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RESEARCH ARTICLE

Confidence Level Aggregation Operators Based on Intuitionistic Fuzzy Rough Sets With Application in Medical Diagnosis

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ABSTRACT In recent days, due to the complexities of different diseases of similar types, it becomes very difficult to diagnose an accurate type of disease, and so medical diagnosis becomes a difficult task for the experts working in health departments. Many researchers try to develop new methods and techniques to over the difficulties that come across in the way of medical diagnosis. In this paper, we try to develop some novel techniques that will help experts to diagnose diseases accurately. Based on a more advanced structure of intuitionistic fuzzy rough sets, in this article, we establish confidence-level intuitionistic fuzzy average/geometric aggregation operators to incorporate the familiarity degree of experts with evaluated objects for an initial assessment while intuitionistic fuzzy rough average/geometric aggregation operators cannot do so. Moreover, we have given some basic properties of the initiated operators. To show the effective use of these operators we have proposed an algorithm with an illustrative example. Furthermore, based on the intuitionistic fuzzy rough model, we have also established a medical diagnosis model to incorporate the difficulty that occurs in the diagnosis of disease. Furthermore, a comparative analysis demonstrates the efficiency of our proposed methods.

INDEX TERMS Fuzzy sets, intuitionistic fuzzy sets, rough sets, intuitionistic fuzzy rough sets, confidence-level aggregation operators, medical diagnosis.

I. INTRODUCTION

The earliest doctors in ancient times made the diagnosis and suggested treatment based on observation of clinical symptoms. Ancient physicians' observations with their eyes, and ears, and occasionally by examining human specimens were the basis for the first medical diagnoses given by humans. The late medieval era saw the widespread use of the diagnostic phase by medical professionals. The ancient Greeks believed that all sickness was caused by disturbances of body fluids called humours. Later, the microscope revealed both the cellular makeup of human tissue and the pathogenic microorganisms. It wasn't until the end of the 19th century that more advanced diagnostic methods and instruments, including the thermometer for detecting temperature and the stethoscope for measuring heart rate, became widely used. In medicine, the clinical laboratory would not be widely used until the turn of the 20th century.

It is necessary to be aware of any unexpected changes to your body. This could be an endless cough or even an enlarging waistline. Syndromes are often nothing to distress but sometimes need further inspection. Diagnosis is the procedure of deciding the nature of a disease and separating it from other possible conditions. Diagnosis is a Greek term that comes from a gnosis called knowledge. Symptoms that appear early in the course of the disease are vague. It is very difficult in this situation to make an accurate diagnosis. Reach an accurate decision depends on the medical history and risk factors for a certain disease.

Fuzzy sets (FSs), defined by Zadeh [1], are an efficient technique that generalizes the classical set theory (CST) where any element has membership grade (MG) belonging

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to [0, 1]. Similar to CST, operations, and relations can be defined for FSs. Since their appearance in 1965, FSs had their applications in a diversity of ways and any discipline. Uses of FSs can be seen in artificial intelligence [2], medicine [3], statistics [4], medical diagnosis [5], and clustering [6]. Some researchers proposed this notion to aggregation operators (AOs) like Fahmi et al. [7] proposed cubic fuzzy Einstein AOs and their application to decision making (DM) problems.

An intuitionistic fuzzy set (IFS) was established by Atanassov [8] in the structure of combining both MG and non-membership grade (NMG). IFS uses the constraint that the sum (MG, NMG) belongs to [0, 1]. It is noticed that IFS is a very valuable structure and it can provide a two-dimensional scenario in decision-making problems. Based on this notion many researchers have developed the methods and applications of IFS in different fields. De et al. [9] established the use of the notion of IFS in medical diagnosis. Moreover, Xiao [10] proposed a distance measure based on IFS and applied it to the pattern classification problem. Yang et al. [11] proposed belief and plausibility measures on IFSs with the belief-plausibility TOPSIS. Moreover, based on IFSs, many new theories like similarity measures and aggregation operators had been established by researchers. Xu and Yager [12] introduced some geometric AOs based on IFSs. Also, some average AOs based on IFSs were introduced by Xu [13]. Hwang and Yang [14] gave some similarity measures between IFSs. Nagoorgani et al. [15] introduced the idea of double domination on IF graphs. Sheikh and Mandal [16] proposed some Dombi aggregation operators based on IFS.

The rough sets (RSs) model started by Pawlak [17] is a pair of precise sets called lower and upper approximation of RSs. A lot of researchers utilize the conception of RSs in many areas [18], [19]. Afterward, Ayub et al. [20] initiated the conception of linear Diophantine fuzzy RSs and provide its application to DM issues. Many researchers had developed the combined concept of RSs and FSs theory, such as the idea of the fuzzy rough sets (FRSs) initiated by Dubois and Prade [21]. Qureshi and Shabir [22] initiated the generalized rough fuzzy ideals of quantale and roughness in quantale module.

The idea of intuitionistic fuzzy rough sets (IFRSs) proposed by Cornelis et al. [23] is the generalization for FRSs. Chowdhary and Acharjya [24] used the IFR notion for the detection of breast cancer. Chinram et al. [25] used the concept of IFRSs for AOs and established the notion of IFR average AOs. Based on IFRSs, in this article, we propose some new AOs like confidence-level IFR average (CIFRA) AOs and confidence-level IFR geometric (CIFRG) AOs because of the following reasons:

1. IFRS anticipates more space for decision-makers due to the combined notion of IFS and RS.
2. IFRS uses upper and lowers approximation spaces that property lacks in IFS. It means that when decision-makers come up with data in the form of upper and lower approximations, then these kinds of data cannot be handled by simple intuitionistic fuzzy numbers in many decision-making problems related to medical diagnosis where experts use the data in the form of intuitionistic fuzzy rough number (IFRN) to diagnose a disease. So there is a need to develop the notions of confidence IFR aggregation operators.
3. IFRS uses the advance condition that the sum (MG, NMG) of upper and lower approximations must belong to [0, 1].
4. CIFRA and CIFRG AOs can incorporate the familiarity degree of experts with evaluated objects for initial assessment and that property lacking in IFWA and IFWG aggregation operators.
5. Taking into account that CIFRA and CIFRG operators are straightforward and cover the decision-making approach, this article aims to cover more advance and complex data.
6. The proposed work covers the limitation of all existing drawbacks.

TABLE 1. Abbreviations used throughout the article.

Full Name	Abbreviation
Membership grade	MG
Non-membership grade	NMG
Fuzzy set	FS
Intuitionistic fuzzy set	IFS
Rough set	RS
Fuzzy rough set	FRS
Intuitionistic fuzzy rough set	IFRS
Intuitionistic fuzzy rough number	IFRN
Confidence intuitionistic fuzzy rough weighted average	CIFRWA
Confidence intuitionistic fuzzy rough ordered weighted average	CIFROWA
Confidence intuitionistic fuzzy rough weighted geometric	CIFRWG
Confidence intuitionistic fuzzy rough ordered weighted geometric	CIFROWG
Aggregation operator	AO
Multi-criteria decision making	MCDM

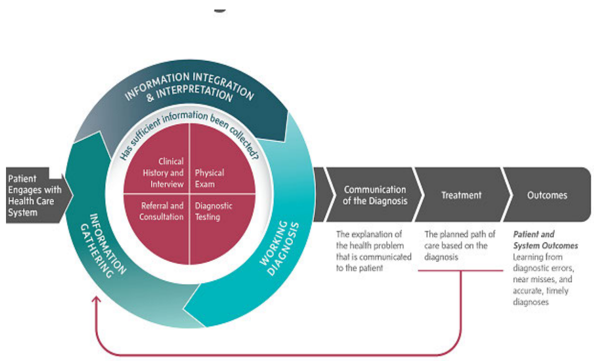

FIGURE 1. **Diagnosis procedure.**

To have clear abbreviations used throughout the paper, we give Table 1 with these abbreviations. The contributions of this study are described as follows:

1. To initiate the conception of new AOs like confidence intuitionistic fuzzy rough average and confidence intuitionistic fuzzy rough geometric AOs.
2. Properties of these aggregation operations have been proposed.
3. An algorithm for the MCDM approach has been given developed to cover more advanced data.
4. Also, we have proposed a medical diagnosis algorithm based on IFRS and an illustrative example is given to show the effective define algorithm.

This article is organized as follows. In Section II, we examine the basic conception of FS, IFS, RS, IFRS, and some basic operational laws. In Section III, we introduce new aggregation operators like CIFRWA and CIFROWA. In Section IV, we discuss the basic notion of CIFRWG and CIFROWG operators. In Section V, we give an algorithm for the proposed methods along with numerical examples. In Section VI, we deal with the medical diagnosis algorithm for IFRSs. In Section VII, we have a comparative analysis of the proposed methods with some existing methods. Finally, we make conclusions in Section VIII.

II. PRELIMINARIES

Medical diagnosis is the way to take suitable decisions about certain diseases based on symptoms. Due to complications in various diseases, health physicians find some problems in handling the more complex diseases. To reach an accurate decision in the medical field is very important for patients and doctors as well for the survival of medical theory. Based on the complication of these problems, FS theory takes its part in this field and it has a wide range of applications cited in [5]. The overall overview of the medical diagnosis procedure has been presented in Fig. 1.

In the following, we will overview the basic ideas for IFSs, IFRSs, score function (SF), and accuracy function (AF). Furthermore, in a later discussion, we will adopt medical diagnosis procedures as well.

Definition 1 [10]: Let X be a general set. An intuitionistic fuzzy set is the notion of the form

$$\{(\top, \text{ꝑ}(\top), \Phi(\top)) \,|\, \top \in X\}$$

with the condition that $sum(\text{ꝑ}(\top), \Phi(\top)) \in [0,1]$. Also $\text{ꝑ}(\top), \Phi(\top)$ are MG and NMG, respectively.

Definition 2 [23]: Let X be a general set and $\text{ɋ} \in IFS(X \times X)$ be intuitionistic fuzzy relation. Then, the pair $(\text{ɋ}, X)$ is called IF approximation space. Now for any $\uplus \subseteq X$, the lower rough approximation (LRA) and upper rough approximation (URA) of $\uplus$ w.r.t $(X, \text{ɋ})$ are given by

$$\underline{\text{ɋ}}(\uplus) = \left\{\left\langle\left(\top : \underline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top), \underline{\Phi_{\text{ɋ}_{\uplus}}}(\top)\right) | \top \in X\right\rangle\right\}$$
$$\overline{\text{ɋ}}(\uplus) = \left\{\left\langle\left(\top : \overline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top), \overline{\Phi_{\text{ɋ}_{\uplus}}}(\top)\right) | \top \in X\right\rangle\right\}$$

where $\underline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top) = \bigwedge_{c\in X}[\text{ꝑ}_{\text{ɋ}}(\top, c) \bigwedge \text{ꝑ}_{\uplus}(c)]$, $\underline{\Phi_{\text{ɋ}_{\uplus}}}(\top) = \bigvee_{c\in X}[\Phi_{\text{ɋ}}(\top, c) \bigvee \Phi_{\uplus}(c)]$ and $\overline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top) = \bigvee_{c\in X}\left[\text{ꝑ}_{\text{ɋ}}(\top, c) \bigvee \text{ꝑ}_{\uplus}(c)\right]$, $\overline{\Phi_{\text{ɋ}_{\uplus}}}(\top) = \bigwedge_{c\in X}[\Phi_{\text{ɋ}}(\top, c) \bigwedge \Phi_{\uplus}(c)]$ with $0 \le \overline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top) + \overline{\Phi_{\text{ɋ}_{\uplus}}}(\top) \le 1, 0 \le \underline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top) + \underline{\Phi_{\text{ɋ}_{\uplus}}}(\top) \le 1$. As $\underline{\text{ɋ}}(\uplus)$ and $\overline{\text{ɋ}}(\uplus)$ are IFSs, so, $\underline{\text{ɋ}}(\uplus), \overline{\text{ɋ}}(\uplus) : IFS(X) \to IFS(X)$ are LR and UR, approximation operators. Then, $\text{ɋ}(\uplus) = \left(\underline{\text{ɋ}}(\uplus), \overline{\text{ɋ}}(\uplus)\right) = \left\{\top : \left(\underline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top), \underline{\Phi_{\text{ɋ}_{\uplus}}}(\top)\right), \left(\overline{\text{ꝑ}_{\text{ɋ}_{\uplus}}}(\top), \overline{\Phi_{\text{ɋ}_{\uplus}}}(\top)\right) | \top \in X\right\}$ is called IFRS.

Definition 3 [25]: Let $F = \left\{\left(\underline{\text{ꝑ}}_{\text{ʆ}}, \underline{\Phi}_{\text{ʆ}}\right), \left(\overline{\text{ꝑ}}_{\text{ʆ}}, \overline{\Phi}_{\text{ʆ}}\right)\right\}$ be an intuitionistic fuzzy rough number (IFRN). Then, the score function (SF) and accuracy function (AF) are given by

$$Sc(\text{ʆ}) = \frac{1}{4}\left(2 + \underline{\text{ꝑ}}_{\text{ʆ}} + \overline{\text{ꝑ}}_{\text{ʆ}} - \underline{\Phi}_{\text{ʆ}} - \overline{\Phi}_{\text{ʆ}}\right), S(\text{ʆ}) \in [0,1]$$
$$Ac(\text{ʆ}) = \frac{1}{4}\left(2 + \underline{\text{ꝑ}}_{\text{ʆ}} + \overline{\text{ꝑ}}_{\text{ʆ}} + \underline{\Phi}_{\text{ʆ}} + \overline{\Phi}_{\text{ʆ}}\right), A(\text{ʆ}) \in [0,1].$$

Definition 4 [25]: For two IFRNs $\text{ʆ}_1 = \left\{\left(\underline{\text{ꝑ}}_{\text{ʆ}_1}, \underline{\Phi}_{\text{ʆ}_1}\right), \left(\overline{\text{ꝑ}}_{\text{ʆ}_1}, \overline{\Phi}_{\text{ʆ}_1}\right)\right\}$ and $\text{ʆ}_2 = \left\{\left(\underline{\text{ꝑ}}_{\text{ʆ}_2}, \underline{\Phi}_{\text{ʆ}_2}\right), \left(\overline{\text{ꝑ}}_{\text{ʆ}_2}, \overline{\Phi}_{\text{ʆ}_2}\right)\right\}$, we have the following results:

1) If $S(\text{ʆ}_1) > S(\text{ʆ}_2)$ then $\text{ʆ}_1 > \text{ʆ}_2$,
2) If $S(\text{ʆ}_1) < S(\text{ʆ}_2)$ then $\text{ʆ}_1 < \text{ʆ}_2$,
3) If $S(\text{ʆ}_1) = S(\text{ʆ}_2)$ then
   i. If $A(\text{ʆ}_1) > A(\text{ʆ}_2)$ then $\text{ʆ}_1 > \text{ʆ}_2$,
   ii. If $A(\text{ʆ}_1) < A(\text{ʆ}_2)$ then $\text{ʆ}_1 < \text{ʆ}_2$,
   iii. If $A(\text{ʆ}_1) = A(\text{ʆ}_2)$ then $\text{ʆ}_1 = \text{ʆ}_2$.

III. CONFIDENCE INTUITIONISTIC FUZZY ROUGH (CIFR) AVERAGE AGGREGATION OPERATORS

All of the researchers in the aforementioned literatures believed that the decision-makers are certain experts and aware of the choices that are being examined. However, there are multiple cases in which this concept fails in problems from daily life. Some researchers created the concept of confidence level and also provided some AOs based on confidence level in order to overcome and regulate this type of constraint. In this section, we discuss confidence intuitionistic fuzzy rough average (CIFRA) AOs. We also discuss the basic properties of the operators.

A. CIFR WEIGHTED AVERAGE (CIFRWA) AGGREGATION OPERATORS

We first discuss confidence intuitionistic fuzzy rough weighted average (CIFRWA) AOs.

Definition 5: Let $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right)$, $i = 1, 2, \ldots, n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence level (CL) of $\mathfrak{z}_i$ with $0 \le \mathbb{l}_i \le 1$. Let $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n} \psi_i = 1$. Then, the mapping *CIFRWA* : $\circledcirc^n \to \circledcirc$ operator is given as $CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) = \oplus_{i=1}^{n} \psi_i (\mathbb{l}_i \mathfrak{z}_i) = \psi_1 (\mathbb{l}_1 \mathfrak{z}_1) \oplus \psi_2 (\mathbb{l}_2 \mathfrak{z}_2) \oplus \psi_3 (\mathbb{l}_3 \mathfrak{z}_3) \oplus \ldots \oplus \psi_n (\mathbb{l}_n \mathfrak{z}_n)$.

It is called the confidence intuitionistic fuzzy rough weighted average (CIFRWA) operator.

Theorem 1: Let $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right), i = 1, 2, \ldots, n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence level of $\mathfrak{z}_i$ with $0 \le \mathbb{l}_i \le 1$. Let $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^T$ be the weight vectors for the IFRNs with the condition $\sum_{i=1}^{n} \psi_i = 1$. Then

$$CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) = \left(\begin{array}{c} \left(1 - \prod_{i=1}^{n} \left(1 - \underline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{n} \left(\underline{\Phi}_i\right)^{\mathbb{l}_i \psi_i}\right), \\ 1 - \prod_{i=1}^{n} \left(1 - \overline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{n} \left(\overline{\Phi}_i\right)^{\mathbb{l}_i \psi_i} \end{array}\right) \quad (1)$$

Proof: For $n = 2$, we have $CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2)\right) = \psi_1 (\mathbb{l}_1 \mathfrak{z}_1) \oplus \psi_2 (\mathbb{l}_2 \mathfrak{z}_2)$. By using the operational laws for IFRNs, we get $\mathbb{l}_1 \mathfrak{z}_1 = \left(\begin{array}{c} \left(1 - \left(1 - \underline{p}_1\right)^{\mathbb{l}_1}, \underline{\Phi}_1^{\mathbb{l}_1}\right), \\ 1 - \left(1 - \overline{p}_1\right)^{\mathbb{l}_1}, \overline{\Phi}_1^{\mathbb{l}_1} \end{array}\right) = \left(\left(\underline{\mathfrak{d}}_1, \underline{\mathfrak{e}}_1\right), \left(\overline{\mathfrak{d}}_1, \overline{\mathfrak{e}}_1\right)\right)$. Then

$$\begin{aligned} &\psi_1 (\mathbb{l}_1 \mathfrak{z}_1) \\ &= \left(\begin{array}{c} \left(1 - \left(1 - \underline{\mathfrak{d}}_1\right)^{\mathbb{l}_1}, \underline{\mathfrak{e}}_1^{\mathbb{l}_1}\right), \\ 1 - \left(1 - \overline{\mathfrak{d}}_1\right)^{\mathbb{l}_1}, \overline{\mathfrak{e}}_1^{\mathbb{l}_1} \end{array}\right) \\ &= \left(\begin{array}{c} \left(\left(1 - \left[1 - \left\{1 - \left(1 - \underline{p}_1\right)^{\mathbb{l}_1}\right\}\right]^{\psi_1}\right), \left(\underline{\Phi}_1^{\mathbb{l}_1}\right)^{\psi_1}\right), \\ \left(\left(1 - \left[1 - \left\{1 - \left(1 - \overline{p}_1\right)^{\mathbb{l}_1}\right\}\right]^{\psi_1}\right), \left(\overline{\Phi}_1^{\mathbb{l}_1}\right)^{\psi_1}\right) \end{array}\right) \\ &= \left(\begin{array}{c} \left(\left(1 - \left(1 - \underline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right), \left(\underline{\Phi}_1^{\psi_1 \mathbb{l}_1}\right)\right), \\ \left(\left(1 - \left(1 - \overline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right), \left(\overline{\Phi}_1^{\psi_1 \mathbb{l}_1}\right)\right) \end{array}\right). \end{aligned}$$

Similarly,we can see that

$$\psi_2 (\mathbb{l}_2 \mathfrak{z}_2) = \left(\begin{array}{c} \left(\left(1 - \left(1 - \underline{p}_2\right)^{\psi_2 \mathbb{l}_2}\right), \left(\underline{\Phi}_2^{\psi_2 \mathbb{l}_2}\right)\right), \\ \left(\left(1 - \left(1 - \overline{p}_2\right)^{\psi_2 \mathbb{l}_2}\right), \left(\overline{\Phi}_2^{\psi_2 \mathbb{l}_2}\right)\right) \end{array}\right).$$

Then, *CIFRWA*

$$\begin{aligned} &\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2)\right) \\ &= \psi_1 (\mathbb{l}_1 \mathfrak{z}_1) \oplus \psi_2 (\mathbb{l}_2 \mathfrak{z}_2) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \left(1 - \left(1 - \underline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right) + \left(1 - \left(1 - \underline{p}_2\right)^{\psi_2 \mathbb{l}_2}\right) \\ - \left(1 - \left(1 - \underline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right) \left(1 - \left(1 - \underline{p}_1\right)^{\psi_2 \mathbb{l}_2}\right), \\ \left(\underline{\Phi}_1^{\psi_1 \mathbb{l}_1}\right) \left(\underline{\Phi}_2^{\psi_2 \mathbb{l}_2}\right) \end{array}\right), \\ \left(\begin{array}{c} \left(1 - \left(1 - \overline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right) + \left(1 - \left(1 - \overline{p}_2\right)^{\psi_2 \mathbb{l}_2}\right) \\ - \left(1 - \left(1 - \overline{p}_1\right)^{\psi_1 \mathbb{l}_1}\right) \left(1 - \left(1 - \overline{p}_2\right)^{\psi_2 \mathbb{l}_2}\right), \\ \left(\overline{\Phi}_1^{\psi_1 \mathbb{l}_1}\right) \left(\overline{\Phi}_2^{\psi_2 \mathbb{l}_2}\right) \end{array}\right) \end{array}\right). \end{aligned}$$

Thus,

$$\begin{aligned} &CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2)\right) \\ &= \left(\begin{array}{c} \left(1 - \prod_{i=1}^{2} \left(1 - \underline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{2} \left(\underline{\Phi}_i\right)^{\mathbb{l}_i \psi_i}\right), \\ 1 - \prod_{i=1}^{2} \left(1 - \overline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{2} \left(\overline{\Phi}_i\right)^{\mathbb{l}_i \psi_i} \end{array}\right). \end{aligned}$$

Suppose the result is valid for $= \phi$, that is,

$$\begin{aligned} &CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_\phi, \mathbb{l}_\phi)\right) \\ &= \left(\begin{array}{c} \left(1 - \prod_{i=1}^{\phi} \left(1 - \underline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{\phi} \left(\underline{\Phi}_i\right)^{\mathbb{l}_i \psi_i}\right), \\ 1 - \prod_{i=1}^{\phi} \left(1 - \overline{p}_i\right)^{\mathbb{l}_i \psi_i}, \prod_{i=1}^{\phi} \left(\overline{\Phi}_i\right)^{\mathbb{l}_i \psi_i} \end{array}\right). \end{aligned}$$

Then, for $n = \phi + 1$, we get CIFRWA as shown in the equation at the bottom of the next page. Hence the result is valid for $n = \phi + 1$. Therefore, the result is valid for any number of IFRNs. ∎

Example 1: Suppose $\mathfrak{z}_1 = \left(((0.3, 0.4), 0.5), ((0.5, 0.2), 0.4)\right)$, $\mathfrak{z}_2 = \left(((0.1, 0.8), 0.6), ((0.2, 0.7), 0.4)\right)$, $\mathfrak{z}_3 = \left(((0.5, 0.3), 0.3), ((0.4, 0.2), 0.1)\right)$ and $\mathfrak{z}_4 = \left(((0.7, 0.2), 0.4), ((0.1, 0.6), 0.3)\right)$ are four *IFRNs* along with their confidence level. If $\psi = (0.29, 0.25, 0.22, 0.24)$, then

$$\begin{aligned} &CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), (\mathfrak{z}_3, \mathbb{l}_3), (\mathfrak{z}_4, \mathbb{l}_4)\right) \\ &= \left(\begin{array}{c} \left(\begin{array}{c} \left(\begin{array}{c} 1 - (1 - 0.3)^{0.5 \times 0.29} \times (1 - 0.1)^{0.6 \times 0.25} \times \\ (1 - 0.5)^{0.3 \times 0.22} \times (1 - 0.7)^{0.4 \times 0.24} \end{array}\right), \\ (0.4)^{0.5 \times 0.29} \times (0.8)^{0.6 \times 0.25} \times \\ (0.3)^{0.3 \times 0.22} \times (0.2)^{0.4 \times 0.24} \end{array}\right), \\ \left(\begin{array}{c} \left(\begin{array}{c} 1 - (1 - 0.5)^{0.4 \times 0.29} \times (1 - 0.2)^{0.4 \times 0.25} \times \\ (1 - 0.4)^{0.1 \times 0.22} \times (1 - 0.1)^{0.3 \times 0.24} \end{array}\right), \\ (0.2)^{0.4 \times 0.29} \times (0.7)^{0.4 \times 0.25} \times \\ (0.2)^{0.1 \times 0.22} \times (0.6)^{0.3 \times 0.24} \end{array}\right) \end{array}\right) \\ &= \left((0.2045, 0.6701), (0.1144, 0.7448)\right). \end{aligned}$$

For a family of IFRNs $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right)$ where $i = 1, 2, \ldots, n$ and $\mathbb{l}_i$ being the confidence level of $\mathfrak{z}_i$ with $0 \le \mathbb{l}_i \le 1$. Let $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n} \psi_i = 1$. Then CIFRWA AOs have the following properties:

1) **Idempotency:** If for all i $(\mathfrak{z}_i, \mathbb{l}_i) = (\mathfrak{z}, \mathbb{l})$, i.e., $\underline{p}_i = \underline{p}, \overline{p}_i = \overline{p}$, $\underline{\Phi}_i = \underline{\Phi}$ *and* $\overline{\Phi}_i = \overline{\Phi}$, $\mathbb{l}_i = \mathbb{l}$, then

$$CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) = \mathbb{l}\mathfrak{z}$$

Proof: If $(\mathfrak{z}_i, \mathbb{l}_i) = (\mathfrak{z}, \mathbb{l})$, then by using Theorem 1, we get

$$CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right)$$

$$= \begin{pmatrix} \left(1-\prod_{i=1}^{n}(1-\underline{p})^{\mathbb{l}\psi_i}, \prod_{i=1}^{n}(\underline{\Phi})^{\mathbb{l}\psi_i}\right), \\ 1-\prod_{i=1}^{n}(1-\overline{p})^{\mathbb{l}\psi_i}, \prod_{i=1}^{n}(\overline{\Phi})^{\mathbb{l}\psi_i} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-(1-\underline{p})^{\mathbb{l}\sum_{i=1}^{n}\psi_i}, (\underline{\Phi})^{\mathbb{l}\sum_{i=1}^{n}\psi_i}\right), \\ 1-(1-\overline{p})^{\mathbb{l}\sum_{i=1}^{n}\psi_i}, (\overline{\Phi})^{\mathbb{l}\sum_{i=1}^{n}\psi_i} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-(1-\underline{p})^{\mathbb{l}}, (\underline{\Phi})^{\mathbb{l}}\right), \\ 1-(1-\overline{p})^{\mathbb{l}}, (\overline{\Phi})^{\mathbb{l}} \end{pmatrix} = \mathbb{l}\mathfrak{z}.$$

■

2. **(Boundedness):** Let $\mathfrak{z}^{-}_i = \left(\left(\underline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$ and $\mathfrak{z}^{+}_i = \left(\left(\underline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$. Then, for all ψ_i, we have $\mathfrak{z}^{-}_i \leq CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) \leq \mathfrak{z}^{+}_i$.

Proof: For every i, $\min(\underline{p}_i) \leq \underline{p}_i \leq \max(\underline{p}_i) \Longrightarrow 1-\max(\underline{p}_i) \leq 1-\underline{p}_i \leq 1-\min(\underline{p}_i)$. Now for every ψ, we get $\prod_{i=1}^{n}(1-\max(\underline{p}_i))^{(max\mathbb{l}_i)\psi_i} \leq \prod_{i=1}^{n}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i} \leq \prod_{i=1}^{n}(1-\min(\underline{p}_i))^{(min\mathbb{l}_i)\psi_i} \Longrightarrow (1-\max(\underline{p}_i))^{(max\mathbb{l}_i)\sum_{i=1}^{n}\psi_i} \leq \prod_{i=1}^{n}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i} \leq (1-\min(\underline{p}_i))^{(min\mathbb{l}_i)\sum_{i=1}^{n}\psi_i} \Longrightarrow 1-(1-\min(\underline{p}_i))^{(min\mathbb{l}_i)} \leq 1-\prod_{i=1}^{n}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i} \leq 1-(1-\max(\underline{p}_i))^{(max\mathbb{l}_i)}\ \underline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq 1-\prod_{i=1}^{n}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i} \leq \underline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}$. Similarly, for every i, $\min(\overline{p}_i) \leq \overline{p}_i \leq \max(\overline{p}_i) \Longrightarrow 1-\max(\overline{p}_i) \leq 1-\overline{p}_i \leq 1-\min(\overline{p}_i)$. Now for every ψ, we get $\prod_{i=1}^{n}(1-\max(\overline{p}_i))^{(max\mathbb{l}_i)\psi_i} \leq \prod_{i=1}^{n}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i} \leq \prod_{i=1}^{n}(1-\min(\overline{p}_i))^{(min\mathbb{l}_i)\psi_i} \Longrightarrow (1-\max(\overline{p}_i))^{(max\mathbb{l}_i)\sum_{i=1}^{n}\psi_i} \leq \prod_{i=1}^{n}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i} \leq (1-\min(\overline{p}_i))^{(min\mathbb{l}_i)\sum_{i=1}^{n}\psi_i} \Longrightarrow 1-(1-\min(\overline{p}_i))^{(min\mathbb{l}_i)} \leq 1-\prod_{i=1}^{n}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i} \leq 1-(1-\max(\overline{p}_i))^{(max\mathbb{l}_i)}\ \overline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq 1-\prod_{i=1}^{n}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i} \leq \overline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}$. Also, $\min(\underline{\Phi}_i) \leq \underline{\Phi}_i \leq \max(\underline{\Phi}_i) \Longleftrightarrow (\min(\underline{\Phi}_i))^{min\mathbb{l}_i} \leq \prod_{i=1}^{n}(\underline{\Phi})^{\mathbb{l}_i\psi_i} \leq (\max(\underline{\Phi}_i))^{max\mathbb{l}_i} \Longrightarrow \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq \prod_{i=1}^{n}(\underline{\Phi})^{\mathbb{l}_i\psi_i} \leq \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}$, and $\min(\overline{\Phi}_i) \leq \overline{\Phi}_i \leq \max(\overline{\Phi}_i) \Longleftrightarrow (\min(\overline{\Phi}_i))^{min\mathbb{l}_i} \leq \prod_{i=1}^{n}(\overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \leq (\max(\overline{\Phi}_i))^{max\mathbb{l}_i} \Longrightarrow \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq \prod_{i=1}^{n}(\overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \leq \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}$. If $CIFRWA((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)) = \mathfrak{z} = \left((\underline{p}, \underline{\Phi}), (\overline{p}, \overline{\Phi})\right)$, then from the above analysis, we get $\underline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq \underline{p}_{\mathfrak{z}_i} \leq \underline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq \overline{p}_{\mathfrak{z}_i} \leq \overline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq \underline{\Phi}_{\mathfrak{z}_i} \leq \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i} \leq (\overline{\Phi}_{\mathfrak{z}_i}) \leq \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}$. Then, by using the definition of SF, we can conclude that $\mathfrak{z}^{-}_i \leq CIFRWA((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)) \leq \mathfrak{z}^{+}_i$. ■

3) **Monotonicity:** Let $\ddot{\mathfrak{z}}_i = \left(\left(\underline{p}_{\ddot{\mathfrak{z}}_i}, \underline{\Phi}_{\ddot{\mathfrak{z}}_i}\right), \left(\overline{p}_{\ddot{\mathfrak{z}}_i}, \overline{\Phi}_{\ddot{\mathfrak{z}}_i}\right)\right)$ $(i = 1, 2, 3, \ldots, n)$ be another family of IFRNs such that $\underline{p}_{\mathfrak{z}_i} \leq \underline{p}_{\ddot{\mathfrak{z}}_i}, \underline{\Phi}_{\mathfrak{z}_i} \geq \underline{\Phi}_{\ddot{\mathfrak{z}}_i}, \overline{p}_{\mathfrak{z}_i} \leq \overline{p}_{\ddot{\mathfrak{z}}_i}, \overline{\Phi}_{\mathfrak{z}_i} \geq \overline{\Phi}_{\ddot{\mathfrak{z}}_i}$ for all ψ_i. Then

$$CIFRWA((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)) \leq CIFRWA\left(\left(\ddot{\mathfrak{z}}_1, \mathbb{l}_1\right), \left(\ddot{\mathfrak{z}}_2, \mathbb{l}_2\right), \ldots, \left(\ddot{\mathfrak{z}}_n, \mathbb{l}_n\right)\right).$$

Proof: Since $\underline{p}_{\mathfrak{z}_i} \leq \underline{p}_{\ddot{\mathfrak{z}}_i}, \underline{\Phi}_{\mathfrak{z}_i} \geq \underline{\Phi}_{\ddot{\mathfrak{z}}_i}, \overline{p}_{\mathfrak{z}_i} \leq \overline{p}_{\ddot{\mathfrak{z}}_i}, \overline{\Phi}_{\mathfrak{z}_i} \geq \overline{\Phi}_{\ddot{\mathfrak{z}}_i}$ for all i, $1-\underline{p}_{\ddot{\mathfrak{z}}_i} \leq 1-\underline{p}_{\mathfrak{z}_i} \Longrightarrow \prod_{i=1}^{n}\left(1-\underline{p}_{\ddot{\mathfrak{z}}_i}\right)^{\mathbb{l}_i\psi_i} \leq \prod_{i=1}^{n}\left(1-\underline{p}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \Longrightarrow 1-\prod_{i=1}^{n}\left(1-\underline{p}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \leq 1-\prod_{i=1}^{n}\left(1-\underline{p}_{\ddot{\mathfrak{z}}_i}\right)^{\mathbb{l}_i\psi_i}$. Also,

$$CIFRWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_k, \mathbb{l}_k), (\mathfrak{z}_{k+1}, \mathbb{l}_{k+1})\right)$$

$$= \begin{pmatrix} \left(1-\prod_{i=1}^{k}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i}, \prod_{i=1}^{k}(\underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right), \\ 1-\prod_{i=1}^{k}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i}, \prod_{i=1}^{k}(\overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left(\left(1-\left(1-\underline{p}_{k+1}\right)^{\psi_{k+1}\mathbb{l}_{k+1}}\right), \left(\underline{\Phi}_{k+1}{}^{\psi_{k+1}\mathbb{l}_{k+1}}\right)\right), \\ \left(\left(1-\left(1-\overline{p}_{k+1}\right)^{\psi_{k+1}\mathbb{l}_{k+1}}\right), \left(\overline{\Phi}_{k+1}{}^{\psi_{k+1}\mathbb{l}_{k+1}}\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} \left(1-\prod_{i=1}^{k}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i}\right)+\left(1-\left(1-\underline{p}_{k+1}\right)^{\psi_{k+1}\mathbb{l}_{k+1}}\right)-\left(1-\prod_{i=1}^{k}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i}\right)\left(1-\left(1-\underline{p}_{k+1}\right)^{\psi_{k+1}\mathbb{l}_{k+1}}\right), \\ \left(\prod_{i=1}^{k}(\underline{\Phi}_i)^{\mathbb{l}_i\psi_i}.\underline{\Phi}_{k+1}{}^{\psi_{k+1}\mathbb{l}_{k+1}}\right) \end{pmatrix}, \\ \begin{pmatrix} \left(1-\prod_{i=1}^{k}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i}\right)+\left(1-(1-\overline{p}_{k+1})^{\psi_{k+1}\mathbb{l}_{k+1}}\right)-\left(1-\prod_{i=1}^{k}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i}\right)\left(1-(1-\overline{p}_{k+1})^{\psi_{k+1}\mathbb{l}_{k+1}}\right), \\ \left(\prod_{i=1}^{k}(\overline{\Phi}_i)^{\mathbb{l}_i\psi_i}.\overline{\Phi}_{k+1}{}^{\psi_{k+1}\mathbb{l}_{k+1}}\right) \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-\prod_{i=1}^{k+1}(1-\underline{p}_i)^{\mathbb{l}_i\psi_i}, \prod_{i=1}^{k+1}(\underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right), \\ 1-\prod_{i=1}^{k+1}(1-\overline{p}_i)^{\mathbb{l}_i\psi_i}, \prod_{i=1}^{k+1}(\overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \end{pmatrix}.$$

$1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i} \le 1-\overline{\mathfrak{p}}_{\mathfrak{z}_i} \implies \prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i} \le \prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \implies 1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \le 1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$, and $\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \ge \prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$, $\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} \ge \prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$. If $CIFRWA\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right) = \left(\left(\underline{\mathfrak{p}}_{\mathfrak{z}},\underline{\Phi}_{\mathfrak{z}}\right),\left(\overline{\mathfrak{p}}_{\mathfrak{z}},\overline{\Phi}_{\mathfrak{z}}\right)\right) = \mathfrak{z}$ and $CIFRWA\left((\mathfrak{z}^{\because}_1,\mathbb{l}_1),(\mathfrak{z}^{\because}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}^{\because}_n,\mathbb{l}_n)\right) = \left(\left(\underline{\mathfrak{p}}_{\mathfrak{z}^{\because}},\underline{\Phi}_{\mathfrak{z}^{\because}}\right),\left(\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}},\overline{\Phi}_{\mathfrak{z}^{\because}}\right)\right) = \mathfrak{z}^{\because}$, then we get $SF(\mathfrak{z}) \le SF(\mathfrak{z}^{\because})$. We have two cases:

Case 1: If $SF(\mathfrak{z}) < SF(\mathfrak{z}^{\because})$, using SF we get

$$\begin{aligned}&CIFRWA\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\\&\quad< CIFRWA\left(\left(\mathfrak{z}^{\because}_1,\mathbb{l}_1\right),\left(\mathfrak{z}^{\because}_2,\mathbb{l}_2\right),\ldots,\left(\mathfrak{z}^{\because}_n,\mathbb{l}_n\right)\right)\end{aligned}$$

Case 2: If $SF(\mathfrak{z}) = SF(\mathfrak{z}^{\because})$, using SF we get

$$\begin{aligned}SF(\mathfrak{z}) = \frac{1}{4}\Bigg(2&+\left(1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&+\left(1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&-\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}-\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\Bigg).\end{aligned}$$

$$\begin{aligned}SF\left(\mathfrak{z}^{\because}\right) = \frac{1}{4}\Bigg(2&+\left(1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&+\left(1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&-\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}-\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\Bigg).\end{aligned}$$

Since we have $\underline{\mathfrak{p}}_{\mathfrak{z}_i} \le \underline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}, \underline{\Phi}_{\mathfrak{z}_i} \ge \underline{\Phi}_{\mathfrak{z}^{\because}_i}, \overline{\mathfrak{p}}_{\mathfrak{z}_i} \le \overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}, \overline{\Phi}_{\mathfrak{z}_i} \ge \overline{\Phi}_{\mathfrak{z}^{\because}_i}$ for all i, we have $1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} = 1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$, $1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} = 1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$ and $\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} = \prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$, $\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i} = \prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}$. Now using the definition of AF, we get

$$\begin{aligned}&AC(\mathfrak{z})\\&=\frac{1}{4}\left(2+\left(1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)\right.\\&\quad+\left(1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&\quad\left.+\left(\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)+\left(\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}_i}\right)^{\mathbb{l}_i\psi_i}\right)\right)\\&=\frac{1}{4}\left(2+\left(1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)\right.\\&\quad+\left(1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)+\left(\prod_{i=1}^{n}\left(\underline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)\\&\quad\left.+\left(\prod_{i=1}^{n}\left(\overline{\Phi}_{\mathfrak{z}^{\because}_i}\right)^{\mathbb{l}_i\psi_i}\right)\right)=AC\left(\mathfrak{z}^{\because}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&CIFRWA\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\\&\quad\le CIFRWA\left(\left(\mathfrak{z}^{\because}_1,\mathbb{l}_1\right),\left(\mathfrak{z}^{\because}_2,\mathbb{l}_2\right),\ldots,\left(\mathfrak{z}^{\because}_n,\mathbb{l}_n\right)\right).\end{aligned}$$

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B. CIFR ORDERED WEIGHTED AVERAGE (CIFROWA) AGGREGATION OPERATORS

In this part, we discuss the basic definition of a CIFROWA operator. Furthermore, we will discuss the basic properties of this operator in detail.

Definition 6: Let $\mathfrak{z}_i = \left(\left(\underline{\mathfrak{p}}_i,\underline{\Phi}_i\right),\left(\overline{\mathfrak{p}}_i,\overline{\Phi}_i\right)\right)$, $i = 1,2,\ldots,n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence level of $\mathfrak{z}_i$ with $0 \le \mathbb{l}_i \le 1$. Let $\psi = (\psi_1,\psi_2,\psi_3,\ldots,\psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n}\psi_i = 1$. Then the mapping $CIFROWA: \circledcirc^{n} \to \circledcirc$ operator is given as

$$\begin{aligned}&CIFROWA\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\\&\quad=\psi_1\left(\mathbb{l}_{\ltimes(1)}\mathfrak{z}_{\ltimes(1)}\right)\oplus\psi_2\left(\mathbb{l}_{\ltimes(2)}\mathfrak{z}_{\ltimes(2)}\right)\oplus\psi_3\left(\mathbb{l}_{\ltimes(3)}\mathfrak{z}_{\ltimes(3)}\right)\\&\qquad\oplus\ldots\oplus\psi_n\left(\mathbb{l}_{\ltimes(n)}\mathfrak{z}_{\ltimes(n)}\right).\end{aligned}$$

where $(\ltimes(1),\ltimes(2),\ltimes(3),\ldots,\ltimes(n))$ is the permutation of $(1,2,3,..,n)$ such that for all i, $\mathfrak{z}_{\ltimes(i-1)} \ge \mathfrak{z}_{\ltimes(i)}$.

Theorem 2: Let $\mathfrak{z}_i = \left(\left(\underline{\mathfrak{p}}_i,\underline{\Phi}_i\right),\left(\overline{\mathfrak{p}}_i,\overline{\Phi}_i\right)\right)$, $i = 1,2,\ldots,n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence level of $\mathfrak{z}_i$ with $0 \le \mathbb{l}_i \le 1$. Let $\psi = (\psi_1,\psi_2,\psi_3,\ldots,\psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n}\psi_i = 1$. Then

$$\begin{aligned}&CIFROWA\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\\&=\left(\begin{array}{c}\left(1-\prod_{i=1}^{n}\left(1-\underline{\mathfrak{p}}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},\prod_{i=1}^{n}\left(\underline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\right),\\1-\prod_{i=1}^{n}\left(1-\overline{\mathfrak{p}}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},\prod_{i=1}^{n}\left(\overline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\end{array}\right)\end{aligned}\tag{2}$$

Proof: The proof is similar to the proof of Theorem 1.

Example 2: Consider the data from Example 1 and calculate the score values for each IFRN. The score values are given by $Sc(\mathfrak{z}_1) = 0.55$, $Sc(\mathfrak{z}_2) = 0.2$, $Sc(\mathfrak{z}_3) = 0.6$, $Sc(\mathfrak{z}_4) = 0.5$. Thus, $\mathfrak{z}_{\ltimes(1)} = (((0.5,0.3),0.3),((0.4,0.2),0.1))$, $\mathfrak{z}_{\ltimes(2)} = (((0.3,0.4),0.5),((0.5,0.2),0.4))$, $\mathfrak{z}_{\ltimes(3)} = (((0.7,0.2),0.4),((0.1,0.6),0.3))$, $\mathfrak{z}_{\ltimes(4)} = (((0.1,0.8),0.6),((0.2,0.7),0.4))$. We use Definition 2 to find the aggregated result for the above-given data as shown in the equation at the bottom of the next page.

Here, we discuss the properties of the CIFROWA operator.

1) **Idempotency:** If for all i $(\mathfrak{z}_i, \mathbb{l}_i) = (\mathfrak{z}, \mathbb{l})$, i.e., $\underline{\mathfrak{p}}_i = \underline{\mathfrak{p}}, \overline{\mathfrak{p}}_i = \overline{\mathfrak{p}}, \underline{\Phi}_i = \underline{\Phi}$ *and* $\overline{\Phi}_i = \overline{\Phi}, \mathbb{l}_i = \mathbb{l}$, then $CIFROWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) = \mathbb{l}\mathfrak{z}$.
2) **Boundedness:** Let $\mathfrak{z}^-{}_i = \left(\left(\underline{\mathfrak{p}^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{\mathfrak{p}^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$ and $\mathfrak{z}^+{}_i = \left(\left(\underline{\mathfrak{p}^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{\mathfrak{p}^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$. Then for all $\psi_i, \mathfrak{z}^-{}_i \leq CIFROWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) \leq \mathfrak{z}^+{}_i$.
3) **Monotonicity:** Let $\mathfrak{z}^{\cdot\cdot}{}_i = \left(\left(\underline{\mathfrak{p}}_{\mathfrak{z}^{\cdot\cdot}{}_i}, \underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}{}_i}\right), \left(\overline{\mathfrak{p}}_{\mathfrak{z}^{\cdot\cdot}{}_i}, \overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}{}_i}\right)\right)$ $(i = 1, 2, 3, \ldots, n)$ be another family of IFRNs such that $\underline{\mathfrak{p}}_{\mathfrak{z}_i} \leq \underline{\mathfrak{p}}_{\mathfrak{z}^{\cdot\cdot}{}_i}, \underline{\Phi}_{\mathfrak{z}_i} \geq \underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}{}_i}, \overline{\mathfrak{p}}_{\mathfrak{z}_i} \leq \overline{\mathfrak{p}}_{\mathfrak{z}^{\cdot\cdot}{}_i}, \overline{\Phi}_{\mathfrak{z}_i} \geq \overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}{}_i}$ for all ψ_i. Then $CIFROWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) \leq CIFROWA\left((\mathfrak{z}^{\cdot\cdot}{}_1, \mathbb{l}_1), (\mathfrak{z}^{\cdot\cdot}{}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}^{\cdot\cdot}{}_n, \mathbb{l}_n)\right)$.

IV. CONFIDENCE INTUITIONISTIC FUZZY ROUGH (CIFR) GEOMETRIC AGGREGATION OPERATORS

In this part, we discuss CIFR geometric AOs. Also, we will discuss the basic properties of the operators.

A. CIFR WEIGHTED GEOMETRIC (CIFRWG) AGGREGATION OPERATORS

Definition 7: Let $\mathfrak{z}_i = \left(\left(\underline{\mathfrak{p}}_i, \underline{\Phi}_i\right), \left(\overline{\mathfrak{p}}_i, \overline{\Phi}_i\right)\right)$, $i = 1, 2, \ldots, n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence level of $\mathfrak{z}_i$ with $0 \leq \mathbb{l}_i \leq 1$. Let $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n} \psi_i = 1$. Then the mapping $CIFRWG : \circledcirc^{n} \rightarrow \circledcirc$ operator is given as

$$CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right) = \otimes_{i=1}^{n}\left(\mathfrak{z}_i^{\mathbb{l}_i}\right)^{\psi_i}$$
$$= \left(\mathfrak{z}_1^{\mathbb{l}_1}\right)^{\psi_1} \otimes \left(\mathfrak{z}_2^{\mathbb{l}_2}\right)^{\psi_2} \otimes \left(\mathfrak{z}_3^{\mathbb{l}_3}\right)^{\psi_3} \otimes \ldots \otimes \left(\mathfrak{z}_n^{\mathbb{l}_n}\right)^{\psi_n}.$$

It is called the confidence intuitionistic fuzzy rough weighted geometric (CIFRWG) operator.

Theorem 3: Let $\mathfrak{z}_i = \left(\left(\underline{\mathfrak{p}}_i, \underline{\Phi}_i\right), \left(\overline{\mathfrak{p}}_i, \overline{\Phi}_i\right)\right)$ be a collection of IFRNs where $i = 1, 2, \ldots, n$ and $\mathbb{l}_i$ be the confidence level of $\mathfrak{z}_i$ with $0 \leq \mathbb{l}_i \leq 1$. Let $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n} \psi_i = 1$. Then

$$CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots, (\mathfrak{z}_n, \mathbb{l}_n)\right)$$
$$= \left(\begin{array}{c}\left(\prod_{i=1}^{n}\left(\underline{\mathfrak{p}}_i\right)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{n}\left(1 - \underline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\right), \\ \prod_{i=1}^{n}\left(\overline{\mathfrak{p}}_i\right)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{n}\left(1 - \overline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\end{array}\right) \quad (3)$$

Proof: For $n = 2$, we have $CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2)\right) = \left(\mathfrak{z}_1^{\mathbb{l}_1}\right)^{\psi_1} \otimes \left(\mathfrak{z}_2^{\mathbb{l}_2}\right)^{\psi_2}$.By using the operational laws for IFRNs we will get

$$\mathfrak{z}_1^{\mathbb{l}_1} = \left(\begin{array}{c}\left(\underline{\mathfrak{p}}_1^{\mathbb{l}_1}, 1 - \left(1 - \underline{\Phi}_1\right)^{\mathbb{l}_1}\right), \\ \left(\overline{\mathfrak{p}}_1^{\mathbb{l}_1}, 1 - \left(1 - \overline{\Phi}_1\right)^{\mathbb{l}_1}\right)\end{array}\right)$$
$$= \left(\left(\underline{\mathfrak{d}}_1, \underline{\mathfrak{e}}_1\right), \left(\overline{\mathfrak{d}}_1, \overline{\mathfrak{e}}_1\right)\right) \Rightarrow \left(\mathfrak{z}_1^{\mathbb{l}_1}\right)^{\psi_1}$$
$$= \left(\begin{array}{c}\left(\underline{\mathfrak{d}}_1^{\mathbb{l}_1}, 1 - \left(1 - \underline{\mathfrak{e}}_1\right)^{\mathbb{l}_1}\right), \\ \left(\overline{\mathfrak{d}}_1^{\mathbb{l}_1}, 1 - \left(1 - \overline{\mathfrak{e}}_1\right)^{\mathbb{l}_1}\right)\end{array}\right)$$
$$= \left(\begin{array}{c}\left(\left(\underline{\mathfrak{p}}_1^{\mathbb{l}_1}\right)^{\psi_1}, \left(1 - \left[1 - \left\{1 - \left(1 - \underline{\Phi}_1\right)^{\mathbb{l}_1}\right\}\right]^{\psi_1}\right)\right), \\ \left(\left(\overline{\mathfrak{p}}_1^{\mathbb{l}_1}\right)^{\psi_1}, \left(1 - \left[1 - \left\{1 - \left(1 - \overline{\Phi}_1\right)^{\mathbb{l}_1}\right\}\right]^{\psi_1}\right)\right)\end{array}\right)$$
$$= \left(\begin{array}{c}\left(\left(\underline{\mathfrak{p}}_1^{\psi_1\mathbb{l}_1}\right), \left(1 - \left(1 - \underline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right)\right), \\ \left(\left(\overline{\mathfrak{p}}_1^{\psi_1\mathbb{l}_1}\right), \left(1 - \left(1 - \overline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right)\right)\end{array}\right).$$

Similarly, we can see that

$$\left(\mathfrak{z}_2^{\mathbb{l}_2}\right)^{\psi_2} = \left(\begin{array}{c}\left(\left(\underline{\mathfrak{p}}_2^{\psi_2\mathbb{l}_2}\right), \left(1 - \left(1 - \underline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right)\right), \\ \left(\left(\overline{\mathfrak{p}}_2^{\psi_2\mathbb{l}_2}\right), \left(1 - \left(1 - \overline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right)\right)\end{array}\right).$$

Now,

$$CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2)\right)$$
$$= \left(\mathfrak{z}_1^{\mathbb{l}_1}\right)^{\psi_1} \otimes \left(\mathfrak{z}_2^{\mathbb{l}_2}\right)^{\psi_2}$$
$$= \left(\begin{array}{c}\left(\begin{array}{c}\left(\underline{\mathfrak{p}}_1^{\psi_1\mathbb{l}_1}\right)\left(\underline{\mathfrak{p}}_2^{\psi_2\mathbb{l}_2}\right), \\ \left(1 - \left(1 - \underline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right) + \left(1 - \left(1 - \underline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right) \\ - \left(1 - \left(1 - \underline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right)\left(1 - \left(1 - \underline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right)\end{array}\right), \\ \left(\begin{array}{c}\left(\overline{\mathfrak{p}}_1^{\psi_1\mathbb{l}_1}\right)\left(\overline{\mathfrak{p}}_2^{\psi_2\mathbb{l}_2}\right), \\ \left(1 - \left(1 - \overline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right) + \left(1 - \left(1 - \overline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right) \\ - \left(1 - \left(1 - \overline{\Phi}_1\right)^{\psi_1\mathbb{l}_1}\right)\left(1 - \left(1 - \overline{\Phi}_2\right)^{\psi_2\mathbb{l}_2}\right)\end{array}\right)\end{array}\right).$$

$$CIFROWA\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), (\mathfrak{z}_3, \mathbb{l}_3), (\mathfrak{z}_4, \mathbb{l}_4)\right)$$
$$= \left(\begin{array}{c}\left(\begin{array}{c}\left(\begin{array}{c}1 - (1 - 0.5)^{0.3\times0.29} \times (1 - 0.3)^{0.5\times0.25} \times \\ (1 - 0.7)^{0.4\times0.22} \times (1 - 0.1)^{0.6\times0.24}\end{array}\right), \\ (0.3)^{0.3\times0.29} \times (0.4)^{0.5\times0.25} \times (0.2)^{0.4\times0.22} \times (0.8)^{0.6\times0.24}\end{array}\right), \\ \left(\begin{array}{c}\left(\begin{array}{c}1 - (1 - 0.4)^{0.1\times0.29} \times (1 - 0.5)^{0.4\times0.25} \times \\ (1 - 0.1)^{0.3\times0.22} \times (1 - 0.2)^{0.4\times0.24}\end{array}\right), \\ (0.2)^{0.1\times0.29} \times (0.2)^{0.4\times0.25} \times (0.6)^{0.3\times0.22} \times (0.7)^{0.4\times0.24}\end{array}\right)\end{array}\right)$$
$$= \left((0.2022, 0.6749), (0.1064, 0.7591)\right).$$

Thus, we have that

$$CIFRWG\left((\mathfrak{z}_1,\ \mathbb{l}_1),\ \ (\mathfrak{z}_2,\ \mathbb{l}_2)\right) = \left(\begin{array}{c}\left(\prod_{i=1}^{2}\left(\underline{p}_i\right)^{\mathbb{l}_i\psi_i},\ 1-\prod_{i=1}^{2}\left(1-\underline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\right),\\ \prod_{i=1}^{2}\left(\overline{p}_i\right)^{\mathbb{l}_i\psi_i},\ 1-\prod_{i=1}^{2}\left(1-\overline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\end{array}\right).$$

Suppose the result is true for $n=\phi$, that is,

$$CIFRWG\left((\mathfrak{z}_1,\ \mathbb{l}_1),\ \ (\mathfrak{z}_2,\ \mathbb{l}_2),\ \ldots,\ (\mathfrak{z}_\phi,\ \mathbb{l}_\phi)\right) = \left(\begin{array}{c}\left(\prod_{i=1}^{\phi}\left(\underline{p}_i\right)^{\mathbb{l}_i\psi_i},\ 1-\prod_{i=1}^{\phi}\left(1-\underline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\right),\\ \prod_{i=1}^{\phi}\left(\overline{p}_i\right)^{\mathbb{l}_i\psi_i},\ 1-\prod_{i=1}^{\phi}\left(1-\overline{\Phi}_i\right)^{\mathbb{l}_i\psi_i}\end{array}\right).$$

Now, for $n=\phi+1$, we get as shown in the equation at the bottom of the next page. Hence, the result is true for $n=\phi+1$. Thus, the result is true for any number of IFRNs. ■

Example 3: Using the data of Example 1 with $\mathfrak{z}_1 = (((0.3, 0.4), 0.5), ((0.5, 0.2), 0.4))$, $\mathfrak{z}_2 = (((0.1, 0.8), 0.6), ((0.2, 0.7), 0.4))$, $\mathfrak{z}_3 = (((0.5, 0.3), 0.3), ((0.4, 0.2), 0.1))$ and $\mathfrak{z}_4 = (((0.7, 0.2), 0.4), ((0.1, 0.6), 0.3))$ being the four IFRNs along with their CL. If $\psi = (0.29, 0.25, 0.22, 0.24)$, then as shown in the equation at the bottom of the next page.

For a family of IFRNs $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right)$ where $i=1,2,\ldots,n$ and $\mathbb{l}_i$ being the confidence level of $\mathfrak{z}_i$ with $0\le\mathbb{l}_i\le1$. Let $\psi=(\psi_1,\psi_2,\psi_3,\ldots,\psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n}\psi_i=1$. Then, CIFRWG AOs have the following properties:

1) **Idempotency:** If for all i $(\mathfrak{z}_i,\mathbb{l}_i)=(\mathfrak{z},\mathbb{l})$, i.e., $\underline{p}_i=\underline{p}, \overline{p}_i=\overline{p},\ \underline{\Phi}_i=\underline{\Phi}$ *and* $\overline{\Phi}_i=\overline{\Phi}, \mathbb{l}_i=\mathbb{l}$, then

$$CIFRWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)=\mathbb{l}\mathfrak{z}$$

2) **Boundedness:** Let $\mathfrak{z}^-_i = \left(\left(\underline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$ and $\mathfrak{z}^+_i = \left(\left(\underline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$. Then, for all ψ_i, $\mathfrak{z}^-_i \le CIFRWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\le\mathfrak{z}^+_i$.

3) **Monotonicity:** Let $\mathfrak{z}^{\cdot\cdot}_i = \left(\left(\underline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}\right), \left(\overline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}\right)\right)$ $(i=1,2,3,\ldots,n)$ be another family of IFRNs such that $\underline{p}_{\mathfrak{z}_i}\le\underline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \underline{\Phi}_{\mathfrak{z}_i}\ge\underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{p}_{\mathfrak{z}_i}\le\overline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{\Phi}_{\mathfrak{z}_i}\ge\overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}$ for all ψ_i. Then, $CIFRWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\le CIFRWG\left((\mathfrak{z}^{\cdot\cdot}_1,\mathbb{l}_1),(\mathfrak{z}^{\cdot\cdot}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}^{\cdot\cdot}_n,\mathbb{l}_n)\right)$.

B. CIFR ORDERED WEIGHTED GEOMETRIC (CIFROWG) AGGREGATION OPERATORS

In this part of the article, we discuss the basic definition of a CIFROWG operator. Furthermore, we discuss the basic properties of these operator in detail.

Definition 8: Let $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right)$, $i=1,2,\ldots,n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence levelof $\mathfrak{z}_i$ with $0\le\mathbb{l}_i\le1$. Let $\psi=(\psi_1,\psi_2,\psi_3,\ldots,\psi_n)^T$ be the weight vectors for IFRNs with the condition that $\sum_{i=1}^{n}\psi_i=1$. Then, the mapping $CIFROWG:\circledcirc^n\to\circledcirc$ operator is given as

$$CIFROWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right) = \left(\mathfrak{z}_{\ltimes(1)}{}^{\mathbb{l}_{\ltimes(1)}}\right)^{\psi_1}\otimes\left(\mathfrak{z}_{\ltimes(2)}{}^{\mathbb{l}_{\ltimes(2)}}\right)^{\psi_2}\otimes\left(\mathfrak{z}_{\ltimes(3)}{}^{\mathbb{l}_{\ltimes(3)}}\right)^{\psi_3}\otimes\ldots\otimes\left(\mathfrak{z}_{\ltimes(n)}{}^{\mathbb{l}_{\ltimes(n)}}\right)^{\psi_n}.$$

where $(\ltimes(1),\ltimes(2),\ltimes(3),\ldots,\ltimes(n))$ is the permutation of $(1,2,3,..,n)$ such that for all i, $\mathfrak{z}_{\ltimes(i-1)}\ge\mathfrak{z}_{\ltimes(i)}$.

Theorem 4: Let $\mathfrak{z}_i = \left(\left(\underline{p}_i, \underline{\Phi}_i\right), \left(\overline{p}_i, \overline{\Phi}_i\right)\right)$, $i=1,2,\ldots,n$ be a family of IFRNs and $\mathbb{l}_i$ be the confidence levelof $\mathfrak{z}_i$ with $0\le\mathbb{l}_i\le1$. Let $\psi=(\psi_1,\psi_2,\psi_3,\ldots,\psi_n)^T$ be the weight vectors for IFRNs with the condition $\sum_{i=1}^{n}\psi_i=1$. Then

$$CIFROWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right) = \left(\begin{array}{c}\left(\prod_{i=1}^{n}\left(\underline{p}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},1-\prod_{i=1}^{n}\left(1-\underline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\right),\\ \prod_{i=1}^{n}\left(\overline{p}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},1-\prod_{i=1}^{n}\left(1-\overline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\end{array}\right) \tag{4}$$

Proof: The proof is similar to the proof of Theorem 3. ■

Example 4: Consider the ordered data from Example 1 and use Equation (4) to find the aggregated result for the above-given data. We obtain

$$CIFROWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),(\mathfrak{z}_3,\mathbb{l}_3),(\mathfrak{z}_4,\mathbb{l}_4)\right) = \left(\begin{array}{c}\left(\prod_{i=1}^{4}\left(\underline{p}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},1-\prod_{i=1}^{4}\left(1-\underline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\right),\\ \prod_{i=1}^{4}\left(\overline{p}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i},1-\prod_{i=1}^{4}\left(1-\overline{\Phi}_{\ltimes(i)}\right)^{\mathbb{l}_{\ltimes(i)}\psi_i}\end{array}\right) = ((0.5634,\ 0.2835),(0.6687,\ 0.1852)).$$

We next discuss the properties of the CIFROWG operator.

1) **Idempotency:** If for all i $(\mathfrak{z}_i,\mathbb{l}_i)=(\mathfrak{z},\mathbb{l})$, i.e., $\underline{p}_i=\underline{p}, \overline{p}_i=\overline{p},\ \underline{\Phi}_i=\underline{\Phi}$ *and* $\overline{\Phi}_i=\overline{\Phi}, \mathbb{l}_i=\mathbb{l}$, then $CIFROWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)=\mathbb{l}\mathfrak{z}$.

2) **Boundedness:** Let $\mathfrak{z}^-_i = \left(\left(\underline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$ and $\mathfrak{z}^+_i = \left(\left(\underline{p^{max}}_{\mathbb{l}_i\mathfrak{z}_i}, \underline{\Phi^{max}}_{\mathbb{l}_i\mathfrak{z}_i}\right), \left(\overline{p^{min}}_{\mathbb{l}_i\mathfrak{z}_i}, \overline{\Phi^{min}}_{\mathbb{l}_i\mathfrak{z}_i}\right)\right)$. Then, for all ψ_i, $\mathfrak{z}^-_i \le CIFROWG\left((\mathfrak{z}_1,\ \mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\le\mathfrak{z}^+_i$.

3) **Monotonicity:** Let $\mathfrak{z}^{\cdot\cdot}_i = \left(\left(\underline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}\right), \left(\overline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}\right)\right)$ $(i=1,2,3,\ldots,n)$ be another family of IFRNs such that $\underline{p}_{\mathfrak{z}_i}\le\underline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \underline{\Phi}_{\mathfrak{z}_i}\ge\underline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{p}_{\mathfrak{z}_i}\le\overline{p}_{\mathfrak{z}^{\cdot\cdot}_i}, \overline{\Phi}_{\mathfrak{z}_i}\ge\overline{\Phi}_{\mathfrak{z}^{\cdot\cdot}_i}$ for all ψ_i. Then

$$CIFROWG\left((\mathfrak{z}_1,\mathbb{l}_1),(\mathfrak{z}_2,\mathbb{l}_2),\ldots,(\mathfrak{z}_n,\mathbb{l}_n)\right)\le CIFROWG\left(\left(\mathfrak{z}^{\cdot\cdot}_1,\mathbb{l}_1\right),\left(\mathfrak{z}^{\cdot\cdot}_2,\mathbb{l}_2\right),\ldots,\left(\mathfrak{z}^{\cdot\cdot}_n,\mathbb{l}_n\right)\right).$$

V. MULTI-ATTRIBUTE DECISION MAKING (MADM) TECHNIQUE BASED ON CIFR AVERAGE/GEOMETRIC AOs

Multi-criteria decision-making technique is an effective way of selecting the best alternative corresponding to their criteria. In this section, we will study the application of the introduced operators. So, we develop an MCDM algorithm to show the effectiveness and usefulness of the proposed work. Let $\mathbb{G}^{*} = \left\{\mathbb{G}_1^{*}, \mathbb{G}_2^{*}, \mathbb{G}_3^{*}, \ldots, \mathbb{G}_m^{*}\right\}$ denote the collection of alternatives and $\mathbb{C} = \{\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \ldots, \mathbb{C}_n\}$ denote the collection of criteria. Also, suppose that $\psi = (\psi_1, \psi_2, \psi_3, \ldots, \psi_n)^{\mathbb{T}}$ is the weight vector of criteria set with the condition that $\sum_{i=1}^{n} \psi_i = 1$ *and* $\psi_i > 0$. Let $\mathfrak{G}_e = \left\{\mathfrak{G}^1{}_e, \mathfrak{G}^2{}_e, \mathfrak{G}^3{}_e, \ldots, \mathfrak{G}^f{}_e\right\}$ be the set of 'f' experts with the weight vector $\varrho_v = \left(\varrho^1{}_v, \varrho^2{}_v, \varrho^3{}_v, \ldots, \varrho^f{}_v\right)$ using condition that $\sum_{s=1}^{f} \varrho_v{}^s = 1$ and $\varrho_v{}^s > 0$. Suppose experts provide their assessment for each alternative concerning each criterion in the form of IFRNs $\left(\mathfrak{z}_{ij}{}^s\right)_{m\times n} = \left(\left(\underline{p}^s{}_{ij}, \underline{\Phi}^s{}_{ij}\right), \left(\overline{p}^s{}_{ij}, \overline{\Phi}^s{}_{ij}\right), \mathbb{l}_{ij}{}^s\right)$. To use the notion of CL, the experts provide that they are familiar with evaluated alternatives and assign the CL with $\mathbb{l}_{ij}{}^s \left(0 \le \mathbb{l}_{ij}{}^s \le 1\right)$. Now, we have to follow the following steps:

$$
\begin{aligned}
&CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), \ldots., (\mathfrak{z}_\phi, \mathbb{l}_\phi), (\mathfrak{z}_{\phi+1}, \mathbb{l}_{\phi+1})\right)\\
&= \begin{pmatrix} \left(\prod_{i=1}^{\phi} (\underline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{\phi} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right), \\ \prod_{i=1}^{\phi} (\overline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{\phi} (1 - \overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \end{pmatrix}\\
&\quad \otimes \begin{pmatrix} \left(\left(\underline{p}_{\phi+1}{}^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right), \left(1 - (1 - \underline{\Phi}_{\phi+1})^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right)\right), \\ \left(\left(\overline{p}_{\phi+1}{}^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right), \left(1 - \left(1 - \overline{\Phi_{\phi+1}}\right)^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right)\right) \end{pmatrix}\\
&= \left(\begin{pmatrix} \left(\prod_{i=1}^{\phi} (\underline{p}_i)^{\mathbb{l}_i\psi_i} \times \underline{p}_{\phi+1}{}^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right), \\ \left(1 - \prod_{i=1}^{\phi} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right) + \left(1 - (1 - \underline{\Phi}_{\phi+1})^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right) \\ -\left(1 - \prod_{i=1}^{\phi} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right)\left(1 - (1 - \underline{\Phi}_{\phi+1})^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right) \end{pmatrix}, \begin{pmatrix} \left(\prod_{i=1}^{\phi} (\underline{p}_i)^{\mathbb{l}_i\psi_i} \times \underline{p}_{\phi+1}{}^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right), \\ \left(1 - \prod_{i=1}^{\phi} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right) + \left(1 - (1 - \underline{\Phi}_{\phi+1})^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right) \\ -\left(1 - \prod_{i=1}^{\phi} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right)\left(1 - (1 - \underline{\Phi}_{\phi+1})^{\psi_{\phi+1}\mathbb{l}_{\phi+1}}\right) \end{pmatrix}\right)\\
&= \begin{pmatrix} \left(\prod_{i=1}^{\phi+1} (\underline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{\phi+1} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right), \\ \prod_{i=1}^{\phi+1} (\overline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{\phi+1} (1 - \overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \end{pmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
&CIFRWG\left((\mathfrak{z}_1, \mathbb{l}_1), (\mathfrak{z}_2, \mathbb{l}_2), (\mathfrak{z}_3, \mathbb{l}_3), (\mathfrak{z}_4, \mathbb{l}_4)\right)\\
&= \begin{pmatrix} \left(\prod_{i=1}^{4} (\underline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{4} (1 - \underline{\Phi}_i)^{\mathbb{l}_i\psi_i}\right), \\ \prod_{i=1}^{4} (\overline{p}_i)^{\mathbb{l}_i\psi_i}, 1 - \prod_{i=1}^{4} (1 - \overline{\Phi}_i)^{\mathbb{l}_i\psi_i} \end{pmatrix}\\
&= \left(\begin{pmatrix} (0.3)^{0.5\times0.29} \times (0.1)^{0.6\times0.25} \times (0.5)^{0.3\times0.22} \times (0.7)^{0.4\times0.24}, \\ \begin{pmatrix} 1 - (1 - 0.4)^{0.5\times0.29} \times (1 - 0.8)^{0.6\times0.25} \times \\ (1 - 0.3)^{0.3\times0.22} \times (1 - 0.2)^{0.4\times0.24} \end{pmatrix} \end{pmatrix}, \begin{pmatrix} (0.5)^{0.4\times0.29} \times (0.2)^{0.4\times0.25} \times \\ (0.4)^{0.1\times0.22} \times (0.1)^{0.3\times0.24}, \\ \begin{pmatrix} 1 - (1 - 0.2)^{0.4\times0.29} \times (1 - 0.7)^{0.4\times0.25} \times \\ (1 - 0.2)^{0.1\times0.22} \times (1 - 0.6)^{0.3\times0.24} \end{pmatrix} \end{pmatrix}\right)\\
&= ((0.5488, 0.3026), (0.6522, 0.1952)).
\end{aligned}
$$

Step 1: Collect the expert's data given in the form of IFRNs along with their CL and then construct the expert's assessment matrix as$[\mathrm{M}^s]_{m\times n} = \left(\left(\underline{\mathfrak{p}}^s{}_{ij}, \underline{\Phi}^s{}_i\right), \left(\overline{\mathfrak{p}}^s{}_i, \overline{\Phi}^s{}_i\right), \mathbb{l}_{ij}{}^s\right)$.

Step 2: **Use** the notion of CIFRWA or (CIFRWG) to combine all individual matrices of experts into a collective judgment matrix $[\mathrm{M}]_{m\times n}$. That is,

$$\mathfrak{z}_{ij} = CIFRWA\left(\left(\mathfrak{z}^1{}_{ij}, \mathbb{l}^1{}_{ij}\right), \left(\mathfrak{z}^2{}_{ij}, \mathbb{l}^2{}_{ij}\right), \ldots, \left(\mathfrak{z}^s{}_{ij}, \mathbb{l}^s{}_{ij}\right)\right) = \left(\begin{array}{c}\left(1-\prod_{s=1}^f\left(1-\underline{\mathfrak{p}}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}, \prod_{s=1}^f\left(\underline{\Phi}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}\right), \\ 1-\prod_{s=1}^f\left(1-\overline{\mathfrak{p}}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}, \prod_{s=1}^f\left(\overline{\Phi}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}\end{array}\right),$$

or

$$CIFRWA\left(\left(\mathfrak{z}^1{}_{ij}, \mathbb{l}^1{}_{ij}\right), \left(\mathfrak{z}^2{}_{ij}, \mathbb{l}^2{}_{ij}\right), \ldots, \left(\mathfrak{z}^s{}_{ij}, \mathbb{l}^s{}_{ij}\right)\right) = \left(\begin{array}{c}\left(\prod_{s=1}^f\left(\underline{\mathfrak{p}}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}, 1-\prod_{s=1}^f\left(1-\underline{\Phi}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}\right), \\ \prod_{s=1}^f\left(\overline{\mathfrak{p}}^s{}_{ij}\right)^{\mathbb{l}_{ij}{}^s\varrho_i}, 1-\prod_{s=1}^f\left(1-\overline{\Phi}^s{}_i\right)^{\mathbb{l}_{ij}{}^s\varrho_i}\end{array}\right).$$

Step 3: Using the notion of IFRWA or IFRWG operator to aggregate the execution of alternative of the matrix $[\mathrm{M}]_{m\times n}$ as $\mathfrak{z}_i = IFRWA(\mathfrak{z}_{i1}, \mathfrak{z}_{i2}, \ldots, \mathfrak{z}_{in}) = \left(\begin{array}{c}\left(1-\prod_{j=1}^n\left(1-\underline{\mathfrak{p}}_{ij}\right)^{\psi_j}, \prod_{j=1}^n\left(\underline{\Phi}_{ij}\right)^{\psi_j}\right), \\ 1-\prod_{j=1}^n\left(1-\overline{\mathfrak{p}}_{ij}\right)^{\psi_j}, \prod_{j=1}^n\left(\overline{\Phi}_{ij}\right)^{\psi_j}\end{array}\right)$ or

$$IFRWG(\mathfrak{z}_{i1}, \mathfrak{z}_{i2}, \ldots, \mathfrak{z}_{in}) = \left(\begin{array}{c}\left(\prod_{j=1}^n\left(\underline{\mathfrak{p}}_{ij}\right)^{\psi_j}, 1-\prod_{j=1}^n\left(1-\underline{\Phi}_{ij}\right)^{\psi_j}\right), \\ \prod_{j=1}^n\left(\overline{\mathfrak{p}}_{ij}\right)^{\psi_j}, 1-\prod_{j=1}^n\left(1-\overline{\Phi}_{ij}\right)^{\psi_j}\end{array}\right).$$

Step 4: Compute the score values for each alternative $\mathfrak{z}_i\ (i = 1, 2, 3, \ldots, m)$ by using Definition 5 and ranking the results.

A. ILLUSTRATIVE EXAMPLE BY USING CIFRWA OPERATOR

Suppose a person wants to buy the best cell phone from a set of four alternatives $\mathbb{G}^* = \left\{\mathbb{G}_1^*, \mathbb{G}_2^*, \mathbb{G}_3^*, \mathbb{G}_4^*\right\}$ based on four criteria$\mathbb{C} = \{\mathbb{C}_1 = Internet\ storage, \mathbb{C}_2 = Hardware, \mathbb{C}_3 = long\ lasting\ battry, \mathbb{C}_4 = crystal\ clear\ display\}$. Let $\psi = (0.32, 0.30, 0.20, 0.18)^{\mathbb{T}}$ and suppose three experts with their heir weight vector $\varrho_v = (0.32, 0.42, 0.36)$. In this subsection, we consider the illustrative example by using the CIFRWA operator.

Step 1: Suppose experts provide their assessment values in the shape of IFRNs along with their CL and then construct the expert's assessment matrix with $[\mathrm{M}^s]_{4\times 4} = \left(\left(\underline{\mathfrak{p}}^s{}_{ij}, \underline{\Phi}^s{}_i\right), \left(\overline{\mathfrak{p}}^s{}_i, \overline{\Phi}^s{}_i\right), \mathbb{l}_{ij}{}^s\right)$ $(s = 1, 2, 3)$, as shown in Tables 2-4.

Step 2: Using the notion of CIFRWA to combine all individual matrices of exerts into a collective judgment matrix as given in Table 5.

Step 3: Use the notion of the IFRWA operator for data given in Table 5 to aggregate the performance of the alternative by using the formula given below

$$\mathfrak{z}_i = IFRWA(\mathfrak{z}_{i1}, \mathfrak{z}_{i2}, \ldots, \mathfrak{z}_{in}) = \left(\begin{array}{c}\left(\mathbf{1}-\prod_{j=1}^n\left(\mathbf{1}-\underline{\mathfrak{p}}_{ij}\right)^{\psi_j}, \prod_{j=1}^n\left(\underline{\Phi}_{ij}\right)^{\psi_j}\right), \\ \mathbf{1}-\prod_{j=1}^n\left(\mathbf{1}-\overline{\mathfrak{p}}_{ij}\right)^{\psi_j}, \prod_{j=1}^n\left(\overline{\Phi}_{ij}\right)^{\psi_j}\end{array}\right).$$

Step 4: Compute the score values for each alternative $\mathfrak{z}_i\ (i = 1, 2, 3, \ldots, m)$ by using Definition 5 and ranking the results, as shown in Table 6.

B. ILLUSTRATIVE EXAMPLE BY USING CIFRWG OPERATOR

In this subsection, we consider the illustrative example by using the CIFRWG operator.

Step 1: Same as above

Step 2: Using the notion of CIFRWG to combine all individual matrices of experts into a collective judgment matrix as given in Table 7.

Step 3: Use the notion of the IFRWG operator for data given in Table 7 to aggregate the performance of the alternative.

Step 4: Compute the score values for each alternative $\mathfrak{z}_i\ (i = 1, 2, 3, \ldots, m)$ by using Definition 7 and ranking the results, as shown in Table 8.

VI. MEDICAL DIAGNOSIS APPROACH BASED ON IFRNs

As medical diagnosis is an effective process for the assessment of some diseases, in this section, we elaborate on an algorithm of medical diagnosis based on IFRNs.

Definition 9: Let C and D be two sets. An intuitionistic fuzzy rough relation (IFRR) R_e from C to D is an IFRS of $C \times D$ characterized by MG $\overline{\mathfrak{p}}_{R_e}, \underline{\mathfrak{p}}_{R_e}$ and NMG$\underline{\Phi}_{R_e}, \overline{\Phi}_{R_e}$. An IFRR from C to D is denoted by $R_e\,(C \to D)$.

Definition 10: If J is an IFRS of C, then the max-min-max composition of IFRR $R_e\,(C \to D)$ with J is an IFRS Q of D denoted by $Q = CoD$ and is defined by $\underline{\mathfrak{p}}_{R_eoJ}(d) = \vee_c\left[\underline{\mathfrak{p}}_J(c)\wedge\underline{\mathfrak{p}}_{R_e}(c,d)\right], \overline{\mathfrak{p}}_{R_eoJ}(d) = \vee_c\left[\overline{\mathfrak{p}}_J(c)\wedge\overline{\mathfrak{p}}_{R_e}(c,d)\right]$ and $\underline{\Phi}_{R_eoJ}(d) = \wedge_c\left[\underline{\Phi}_J(c)\vee\underline{\Phi}_{R_e}(c,d)\right], \overline{\Phi}_{R_eoJ}(d) = \wedge_c\left[\overline{\Phi}_J(c)\vee\overline{\Phi}_{R_e}(c,d)\right]$ for all $d\in D$, where $(\vee = \max$ and $\wedge = min)$.

Definition 11: Let $R_e\,(C \to D)$ and $R_e{}^*\,(D \to E)$ be two IFRRs. The max-min-max composition $R_eoR_e{}^*$ is an IFRR from C to E defined by

$$\underline{\mathfrak{p}}_{R_eoR_e{}^*}(c,e) = \vee_d\left[\underline{\mathfrak{p}}_{R_e}(c,d)\wedge\underline{\mathfrak{p}}_{R_e{}^*}(d,e)\right],$$
$$\overline{\mathfrak{p}}_{R_eoR_e{}^*}(c,e) = \vee_d\left[\overline{\mathfrak{p}}_{R_e}(c,d)\wedge\overline{\mathfrak{p}}_{R_e{}^*}(d,e)\right],$$
$$\underline{\Phi}_{R_eoR_e{}^*}(c,e) = \wedge_d\left[\underline{\Phi}_{R_e}(c,d)\vee\underline{\Phi}_{R_e{}^*}(d,e)\right], \overline{\Phi}_{R_eoR_e{}^*}(c,e) = \wedge_d\left[\overline{\Phi}_{R_e}(c,d)\vee\overline{\Phi}_{R_e{}^*}(d,e)\right].$$

Now, we present the uses of IFRSs theory in Sanchez's approach [26], [27] for medical diagnosis. Let $S^{\cdot}$ denote the set of symptoms and D^* denote the set of diagnoses and P° denote the set of patients. We define "IFR medical knowledge" as an IFRR R_e from the set of symptoms $S^{\cdot}$ to the set of diagnosis D^* that reveals the degree of association and

TABLE 2. IFR data by expert 1.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.21,0.22),0.3),\\((0.19,0.18),0.4)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.19,0.20),0.2),\\((0.51,0.26),0.5)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.31,0.42),0.3),\\((0.51,0.12),0.6)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.14,0.17),0.2),\\((0.58,0.18),0.4)\end{matrix}\right\}$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.11,0.12),0.5),\\((0.25,0.30),0.7)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.41,0.42),0.7),\\((0.38,0.32),0.4)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.60,0.30),0.4),\\((0.51,0.12),0.6)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.35,0.36),0.3),\\((0.17,0.19),0.5)\end{matrix}\right\}$
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.18,0.35),0.2),\\((0.29,0.38),0.1)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.31,0.37),0.1),\\((0.19,0.12),0.7)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.41,0.32),0.5),\\((0.31,0.37),0.4)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.49,0.21),0.6),\\((0.27,0.26),0.7)\end{matrix}\right\}$
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.37,0.41),0.8),\\((0.61,0.24),0.5)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.10,0.20),0.6),\\((0.41,0.52),0.9)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.49,0.30),0.1),\\((0.51,0.42),0.6)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.37,0.50),0.8),\\((0.49,0.46),0.6)\end{matrix}\right\}$

TABLE 3. IFR data by expert 2.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.12,0.22),0.13),\\((0.10,0.16),0.14)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.29,0.30),0.21),\\((0.41,0.36),0.53)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.21,0.62),0.31),\\((0.41,0.42),0.61)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.15,0.15),0.42),\\((0.54,0.28),0.34)\end{matrix}\right\}$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.18,0.22),0.25),\\((0.26,0.40),0.27)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.21,0.12),0.71),\\((0.58,0.16),0.14)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.50,0.29),0.14),\\((0.66,0.18),0.36)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.30,0.31),0.33),\\((0.27,0.39),0.45)\end{matrix}\right\}$
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.28,0.55),0.32),\\((0.39,0.17),0.41)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.64,0.27),0.41),\\((0.49,0.42),0.27)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.31,0.62),0.55),\\((0.21,0.27),0.64)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.59,0.11),0.26),\\((0.17,0.16),0.67)\end{matrix}\right\}$
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.17,0.51),0.38),\\((0.31,0.34),0.45)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.50,0.30),0.26),\\((0.51,0.22),0.39)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.39,0.30),0.71),\\((0.41,0.32),0.76)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.27,0.20),0.48),\\((0.29,0.26),0.56)\end{matrix}\right\}$

TABLE 4. IFR data by expert 3.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.22,0.37),0.31),\\((0.17,0.19),0.24)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.19,0.20),0.23),\\((0.31,0.46),0.53)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.31,0.32),0.32),\\((0.44,0.45),0.41)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.16,0.17),0.32),\\((0.14,0.15),0.54)\end{matrix}\right\}$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.38,0.42),0.45),\\((0.24,0.46),0.37)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.51,0.42),0.75),\\((0.68,0.26),0.16)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.40,0.39),0.24),\\((0.36,0.48),0.56)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.10,0.21),0.63),\\((0.20,0.29),0.15)\end{matrix}\right\}$
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.78,0.10),0.62),\\((0.38,0.19),0.51)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.34,0.47),0.47),\\((0.40,0.52),0.28)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.11,0.19),0.65),\\((0.18,0.28),0.74)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.39,0.41),0.16),\\((0.57,0.17),0.47)\end{matrix}\right\}$
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.17,0.21),0.28),\\((0.33,0.34),0.45)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.60,0.32),0.20),\\((0.41,0.42),0.30)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.30,0.40),0.41),\\((0.51,0.12),0.36)\end{matrix}\right\}$	$\left\{\begin{matrix}((0.17,0.26),0.68),\\((0.28,0.46),0.16)\end{matrix}\right\}$

degree of non-association. We next present our intuitionistic fuzzy rough medical diagnosis. We have the following main three steps:

1) Determine the symptoms.
2) Development of medical diagnosis based on IFRR.
3) Determine the diagnosis based on the composition of IFRR.

Let "J" be an IFRS of the set $S^\cdot$ and R_e be an IFRR form $S^\cdot$ to D^*. Then, the max-min-max composition of IFRS Q of IFRS "J" with IFRR $R_e\,(S^\cdot \to D^*)$ denoted by $Q = JoR_e$ signifies the state of patients in the form of diagnosis as an IFRS "Q" of D^* with MG $\underline{\mathfrak{p}}_Q(d) = \vee_{s\in S^\cdot}\left[\underline{\mathfrak{p}}_J(s)\wedge\underline{\mathfrak{p}}_{R_e}(s,d)\right]$, $\overline{\mathfrak{p}}_Q(d) = \vee_{s\in S^\cdot}\left[\overline{\mathfrak{p}}_j(s)\wedge\overline{\mathfrak{p}}_{R_e}(s,d)\right]$ and $\underline{\Phi}_Q(d) = \wedge_{s\in S^\cdot}\left[\underline{\Phi}_J(s)\vee\underline{\Phi}_{R_e}(s,d)\right]$, $\overline{\Phi}_Q(d) = \wedge_{s\in S^\cdot}\left[\overline{\Phi}_J(s)\vee\overline{\Phi}_{R_e}(s,d)\right]$ for all $d\in D^*$.

A. ALGORITHM

Let $P^\circ = \{p_1^\circ, p_2^\circ, p_3^\circ, \ldots, p_n^\circ\}\ (i = 1, 2, 3, \ldots, n)$ denote the set of "n" patients in a hospital. Let R_e be an IFRR $(S^\cdot \to D^*)$ and construct an IFRR $R_e{}^*$ from a set of patients to a set of symptoms. The composition $T^\sim = R_eoR_e{}^*$ describes the state of the patients p_i° in terms

TABLE 5. IFR combined expert assessments matrix.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.0557, 0.7124)),\\((0.0480, 0.6245))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.0591, 0.7100)),\\((0.2609, 0.5537))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1033, 0.7582)),\\((0.3007, 0.4736))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.0567, 0.5209)),\\((0.2221, 0.4629))\end{matrix}\right\}$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.1103, 0.5279)),\\((0.1264, 0.6206))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.3168, 0.3461)),\\((0.1629, 0.7180))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1830, 0.7347)),\\((0.3229, 0.4429))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1083, 0.5409)),\\((0.0963, 0.6001))\end{matrix}\right\}$
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.3261, 0.5160)),\\((0.1679, 0.5267))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.2274, 0.6804)),\\((0.1605, 0.5276))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1791, 0.5059)),\\((0.1510, 0.4411))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.2251, 0.5532)),\\((0.2333, 0.3271))\end{matrix}\right\}$
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.1536, 0.6107)),\\((0.2484, 0.5449))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1494, 0.5930)),\\((0.2780, 0.5885))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.1987, 0.5869)),\\((0.3107, 0.4825))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.2033, 0.4353)),\\((0.2044, 0.6001))\end{matrix}\right\}$

TABLE 6. Aggregated values by using the IFRWA operator.

Alternatives	Aggregated results	Score values	Ranking results
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.0668, 0.6811)),\\((0.20086, 0.5400))\end{matrix}\right\}$	$Sc(\mathbb{G}_1^*) = 0.2613$	$\mathbb{G}_3^* \geq \mathbb{G}_4^* \geq \mathbb{G}_2^* \geq \mathbb{G}_1^*$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.1916, 0.4991)),\\((0.1753, 0.6024))\end{matrix}\right\}$	$Sc(\mathbb{G}_2^*) = 0.3163$	
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.2511, 0.5655)),\\((0.1746, 0.4668))\end{matrix}\right\}$	$Sc(\mathbb{G}_3^*) = 0.3483$	
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.1707, 0.5650)),\\((0.2626, 0.5538))\end{matrix}\right\}$	$Sc(\mathbb{G}_4^*) = 0.3286$	

TABLE 7. IFR combined expert's assessments matrix.

	$\mathbb{C}_1$	$\mathbb{C}_2$	$\mathbb{C}_3$	$\mathbb{C}_4$
$\mathbb{G}_1^*$	$\left\{\begin{matrix}((0.6475, 0.0851)),\\((0.6058, 0.0524))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.7025, 0.0621)),\\((0.5881, 0.2328))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.6372, 0.1996)),\\((0.6194, 0.2230))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.5108, 0.0601)),\\((0.5827, 0.0986))\end{matrix}\right\}$
$\mathbb{G}_2^*$	$\left\{\begin{matrix}((0.5015, 0.1260)),\\((0.5202, 0.1974))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.4287, 0.2645)),\\((0.8368, 0.0740))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.8308, 0.1028)),\\((0.6716, 0.1700))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.4538, 0.1373)),\\((0.5390, 0.1355))\end{matrix}\right\}$
$\mathbb{G}_3^*$	$\left\{\begin{matrix}((0.7144, 0.1464)),\\((0.6842, 0.0824))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.7431, 0.1617)),\\((0.5796, 0.1515))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.3946, 0.2843)),\\((0.3583, 0.2064))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.7797, 0.0845)),\\((0.4118, 0.1376))\end{matrix}\right\}$
$\mathbb{G}_4^*$	$\left\{\begin{matrix}((0.4887, 0.2386)),\\((0.6187, 0.1728))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.5743, 0.1037)),\\((0.6291, 0.2672))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.6179, 0.1756)),\\((0.6207, 0.2131))\end{matrix}\right\}$	$\left\{\begin{matrix}((0.3858, 0.2563)),\\((0.6056, 0.2018))\end{matrix}\right\}$

of diagnosis as an IFRR from P° to D^* given by MG as $\underline{\mathfrak{p}}_T\left(p_i^{\circ}, d\right) = \vee_{s\in S^{\cdot}}\left[\underline{\mathfrak{p}}_{R_e^*}\left(p_i^{\circ}, s\right) \wedge \underline{\mathfrak{p}}_{R_e}\left(s, d\right)\right]$, $\overline{\mathfrak{p}}_T\left(p_i^{\circ}, d\right) = \vee_{s\in S^{\cdot}}\left[\overline{\mathfrak{p}}_{R_e^*}\left(p_i^{\circ}, s\right) \wedge \overline{\mathfrak{p}}_{R_e}\left(s, d\right)\right]$ and NMG as $\underline{\Phi}_T\left(p_i^{\circ}, d\right) = \wedge_{s\in S^{\cdot}}\left[\underline{\Phi}_{R_e^*}\left(p_i^{\circ}, s\right) \vee \underline{\Phi}_{R_e}\left(s, d\right)\right]$, $\overline{\Phi}_T\left(p_i^{\circ}, d\right) = \wedge_{s\in S^{\cdot}}\left[\overline{\Phi}_{R_e^*}\left(p_i^{\circ}, s\right) \vee \overline{\Phi}_{R_e}\left(s, d\right)\right]$ for all $d\in D^*$. For given R_e *and* $R_e{}^*$, the relation $T^{\sim} = R_e o R_e{}^*$ can be calculated. From the information of R_e and $T^{\sim}$, one can calculate an improved version of IFRR for which the following holds.

1) $S_{R_e} = \frac{1}{4}\left(2 + \underline{\mathfrak{p}}_{R_e} + \overline{\mathfrak{p}}_{R_e} - \underline{\Phi}_{R_e} - \overline{\Phi}_{R_e}\right)$ is the greatest.
2) The equality $T^{\sim} = R_e o R_e{}^*$ is retained.

Example 5: Suppose we have four patients: Ali, Jabir, Ubaid, and Umar in the hospital. Their symptoms are headache, temperature, stomach pain, cough, and chest pain. Then, $P^{\circ} = \{Ali, Jabir, Ubaid \text{ } and \text{ } Umar\}$ is the set of patients, $S^{\cdot} = \{Headache, temperature, stomach\ pain, cough, chets\ pain\}$. Now, IFRR $R_e\left(P^{\circ} \to S^{\cdot}\right)$ is given in Table 9. Let $D^* = \{Fever, Malaria, Typhoid, Stomach\ issues, Heart\ issues\}$. IFRR $R_e{}^*\left(S^{\cdot} \to D^*\right)$ is given in Table 10. Therefore, the composition $T^{\sim} = R_e o R_e{}^*$ is given in Table 11. We calculate S_{R_e} as given in Table 12.

From Table 12, we can observe that *Ali* is suffering from fever, *Jabir and Umar* face stomach issue, and *Ubaid* is suffering from typhoid. Note that, based on the more advanced structure of IFRSs, we can perform the applications of medical diagnosis. IFRS covers the issues of data loss in terms of considering the upper and lower operators and provides

TABLE 8. Aggregated values by using the IFRWG operator.

Alternatives	Aggregated results	Score values	Ranking result
$\mathbb{G}_1^*$	$\begin{Bmatrix}((0.6338, 0.0982)),\\((0.5991, 0.1528))\end{Bmatrix}$	$Sc(\mathbb{G}_1^*) = 0.7454$	$\mathbb{G}_1^* \geq \mathbb{G}_3^* \geq \mathbb{G}_2^* \geq \mathbb{G}_4^*$
$\mathbb{G}_2^*$	$\begin{Bmatrix}((0.5199, 0.1677)),\\((0.6354, 0.1452))\end{Bmatrix}$	$Sc(\mathbb{G}_2^*) = 0.7106$	
$\mathbb{G}_3^*$	$\begin{Bmatrix}((0.6522, 0.1700)),\\((0.5220, 0.1390))\end{Bmatrix}$	$Sc(\mathbb{G}_3^*) = 0.7162$	
$\mathbb{G}_4^*$	$\begin{Bmatrix}((0.5152, 0.1910)),\\((0.6198, 0.2152))\end{Bmatrix}$	$Sc(\mathbb{G}_4^*) = 0.6821$	

TABLE 9. Intuitionistic fuzzy rough relation.

R_e	*Headache*	*Temperature*	*Stomach pain*	*Cough*	*Chest pain*
Ali	$\begin{Bmatrix}(0.3, 0.4),\\(0.2, 0.5)\end{Bmatrix}$	$\begin{Bmatrix}(0.2, 0.1),\\(0.1, 0.5)\end{Bmatrix}$	$\begin{Bmatrix}(0.6, 0.2),\\(0.5, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.51, 0.14),\\(0.12, 0.35)\end{Bmatrix}$	$\begin{Bmatrix}(0.28, 0.14),\\(0.12, 0.65)\end{Bmatrix}$
Jabir	$\begin{Bmatrix}(0.6, 0.1),\\(0.5, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.3, 0.5),\\(0.2, 0.5)\end{Bmatrix}$	$\begin{Bmatrix}(0.31, 0.4),\\(0.22, 0.5)\end{Bmatrix}$	$\begin{Bmatrix}(0.28, 0.24),\\(0.32, 0.35)\end{Bmatrix}$	$\begin{Bmatrix}(0.12, 0.24),\\(0.52, 0.15)\end{Bmatrix}$
Ubaid	$\begin{Bmatrix}(0.4, 0.4),\\(0.1, 0.8)\end{Bmatrix}$	$\begin{Bmatrix}(0.5, 0.2),\\(0.1, 0.4)\end{Bmatrix}$	$\begin{Bmatrix}(0.30, 0.42),\\(0.26, 0.51)\end{Bmatrix}$	$\begin{Bmatrix}(0.36, 0.34),\\(0.12, 0.15)\end{Bmatrix}$	$\begin{Bmatrix}(0.17, 0.27),\\(0.28, 0.37)\end{Bmatrix}$
Umar	$\begin{Bmatrix}(0.4, 0.5),\\(0.3, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.1, 0.3),\\(0.2, 0.2)\end{Bmatrix}$	$\begin{Bmatrix}(0.22, 0.54),\\(0.24, 0.25)\end{Bmatrix}$	$\begin{Bmatrix}(0.11, 0.10),\\(0.12, 0.19)\end{Bmatrix}$	$\begin{Bmatrix}(0.39, 0.40),\\(0.21, 0.35)\end{Bmatrix}$

TABLE 10. Intuitionistic fuzzy rough relation.

${R_e}^*$	*Fever*	*Malaria*	*Typhoid*	*Stomach issues*	*Heart issues*
Headache	$\begin{Bmatrix}(0.31, 0.44),\\(0.29, 0.56)\end{Bmatrix}$	$\begin{Bmatrix}(0.47, 0.21),\\(0.11, 0.65)\end{Bmatrix}$	$\begin{Bmatrix}(0.17, 0.28),\\(0.19, 0.27)\end{Bmatrix}$	$\begin{Bmatrix}(0.63, 0.10),\\(0.22, 0.15)\end{Bmatrix}$	$\begin{Bmatrix}(0.26, 0.24),\\(0.22, 0.61)\end{Bmatrix}$
Temperature	$\begin{Bmatrix}(0.6, 0.1),\\(0.52, 0.38)\end{Bmatrix}$	$\begin{Bmatrix}(0.37, 0.25),\\(0.21, 0.55)\end{Bmatrix}$	$\begin{Bmatrix}(0.16, 0.26),\\(0.60, 0.25)\end{Bmatrix}$	$\begin{Bmatrix}(0.18, 0.46),\\(0.21, 0.31)\end{Bmatrix}$	$\begin{Bmatrix}(0.19, 0.14),\\(0.15, 0.18)\end{Bmatrix}$
Stomach pain	$\begin{Bmatrix}(0.54, 0.45),\\(0.31, 0.18)\end{Bmatrix}$	$\begin{Bmatrix}(0.15, 0.27),\\(0.16, 0.47)\end{Bmatrix}$	$\begin{Bmatrix}(0.50, 0.12),\\(0.13, 0.16)\end{Bmatrix}$	$\begin{Bmatrix}(0.41, 0.51),\\(0.19, 0.13)\end{Bmatrix}$	$\begin{Bmatrix}(0.17, 0.17),\\(0.16, 0.36)\end{Bmatrix}$
Cough	$\begin{Bmatrix}(0.24, 0.35),\\(0.37, 0.38)\end{Bmatrix}$	$\begin{Bmatrix}(0.18, 0.38),\\(0.19, 0.39)\end{Bmatrix}$	$\begin{Bmatrix}(0.14, 0.15),\\(0.20, 0.29)\end{Bmatrix}$	$\begin{Bmatrix}(0.16, 0.13),\\(0.15, 0.17)\end{Bmatrix}$	$\begin{Bmatrix}(0.19, 0.41),\\(0.31, 0.30)\end{Bmatrix}$
Chest pain	$\begin{Bmatrix}(0.13, 0.14),\\(0.12, 0.15)\end{Bmatrix}$	$\begin{Bmatrix}(0.36, 0.47),\\(0.17, 0.36)\end{Bmatrix}$	$\begin{Bmatrix}(0.23, 0.39),\\(0.29, 0.51)\end{Bmatrix}$	$\begin{Bmatrix}(0.37, 0.47),\\(0.29, 0.41)\end{Bmatrix}$	$\begin{Bmatrix}(0.13, 0.34),\\(0.21, 0.53)\end{Bmatrix}$

TABLE 11. Intuitionistic fuzzy rough relation.

$T^{\sim} = R_e o {R_e}^*$	*Fever*	*Malaria*	*Typhoid*	*Stomach issues*	*Heart issues*
Ali	$\begin{Bmatrix}(0.54, 0.1),\\(0.31, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.3, 0.27),\\(0.16, 0.39)\end{Bmatrix}$	$\begin{Bmatrix}(0.5, 0.15),\\(0.19, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.41, 0.14),\\(0.2, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.26, 0.14),\\(0.2, 0.35)\end{Bmatrix}$
Jabir	$\begin{Bmatrix}(0.31, 0.24),\\(0.32, 0.15)\end{Bmatrix}$	$\begin{Bmatrix}(0.47, 0.21),\\(0.2, 0.36)\end{Bmatrix}$	$\begin{Bmatrix}(0.31, 0.24),\\(0.29, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.6, 0.1),\\(0.29, 0.3)\end{Bmatrix}$	$\begin{Bmatrix}(0.26, 0.24),\\(0.31, 0.35)\end{Bmatrix}$
Ubaid	$\begin{Bmatrix}(0.5, 0.2),\\(0.26, 0.37)\end{Bmatrix}$	$\begin{Bmatrix}(0.4, 0.38),\\(0.17, 0.37)\end{Bmatrix}$	$\begin{Bmatrix}(0.5, 0.26),\\(0.28, 0.29)\end{Bmatrix}$	$\begin{Bmatrix}(0.4, 0.34),\\(0.28, 0.17)\end{Bmatrix}$	$\begin{Bmatrix}(0.26, 0.2),\\(0.21, 0.3)\end{Bmatrix}$
Umar	$\begin{Bmatrix}(0.31, 0.3),\\(0.29, 0.25)\end{Bmatrix}$	$\begin{Bmatrix}(0.4, 0.38),\\(0.2, 0.36)\end{Bmatrix}$	$\begin{Bmatrix}(0.23, 0.15),\\(0.21, 0.25)\end{Bmatrix}$	$\begin{Bmatrix}(0.4, 0.13),\\(0.22, 0.19)\end{Bmatrix}$	$\begin{Bmatrix}(0.26, 0.3),\\(0.22, 0.2)\end{Bmatrix}$

more space to analyze more complex data. So, based on these observations, we can say that this developed algorithm is stronger, more efficient, and can help in many medical diagnosis problems.

TABLE 12. Intuitionistic fuzzy rough relation.

$R_e{}^*$	*Fever*	*Malaria*	*Typhoid*	*Stomach issues*	*Hear issues*
Ali	0.6125	0.45	0.56	0.5425	0.4925
Jabir	0.56	0.525	0.515	0.6225	0.495
Ubaid	0.5475	0.455	0.5575	0.5425	0.4925
Umar	0.5125	0.465	0.51	0.575	0.495

TABLE 13. Comparative analysis.

Methods	Score values	Ranking result
IFWA operators [13]	×	No result
IFWG operators [12]	×	No result
IFRWA operators [25]	×	No result
IFRWG operators [25]	×	No result
IFDWA [16]	×	No result
IFDWG [16]	×	No result
CIFRWA operators (Proposed)	$Sc(\mathbb{G}_1^*) = 0.2613$, $Sc(\mathbb{G}_2^*) = 0.3163$, $Sc(\mathbb{G}_3^*) = 0.3483$, $Sc(\mathbb{G}_4^*) = 0.3286$	$\mathbb{G}_3^* \geq \mathbb{G}_4^* \geq \mathbb{G}_2^* \geq \mathbb{G}_1^*$
CIFRWG operators (Proposed)	$Sc(\mathbb{G}_1^*) = 0.7454$, $Sc(\mathbb{G}_2^*) = 0.7106$, $Sc(\mathbb{G}_3^*) = 0.7162$ $Sc(\mathbb{G}_4^*) = 0.6821$	$\mathbb{G}_1^* \geq \mathbb{G}_3^* \geq \mathbb{G}_2^* \geq \mathbb{G}_4^*$

VII. COMPARATIVE ANALYSIS

In this part of the article, we study the comparative analysis of our initiated work along with some existing methods to show the superiority and reliability of our initiated work. Here, we will compare our work with IFWA [13], IFWG [12], IFDWA [16], IFDWG [16], IFRWA [25], and IFRWG [25] operators. The overall analysis of the comparative study is given in Table 13. From the analysis, we can observe that

1. The IFWA [13], IFWG [12], IFDWA [16], IFDWG [16], IFRWA [25], and IFRWG [25] operators can only deal with IF data, and these notions lack the extra characteristic of handling rough information. While our initiated work has the advantages of using rough information in their stricture.
2. Moreover, we can see that established work can provide more space to decision-makers in the form of an IFR structure. While existing notions cannot do so. So our initiated work is more general.

VIII. CONCLUSION

In our daily life, we have to deal with complicated and advanced data. For this type of data, we need to make such types of methods and tools that can ease our work and compute overall information. As aggregation operates are fundamental tools to convert the overall information into a single value. So, here in this article, we establish some new aggregation operators (AOs) called confidence level intuitionistic fuzzy rough (CLIFR) weighted average, CLIFR ordered weighted average, CLIFR weighted geometric, and CLIFR ordered weighted geometric AOs. Furthermore, the properties of these operators have been discussed in detail. We also initiate an MCDM algorithm based on the notion of our proposed work. Moreover, an illustrative example shows the effective use of these initiated notions in daily life, such as medical diagnosis. Also, we have given a comparative analysis of our proposed work with some existing methods to show the reliability of the established work.

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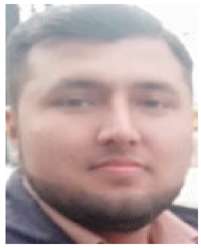

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