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RESEARCH ARTICLE

Patched Network and Its Vertex-Edge Metric-Based Dimension

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ABSTRACT The p-type networks are designed with the help of CVNET at topo group Cluj and also given support by nano studio. Such networks develop new p-type surfaces and also represent the decorations of the surfaces. This patched network is designed by two repeated units. The first one is triphenylene having a Z-pen formula and the second one is triphenylene with A-phe. Furthermore, these decorations are acquired as the result of map operations represented in the CVNET software, while its assembling is conducted with the help of the nano studio program. In the literature, its topology is discussed by Omega polynomials which is an applied graph theory topic. Another most applied topic of graph theory is known as the resolvability parameter. So this article studied the resolvability parameters of patched networks, such as metric dimension, and edge metric dimension. These parameters are defined as a resolving set is a subset of vertices of a graph with a condition that each vertex of that graph has a unique code or representation with respect to the chosen subset. Its minimum cardinality is known as metric dimension, while the edge metric dimension is defined by the minimum count of members in the edge resolving set and this set is defined as according to a chosen subset each edge of a graph has unique representations, then this set is known as edge resolving set. A resolving set is a subset of vertices of a graph with a condition that each edge of that graph has a unique code or representation with respect to the chosen subset. It is minimum cardinality is known as the edge metric dimension.

INDEX TERMS Patched network, p-type network, metric dimension, edge metric dimension, resolving set, chemical graph.

I. INTRODUCTION

Graph theory has a branch which is chemical graph theory, this branch deals with mathematics and chemistry. Different chemical structures are studied under this branch from the perspective of graphs. It is not easy to study complex and huge chemical structures in their original shape. Chemical graph theory is used to understand such complex structures. The atoms and bonds between the atoms of a chemical structure become vertices and edges in the graph, respectively, This implementation is known as a molecular graph. [1].

The physical properties of a chemical structure are represented in a unique mathematical form. In the structure, each atom or vertex has its unique position or identification. In order to provide a unique identification for the complete vertex set, some atoms or vertices are picked, allowing a group of atoms to have a specific position on those vertices. This idea is known as metric basis in graph theory [2] and resolving or locating set in applied graph theory [3].

The computational complexity and NP-hardness of the resolvability parameters are explored in [4], [5], [6], and [7]. Different practical applications of daily life are in the metric dimension, researchers are motivated by this and studied widely. Metric dimension is used in a variety of scientific fields, including facility location problems, sonar, and coast guard Loran [2], computer networks [8], combinatorial optimization [9], robot navigation [10], pharmaceutical chemistry [5], weighing problems [11], and image

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processing [12], [13]. In various real-world applications partition dimension also has many applications such as strategies, coding, and decoding of master mind games [14],the relation of Djokovic-Winkler [15], and to discover and verify the network [16], to explore more, we suggest to see [17], [18], [19], [20], [43]

Due to the numerous uses of metric dimensions in chemistry, this characteristic was investigated for a variety of chemical structures.metric dimensions of H-naphtalenic and nanotubes VC_5C_7 are discussed in [21], cellulose networks' upper bounds of metric dimension discussed in [22], silicate star's resolving sets are discussed in [23], the 2D lattice of alpha-boron nanotubes' metric basis are discussed in [24]. Resolving partition set and partition dimension also appeared in the literature, as in [25]. Constant metric dimension of some convex strcutures are found in [44] and [45]. The partition dimension $n-3$ of graphs, the partition dimension of the(4, 6) fullerene and it has bounded partition dimension is proved in [26]. For the latest literature and results one can see [27], [28], [29]. The barycentric split of the edge metric dimension in the Cayley graph is discussed in [30]. Some chemical compounds linked to wheel graphs, and also a comparison of metric and edge metric dimensions is discussed in [31]. Adding more to it, a few interesting studies related to metric dimension can be found in [32], [33], [34], [35], and [36]. More related topics are discussed in [37], [38], [39], [40], and [41].

Basic concepts used in this research are given below.

Definition 1: The collection of vertices $V(G)$ and the set of edges $E(G)$ make up the graph $G(V, E)$. A graph G is a simple, undirected graph. The shortest path between two vertices (a, b) is symbolized by $d(a, b)$, wjich is the minimum count of edges between (a, b).

Definition 2: The distance between two primary nodes $v_1, v_2 \in V(G)$, abbreviated as $d(v_1, v_2)$, is the least number of edges between the v_1, v_2 route. Assume $R \subset V(G)$ is the subset of principal nodes defined by $R = \{v_1, v_2, \ldots, v_s\}$, and consider a principal node $v \in V(G)$. A primary node's identification or position $r(v|R)$ with regard to R is really a distances $(d(v, v_1), d(v, v_2), \ldots, d(v, v_s))$. If each primary node in $V(G)$ has a unique identity according to the ordered subset R, then this subset is termed a network resolving set. The metric dimension of is really the minimal number of elements in the subset R, which is indicated by the word dim (G).

Definition 3: The distance between an edge $e = v_1v_2 \in E(G)$, and a node $v \in V(G)$ is counted by the relation $d(e, v) = min\{d(v_1, v), d(v_2, v)\}$. Taking a subset R_e of $V(G)$, if R_e has a unique representation for all edges belonging to $E(G)$, then R_e is a resolving set. The edge metric dimension is the minimum number of members of R_e, denoted by $dim_e(G)$.

II. CONSTRUCTION OF THE PATCHED NETWORK

The network shown in the Figure 2, is a patched network containing pentagons, hexagons, and octagons. By assuming some transformations we translated this particular patched network into a graph. We use $PN_{m,n}$ notation to represent the graph of the patched network. In this structure, there are 12 pentagons, 4 hexagons, and 4 octagons for the particular

$$
\begin{aligned}
V\left(PN_{m,n}\right) &= \{a_{i,j}^{l,k} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ l = 1, 2, \ldots, 5,\ k = 1, 2, \ldots, 12\}.\\
E\left(PN_{m,n}\right) &= \{a_{i,j}^{l,k} a_{i,j}^{l+1,k} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ l = 1, 2, \ldots, 5,\ k = 1, 2, \ldots, 12\}\\
&\cup \{a_{i,j}^{l,k} a_{i,j}^{l-4,k} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ l = 5,\ k = 1, 2, \ldots, 12\}\\
&\cup \{a_{i,j}^{l,k} a_{i,j}^{l-3,k+2} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 4,\ \text{then } k = 1, 2,\\
&\quad \text{if } l = 5,\ \text{then } k = 4, 5\} \cup \{a_{i,j}^{l,k} a_{i,j}^{l-2,k+1} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\\
&\quad \text{if } l = 3,\ \text{then } k = 1, 11,\ \text{if } l = 4,\ \text{then } k = 4,\ \text{if } l = 5,\ \text{then } k = 7\}\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+3,k+2} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 1,\ \text{then } k = 7,\\
&\quad \text{if } l = 2,\ \text{then } k = 10\} \cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+3,k+1} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\\
&\quad \text{if } l = 2,\ \text{then } k = 2,\ \text{if } l = 1,\ \text{then } k = 10\}\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l-3,k+10} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 5,\ \text{then } k = 1\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l-1,k+3} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 4,\ \text{then } k = 6\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+2,k-3} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 1,\ \text{then } k = 3\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l-3,k+9} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 5,\ \text{then } k = 3\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+2,k+2} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 1,\ \text{then } k = 8\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+3,k+3} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 1,\ \text{then } k = 9\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l-3,k+1} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 5,\ \text{then } k = 2\},\\
&\cup \{a_{i,j}^{l,k} {a_{i,j}}^{l+2,k+1} : i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,\ \text{if } l = 1,\ \text{then } k = 5\}.
\end{aligned}
$$

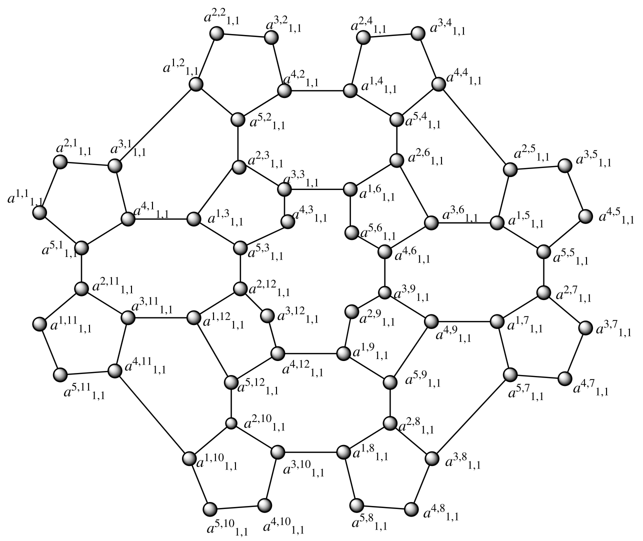

FIGURE 1. Patched network with $m = n = 1$ or $PN_{1,1}$.

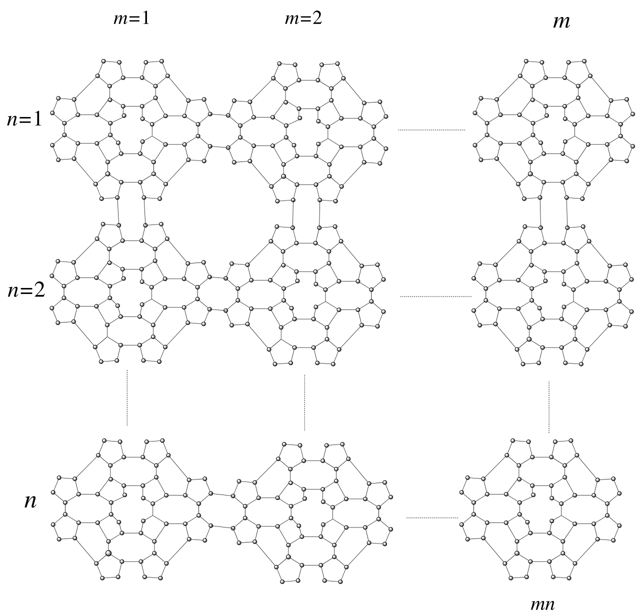

FIGURE 2. Generalized patched network $PN_{m,n}$ with $m, n \geq 1$.

value of $m = 1 = n$, and this is shown in the Figure 1. Now, in general, $12mn$ numbers of pentagons, $4mn$ numbers of hexagons, and $4mn$ numbers of octagons in $PN_{m,n}$. In total there are $|V(PN_{m,n})| = 60mn$ of vertices or atoms and $|V(PN_{m,n})| = 84mn - 2(m - n)$ count of edges or relation between bonds. Further graphical detail of this strcuture found in [42]. For the use of in our result, we apply the labeling on vertices shown in Figure 1. Moreover, the vertex and edge sets are described as shown in the equation at the bottom of the previous page.

This network is also known as P-type network.

Theorem 1: Let $PN_{m,n}$ be a patched network, with $m = n = 1$. Then $dim(PN_{m,n}) = 3$.

Proof: As we know by definition of resolving set, that few vertices are chosen to get whole network into a unique form. To prove that $dim(PN_{m,n}) \leq 3$. Assume a resolving set with cardinality three. Let $R = \{a^{1,1}_{1,1}, a^{3,2}_{1,1}, a^{3,7}_{1,1}\}$. Now, to fulfill the requirement of resolving set. Lets check the code representation of each vertex of $V(PN_{m,n})$.

$$r(a^{l,1}_{1,1}|R) = \begin{cases} (-1+\zeta, 6-l, 11+l), & \text{if } l = 1, \\ (-1+l, 6-l, 14-l), & \text{if } l = 2, 3, \\ (-3+l, l, 14-l), & \text{if } l = 4, \\ (-3+l, l, 3+l), & \text{if } l = 5. \end{cases}$$

There are twelve pentagons in our particular network $(PN_{1,1})$. We can see the representation of first pentagon given above, and there are unique, also fulfilling the definition of resolving set. So we can say that the chosen subset $R = \{a^{1,1}_{1,1}, a^{3,2}_{1,1}, a^{3,7}_{1,1}\}$ is a possible nominee for the resolving set with cardinality three.

Similarly, we will check the remaining pentagonal vertices representation.

$$r(a^{l,2}_{1,1}|R) = \begin{cases} (l+2, 3-l, l+9), & \text{if } l = 1, \\ (l+2, 3-l, 12-l), & \text{if } l = 2, 3, \\ (l, -3+l, 12-l), & \text{if } l = 4, \\ (l, -3+l, 4+l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,3}_{1,1}|R) = \begin{cases} (2+l, 3+l, 10-l), & \text{if } l = 1, \\ (2+l, 1+l, 10-l), & \text{if } l = 2, 3, \\ (l, 1+l, 4+l), & \text{if } l = 4, \\ (l, l, 4+l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,4}_{1,1}|R) = \begin{cases} (5+l, 1+l, 6+l), & \text{if } l = 1, \\ (5+l, 1+l, 9-l), & \text{if } l = 2, 3, \\ (3+l, -1+l, 9-l), & \text{if } l = 4, \\ (3+l, -1+l, 1+l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,5}_{1,1}|R) = \begin{cases} (8+l, 5+l, 2+l), & \text{if } l = 1, \\ (8+l, 3+l, 2+l), & \text{if } l = 2, \\ (8+l, 3+l, 7-l), & \text{if } l = 3, \\ (6+l, 3+l, 7-l), & \text{if } l = 4, \\ (6+l, 2+l, 7-l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,6}_{1,1}|R) = \begin{cases} (5+l, 4+l, 7-l), & \text{if } l = 1, \\ (5+l, 2+l, 7-l), & \text{if } l = 2, 3, \\ (3+l, 2+l, 1+l), & \text{if } l = 4, \\ (3+l, 1+l, 1+l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,7}_{1,1}|R) = \begin{cases} (l+9, l+8, l+9), & \text{if } l = 1, \\ (l+9, l+6, l+9), & \text{if } l = 2, 3, \\ (l+7, l+6, 16-l), & \text{if } l = 4, \\ (l+7, l+5, 16-l), & \text{if } l = 5. \end{cases}$$

$$r(a^{l,8}_{1,1}|R) = \begin{cases} (l+7, 12-l, 6-l), & \text{if } l = 1, 2, \\ (l+7, l+8, 8-l), & \text{if } l = 3, \\ (l+5, l+8, 8-l), & \text{if } l = 4, \\ (l+5, l+7, 8-l), & \text{if } l = 5. \end{cases}$$

$$r(a_{1,1}^{l,9}|R) = \begin{cases} (l+7, 10-l, l+4), & \text{if } l=1, \\ (l+7, 10-l, 7-l), & \text{if } l=2,3, \\ (l+5, l+4, 7-l), & \text{if } l=4, \\ (l+5, l+4, -1+l), & \text{if } l=5. \end{cases}$$

$$r(a_{1,1}^{l,10}|R) = \begin{cases} (4+l, 11-l, 9-l), & \text{if } l=1,2, \\ (4+l, 7+l, 9-l), & \text{if } l=3, \\ (2+l, 7+l, 3+l), & \text{if } l=4, \\ (2+l, 6+l, 3+l), & \text{if } l=5. \end{cases}$$

$$r(a_{1,1}^{l,11}|R) = \begin{cases} (l+1, 8-l, 12-l), & \text{if } l=1,2, \\ (l+1, l+4, 12-l), & \text{if } l=3, \\ (-1+l, l+4, l+6), & \text{if } l=4, \\ (-1+l, l+3, l+6), & \text{if } l=5. \end{cases}$$

$$r(a_{1,1}^{l,12}|R) = \begin{cases} (3+l, 8-l, 7+l), & \text{if } l=1, \\ (3+l, 8-l, 10-l), & \text{if } l=2, \\ (3+l, 4+l, 10-l), & \text{if } l=3, \\ (1+l, 4+l, 10-l), & \text{if } l=4, \\ (1+l, 3+l, 2+l), & \text{if } l=5. \end{cases}$$

All the pentagons have unique representation and there are no two vertices with same representation. So we can conclude that the chosen subset is an actual possible nominee for the resolving set with three cardinality.

This implies that

$$dim\left(PN_{m,n}\right) \leq 3. \tag{1}$$

Now, let $dim\left(PN_{m,n}\right) \geq 3$, on contrary $dim\left(PN_{m,n}\right) = 2$. To support this statement, there are few choices of resolving sets with two cardinality.

Case 1: Let $R' \subset \{a_{1,1}^{l,k} : 1 \leq l \leq 5, k = 1, 2, 4, 6, 12\}$ is a possible nominee for a two cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{3,k}|R') = r(a_{1,1}^{4,k}|R')$

Case 2: Let $R' \subset \{a_{1,1}^{l,k} : 1 \leq l \leq 5, k = 5, 8, 10, 11\}$ is a possible nominee for a two cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,k}|R') = r(a_{1,1}^{3,k}|R')$

Case 3: Let $R' \subset \{a_{1,1}^{l,k} : 1 \leq l \leq 5, k = 3, 6\}$ is a possible nominee for a two cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{1,k}|R') = r(a_{1,1}^{2,k}|R')$

Case 4: Let $R' \subset \{a_{1,1}^{l,k} : 1 \leq l \leq 5, k = 7, 9\}$ is a possible nominee for a two cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{4,k}|R') = r(a_{1,1}^{5,k}|R')$

Case 5: Let $R' \subset \{a_{1,1}^{l,k} : 1 \leq l \leq 5, k = 2, 8, 9, 10\}$ is a possible nominee for a two cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{1,k}|R') = r(a_{1,1}^{5,k}|R')$

We can see from above discussion, that there is no subset with two cardinaliy for resolving set possible nominee. So we can conclude that

$$dim\left(PN_{m,n}\right) \geq 3. \tag{2}$$

Now, by inequalities 1 and 2 we can say that $dim\left(PN_{m,n}\right) = 3$. ☐

Theorem 2: Let $PN_{m,n}$ be a patched network with $m, n \geq 1$. Then $dim\left(PN_{m,n}\right) = mn + 2$.

Proof: To show that $dim\left(PN_{m,n}\right) = mn+2$, we will use the induction method on m and n which are the horizontal and vertical copies of base graph of patched network. The initial case is proved for $m = 1 = n$ in the Theorem 1, and concluded that $dim\left(PN_{1,1}\right) = 3$. Now, we will show that it is true for $m = \kappa, n = 1 : dim\left(PN_{\kappa,1}\right) = \kappa + 2 \rightarrow$ (i). We will show that its true for $m = \kappa + 1, n = 1$. As $dim\left(PN_{\kappa+1,1}\right) = dim\left(PN_{\kappa,1}\right) + 1 \rightarrow$ (ii). Now putting (i) and (ii) in $dim\left(PN_{\kappa+1,1}\right) = \kappa + 2 + 1 \rightarrow$ (iii). This implies that $dim\left(PN_{\kappa+1,1}\right) = \kappa+3$. Similarly we will get, for $m = 1$, and $n = \kappa$. This will be resulted in $dim\left(PN_{1,\kappa+1}\right) = \kappa+3$. So the final result will be concluded as $dim\left(PN_{m,n}\right) = mn + 2$, which is true for all the positive values of m and n. ☐

Theorem 3: Let $PN_{m,n}$be a patched network, with $m = n = 1$. Then $dim_e\left(PN_{m,n}\right) = 4$.

Proof: As we know by definition of resolving set, that few vertices are chosen to get whole network into a unique form. To prove that $dim_e\left(PN_{m,n}\right) \leq 4$. Assume a resolving set with cardinality four. Let $R = \{a_{1,1}^{1,1}, a_{1,1}^{2,2}, a_{1,1}^{3,7}, a_{1,1}^{4,8}\}$. Now, to fulfill the requirement of resolving set. Lets check the code representation of each edge of $E\left(PN_{m,n}\right)$.

There are twelve pentagons in our particular network $\left(PN_{1,1}\right)$. We can see the representation of the first pentagon given above, and they are unique, also fulfilling the definition of resolving set. So we can say that the chosen subset $R = \{a_{1,1}^{1,1}, a_{1,1}^{2,2}, a_{1,1}^{3,7}, a_{1,1}^{4,8}\}$ is a possible nominee for the resolving set with cardinality four.

Similarly, we will check the remaining pentagonal edge representation.

All the edges have unique representation and there are no three edges with same representation. So we can

$$r(a_{1,1}^{l_1,1}a_{1,1}^{l_2,1}|R_e) = \begin{cases} (-1+l_1, 2+l_1, 10+l_1, 8+l_1), & \text{if } l_1=1, l_2=1+l_1, \\ (-1+l_1, l_1, 9+l_1, 8+l_1), & \text{if } l_1=2, l_2=1+l_1, \\ (-1+l_1, -1+l_1, 7+l_1, 8+l_1), & \text{if } l_1=3, l_2=1+l_1, \\ (-3+l_1, -1+l_1, 6+l_1, 5+l_1), & \text{if } l_1=4, l_2=1+l_1, \\ (-5+l_1, -1+l_1, 7+l_1, 3+l_1), & \text{if } l_1=5, l_2=-4+l_1. \end{cases}$$

conclude that the chosen subset is an actual possible nominee for the resolving set with four cardinality. This implies that $dim_e\left(PN_{m,n}\right) \leq 4$. Now, let $dim_e\left(PN_{m,n}\right) \geq 4$, on contrary $dim_e\left(PN_{m,n}\right) = 3$. To support this statement, there are few choices of resolving sets with three cardinality.

$$r(a_{1,1}^{l_1,2}a_{1,1}^{l_2,2}|R_e) = \begin{cases} (2+l_1, -1+l_1, 9+l_1, 11+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (2+l_1, -2+l_1, 7+l_1, 10+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (2+l_1, -2+l_1, 5+l_1, 8+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (l_1, -2+l_1, 4+l_1, 7+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (-2+l_1, -4+l_1, 4+l_1, 7+l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,3}a_{1,1}^{l_2,3}|R_e) = \begin{cases} (3+l_1, 2+l_1, 7+l_1, 8+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (3+l_1, 1+l_1, 5+l_1, 8+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (2+l_1, 2+l_1, 4+l_1, 6+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (l_1, 1+l_1, 4+l_1, 4+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (-2+l_1, -1+l_1, 4+l_1, 3+l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,4}a_{1,1}^{l_2,4}|R_e) = \begin{cases} (6+l_1, 2+l_1, 6+l_1, 9+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (6+l_1, 2+l_1, 4+l_1, 7+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (6+l_1, 2+l_1, 2+l_1, 5+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (4+l_1, l_1, l_1, 4+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (2+l_1, -2+l_1, 1+l_1, 4+l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,5}a_{1,1}^{l_2,5}|R_e) = \begin{cases} (9+l_1, 5+l_1, 2+l_1, 5+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (8+l_1, 4+l_1, 2+l_1, 5+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (8+l_1, 4+l_1, l_1, 3+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (8+l_1, 4+l_1, -2+l_1, 1+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (5+l_1, 2+l_1, -3+l_1, l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,6}a_{1,1}^{l_2,6}|R_e) = = \begin{cases} (5+l_1, 4+l_1, 4+l_1, 7+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (6+l_1, 3+l_1, 2+l_1, 5+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (5+l_1, 3+l_1, 1+l_1, 3+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (3+l_1, 2+l_1, 1+l_1, 2+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (1+l_1, l_1, 1+l_1, 2+l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,7}a_{1,1}^{l_2,7}|R_e) = \begin{cases} (10+l_1, 8+l_1, l_1, 2+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (10+l_1, 7+l_1, -2+l_1, 2+l_1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (8+l_1, 5+l_1, -3+l_1, l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (8+l_1, 7+l_1, -3+l_1, -2+l_1), & \text{if } l_1 = 4, l_2 = l_1+1, \\ (6+l_1, 5+l_1, -4+l_1, -3+l_1), & \text{if } l_1 = 5, l_2 = -4+l_1. \end{cases}$$

$$r(a_{1,1}^{l_1,8}a_{1,1}^{l_2,8}|R_e) = \begin{cases} (7+l_1, 10+l_1, 3+l_1, l_1+1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (7+l_1, 9+l_1, 1+l_1, l_1-1), & \text{if } l_1 = 2, l_2 = 1+l_1, \\ (7+l_1, 9+l_1, l_1, -3+l_1), & \text{if } l_1 = 3, l_2 = 1+l_1, \\ (5+l_1, 8+l_1, l_1, -4+l_1), & \text{if } l_1 = 4, l_2 = 1+l_1, \\ (l_1+3, l_1+6, l_1, l_1-4), & \text{if } l_1 = 5, l_2 = l_1-4. \end{cases}$$

$$r(a_{1,1}^{l_1,9}a_{1,1}^{l_2,9}|R_e) = \begin{cases} (6+l_1, 8+l_1, 4+l_1, 3+l_1), & \text{if } l_1 = 1, l_2 = 1+l_1, \\ (l_1+6, l_1+6, l_1+2, l_1+3), & \text{if } l_1 = 2, l_2 = l_1+1, \\ (l_1+6, l_1+5, l_1, l_1+1), & \text{if } l_1 = 3, l_2 = l_1+1, \\ (l_1+4, l_1+5, l_1-1, l_1-1), & \text{if } l_1 = 4, l_2 = l_1+1, \\ (l_1+2, l_1+5, l_1-1, l_1-2), & \text{if } l_1 = 5, l_2 = l_1-4. \end{cases}$$

Case 1: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 2, l_3 = 3, k_1 = k_2 = k_3 = 1\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,7}a_{1,1}^{3,7}|R') = r(a_{1,1}^{3,7}a_{1,1}^{4,7}|R')$

Case 2: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 4, l_3 = 5, k_1 = k_2 = k_3 = 4\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{3,10}a_{1,1}^{4,10}|R') = r(a_{1,1}^{4,10}a_{1,1}^{5,10}|R')$

Case 3: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 4, l_2 = 1, l_3 = 5, k_1 = 2, k_2 = k_3 = 4\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,10}a_{1,1}^{3,10}|R') = r(a_{1,1}^{1,8}a_{1,1}^{2,8}|R')$

Case 4: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 4, l_3 = 1, k_1 = k_2 = 3, k_3 = 11\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,7}a_{1,1}^{3,7}|R') = r(a_{1,1}^{3,7}a_{1,1}^{4,7}|R')$

Case 5: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 2, l_2 = 3, l_3 = 4, k_1 = k_2 = k_3 = 2\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{3,8}a_{1,1}^{4,8}|R') = r(a_{1,1}^{4,8}a_{1,1}^{5,8}|R')$

Case 6: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 2, l_3 = 3, k_1 = k_2 = k_3 = 5\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,8}a_{1,1}^{3,8}|R') = r(a_{1,1}^{3,8}a_{1,1}^{4,8}|R')$

Case 7: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 2, l_3 = 3, k_1 = k_2 = k_3 = 3\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{1,12}a_{1,1}^{2,12}|R') = r(a_{1,1}^{2,12}a_{1,1}^{3,12}|R')$

Case 8: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 2, l_2 = 3, l_3 = 4, k_1 = k_2 = k_3 = 12\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{1,3}a_{1,1}^{5,3}|R') = r(a_{1,1}^{5,3}a_{1,1}^{4,3}|R')$

$$r(a_{1,1}^{l_1,10}a_{1,1}^{l_2,10}|R_e) = \begin{cases} (l_1+4, l_1+8, l_1+6, l_1+3), & \text{if } l_1 = 1, l_2 = l_1+1, \\ (l_1+4, l_1+7, l_1+4, l_1+1), & \text{if } l_1 = 2, l_2 = l_1+1, \\ (l_1+4, l_1+7, l_1+3, l_1), & \text{if } l_1 = 3, l_2 = l_1+1, \\ (l_1+2, l_1+5, l_1+3, l_1), & \text{if } l_1 = 4, l_2 = l_1+1, \\ (l_1, l_1+3, l_1+3, l_1), & \text{if } l_1 = 5, l_2 = l_1-4. \end{cases}$$

$$r(a_{1,1}^{l_1,11}a_{1,1}^{l_2,11}|R_e) = \begin{cases} (1+l_1, 4+l_1, 10+l_1, l_1+7), & \text{if } l_1 = 1, l_2 = l_1+1, \\ (l_1, l_1+3, l_1+7, l_1+5), & \text{if } l_1 = 2, l_2 = l_1+1, \\ (l_1, l_1+3, l_1+6, l_1+3), & \text{if } l_1 = 3, l_2 = l_1+1, \\ (l_1, l_1+3, l_1+5, l_1+2), & \text{if } l_1 = 4, l_2 = l_1+1, \\ (l_1-2, l_1+1, l_1+5, l_1+2), & \text{if } l_1 = 5, l_2 = l_1-4. \end{cases}$$

$$r(a_{1,1}^{l_1,12}a_{1,1}^{l_2,12}|R_e) = \begin{cases} (l_1+3, l_1+5, l_1+8, l_1+5), & \text{if } l_1 = 1, l_2 = l_1+1, \\ (l_1+3, l_1+4, l_1+5, l_1+4), & \text{if } l_1 = 2, l_2 = l_1+1, \\ (l_1+3, l_1+4, l_1+3, l_1+2), & \text{if } l_1 = 3, l_2 = l_1+1. \end{cases}$$

$$r(a_{1,1}^{l_1,k_1}a_{1,1}^{l_2,k_2}|R_e) = \begin{cases} (l_1-1, l_1-2, l_1+7, l_1+8), & \text{if } l_1 = 3, l_2 = l_1-2, k_1 = 1, k_2 = 2, \\ (l_1-2, l_1-1, l_1+5, l_1+5), & \text{if } l_1 = 4, l_2 = l_1-3, k_1 = 1, k_2 = 3, \\ (l_1-4, l_1-2, l_1+6, l_1+3), & \text{if } l_1 = 5, l_2 = l_1-3, k_1 = 1, k_2 = 11, \\ (l_1+2, l_1-2, l_1+3, l_1+6), & \text{if } l_1 = 4, l_2 = l_1-3, k_1 = 2, k_2 = 4, \\ (l_1-1, l_1-3, l_1+3, l_1+5), & \text{if } l_1 = 5, l_2 = l_1-3, k_1 = 2, k_2 = 3, \\ (l_1+5, l_1+1, l_1, l_1+3), & \text{if } l_1 = 4, l_2 = l_1-2, k_1 = 4, k_2 = 5, \\ (l_1+3, l_1-1, l_1, l_1+3), & \text{if } l_1 = 5, l_2 = l_1+3, k_1 = 4, k_2 = 6, \\ (l_1+8, l_1+5, l_1+2, l_1+5), & \text{if } l_1 = 1, l_2 = l_1+1, k_1 = 5, k_2 = 6, \\ (l_1+6, l_1+3, l_1-4, l_1-1), & \text{if } l_1 = 5, l_2 = l_1-3, k_1 = 5, k_2 = 7, \\ (l_1+4, l_1+3, l_1, l_1+1), & \text{if } l_1 = 4, l_2 = l_1-1, k_1 = 6, k_2 = 9, \\ (l_1+8, l_1+8, l_1+2, l_1+2), & \text{if } l_1 = 1, l_2 = l_1+3, k_1 = 7, k_2 = 9, \\ (l_1+5, l_1+6, l_1-3, l_1-4), & \text{if } l_1 = 5, l_2 = l_1-2, k_1 = 7, k_2 = 8, \\ (l_1+6, l_1+8, l_1+2, l_1), & \text{if } l_1 = 2, l_2 = l_1+3, k_1 = 8, k_2 = 9, \\ (l_1+3, l_1+6, l_1+7, l_1+5), & \text{if } l_1 = 1, l_2 = l_1+3, k_1 = 10, k_2 = 11, \\ (l_1+3, l_1+6, l_1+5, l_1+5), & \text{if } l_1 = 2, l_2 = l_1+3, k_1 = 10, k_2 = 12, \\ (l_1, l_1+3, l_1+6, l_1+3), & \text{if } l_1 = 3, l_2 = l_1-2, k_1 = 11, k_2 = 12. \end{cases}$$

Case 9: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 2, l_3 = 3, k_1 = k_2 = k_3 = 6\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,9}a_{1,1}^{3,9}|R') = r(a_{1,1}^{3,9}a_{1,1}^{4,9}|R')$

Case 10: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 2, l_3 = 3, k_1 = k_2 = k_3 = 9\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{5,6}a_{1,1}^{4,6}|R') = r(a_{1,1}^{4,6}a_{1,1}^{3,6}|R')$

Case 11: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 2, l_2 = 3, l_3 = 4, k_1 = k_2 = k_3 = 10\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{1,6}a_{1,1}^{2,6}|R') = r(a_{1,1}^{2,6}a_{1,1}^{5,4}|R')$

Case 12: Let $R' = \{a_{1,1}^{l_1,k_1}, a_{1,1}^{l_2,k_2}, a_{1,1}^{l_3,k_3} : l_1 = 1, l_2 = 5, l_3 = 3, k_1 = k_2 = 8, k_3 = 10\}$ is a possible nominee for a three cardinality resolving set, but not fulfilling the criteria of definition and resulted in $r(a_{1,1}^{2,6}a_{1,1}^{3,6}|R') = r(a_{1,1}^{1,5}a_{1,1}^{3,6}|R')$

We can see from above discussion, that there is no subset with three cardinaliy for resolving set possible nominee. So we can conclude that $dim_e\left(PN_{m,n}\right) \geq 4$. Now, by equation we can say that $dim\left(PN_{m,n}\right) = 4$. □

Theorem 4: Let $PN_{m,n}$ be a patched network with $m, n \geq 2$. Then $dim_e\left(PN_{m,n}\right) = mn + 3$.

Proof: To show that $dim_e\left(PN_{m,n}\right) = mn + 3$, we will use the induction method on m and n which are the horizontal and vertical copies of base graph of patched network. The initial case is proved for $m = 1 = n$ in the Theorem 3, and concluded that $dim_e\left(PN_{1,1}\right) = 4$. Now, we will show that it is true for $m = \kappa, n = 1 : dim_e\left(PN_{\kappa,1}\right) = \kappa + 3 \rightarrow$ (i). We will show that its true for $m = \kappa + 1, n = 1$. As $dim_e\left(PN_{\kappa+1,1}\right) = dim_e\left(PN_{\kappa,1}\right) + 1 \rightarrow$ (ii). Now putting (i) and (ii) in $dim_e\left(PN_{\kappa+1,1}\right) = \kappa + 3 + 1 \rightarrow$ (iii). This implies that $dim_e\left(PN_{\kappa+1,1}\right) = \kappa + 4$. Similarly we will get, for $m = 1$, and $n = \kappa$. This will be resulted in $dim_e\left(PN_{1,\kappa+1}\right) = \kappa + 4$. So the final result will be concluded as $dim_e\left(PN_{m,n}\right) = mn + 3$, which is true for all the positive values of m and n. □

III. CONCLUSION

The p-type networks are created with the use of nano studio and CVNET at the topo group in Cluj. Such networks create new p-type surfaces and also serve to represent the surface embellishments. Two repeating units created this patched network. To put more lights on this network, we discussed this structure in the form of one of the most useful parameter of graph theory which is known as metric dimension. We also discussed the generalization of vertex-based metric dimension which is known as edge metric dimension for the patched network $PN_{m,n}$. Both parameters (metric and edge metric dimension) are highly depended on the order of graph and also varies by varying horizontal and vertical copies of base graph. We concluded that $dim\left(PN_{m,n}\right) = mn + 2$, and $dim_e\left(PN_{m,n}\right) = dim\left(PN_{m,n}\right) + 1$.

AUTHORS CONTRIBUTION

Muhammad Azeem conceived of the presented idea. Sidra Bukhari developed the theory and performed the computations. Muhammad Kamran Jamil verified the analytical methods, and Sensie Swaray investigated and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

DATA AVAILABILITY

All the data supporting the results are included in the manuscript.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest.

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