipschitz. Thus, if the reference/disturbance signals are assumed smooth and bounded, it is possible to construct a High-Gain observer for each of them. On the other hand, classical results on output regulation require the fully modeled dynamics for the reference to track making it impractical for unmodelled references. Thus, the problem consists of solving some modified regulation equations which consider an exosystem constructed upon High-Gain observers.

So, based on blending both the output regulation and the High-Gain Observer theories, the main contribution of the work is to provide an alternative method for tracking signals whose dynamics are unknown, extending, in this way, the classical regulation results. Thus, the output regulation is achieved through the modified regulation equation when a High-Gain Observer is used to replace the missing exosystem. Sufficient conditions for the existence of the linear and the nonlinear regulators are given. Besides, it is shown that the proposed controller can be obtained readily, and that it can be easily implemented in different systems such as the Furuta pendulum.

The rest of this work is organized as follows. The studied problem is defined in Section II. The High-Gain Observers are briefly reminded in Section III. Also, in Section III, the modified regulation equations are presented. The numerical and real-time results are given in Sections IV and V, respectively. Finally, the concluding remarks appear in Section VI.

II. PROBLEM STATEMENT

Consider the plant for output regulation as a nonlinear system described by

$$\dot{x}(t)=f(x,w,u), \qquad y(t)=h(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^o$ is the output vector, and $w \in \mathbb{R}^q$ is the state vector of the exosystem, to be defined later, which generates the reference and/or the perturbation signals; $f(\cdot,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^m \mapsto \mathbb{R}^n$ and $h(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^o$ are sufficiently smooth vector fields such that $f(0,0,0) = 0$ and $h(0) = 0$. For the sake of space, the argument t is omitted in some expressions, but it is considered that the states, outputs, and inputs of the plant and the exosystem are time-dependent, as well as the measurable reference/disturbance signals. Now, consider the following exosystem

$$\dot{w}(t) = s(w), \qquad y_r(t) = q(w), \tag{2}$$

where $s(w) : \mathbb{R}^q \mapsto \mathbb{R}^q$ and $q(\cdot) : \mathbb{R}^q \mapsto \mathbb{R}^o$ are sufficiently smooth vector fields holding $s(0) = 0$, and $q(0) = 0$, where $y_r \in \mathbb{R}^o$ is called the reference output. Besides, such an exosystem is Poisson stable. Then, the task is to design a controller such that the tracking error

$$e(t) = y(t) - y_r(t) = h(x) - q(w), \tag{3}$$

goes asymptotically to zero [21].

In [1] and [4], the nonlinear output regulation problem for systems in the form (1)-(3) consists of finding the controller

$$u = K\,(x - \pi(w)) + \gamma(w), \tag{4}$$

with K designed such that the linear approximation of (1) is asymptotically stable, while the nonlinear gains $\pi(w)$, with $\pi(0) = 0$ (the steady-state zero-error manifold) and $\gamma(w)$, with $\gamma(0) = 0$ (the steady-state input) are computed from

$$\frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, \gamma(w)), \tag{5}$$

$$0 = h(\pi(w)) - q(w). \tag{6}$$

The set of equations (5)-(6) is known as Francis-Isidori-Byrnes (FIB) equations.

Clearly, the linear output regulation problem appears when (1), (2) and (3) are linearized around the origin, yielding:

$$\dot{x} = Ax + Bu + Pw, \quad y = Cx, \tag{7}$$

$$\dot{\omega} = S\omega, \tag{8}$$

$$e = Cx - Q\omega. \tag{9}$$

The linear counterpart of the set (5)-(6) is obtained when the nonlinear mappings $x_{ss} = \pi(\omega)$ and $u_{ss} = \gamma(\omega)$ change to their linear version, i.e., $x_{ss} = \Pi\omega$ and $u_{ss} = \Gamma\omega$, respectively. Thus, considering the feedback control $u = Kx + Lw$ has the full access to states x and ω, then the conditions for linear output regulation arises; with $K \in \mathbb{R}^{m\times n}$, $\Gamma \in \mathbb{R}^{m\times q}$, $\Pi \in \mathbb{R}^{n\times q}$, and $L = \Gamma - K\Pi$; with this in mind, the set (5)-(6) turns into:

$$\Pi S = A\Pi + B\Gamma + P,$$
$$C\Pi = Q,$$

and the linear regulation problem is solvable by

$$u(t) = K\,[x - \Pi w] + \Gamma w. \tag{10}$$

The following section presents a proposal to achieve the output regulation using the High-Gain observer as the exosystem, resulting, in this way, the modified regulation equations.

III. HIGH-GAIN OBSERVER

In this section, some well-known results of High-Gain observers are briefly reminded, while their use to construct an exosystem is introduced. Consider, the nonlinear system described by:

$$\dot{x} = A_h x + B_h \Phi(x), \tag{11}$$

$$y = C_h x, \tag{12}$$

with $x \in \mathbb{R}^{\rho}$, bounded for all $t \geq 0$, and $y \in \mathbb{R}$ as the state and output vector, respectively; besides $\Phi(x)$ is a nonlinear function, locally Lipschitz and partially or fully unknown.

Then, the high-gain observer for (11) and (12) is defined as

$$\dot{\hat{x}} = A_h\hat{x} + B_h\Phi_0(\hat{x}) + H\cdot(y - \hat{y}), \tag{13}$$

$$\hat{y} = C_h\hat{x}, \tag{14}$$

where

$$A_h = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{\rho\times\rho}, \tag{15}$$

$$B_h = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T_{1\times\rho}, \tag{16}$$

$$C_h = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}_{1\times\rho}, \tag{17}$$

$\Phi_0(\hat{x})$ is a nominal model for $\Phi(\hat{x})$. By considering the estimation error $\tilde{x} = x - \hat{x}$ and knowing that $y = x_1$ and $\hat{y} = \hat{x_1}$, then equation (13) can be rewritten as:

$$\dot{\tilde{x}} = A_h\tilde{x} + B_h\cdot(\Phi(x) - \Phi_0(\hat{x})) - H\tilde{x}_1, \tag{18}$$

$$= A_0\tilde{x} + B_h\cdot(\Phi(x) - \Phi_0(\hat{x})), \tag{19}$$

where

$$A_0 = \begin{bmatrix} -h_1 & 1 & 0 & \cdots & 0 \\ -h_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -h_{\rho-1} & 0 & 0 & \cdots & 1 \\ -h_\rho & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{20}$$

Notice that in the absence of $B_h\cdot(\Phi(x) - \Phi_0(\tilde{x}))$ the asymptotic error convergence is reached by designing the gains $H = \mathrm{col}(h_1, h_2, \ldots, h_\rho)$ such that A_0 is Hurwitz and H is defined as:

$$H = \begin{bmatrix} \frac{\alpha_1}{\epsilon} & \frac{\alpha_2}{\epsilon^2} & \cdots & \frac{\alpha_\rho}{\epsilon^\rho} \end{bmatrix}^T, \tag{21}$$

where ϵ is a positive number sufficiently small related to the decay rate of the estimation error and $\alpha_1, \ldots, \alpha_p$ are fixed such that

$$s^\rho + \alpha_1 s^{\rho-1} + \alpha_2 s^{\rho-2} + \ldots + \alpha_{\rho-1}s + \alpha_\rho, \tag{22}$$

is Hurwitz. On the other hand, let

$$\eta_1 = \frac{\tilde{x}_1}{\epsilon^{\rho-1}}, \quad \eta_2 = \frac{\tilde{x}_2}{\epsilon^{\rho-2}}, \; \ldots, \; \eta_\rho = \tilde{x}_\rho. \tag{23}$$

Then

$$\epsilon\dot{\eta} = F\eta + \epsilon B_h\cdot(\Phi(x) - \Phi_0(\hat{x})), \tag{24}$$

where

$$F = \begin{bmatrix} -\alpha_1 & 1 & 0 & \ldots & 0 \\ -\alpha_1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\alpha_{\rho-1} & 0 & 0 & \ldots & 1 \\ -\alpha_\rho & 0 & 0 & \ldots & 0 \end{bmatrix},$$

is also Hurwitz because A_0 and F are related by a similarity transformation. Besides, it can be noticed that when ϵ gets smaller, then the second term of (24) becomes negligible, and in case of considering $\Phi_0(\hat{x}) = 0$, then the linear observer can estimate the system (11) and (12) even if $\Phi(x)$ is completely unknown [22].

A. HIGH-GAIN OBSERVER USED AS EXOSYSTEM AND THE MODIFIED FRANCIS-ISIDORI-BYRNES EQUATIONS

Consider a smooth, unmodelled, and continuously differentiable signal $\psi(t)$ as the output of the nonlinear system (11) and (12) with $\rho \geq 1$. Therefore, by considering $\Phi_0(\hat{x}) = 0$ the High-Gain Observer (13) and (14) can be rewritten as

$$\dot{\hat{x}} = A_h\hat{x} + H \cdot (\psi(t) - \hat{y}), \tag{25}$$
$$\hat{y} = C_h\hat{x}, \tag{26}$$

and it is capable of estimating the states of the non-existent system (11) and (12) with $\Phi(x) = \psi^{\rho}(t)$ where $\psi^{\rho}(t)$ is derivative of order ρ of $\psi(t)$ with $\rho \geq 1$. Then, the existence of High-Gain Observer is granted if $\psi^{\rho}(t)$ is Lipschitz. Thus, consider the smooth, unmodelled, and continuously differentiable vector $\Psi(t)$ defined as

$$\Psi^T(t) = \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_m & \cdots & \psi_{m+d} \end{bmatrix}, \tag{27}$$

where $\psi_1, \ldots, \psi_m$ are reference signals to be tracked and $\psi_{m+1}, \ldots, \psi_{m+d}$ are the disturbance signals to be rejected. Then $\Psi(t)$, measurable for $t \geq 0$, can be considered as the output of a non-existent dynamical model, and the High-Gain Observer (25) and (26) can be used to estimate the state and output of such a system, resulting:

$$\dot{w} = S_A w + S_H \cdot (\Psi(t) - y_w), \tag{28}$$
$$y_w = \begin{bmatrix} y_{ref} \\ y_{dis} \end{bmatrix} = \begin{bmatrix} Q_{ref} \\ Q_{dis} \end{bmatrix} w = Q_w w, \tag{29}$$

where $y_{ref} \in \mathbb{R}^m$ are the m references to track and $y_{dis} \in \mathbb{R}^d$ are the d disturbances to reject. Thus, the overall output vector is $y_w \in \mathbb{R}^{m+d}$.

One of the problems that faces this observer is when ρ is to big, because the gains are proportional to powers of $\frac{1}{\epsilon}, \ldots, \frac{1}{\epsilon^{\rho}}$, causing large peaks during the transient state. To avoid this problem and for sake of simplicity, in this work, ρ is considered equal to 2.

On the other hand, the matrices involved in (28)-(29), are related to the m reference and the d disturbance signals, and they are defined as follows:

$$S_A = \begin{bmatrix} A_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & A_{m+d} \end{bmatrix},$$

$$S_H = \begin{bmatrix} H_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & H_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & H_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & H_{m+d} \end{bmatrix},$$

$$Q_{ref} = \begin{bmatrix} Q_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & Q_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & Q_m & \cdots & 0 \end{bmatrix},$$

$$Q_{dis} = \begin{bmatrix} 0 & \cdots & Q_{m+1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & Q_{m+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & Q_{m+d} \end{bmatrix}.$$

with A_i, Q_i and H_i as (15), (17) and (21), respectively, for $i = 1, \ldots, m+d$.

Thus, by virtue of (20), the system (28)-(29) can be rewritten as

$$\dot{w} = Sw + S_H\Psi(t), \tag{30}$$
$$y_w = \begin{bmatrix} y_{ref} \\ y_{dis} \end{bmatrix} = \begin{bmatrix} Q_{ref} \\ Q_{dis} \end{bmatrix} w, \tag{31}$$

where

$$S = \begin{bmatrix} S_1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & S_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & S_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & S_{m+d} \end{bmatrix},$$

with $S_i \in R^{\rho \times \rho}$ are block matrices defined as:

$$S_i = \begin{bmatrix} -\frac{\alpha_1}{\epsilon} & 1 & 0 & \cdots & 0 \\ -\frac{\alpha_2}{\epsilon^2} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\alpha_{\rho-1}}{\epsilon^{\rho-1}} & 0 & 0 & \cdots & 1 \\ -\frac{\alpha_\rho}{\epsilon^\rho} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

for $i = 1, \ldots, m+d$.

Notice that equations (24) to (31) are given for $\rho \geq 1$, which means that the proposed approach can be applied even if observers of higher order are considered.

Now the Nonlinear Regulation Problem for unmodelled reference/disturbance signals, with the exosystem (30) and (31), when (27) is measurable for all $t \geq 0$ can be summarized as follows: 1) To find a gain K such that the linear approximation of (1) is asymptotically stable, and 2) To obtain the nonlinear gains $\pi(w)$, and $\gamma(w)$ from

$$\frac{\partial \pi(w)}{\partial w}(Sw + S_H\Psi(t)) = f(\pi(w), w, \gamma(w)), \tag{32}$$
$$0 = h(\pi(w)) - Q_{ref}(w). \tag{33}$$

Then, the Nonlinear Regulation Problem for unmodeled references is solvable by $u = K(x - \pi(w)) + \gamma(w)$.

The set of equations (32)-(33) are the Francis-Isidori-Byrnes (FIB) equations for unmodeled references.

For the linear counterpart, the regulation problem is described by

$$\dot{x} = Ax + Bu + Pw, \quad y = Cx, \tag{34}$$
$$\dot{\omega} = S\omega + S_H\Psi(t), \tag{35}$$
$$e = Cx - Q_{ref}\omega, \text{ with} \tag{36}$$
$$u = Kx + L\omega. \tag{37}$$

where the mappings are $x_{ss} = \Pi\omega$ and $u_{ss} = \Gamma\omega$, with $K \in \mathbb{R}^{\rho\times n}$, $\Gamma \in \mathbb{R}^{\rho\times\rho(m+d)}$, $\Pi \in \mathbb{R}^{n\times\rho(m+d)}$, and $L = \Gamma - K\Pi$. Thus, by considering the change of coordinates: $\tilde{x} = x - \Pi w$, one has:

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \Pi\dot{w},\\ &= Ax + Bu + Pw - \Pi Sw - \Pi S_H\Psi(t),\\ &= Ax + BKx + BLw + Pw - \Pi Sw - \Pi S_H\Psi(t),\end{aligned}$$

and knowing that $x = \tilde{x} + \Pi w$, the previous system can be expressed as:

$$\begin{aligned}\dot{\tilde{x}} &= A(\tilde{x} + \Pi w) + BK(\tilde{x} + \Pi w) \quad (38)\\ &\quad + BLw + Pw - \Pi Sw - \Pi S_H\Psi(t),\\ &= (A + BK)\tilde{x} \quad (39)\\ &\quad + (A\Pi + BK\Pi + BL + P - \Pi S)w - \Pi S_H\Psi(t).\end{aligned}$$

Hence, the system posses a stable invariant subspace, where Π is a solution of the equation:

$$\Pi S + \Pi S_H\Psi(t) = (A + BK)\,\Pi + (BL + P)\,. \tag{40}$$

On the other hand, the tracking error is

$$\begin{aligned}e = Cx - Q_{ref}w &= C(\tilde{x} + \Pi w) - Q_{ref}w\\ &= C\tilde{x} + (C\Pi - Q_{ref})w. \quad (41)\end{aligned}$$

Therefore, the error goes to zero, i.e., $\lim_{t\to\infty} e(t) = 0$, if Π fulfills $C\Pi - Q_{ref} = 0$. Then, the following conditions directly arise:

- Design K such that $\dot{\tilde{x}} = (A + BK)\tilde{x}$ is stable. To this end, the proposed approach sets a gain K to create a globally attractive steady state by using LMIs; allowing to modify the exponential decay rate or to impose input-output bounds.
- Compute Π and Γ such that the following equations hold:

$$0 = A\Pi + B\Gamma + P - \Pi S - \Pi S_H Q_w, \tag{42}$$
$$0 = C\Pi - Q_{ref}. \tag{43}$$

The previous conditions are the Francis-Isidori-Byrnes for linear systems with unmodelled references/disturbances signals.

The previous discussion is summarized in the flowchart depicted in Figure 1.

TABLE 1. Symbol description rotary inverted pendulum.

Symbol	Description	value
m_p	Mass pendulum	0.027 Kg
M_{arm}	Mass of the arm	0.095 Kg
l_p	Center of mass of pendulum	m
r	Length from shaft to pendulum pivot	0.083 m
J_p	Pendulum moment inertia relative to pivot	1.10×10^{-4}Kg·m^2
J_{arm}	Arm moment inertia relative to shaft	1.23×10^{-4}Kg·m^2
R_m	Motor armature resistance	3.30 Ω
L_m	Motor armature inductance	47.0 mH
K_t	Motor torque constant	0.028 N·m
K_m	Back e.m.f constant	0.028 V/(rad/s)
V_m	Applied supply voltage	- (volts)

IV. NUMERICAL EXAMPLES

A. FURUTA PENDULUM SYSTEM

Figure 2 shows a rotational pendulum mechanism [23], which consists of two beams, the horizontal one has driven by the DC motor while the vertical is joined with the first one, so it can freely rotate. Its dynamic equations can be derived from the Lagrange Equations of motion, where ϕ is the angle of the horizontal beam and θ is the angle between the upright position and the vertical beam $q = [\phi, \theta]$, thus

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \bar{B}u, \tag{44}$$

where

$$M(q) = \begin{bmatrix} J_{arm} + m_p r^2 + m_p l_p^2 \sin^2\theta & m_p r l_p \cos\theta \\ m_p r l_p \cos\theta & J_p + m_p l_p^2 \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} \frac{1}{2}m_p l_p^2\dot{\theta}\sin 2\theta & \frac{1}{2}m_p l_p^2\dot{\phi}\sin 2\theta - m_p r l_p\dot{\theta}\sin\theta \\ \frac{1}{2}m_p l_p^2\sin\theta & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} 0 \\ m_p g l_p \sin 2\theta \end{bmatrix},$$

$$\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u = \frac{K_t(V_m - K_m\dot{\phi})}{R_m}.$$

Consider $x_1 = \phi$, $x_3 = \dot{\phi}$, $x_2 = \theta$ and $x_4 = \dot{\theta}$, its nonlinear model from the Lagrange-Euler equations is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ f_1(x,u) \\ f_2(x,u) \end{bmatrix} \tag{45}$$

with

$$\begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix} = M(x)^{-1}(\bar{B}u - C(x,\dot{x})\dot{x} - G(x)) \tag{46}$$

Figure 2 shows a free body diagram of rotary Inverse Pendulum with pendulum mass m_p and length l_p. The pendulum is connected to the DC motor through an arm of mass m_{arm}. θ and α are the arm angle and the pendulum angle, respectively. In table 1 one can see the terminology used.

By linearizing the nonlinear system (45) around the origin and by considering x_1 as the output to be regulated, results:

$$\dot{x} = Ax + Bu + Pw, \quad y = Cx, \tag{47}$$

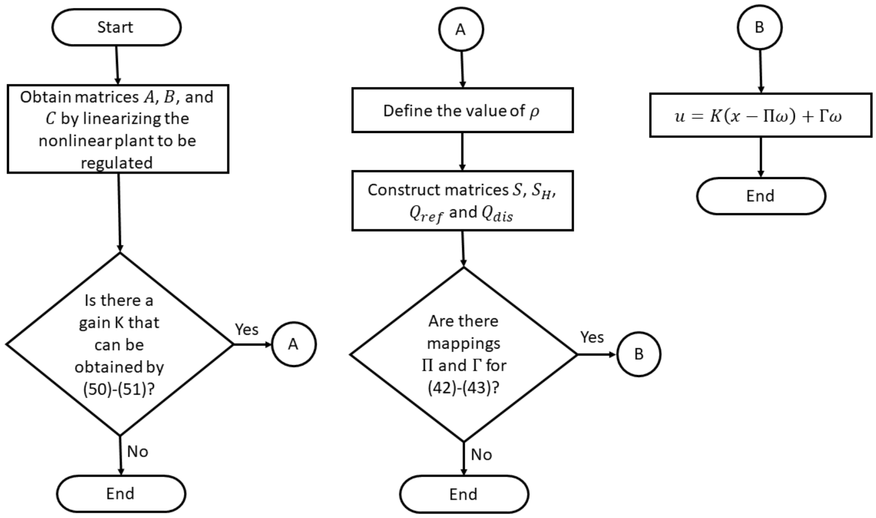


FIGURE 1. Flowchart for the design of the output regulator when the exosystem is a high-gain observer.

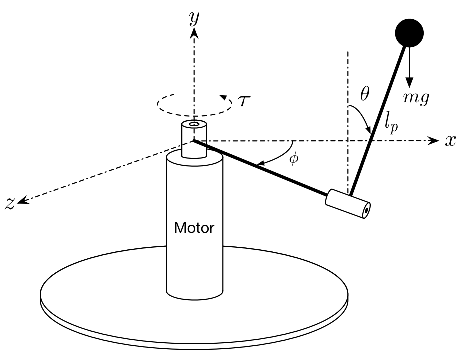


FIGURE 2. The furuta pendulum system.

$$\dot{\omega} = S\omega + S_H Q w, \quad y_{ref} = Q_{ref}\omega, \tag{48}$$

$$e = Cx - Q_{ref}\omega, \tag{49}$$

where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 76.2884 & -0.5689 & 0 \\ 0 & 82.2655 & -0.2399 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 17.0842 & 7.2054 \end{bmatrix}^T, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$Q_{ref} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

According to (30), and by assuming $\rho = 2$, the block matrices S and S_H for $\alpha_1 = 1, \alpha_2 = 1$ and $\epsilon = 0.005$ are:

$$S = \begin{bmatrix} -200 & 1 \\ -40000 & 0 \end{bmatrix}, \quad S_H = \begin{bmatrix} 200 \\ 40000 \end{bmatrix}.$$

The linear mappings $x_{ss} = \Pi\omega$ and $u_{ss} = \Gamma\omega$ are computed by the equations (42)-(43) which leads to:

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 0.0333 \end{bmatrix}.$$

On the other hand, in order to get a K such that the system $A + BK$ be Hurwitz the following Lyapunov function $V(x) = x^T X x$ is proposed with $X = X^T$ and M satisfying

$$X > 0, \quad AX + BM + XA^T + MB^T + 2\alpha X < 0. \tag{50}$$

The previous inequality guarantees that $\dot{V} \leq -2\alpha V$, related to a maximum speed convergence $\alpha > 0$. Nevertheless, in practical it is important to consider a bound on the control input $||u(t)|| \leq \beta, \beta > 0$ so that it can be ensured the actuator security. For control bounding it has

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & X \end{bmatrix} \geq 0 \text{ and } \begin{bmatrix} X & M^T \\ M & \beta^2 \end{bmatrix} \geq 0, \tag{51}$$

with $x(0)$ as initial condition and K computed from $K = MX^{-1}$. Therefore, setting $\alpha = 5.85$, $\mu = 5$volts and $x(0) = [0\ 0\ 0\ 0]^T$ one can obtain

$$K = \begin{bmatrix} 8.7550 & -90.5175 & 3.3777 & -12.1899 \end{bmatrix}.$$

The simulation is carried out using the Matlab software with the following initial conditions $x(0) = [0\ \ 2°\ \ 0\ \ 0]^T$ and $w(0) = [-10°,\ \ 0]^T$. In Figure 3, it can be seen the behavior of state x_1 under the action of the proposed control law. Besides, it is showed how the output tends to the measurable reference signal, which in turn, is estimated by the state ω_1 of the High-Gain Observer, which is used as the exosystem.

The performance of the exosystem through the High-Gain Observer is depicted in Figure 4, for $\psi_1(t) = \pi/12 \sin(\pi/5 \cdot t) - \pi/12 \cos(\pi/3 \cdot t)$ as unmodeled reference signal. It can readly observed that the output w_1 tends to the reference $\psi_1(t)$.

When the controller is applied to the system the states behavior can see in Figure 5 for the tracking signal.

Since the goal in this work is the straightforward implementation in real-time, the use of LMIs is recommended because they are an alternative to provide a decay rate $\alpha \geq 0$ and impose the constrain $|u(t)| \leq \mu$ for an initial condition

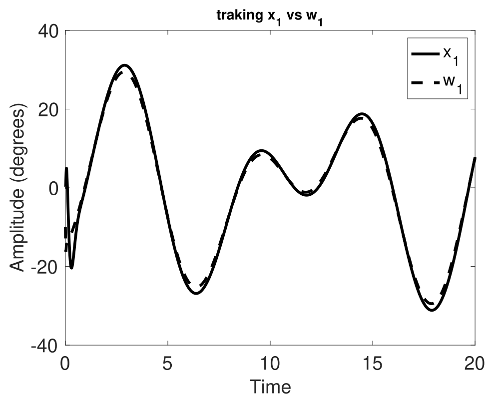


FIGURE 3. Output versus reference (x_1 vs w_1) for the Inverse Rotary Pendulum system.

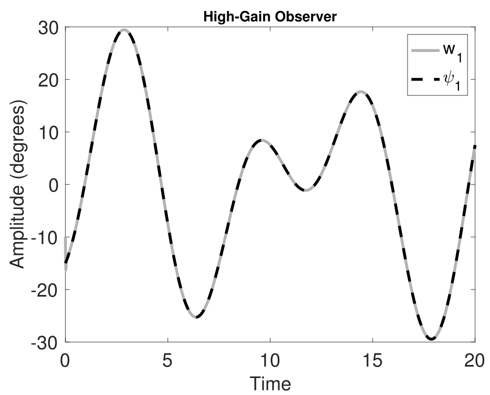


FIGURE 4. Exosystem output signal versus unmodeled reference signal ψ_1.

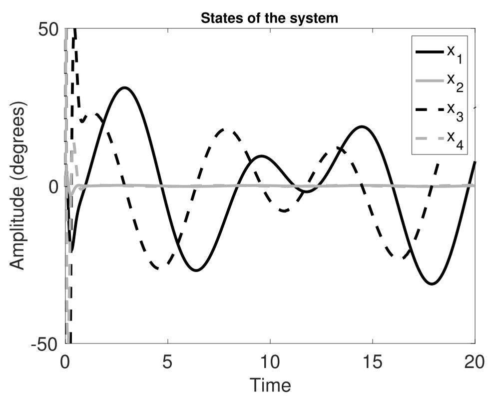


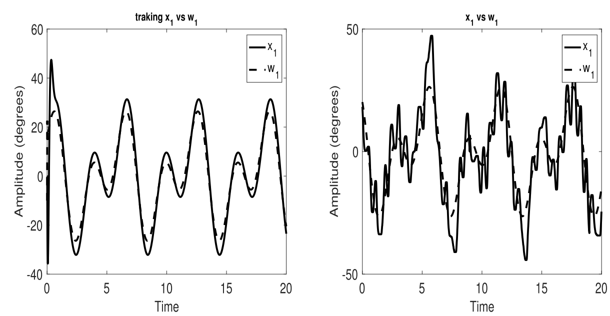
FIGURE 5. States of the Inverse Rotary Pendulum system.

$x(0) = [0\ 2°\ 0\ 0]^T$. Then, large amplitudes are avoided in the control signal.

V. EXPERIMENTAL TEST

This section presents the experimentation equipment and analysis of results. The QNET 2.0 Rotary Pendulum board, Figure 6, is used to verify the proposed control scheme via hardware- in-the-loop experiments

FIGURE 6. Inverse Rotary Pendulum system.

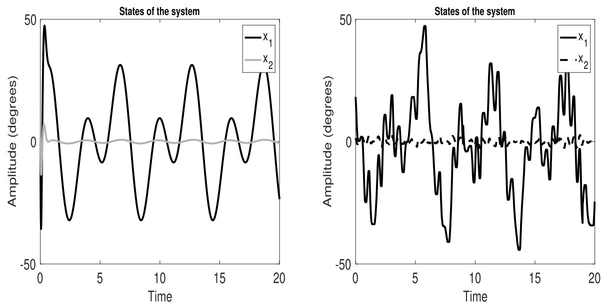


(a) Simulation (b) Real system

FIGURE 7. Output versus reference (x_1 vs w_1) for the Inverse Rotary Pendulum system.

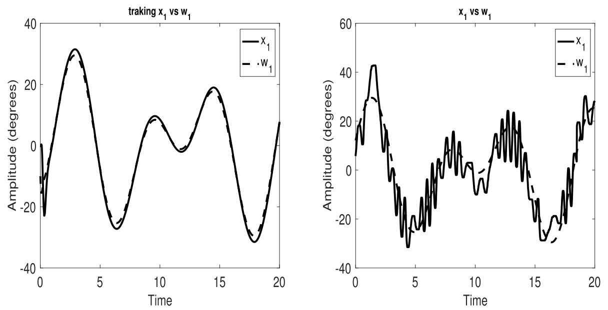
The test is carried out using the LabView software. Therefore, for a reference signal (unmodeled but measurable) $\psi(t) = \pi/12 \sin(2\pi/3 \cdot t) + \pi/12 \cos(\pi/3 \cdot t)$ it can be seen in Figure 7a a simulation of (47) and (48) with initial conditions defined as $x(0) = [0\ 2°\ 0\ 0]^T$ under the influence of the proposed controller. Besides, Figure 7b shows the behavior of the physical plant under the same control and as it can be seen, the output x_1 tends to the reference signal ω_1 in accordance with what was expected in the simulation results.

Even more, the states of the simulated system are depicted in Figure 8a in order to see the upright position (pendulum) and the reference tracking. On the other hand, in Figure 8b, a comparison between the simulation and the in real-time experiment is depicted. As can be noticed, for the reference signal $\psi(t) = \pi/12 \sin(2\pi/3 \cdot t) + \pi/12 \cos(\pi/3 \cdot t)$ the simulation and the real-time results are close enough to validate the efficacy of the regulator.



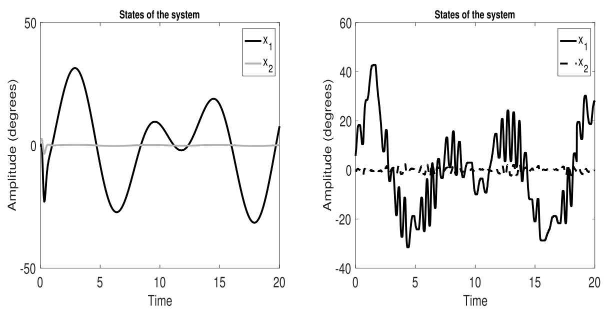
(a) Simulation (b) Real system

FIGURE 8. States for the Inverse Rotary Pendulum system.



(a) Simulation (b) Real system

FIGURE 9. Output versus reference (x_1 vs w_1) for the inverse rotary pendulum system with different reference signal.



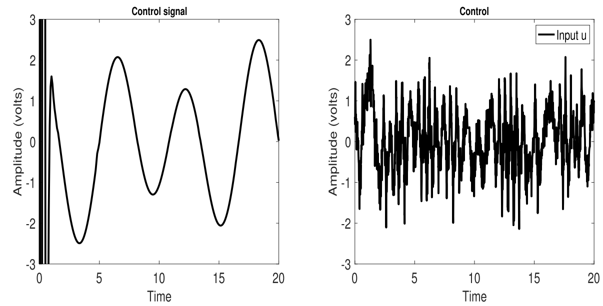
(a) Simulation (b) Real system

FIGURE 10. States for the inverse rotary pendulum system.

In order to verify the performance of the proposed approach; different signals and gains for the observer are considered. Now, the block matrices S and S_H for $\alpha_1 = 1$, $\alpha_2 = 1$, and $\epsilon = 0.05$ are

$$S = \begin{bmatrix} -20 & 1 \\ -400 & 0 \end{bmatrix}, \quad S_H = \begin{bmatrix} 20 \\ 400 \end{bmatrix}.$$

Again, the simulation is carried out using the Matlab software with the following initial conditions $x(0) = [0\ 2^\circ\ 0\ 0]^T$ and $w(0) = [-10^\circ, \quad 0]^T$. Therefore, for a new reference signal defined as $\psi_1(t) = \pi/12 \sin(\pi/5 \cdot t) - \pi/12 \cos(\pi/3 \cdot t)$ for the same system (Furuta pendulum) the simulation behavior can be seen in Figure 9a, while the evolving of the physical system is depicted in Figure 9b. Additionally, as in the previous example, the simulation/real-time system states are given in Figures 10a and 10b, respectively. The corresponding control signals are depicted in Figures 11a and 11b for simulation/real system, respectively. Once more, the efficacy of the approach has been validated by a real-time experiment.



(a) Simulation (b) Real system

FIGURE 11. Control signal for the inverse rotary pendulum system.

VI. CONCLUSION

In this work, an extension of the output regulation theory to the case of unmodeled reference/disturbance signals has been presented. The missing exosystem is constructed based on the well-known High-Gain Observers theory. Roughly speaking, the proposed approach is formed by two parts: 1) a stabilizer, obtained by LMIs, in which certain performances create a globally attractive steady state, and 2) the output regulator for unmodeled reference signals based on new equations, which can be solved practically. As a result, an approach capable of minimizing the tracking error for nonlinear problems has been obtained. This leads to a new opportunity for its application in different platforms as robotics, process control, synchronization, and entertainment, among others. The challenges will consist of the adequate parameterization of trajectories, modeling, and application in real-time. Finally, a Rotary Inverted Pendulum system has been presented to show the advantages of the proposed approach.

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