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RESEARCH ARTICLE

Graphical and Numerical Study of a Newly Developed Root-Finding Algorithm and Its Engineering Applications

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ABSTRACT The primary objective of this paper is to develop a new method for root-finding by combining forward and finite-difference techniques in order to provide an efficient, derivative-free algorithm with a lower processing cost per iteration. This will be accomplished by combining forward and finite-difference techniques. We also detail the convergence criterion that was devised for the root-finding approach, and we show that the method that was recommended is quintic-order convergent. We addressed a few engineering issues in order to illustrate the validity and application of the developed root-finding algorithm. The quantitative results justified the constructed root-finding algorithm's robust performance in comparison to other quintic-order methods that can be found in the literature. For the graphical analysis, we make use of the newly discovered method to plot some novel polynomiographs that are attractive to the eye, and then we evaluate these new plots in relation to previously established quintic-order root-finding strategies. The graphic analysis demonstrates that the newly created method for root-finding has better convergence with the larger area than the other comparable methods do.

INDEX TERMS Computational algorithms, convergence-order, non-linearity, Halley's scheme, dynamical aspects.

I. INTRODUCTION

The use of root-finding algorithms has become more important across a variety of modern scientific fields, particularly in the fields of computational and applied mathematics. In this day and age, a wide range of root-finding algorithms may be carried out with the assistance of a variety of computer tools, including SageMath, MATLAB, Mathcad, Maple, and Mathematica, amongst others. In recent years, mathematicians working in a variety of subfields of mathematics have increased their reliance on computers and the numerous types of software available for use on computers. The quest for the roots of the polynomial is the most important of them, as it has had a significant impact not just on applied mathematics and computational mathematics but also on a wide variety of other subfields within contemporary science. Equations that are not linear represent a number of the technical issues that must be addressed. Iterative algorithms are required to address problems of this technical kind because analytical approaches are typically unsuccessful in solving problems of this nature. Around the end of the 15th century, Newton [1] came up with the following formula, which is now seen as the basis of the well-known classical iterative algorithm:

$$x_{p+1} = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$

which is quadratic-order Newton's algorithm for root-finding of non-linear scalar equations.

Recent years have seen the development of new multi-step algorithms and the application of diverse mathematical methodologies that vastly improve upon the previous approaches. Using decomposition methods, the authors of [2]

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and [3] developed some new algorithms for root-finding of non-linear equations. The authors of [4] and [5] propose and study new three-step optimum methods for solving complex polynomials. A novel high-order and economical family of iterative approaches for nonlinear models was proposed by Behl et al. in [6]. The scalar approach established by Kou et al [7] forms the basis for the evolution of various methods. Multiple second derivative free root-finding methods for non-linear equations were developed by Naseem et al. [8], [9], [10] and then used for various problems in chemical and civil engineering.

In this part of the article, we will talk about several quintic-order multi-step algorithms that can be found in the research literature. These algorithms were developed by a variety of academics in order to locate approximate roots of nonlinear scalar equations. Using the finite difference approach, authors of [11] developed a method with the quintic-order that is second derivative-free. The method is listed below:

$$y_p = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$
$$x_{p+1} = y_p - \frac{2\xi(x_p)\xi(y_p)\xi'(y_p)}{2\xi(x_p)\xi'^2(y_p) - \xi'^2(x_p)\xi(y_p) + \xi'(x_p)\xi'(y_p)\xi(y_p)}.$$

Following that, in the year 2008, authors in [12] suggested a quintic-order Chebyshev-Halley type approach of the following form:

$$y_p = x_p - \frac{2\xi(x_p)\xi'(x_p)}{2\xi'^2(x_p) - \xi(x_p)\xi''(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$
$$x_{p+1} = x_p - \frac{2\xi'(x_p)[\xi(x_p) + \xi(y_p)]}{2\xi'^2(x_p) - \xi''(x_p)[\xi(x_p) + \xi(y_p)]}.$$

A three-step quintic-order iterative algorithm was proposed by Zhanlav et al. [13] in 2010. The algorithm's given form is as follows:

$$y_p = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$
$$w_p = y_p - \frac{\xi(y_p)}{\xi'(y_p)},$$
$$x_{p+1} = y_p - \frac{\xi(y_p) + \xi(w_p)}{\xi'(y_p)}.$$

Nazeer et al. [14] recently used a finite-difference scheme to present a new second derivative-free Householder's method that converges in quintic order.

$$y_p = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$
$$x_{p+1} = y_p - \frac{\xi(y_p)}{\xi'(y_p)}\left[1 - \frac{\xi'(y_p)\xi'(x_p)\xi(y_p) - \xi'^2(x_p)\xi(x_p)}{2\xi'^2(y_p)\xi(x_p)}\right].$$

In this research, we suggest a method that is the most effective for locating the roots of non-linear scalar equations. By combining the forward and finite-difference approaches, we are able to develop an iterative strategy that is based on Halley's algorithm. The primary purpose of utilizing these tactics is to produce an efficient root-finding method and to come up with an approximation for derivatives. We study the convergence condition of the recently proposed method and prove that it converges in quintic order. We solved some engineering problems to illustrate the robustness and validity of the constructed root-finding algorithm. The numeric results justify the robustness of the constructed root-finding algorithm in comparison to other quintic-order methods that are available in the literature. This was done to show that the constructed root-finding algorithm is valid. In order to examine the graphical behavior of polynomiographs, we employ a computer program to plot them on the complex plane using a variety of fifth-order processes.

The remaining parts of the paper are structured as described in the following paragraphs. In the second section II, we present an approach that has proven to be the most effective for locating the roots. In the third section III, the convergence criterion for the newly proposed method was discussed. In the fourth section IV, six engineering problems were successfully solved. The fifth section V includes a graphic analysis that analyses the technique that was described. The conclusion of the paper can be found in the very final section VI wherein some future directions for the improvement of the proposed algorithm are also discussed in detail.

II. MAIN RESULTS

Let $\xi : D \to R$, $D \subset R$ be a function in one variable. With the application of the Taylor's series expansion, the following root-finding algorithm was introduced by Edmond Halley:

$$x_{p+1} = x_p - \frac{2\xi(x_p)\xi'(x_p)}{2\xi'^2(x_p) - \xi(x_p)\xi''(x_p)}, \quad p = 0, 1, 2, 3, \ldots. \tag{1}$$

The aforementioned approach is the cubic-degree technique. The non-linear scalar equations may be solved using Halley's approach [15]. As a solution to the problem, the people who made the method in [11] changed it and came up with the following two-step iteration method:

$$y_p = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots, \tag{2}$$
$$x_{p+1} = y_p - \frac{2\xi(y_p)\xi'(y_p)}{2\xi'^2(y_p) - \xi(y_p)\xi''(y_p)}. \tag{3}$$

Calculating zeros of non-linear scalar equations can be done with the help of the approach described above, which is a two-step iteration strategy. By adding Newton's algorithm, the two-step process described before can become a three-step process, as shown in the following form:

$$y_p = x_p - \frac{\xi(x_p)}{\xi'(x_p)}, \quad p = 0, 1, 2, 3, \ldots,$$
$$w_p = y_p - \frac{\xi(y_p)}{\xi'(y_p)},$$
$$x_{p+1} = w_p - \frac{2\xi(w_p)\xi'(w_p)}{2\xi'^2(w_p) - \xi(w_p)\xi''(w_p)}. \tag{4}$$

Because it needs first-order and second-order derivatives in order to have its execution, the method described above has a significant computational overhead associated with each repetition. We estimate its derivatives and make it derivative-free in order to cut down on its processing costs and boost its efficiency. This makes it easy to use with nonlinear scalar functions where the first and higher derivatives are either infinite or do not exist. To estimate $\xi''(w)$, we use the finite-difference approach as follows:

$$\xi''(w_p) = \frac{\xi'(w_p) - \xi'(y_p)}{\xi(w_p) - \xi(y_p)} \tag{5}$$

Now, we apply forward-difference scheme [16], [17] to approximate $\xi'(x)$ as follows:

$$\xi'(x_p) = \frac{\xi(x_p + \xi(x_p)) - \xi(x_p)}{\xi(x_p)} = g(x_p). \tag{6}$$

To approximate $\xi'(y)$ and $\xi'(w)$, we employ the finite-difference scheme:

$$\xi'(y_p) = \frac{\xi(y_p) - \xi(x_p)}{y_p - x_p} = h(x_p, y_p). \tag{7}$$

$$\xi'(w_p) = \frac{\xi(w_p) - \xi(y_p)}{w_p - y_p} = j(y_p, w_p). \tag{8}$$

Using (7) and (8) in (5)

$$\xi''(w_p) = \frac{j(y_p, w_p) - h(x_p, y_p)}{\xi(w_p) - \xi(y_p)} = \kappa(x_p, y_p, w_p) \tag{9}$$

Using (7)–(9) in (4), one can easily obtain the following algorithm:

Algorithm 1: If an initial guess x_0 is provided, one can compute the approximate solution x_{p+1} with the help of following iteration procedures:

$$y_p = x_p - \frac{\xi(x_p)}{g(x_p)},$$
$$w_p = y_p - \frac{\xi(y_p)}{\xi'(y_p)},$$
$$x_{p+1} = y_p - \frac{2\xi(y_p)h(y_p)}{2h^2(y_p) - \xi(y_p)\kappa(x_p, y_p, w_p)}.$$

When it comes to estimating the zeros of nonlinear functions in a single variable, this ground-breaking method of obtaining the optimal root is the strategy to use. The application area handled by the presented method is an important aspect of its design. This is due to the fact that it can handle non-linear functions without the need for first and higher derivatives. Because of the substitutes for the first and second derivatives, the cost of calculating during each iteration is reduced, which leads to a higher efficiency index in comparison to other quintic-order iteration systems that are similar. Based on the results of the test-example simulations, the suggested method is better than similar methods that can be found in the literature.

III. CONVERGENCE ANALYSIS

It is common knowledge that an iterative method for resolving nonlinear equations is not dependable until and until it converges to a single solution. In the next part, we will conduct an abstract analysis in order to investigate the order of convergence of the suggested root-finding method.

Theorem 1: If we assume that β is the simple zero of $\xi(x) = 0$ and that $\xi(x)$ has enough smoothness in the vicinity of the exact zero (root) β, then the proposed approach (II) *has quintic-order convergence. In addition to this, it satisfies the error equation, which is as follows:*

$$e_{p+1} = Ae_p^5 + O(e_p^6),$$

where $A = 3(\frac{\xi''(\beta)}{2\xi'(\beta)})^4$ and e_p represents the error at pth iteration.

Proof: To obtain the proof of the above theorem, assume e_p be an error at pth iteration, it yields to $e_p = x_p - \beta$ and the Taylor's expansion about $x = \beta$ gives the following:

$$\xi(x_p) = \xi'(\beta)e_p + \frac{1}{2!}\xi''(\beta)e_p^2 + \frac{1}{3!}\xi'''(\beta)e_p^3 + \frac{1}{4!}\xi^{(iv)}(\beta)e_p^4 + O(e_p^5),$$
$$\xi(x_p) = \xi'(\beta)[e_p + c_2e_p^2 + c_3e_p^3 + c_4e_p^4 + c_5e_p^5 + O(e_p^6)], \tag{10}$$
$$g(x_p) = \xi'(\beta)[1 + 3c_2e_p + (7c_3 + c_2^2)e_p^2 + (6c_2c_3 + 15c_4)e_p^3 + (18c_2c_4 + 31c_5 + c_3c_2^2 + 5c_3^2)e_p^4 + (50c_5c_2 + 63c_6 + 2c_2c_3^2 + 22c_3c_4 + 7c_2^2c_4)e_p^5 + O(e_p^6)], \tag{11}$$

where

$$c_p = \frac{1}{p!}\frac{\xi^{(q)}(\beta)}{\xi'(\beta)}.$$

With the help of Eqs.(10) and (11), we obtain:

$$y_p = \xi'(\beta)[\beta + 2c_2e_p^2 + (6c_3 - 5c_2^2)e_p^3 + (-26c_2c_3 + 13c_2^3 + 14c_4)e_p^4 + (96c_3c_2^2 - 54c_2c_4 + 30c_5 - 37c_3^2 - 34c_2^4)e_p^5 + (96c_3c_2^2 - 54c_2c_4 + 30c_5 - 37c_3^2 - 34c_2^4)e_p^5 + O(e_p^6)], \tag{12}$$
$$\xi(y_p) = \xi'(\beta)[2c_2e_p^2 + (6c_3 - 5c_2^2)e_p^3 + (17c_2^3 - 26c_2c_3 + 14c_4)e_p^4 + (96c_3c_2^2 - 54c_2c_4 + 30c_5 - 37c_3^2 - 34c_2^4)e_p^5 + (96c_3c_2^2 - 54c_2c_4 + 30c_5 - 37c_3^2 - 34c_2^4)e_p^5 + O(e_p^6)], \tag{13}$$
$$h(x_p, y_p) = \xi'(\beta)[1 + 6c2^2e_p^2 + (18c_2c_3 - 15c_2^3)e_p^3 + (-50c_3c_2^2 + 43c_2^4 + 42c_2c_4)e_p^4(c_6 + 32c_5c_2 - 39c_2c_3^2 + 20c_3c_4 + 89c_3c_2^3 - 55c_2^2c_4 - 34c_2^5)e_p^5 + O(e_p^6)], \tag{14}$$
$$w_p = \xi'(\beta)[\beta + 2c_2e_p^3 + (8c_2c_3 - 7c_2^3)e_p^4 + (-37c_3c_2^2 + 16c_2^4 + 16c_2c_4 + 6c_3^2)e_p^5 + O(e_p^6)], \tag{15}$$
$$\xi(w_p) = \xi'(\beta)[2c_2e_p^3 + (8c_2c_3 - 7c_2^3)e_p^4 + (-37c_3c_2^2$$

$$+16c_2^4+16c_2c_4+6c_3^2)e_p^5+O(e_p^6)], \quad (16)$$

$$j(y_p,w_p)=\xi'(\beta)[1+2c_2^2e^2+(6c_2c_3-3c_2^3)e_p^3 +(-14c_3c_2^2+14c_2c_4+6c_2^4)e_p^4+(43c_3c_2^3 -38c_2^2c_4+30c_5c_2-7c_2c_3^2-18c_2^5)e_p^5 +O(e_p^6)], \quad (17)$$

$$\kappa(x_p,y_p,w_p) =\xi'(\beta)[\frac{1}{2}e_p^{-1}-\frac{c_3}{c_2}+(\frac{3}{4}c_3-\frac{7}{8}c_2^2 -3\frac{c_4}{c_2}+3\frac{c_3^2}{c_2^2})e_p+(\frac{5}{4}c_2c_3-\frac{17}{4}c_4 -\frac{49}{16}c_2^3-7\frac{c_5}{c_2}+5\frac{c_3^2}{c_2}+16\frac{c_3c_4}{c_2^2} -9\frac{c_3^3}{c_2^3})e_p^2+\cdots+O(e_p^6)], \quad (18)$$

With the help of Eqs.(10)–(18) in proposed algorithm (II), we have achieved the equality as given below:

$$x_{p+1}=\beta+3c_2^4e_p^5+O(e^6),$$

which implies that

$$e_{p+1}=3c_2^4e_p^5+O(e^6), \quad (19)$$

Equation (19) proved that the proposed algorithm (II) converges in a quintic order, which is the highest possible order. □

IV. NUMERICAL COMPARISON AND ENGINEERING APPLICATIONS

In this section of the study, we will exemplify the applicability and efficiency of the recently suggested optimum technique by applying it to six different engineering challenges. To show the performance of the suggested method, its comparison is made with the iterative schemes as given below:

- A modified version of Halley's method (MHM) taken from [11].
- Chebyshev-Halley type method (CHM) taken from [12].
- Zhanlav's method (ZM) taken from [13].
- A new Householder's method (NHM) taken from [14].

Example 1: Mathematical Model of Beam from Civil Engineering Beams are horizontal members used in construction to span gaps and support loads, such as the upper portion of a wall made of brick or stone (in which case the beam is referred to as a lintel) (see post-and-lintel system). Depending on whether it's supporting a floor or roof, a beam is referred to as a "floor joist" or "roof joist," respectively. The floor beams are the larger transverse components, while the stringers carry the lighter loads over the bridge deck. Large beams that support the terminal ends of smaller, perpendicular beams are commonly known as girders. Single rolled pieces of metal can be used, or girders can be constructed in the shape of an I by riveting or welding plates and angles together to increase stiffness and lengthen spans. Concrete girders find extensive application as well. In this context, several mathematical models in terms of nonlinear equations have been designed to represent the accurate positioning of the beam. One of such models is given below which has been taken from [18].

$$\xi_1(x)=x^4+4x^3-24x^2+16x+16, \quad (20)$$

which is 4th-order polynomial with the roots 2, 2 and $-4\pm 2\sqrt{3}\approx \pm 0.53$. *We take the initial guess* $x_0=0.50$ *while all the obtained numeric results are listed in Table 3.*

Example 2: Flow Characteristics of Blood The study of the properties of blood flow is the primary focus of blood rheology, which is a subfield of the science of rheology [19]. In order to gain a deeper understanding of the plug flow of fluids, the following function is being utilized in our research on the topic:

$$H=1-\frac{16}{7}\sqrt{x}+\frac{4}{3}x-\frac{1}{21}x^4. \quad (21)$$

In the above model, the flow-rate reduction is determined by H. *If we use* $H=0.40$ *in (21), the following nonlinear model is then created:*

$$\xi_2(x)=\frac{1}{441}x^8-\frac{8}{63}x^5-0.05714285714x^4+\frac{16}{9}x^2 -3.624489796x+0.3. \quad (22)$$

To solve ξ_2, *the initial estimate point was taken as* $x_0 = 0.6$ *and the relevant results are shown in Table 1.*

Example 3: Finding Volume from van der Waal's Nonlinear Model

The standard form of van der Waal nonlinear [20] is given below:

$$(P+\frac{K_1n^2}{V^2})(V-nK_2)=nRT. \quad (23)$$

By choosing the possible values of the parameters in (23), we have:

$$\xi_3(x)=0.986x^3-5.181x^2+9.067x-5.289, \quad (24)$$

To solve ξ_3, *the initial estimate point was taken as* $x_0=0.6$ *and the relevant results are shown in Table 1.*

Example 4: The Law of Plank's Radiation

The standard expression of Plank's Radiation Law given by Planck [21] is:

$$\xi(\sigma)=\frac{8\pi cP}{\sigma^5(e^{\frac{cP}{\sigma kT}}-1)}. \quad (25)$$

By taking $x=\frac{cP}{\sigma kT}$, *we get:*

$$1-\frac{x}{5}=e^{-x}, \quad (26)$$

which implies the following function:

$$\xi_4(x)=e^{-x}+\frac{x}{5}-1. \quad (27)$$

To solve ξ_4, *the initial estimate point was taken as* $x_0 = 6.0$ *and the relevant results are shown in Table 1.*

Example 5: Model for the Permeability of Fluids: The measurement of the flow resistance in a hydraulic system is

TABLE 1. Details on numerical comparison among nonlinear approaches under consideration for $\xi_1 - \xi_6$.

Method	N	x_{p+1}	$\lvert\xi(x_{p+1})\rvert$	$\sigma = \lvert x_{p+1} - x_p\rvert$
$\xi_1(x)$, $x_0 = 0.50$				
MHM	12	2.00000000479207791577223793413759096	$5.511363e-16$	$2.396039e-08$
CHM	14	1.99999998847457411152896659604378345	$3.188051e-15$	$3.841809e-08$
ZM	13	2.00000000439274719599900137152760944	$4.631095e-16$	$1.903524e-08$
NHM	13	2.00000000431931086428734452138927897	$4.477547e-16$	$1.871701e-08$
Algorithm 1	6	−0.53589838486224541173613738954075939	$5.386415e-17$	$8.043248e-05$
$\xi_2(x)$, $x_0 = 2.00$				
MHM	6	0.08643355805229175426168226267521203	$5.073227e-30$	$3.321530e-06$
CHM	3	0.08643355805229175426168099882411671	$4.192636e-24$	$2.227149e-05$
ZM	4	0.08643355805229175426168226267368273	$1.388536e-40$	$2.796234e-07$
NHM	7	0.08643355805229175426168226267368273	$8.130738e-63$	$9.168575e-13$
Algorithm 1	3	0.08643355805229175442054167995685320	$5.269928e-19$	$2.077231e-04$
$\xi_3(x)$, $x_0 = 0.60$				
MHM	8	1.92984624284786221848752742786545655	$5.259251e-36$	$3.944360e-08$
CHM	8	1.92984624284786221848752742787566647	$8.824680e-31$	$3.479212e-07$
ZM	14	1.92984624284787424388447024346403713	$1.039378e-15$	$9.483682e-04$
NHM	12	1.92984624284786221848350208388953535	$3.479182e-22$	$2.291807e-05$
Algorithm 1	9	1.92984624284786229418581459764021326	$6.542748e-18$	$1.013312e-04$
$\xi_4(x)$, $x_0 = 6.00$				
MHM	2	4.96511423174427630369875913132289394	$6.489932e-38$	$6.476027e-07$
CHM	2	4.96511423174427630369875913509072395	$7.272773e-28$	$3.152524e-05$
ZM	2	4.96511423174427630369875913132289394	$3.651973e-65$	$6.050369e-10$
NHM	2	4.96511423174427630369875913132289394	$6.495815e-38$	$6.477201e-07$
Algorithm 1	2	4.96511423174427630369875913132289394	$4.015968e-45$	$4.107847e-08$
$\xi_5(x)$, $x_0 = 0.60$				
MHM	3	0.34264820581144991015211887530820190	$3.184403e-61$	$2.226542e-13$
CHM	3	0.34264820581144991015211887530820190	$1.631743e-51$	$2.053005e-11$
ZM	3	0.34264820581144991015211887530820190	$1.142348e-77$	$3.994024e-14$
NHM	3	0.34264820581144983812328966535267548	$3.418649e-15$	$3.581618e-04$
Algorithm 1	3	0.34264820581145003504509022557429397	$5.927699e-75$	$8.133216e-15$
$\xi_6(x)$, $x_0 = 0.01$				
MHM	4	1.46509122029582464237602090977856610	$2.533087e-42$	$1.316701e-08$
CHM	6	1.46509122029582464237602090977856610	$1.927253e-55$	$1.942495e-11$
ZM	3	1.46509122029582464237602090977856610	$1.593252e-66$	$4.241557e-11$
NHM	4	1.46509122029582464224221468953836004	$1.817358e-18$	$7.771499e-04$
Algorithm 1	3	1.46509122029582464237602090977856610	$3.505722e-39$	$2.609475e-08$

referred to as the hydraulic permeability. It shows that there is a link between the pressure gradient and the speed of the fluid. It can be written as follows:

$$\tau = \frac{r_e x^3}{20(1-x)^2} \tag{28}$$

$$r_e x^3 - 20k(1-x)^2 = 0. \tag{29}$$

The previously mentioned value of tau symbolises the specific hydraulic permeability. The value r_e represents the radius, and the value $0 \leq x \leq 1$ represents the porosity. See [22] for further information and several other citations therein. By setting r_e equal to 100 and setting τ equal to 0.4655 in (29)*, we are able to obtain the problem described above in the following non-linear function*:

$$\xi_5(x) = 100x^3 - 9.31(1-x)^2 \tag{30}$$

In order to solve problem 5 (ξ_5), the initial guess of x 0 = 0.6 was selected for the purpose of initiating the iteration process, and the results are presented in Table 1.

Example 6: Nonlinear Model in Fluid Flow The well-known equation of Manning [23], which describes the flow of water in an open channel under the assumption of uniform flow, reads as follows:

$$WaterFlow = G = \frac{\sqrt{s}ar^{\frac{2}{3}}}{N}. \tag{31}$$

The parameters r, s, and a in (31) *denote the hydraulic-radius, the area, and the slope of the channel, respectively, and N denotes the Manning's roughness coefficient. If we have a channel that is rectangular in shape and has a depth of u and a width of w, then we have*:

$$a = xu, \quad \& \quad r = \frac{wx}{w+2x}. \tag{32}$$

With utilization of the values given in (31)*, one can obtain*:

$$G = \frac{\sqrt{s}wx}{N}\left(\frac{wx}{w+2x}\right)^{\frac{2}{3}}. \tag{33}$$

The aforementioned relationship can be written out as follows in order to calculate the depth of the water:

$$\xi_6(x) = \frac{\sqrt{s}wx}{N}\left(\frac{wx}{w+2x}\right)^{\frac{2}{3}} - G. \tag{34}$$

The following values are assigned to the parameters: $G = 14.15\ m^3/s$, $w = 4.572m$, $s = 0.017$ and $N = 0.0015$. In order to get the iteration process going, we chose the initial value of $x_0 = 10^{-2}$ as the starting point, and the results are recorded in Table 1.

In Table 1, we examine the proposed strategy in relation to a wide variety of well-known iterative approaches. The table's columns show the number of iterations that have been done, the best guess for the root, the absolute value of the function at that root, and the difference between the next two best guesses. We can now confidently state that the developed method outperforms competing root-finding approaches when it comes to accuracy, speed, total iterations, and computing cost. Table 1 shows a summary of how the test cases went and shows that the suggested method works well.

V. GRAPHICAL ANALYSIS VIA POLYNOMIOGRAPHY

In this section, we look at the graphical behavior of the proposed root search method for a variety of complex polynomials in the form of polynomyographs generated during the polynomyography process. These polynomyographs are formed from the polynomyography process. Dr. B. Kalantari [24], [25] is credited with coining the term "polynomyography." He defined it as the process of drawing graphic objects that are aesthetically pleasing by making use of the mathematical convergence properties of iteration functions. Polynomyography is a term that was established by Dr. B. Kalantari. When we talk about iterating functions, we are referring to functions like these. When polynomiography is done, the graphical result is called a polynomiograph.

In the light of theorem of Algebra any polynomial $\mathfrak{p}$ with nth-degree must possess nth number of zeros and can expressed as follows:

$$\mathfrak{p}(z) = d_n z^n + d_{n-1} z^{n-1} + \cdots + d_1 z + d_0. \tag{35}$$

If $z_1, z_2, \ldots, z_{n-1}, z_n$ are the roots (zeros) of $\mathfrak{p}$, then (35) may be rewritten as:

$$\mathfrak{p}(z) = (z - z_1)(z - z_2) \ldots (z - z_p), \tag{36}$$

where $\{d_n, d_{n-1}, \ldots, d_1, d_0\}$ are the complex coefficients. When plotting graphical objects, either of the two expressions of $\mathfrak{p}$ that were discussed earlier can be used, regardless of whether or not they include an iteration process. Algorithm or approach 2 gives the most general form of the algorithm that can be used to plot polynomiographs. We consider an algorithm to have converged when the Convergence Test (z_p+1, z_p, ϵ) produces a result of TRUE, and we consider it to have diverged when the result of Algorithm 2 is FALSE. Most of the time, the following test is used to see if an algorithm converges or diverges:

$$|z_{p+1} - z_p| < \varepsilon, \tag{37}$$

where $\varepsilon > 0$ represents the level of accuracy, and z_p and z_{p+1} represent the two guesses that come immediately after each other in the iterative process. As a second halting condition, we use (37) in this article. Changing the values of the parameters ε and k as well as the iteration strategy enables the plotting of a wide range of color graphical objects. for further information on polynomiography and the applications it has, one can go through the research work carried out by Kalantri et al. [27], Gdawiec et al. [28], Scot et al. [29], Naseem et al. [30], Sharma et al. [31] and Kwun et al. [29]. For the purpose of visualizing graphical objects in the complex plane, we make use of the following four distinct complex polynomials:

$$\mathfrak{p}_1(z) = z^3 - 1, \quad \mathfrak{p}_2(z) = (z^3 - 1)^2,$$
$$\mathfrak{p}_3(z) = z^4 - 1, \quad \mathfrak{p}_4(z) = (z^4 - 1)^2.$$

For the purpose of coloring the iterations, we use the colormap that is shown in figure 1. Moreover, it may be noted that the Figures are designated as follows:

- (i) MHM
- (ii) CHM
- (iii) ZM
- (iv) NHM
- (v) Proposed Method (Algorithm 1)

FIGURE 1. The colormap used for the aesthetically beautiful polynomiographs.

Example 7: Quintic-Order Iterative Approaches for Polynomiographs Towards $\mathfrak{p}_1$ In the first illustration, we explore and compare the dynamical results achieved through several fifth-order iteration methods with the newly devised fifth-order algorithm for the cubic-degree polynomial $\mathfrak{p}^3 - 1$ which possesses three separate simple zeros: $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. This is done in order to demonstrate the use of the recently developed fifth-order algorithm. As a means of locating the simple roots, we put all fifth-order algorithms through their paces; the resulting data may be seen in Figure 2.

Example 8: Quintic-Order Iterative Approaches for Polynomiographs towards $\mathfrak{p}_2$ In the second example, we showed how different iteration strategies with the recently developed fifth-order algorithm led to different graphical and dynamic results. The second example considers the polynomial $(\mathfrak{p}^3 - 1)^2$ which possesses three unique roots: $1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ with multiplicity 2. We run all fifth-order algorithms in order to acquire the simple roots of the given polynomial, and the results can be seen graphically in Figure 3.

Example 9: Quintic-Order Iterative Approaches for Polynomiographs Towards $\mathfrak{p}_3$ In this example, we consider a fourth-degree polynomial $\mathfrak{p}_4$ with four simple roots $1, -1, i$ and $-i$. We run all fifth-order methods to find the simple roots, and the results are shown in Figure 4.

Example 10: Quintic-Order Iterative Approaches for Polynomiographs towards $\mathfrak{p}_4$ In the last example, we consider the fourth-degree polynomial $\mathfrak{p}_4$ with four distinct roots $1, -1, i$ and $-i$, having multiplicity two. We use all fifth-order methods to design the graphical objects, and the results are shown in Figure 5.

In Examples 7-10, we utilized the computer program Mathematica 12.0 to execute all of the comparable fifth-order procedures for constructing aesthetically pleasing polynomiographs. This was accomplished by running the methods on the computer. Using the acquired graphical objects, we are able to quickly and simply investigate the graphical behavior and stability of a variety of methods. It is important to underline the fact that the newly constructed technique has a convergence zone that is noticeably larger than those of the previous methods. The shades of color are a representation of how well the algorithm that was used to draw the

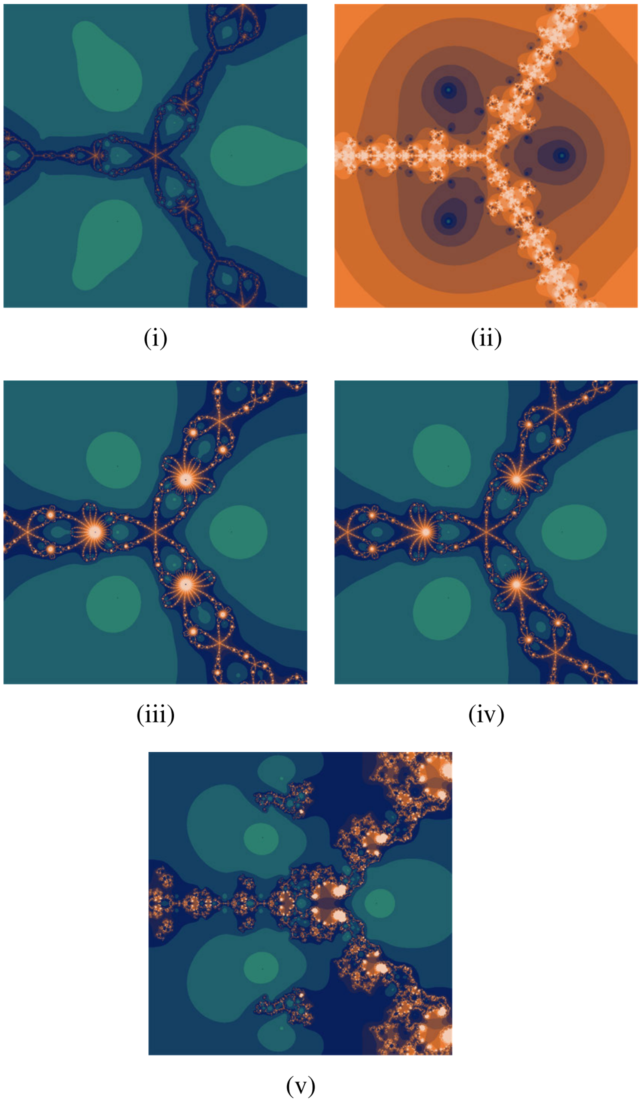

FIGURE 2. Several eye-catching polynomiographs obtained for the cubic-degree polynomial p_1 with the iterative approaches under consideration.

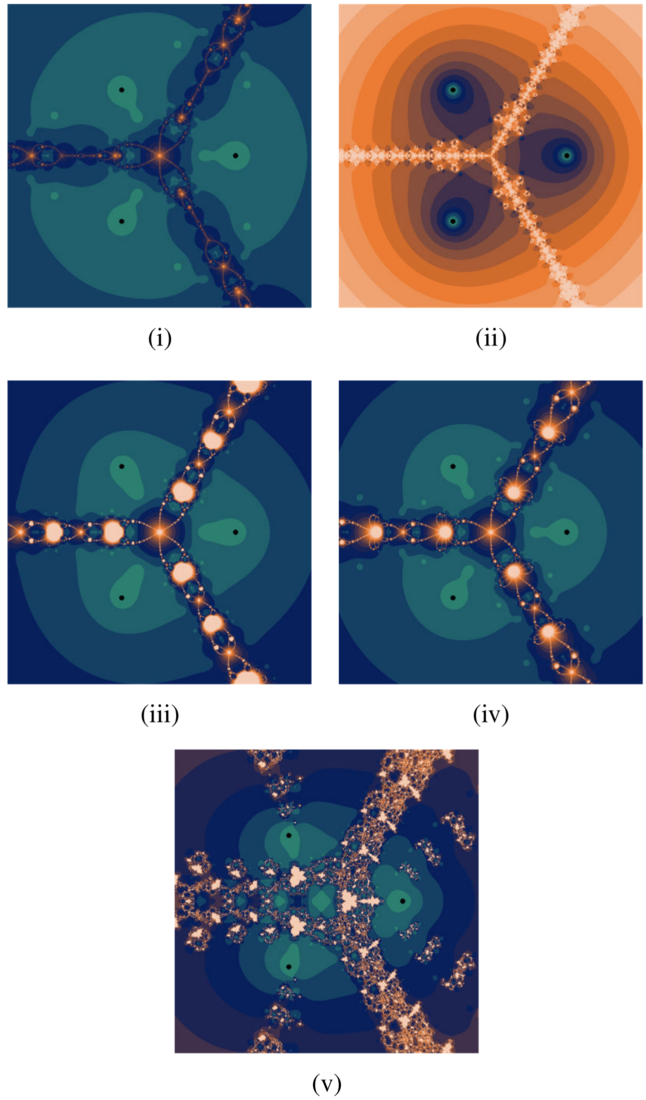

FIGURE 3. Several eye-catching polynomiographs obtained for the sextic-degree polynomial p_2 with the iterative approaches under consideration.

polynomiograph performed. The rate of convergence and the dynamics of the various iteration strategies that were considered in the formation of these graphs are both essential qualities that are indicated by these graphical objects.

The first may be demonstrated by looking at the different color tones used in the image. The richness of color in graphical objects exhibits significant convergence while using fewer repeats of the same color. The second property may be evaluated by looking at the different color combinations that are used in the drawn polynomiographs. Zones with low levels of dynamic activity have small color fluctuation regions, whereas zones with high levels of dynamic activity have vast color variation zones. The regions in the visuals where the solution cannot be reached with the required accuracy in the allotted number of iterations are indicated by the areas of the visuals that have dark shading. The same color on different parts of a graphic object means that the same number of iterations are needed to find the desired solution to the given accuracy.

VI. CONCLUDING REMARKS AND FUTURE ASPECTS

For the purpose of calculating the roots of nonlinear equations using the finite- and forward-difference schemes, an optimum root-finding technique was developed. This approach is presented in this paper. We spoke about the convergence criterion of the devised approach that was provided, and we determined that it converges in quintic order. In order to highlight the great performance and authenticity of the ideal approach that has been offered, we assume a few different technological hurdles. The numerical findings presented in Table 1 demonstrate that the newly developed optimal technique outperforms the older equivalent quintic-order techniques in terms of convergence, precision, and the approximate computational order of convergence. This is demonstrated by the fact that the new technique has a

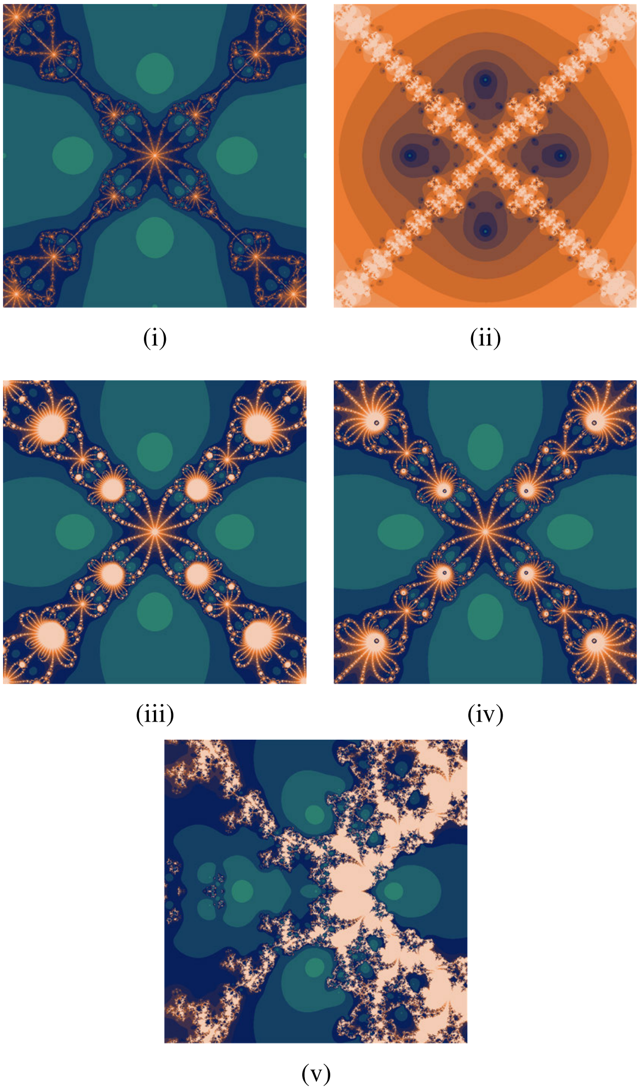

FIGURE 4. Several eye-catching polynomiographs obtained for the quartic-degree polynomial p_3 with the iterative approaches under consideration.

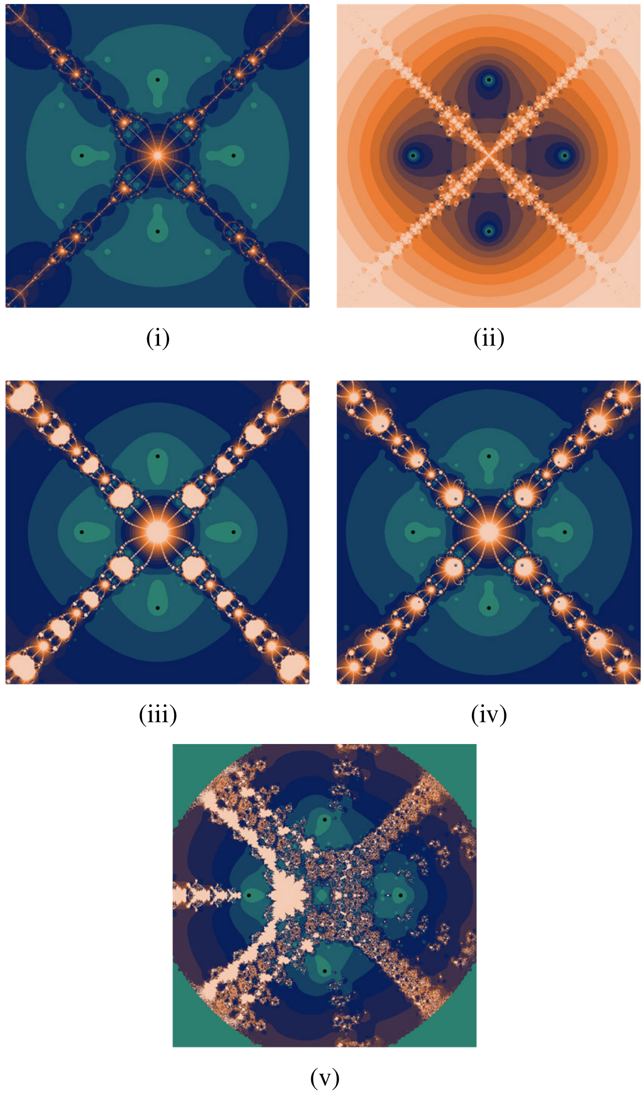

FIGURE 5. Several eye-catching polynomiographs obtained for the octic-degree polynomial p_4 with the iterative approaches under consideration.

higher order of convergence. The accuracy of the successive approximation, which is significantly better than that of the other comparable algorithms, is another evidence of the robust performance of the novel approach.

We employed a piece of software on the computer to generate polynomiographs while taking into consideration a wide variety of complicated polynomials in order to investigate the graphical behavior of the approach that was recommended. The enhanced convergence speed and other graphical characteristics of the constructed optimized approach are demonstrated by the inventive and aesthetically pleasing graphical items that were made, which also showcase the system's potential. Using the same basic idea presented in this article, it is possible to create a new family of better ways to find the roots of nonlinear equations. Moreover, very accurate approximations to solutions are not needed in most practical circumstances. Therefore, it appears that the only people interested in high-order techniques are academics. In comparison to less time-consuming strategies, however, these tend to be fairly sophisticated and provide little value to the production process. This approach is similar to those described in the article that avoid the need for a memory device. Additional processing effort is needed to evaluate Jacobian matrices when dealing with nonlinear systems. Future studies will include a finite-difference approximation with memory, an adjustment we hope will lessen the computing burden of the current approach. The suggested method's semi-local convergence will also be investigated in further detail.

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