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RESEARCH ARTICLE

Optimal Control for a Multistage Uncertain Random System

XIN CHEN AND TING JIN
School of Science, Nanjing Forestry University, Nanjing, Jiangsu 210037, China

Corresponding author: Xin Chen (xchen@njust.edu.cn)

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ABSTRACT Chance theory is a mathematical methodology for modelling complex systems including both uncertainty and randomness. Based on chance theory, this paper introduces the optimal control model for a multistage uncertain random system. For solving such a model, recurrence equations are presented via Bellman's principle. With the aid of recurrence equations, the bang-bang problem of optimal control model and indefinite linear quadratic (LQ) problem of optimal control model are studied. Meanwhile, the exact solutions of such problems are presented. Furthermore, numerical examples are provided on the proposed theorems to show the effectiveness of our results.

INDEX TERMS Chance theory, bang-bang control, indefinite LQ control, uncertain random system, constrained difference equation.

I. INTRODUCTION

Optimal control theory is a discipline which uses to deal with the problem of finding a control law for a given system such that an objective function is optimized. Since the fifties of the last century, the optimal control theory has been an important branch of modern control theory. With the use of methods and results in mathematics and computer science, optimal control theory has made considerable advances. It is widely used in many fields, such as production techniques, finance, and economic management science.

Bellman [1] considered a stochastic optimal control. Kushner [2] used dynamic programming to study the optimal control problem for dynamic systems characterized by Ito stochastic differential equations. For more related works refer to [3], [4], [5], and [6]. With the deepening of human understanding, people found that some imprecise quantities behave neither like randomness nor like fuzziness, such as "about 100 km", "roughly 80 kg", and "low speed". In order to rationally deal with these phenomena, uncertainty theory [7] was developed and refined by Liu [8]. Based on uncertain theory, Zhu [9] first presented an optimal control problem for uncertain continuous-time systems in 2010. Then Xu and Zhu [10], Kang and Zhu [11] discussed uncertain bang-bang optimal control for continuous-time and multistage systems, respectively. Yang and Gao [12] considered an optimal control problem of the uncertain linear quadratic differential game model. Currently, there are many research achievements on uncertain optimal control [13], [14], [15], [16], [17], [18].

Randomness and uncertainty are two kinds of indeterminate events may appear in a complex system simultaneously. For example, we have some old and new components. For the old components probability distributions of lifetimes can be estimated by a large amount of historical data, but for the new components, we can only get the expert's belief degree. Such a system behaves both randomly and uncertainly, and cannot explain clearly by the stochastic system or uncertain system. To handle the phenomena, Liu [19] pioneered the chance theory which is a mathematical methodology for modeling complex systems including both uncertainty and randomness via chance measure, uncertain random variable, chance distribution, expected value, variance, and so on. Then Liu [20] presented the operational laws of uncertain random

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variables. In recent years, considerable related works have been researched [21], [22], [23], [24], [25].

With the development of society and the progress of science and technology, the systems we need to research are more and more complicated. The widespread use of computers has led to the development of control systems, which also increasingly highlights the importance of studying multistage systems. Multistage uncertain random systems as special complicated systems should be studied. And the research on the designing of multistage uncertain random optimal control problems has few results. This study, inspired by previous work, seeks to study the work of an optimal control problem for a multistage uncertain random system.

This paper makes the following contributions. First, many works focus on the stochastic optimal control concerning randomness [3] or uncertain optimal control involving uncertainty [14]. This paper further develops optimal control theory and considers an uncertain random optimal control including uncertainty and randomness. This work provides theoretical support for dynamic optimization and decision-making when the system is disturbed by uncertainty and randomness in production life and other fields [25]. The second contribution is to get recurrence equations that turn the uncertain random optimal control problem into a much easier problem. In comparison to the work in [3] and [14], these recurrence equations presented in Subsection III can tackle more wide optimal control problems. The last one is to present the exact solution of an indefinite LQ optimal control problem. This finding, influenced by the work in [15], is presented in Subsection V.

The remainder of this paper is organized as follows. In Section II, some necessary preliminaries related to the uncertain measure, uncertain variable, and uncertain random variable are reviewed. In Section III, an uncertain random optimal control model is introduced, and the recurrence equations are established for solving such a model. In Section IV, the bang-bang optimal control models for two types of multistage uncertain random systems are investigated. In Section V, an indefinite LQ optimal control model is discussed, whereas the weighting matrices in the objective function are allowed to be indefinite. The last section gives some conclusions.

Notation: $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ matrices, $\boldsymbol{A}^T$ denotes the transpose of a matrix $\boldsymbol{A}$. The Cartesian product $[\alpha, \beta]^n = [\alpha, \beta] \times [\alpha, \beta] \times \cdots \times [\alpha, \beta]$.

II. PRELIMINARY

In this section, we review some fundamental notations and useful concepts in uncertainty theory. For more details about existing measures of uncertainties, the readers may consult Liu [7].

Definition 1 (Liu [7]): Let Γ be a nonempty set, and $\mathcal{L}$ be a σ-algebra over Γ. Each element $\Lambda \in \mathcal{L}$ is called an event. A set function $\mathcal{M}$ defined on the σ-algebra over $\mathcal{L}$ is called an uncertain measure if it satisfies the following three axioms: (1)$\mathcal{M}\{\Gamma\} = 1$; (2)$\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ; (3)$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}$.

The triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. The product uncertain measure was defined by Liu [8] as follows: $\mathcal{M}\left\{\prod_{i=1}^{\infty} \wedge_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\wedge_k\}$ where $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ are uncertainty spaces for $k = 1, 2, \cdots$ and $\wedge_k$ are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Definition 2 (Liu [7]): An uncertain variable is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{\tau \in B\} = \{\gamma \in \Gamma \mid \tau(\gamma) \in B\}$ is an event for any Borel set B of real numbers. The uncertainty distribution of uncertain variable τ is defined as $\Phi(x) = \mathcal{M}\{\tau \leq x\}$ for any $x \in \mathbb{R}$.

Definition 3 (Liu [30]): The uncertain variables $\tau_1, \tau_2, \cdots, \tau_m$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{m}\{\tau_i \in B_i\}\right\} = \min_{1\leq i\leq m} \mathcal{M}\{\tau_i \in B_i\}, \tag{1}$$

for any Borel sets $B_1, B_2, \cdots, B_m$ of real numbers.

Lemma 1 (Zhu [31]): Let τ be an ordinary linear uncertain variable $\mathcal{L}(-1, 1)$ whose uncertain distribution is

$$\Phi(x) = \begin{cases} 0 & \text{if } x < -1, \\ (x+1)/2 & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{2}$$

Then for any real number b, we have

(i) if $b = 0$, $E[\tau^2 + b\tau] = \frac{7}{24}$;
(ii) if $\mid b \mid \geq 2$, $E[\tau^2 + b\tau] = \frac{1}{3}$;
(iii) if $1 \leq \mid b \mid < 2$, $E[\tau^2 + b\tau] = \frac{1}{48}(\mid b \mid^3 - 6 \mid b \mid^2 + 12 \mid b \mid + 8)$;
(iv) if $0 < \mid b \mid < 1$, $E[\tau^2 + b\tau] = \frac{1}{48}(\mid b \mid^3 + 12 \mid b \mid^2 - 12 \mid b \mid + 14)$.

An uncertain random variable is employed to describe a complex system with not only uncertainty but also randomness. Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, Pr)$ a probability space. Then the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, Pr)$ is called a chance space.

Definition 4 (Liu [19]): An uncertain random variable is a function ξ from a chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathrm{Pr})$ to the set of real numbers such that $\{\xi \in B\}$ is an event in $\mathcal{L} \times \mathcal{A}$ for any Borel set B of real numbers. The chance distribution of uncertain random variable ξ is defined as $\Psi(x) = \mathrm{Ch}\{\xi \leq x\}$ where the chance measure of $\Theta \in \mathcal{L} \times \mathcal{A}$ is defined as

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\{\omega \in \Omega | \mathcal{M}\{\gamma \in \Gamma | \xi(\gamma, \omega) \in \Theta\} \geq x\} dx. \tag{3}$$

Definition 5 (Liu [19]): Let ξ be an uncertain random variable. Then its expected value is defined by

$$E[\xi] = \int_0^{+\infty} \mathrm{Ch}\{\xi \geq r\} dr - \int_{-\infty}^0 \mathrm{Ch}\{\xi \leq r\} dr, \tag{4}$$

provided that at least one of the two integrals is finite.

Remark 1: If the uncertain random variable ξ degenerates to a random variable η, then $\mathrm{Ch}\{\xi \geq r\} = \Pr\{\eta \geq r\}$,

$\mathrm{Ch}\{\xi \le r\} = \Pr\{\eta \le r\}$. If the uncertain random variable ξ degenerates to an uncertain variable τ, then $\mathrm{Ch}\{\xi \ge r\} = \mathcal{M}\{\tau \ge r\}$, $\mathrm{Ch}\{\xi \le r\} = \mathcal{M}\{\tau \le r\}$.

Lemma 2 (Liu [20]): Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables, and f be a measurable function. Then

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n), \tag{5}$$

has an expected value

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)] \times d\Psi_1(y_1) d\Psi_2(y_2) \cdots \Psi_m(y_m), \tag{6}$$

where $E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)]$ is the expected value of the uncertain variable $f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)$ for any real numbers $y_1, y_2, \cdots, y_m$.

Remark 2: If $\eta_1, \eta_2, \cdots, \eta_m$ are non-independent random variables with joint distribution $\Psi(y_1, y_2, \cdots, y_m)$, then (6) can be modified as

$$E[\xi] = \int_{\mathbb{R}^m} E[f(y_1, y_2, \cdots, y_m, \tau_1, \tau_2, \cdots, \tau_n)] \times d\Psi(y_1, y_2, \cdots, y_m). \tag{7}$$

Lemma 3 (Liu [19]): Let ξ be an uncertain random variable whose expected value exists. Then for any real numbers a and b, we have $E[a\xi + b] = aE[\xi] + b$.

Utilizing Lemma 3 and Lemma 3.1 in [14], the linearity of expected value operator of a class of uncertain random variables is introduced.

Remark 3: The linearity of expected value operator is not valid for all kinds of uncertain random variables. For example, for uncertain variables as special uncertain random variables, the expected value operator is not necessarily linear if the independence is not assumed [7].

In order to describe the uncertain random vector, we show that the vector is an uncertain vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_p)^T$ if and only if ξ_i, $i = 1, 2, \cdots, p$ are uncertain random variables. The expected value of $\boldsymbol{\xi}$ is provided by $E[\boldsymbol{\xi}] = (E[\xi_1], E[\xi_2], \cdots, E[\xi_p])^T$.

III. OPTIMAL CONTROL MODEL

An optimal control problem for a multistage system is to choose the best decision such that an objective function is optimized subject to a multistage system. In this section, the optimal control model for a multistage uncertain random system is proposed. A linear multistage dynamical system can be written as follows

$$\begin{cases} \boldsymbol{x}(j+1) = \boldsymbol{F}_j\boldsymbol{x}(j) + \boldsymbol{G}_j\boldsymbol{u}(j), \ j = 0, 1, 2, \cdots, N-1, \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{8}$$

where $\boldsymbol{x}(j) \in \mathbb{R}^n$ is the state vector with initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{u}(j) \in \mathbb{R}^r$ is the control vector. The matrices $\boldsymbol{F}_j \in \mathbb{R}^{n\times n}$ and $\boldsymbol{G}_j \in \mathbb{R}^{n\times r}$ are deterministic.

For the system at each stage is perturbed by random variables, a stochastic optimal control problem is found and many works have been done, such as [3] and [33]. Otherwise, considering the system at each stage is disturbed by uncertain variables, uncertain optimal control problem is found and lots of problems have been solved, see [13] and [14]. Different from the stochastic or uncertain situation, we consider a complex system includes both uncertainty and randomness as below

$$\begin{cases} \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j\boldsymbol{F}_j\eta_j + \beta_j\boldsymbol{F}_j\tau_j)\boldsymbol{x}(j) \\ \qquad\qquad + (\boldsymbol{G}_j + \alpha_j\boldsymbol{G}_j\eta_j + \beta_j\boldsymbol{G}_j\tau_j)\boldsymbol{u}(j), \\ \boldsymbol{x}(0) = \boldsymbol{x}_0, j = 0, 1, 2, \cdots, N-1, \end{cases} \tag{9}$$

where $\boldsymbol{x}(j) \in \mathbb{R}^n$ is the state vector with initial state $\boldsymbol{x}_0 \in \mathbb{R}^n$, $\boldsymbol{u}(j) \in \mathbb{R}^r$ is the control vector. The matrices $\boldsymbol{F}_j \in \mathbb{R}^{n\times n}$, $\boldsymbol{G}_j \in \mathbb{R}^{n\times r}$ are deterministic, α_j and β_j are real numbers which satisfy $|\alpha_j| + |\beta_j| \le 1$. In addition, the noises η_j are independent random variables and τ_j are independent uncertain variables, for $j = 0, 1, 2, \cdots, N-1$.

Remark 4: The formulation $|\alpha_j| + |\beta_j| \le 1$ means that at each stage, the effects of the noises are small enough. Systems with noises have a wide range of applications in different fields. For example, the application of systems with uncertain variables [18] was applied to the field of portfolio selection, and the application of systems with uncertain random variables in the field of two-spool turbofan engine.

Remark 5: If $\alpha_j = 0$, then the uncertain random system (9) models an uncertain system. If $\beta_j = 0$, then the uncertain random system (9) becomes a stochastic system. If $\alpha_j = \beta_j = 0$, then the uncertain random system (9) changes into the deterministic system (8).

Based on expected value criterion, the optimal control model for a multistage uncertain random system is formulated as follows

$$\begin{cases} J(\boldsymbol{x}_0, 0) = \min\limits_{\substack{\boldsymbol{u}(j)\in \boldsymbol{U}(j) \\ 0\le j\le N}} E\left[\sum_{j=0}^{N} f(\boldsymbol{x}(j), \boldsymbol{u}(j), j)\right] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j\boldsymbol{F}_j\eta_j + \beta_j\boldsymbol{F}_j\tau_j)\boldsymbol{x}(j) \\ \qquad\qquad + (\boldsymbol{G}_j + \alpha_j\boldsymbol{G}_j\eta_j + \beta_j\boldsymbol{G}_j\tau_j)\boldsymbol{u}(j), \\ \quad j = 0, 1, 2, \cdots, N-1, \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{10}$$

where f is the objective function, $\boldsymbol{U}(j)$ means the control domain for $\boldsymbol{u}(j)$, $J(\boldsymbol{x}_0, 0)$ is the expected optimal reward obtainable from the first stage to the last stage.

Remark 6: In model (10), we construct the optimal control problem with expected value criterion. Of course, we may consider other criteria, e.g., chance measure criteria, optimistic value criteria or pessimistic value criteria. In the aspect of introduction and applications of problems, the difference among them is mainly that they adopt different criteria to compare two different uncertain random variables. For example, in view of the large wage gap from different employees of a company, we can establish the optimal control problem with the optimistic value criterion.

Let $J(\boldsymbol{x}_k, k)$ be the expected optimal reward obtainable in $[k, N]$ for any $0 \le k \le N$ with the condition that at stage k we are in state $\boldsymbol{x}(k) = \boldsymbol{x}_k$. By applying Bellman's principle, the recurrence equations are established to solve the problem (10).

Theorem 1: For the problem (10), we have the following recurrence equations

$$J(\boldsymbol{x}_N, N) = \max_{\boldsymbol{u}(N)\in \boldsymbol{U}(N)} f(\boldsymbol{x}_N, \boldsymbol{u}(N), N), \tag{11}$$

$$J(\boldsymbol{x}_k, k) = \max_{\boldsymbol{u}(k)\in \boldsymbol{U}(k)} E[f(\boldsymbol{x}_k, \boldsymbol{u}(k), k) + J(\boldsymbol{x}(k+1), k+1)], \tag{12}$$

for $k = N-1, N-2, \cdots, 1, 0$.

The proof of Theorem 1 is similar to that in Zhu [29].

Remark 7: Compared with [3], [13], and [14], we use Bellman's principle to solve the optimal control problem. The solution of problem (10) can be deduced by solving the simpler problems (11) and (12) step by step from the last stage to the initial stage. We study optimal control for a multistage uncertain random system which includes both uncertainty and randomness. To some extent, it's a generalization of the stochastic optimal control model and uncertain optimal control model.

IV. BANG-BANG PROBLEM OF OPTIMAL CONTROL MODEL

If the optimal control of an optimal control problem can be expressed by the sign of a function, the problem is called the bang-bang control problem. In this section, let us consider the following optimal control problem

$$\begin{cases} J(\boldsymbol{x}_0, 0) \\ = \min\limits_{\substack{\boldsymbol{u}(j)\in[a,b]^r \\ 0\le j\le N-1}} E\left[\sum_{j=0}^{N-1} (\boldsymbol{a}_j \boldsymbol{x}(j) + \boldsymbol{b}_j \boldsymbol{u}(j)) + \boldsymbol{a}_N \boldsymbol{x}(N)\right] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j \boldsymbol{F}_j \eta_j + \beta_j \boldsymbol{F}_j \tau_j)\boldsymbol{x}(j) \\ \qquad\qquad +(\boldsymbol{G}_j + \alpha_j \boldsymbol{G}_j \eta_j + \beta_j \boldsymbol{G}_j \tau_j)\boldsymbol{u}(j), \\ \quad j = 0, 1, \cdots, N-1,\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{13}$$

where a_j is a vector of dimension n, b_j is a vector of dimension r. Note that in the model (13), the constraint domain of $\boldsymbol{u}(j)$ is $[a, b]^r = [a, b] \times [a, b] \times \cdots \times [a, b]$. With the help of recurrence equations, the optimal results for the optimal control model (13) can be obtained.

Theorem 2: The optimal controls $\boldsymbol{u}^*(k)$ of (13) are provided by

$$u_j^*(k) = \begin{cases} b, & \text{if } (\boldsymbol{\alpha}_k)_j > 0, \\ a, & \text{if } (\boldsymbol{\alpha}_k)_j < 0, \\ \textit{undetermined}, & \text{otherwise}, \end{cases} \tag{14}$$

with $\boldsymbol{\alpha}_k = \boldsymbol{b}_k + (1 + \alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1}$, for $k = 0, 1, 2, \cdots, N-1$, $j = 1, 2, \cdots, r$. The optimal values are

$$J(\boldsymbol{x}_N, N) = \boldsymbol{r}_N^T \boldsymbol{x}_N,\ J(\boldsymbol{x}_k, k) = \boldsymbol{r}_k^T \boldsymbol{x}_k + \sum_{i=k}^{N-1} t_i, \tag{15}$$

where

$$\boldsymbol{r}_N = \boldsymbol{a}_N, \boldsymbol{r}_k = \boldsymbol{a}_k + (1 + \alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{F}_k^T \boldsymbol{r}_{k+1},$$
$$t_k = (\boldsymbol{b}_k + (1 + \alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1})^T \boldsymbol{u}^*(k),$$

and $u_j^*(k)$ and $(\boldsymbol{b}_k + (1+\alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1})_j$ mean the j-th elements in vectors $\boldsymbol{u}^*(k)$ and $\boldsymbol{b}_k + (1 + \alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1}$, respectively, for $k = 0, 1, 2, \cdots, N-1$.

Proof: Denote the optimal controls of problem (13) as $\boldsymbol{u}^*(k) = (u_1^*(k), u_2^*(k), \cdots, u_r^*(k))^T$, for $k = 0, 1, 2, \cdots, N$. By using the recurrence equation (11), we have $J(\boldsymbol{x}_N, N) = \boldsymbol{r}_N^T \boldsymbol{x}_N$. For $k = N-1$, by using the recurrence equation (12), we have

$$\begin{aligned} &J(\boldsymbol{x}_{N-1}, N-1) \\ &= \max_{\boldsymbol{u}(N-1)\in[a,b]^r} \{\boldsymbol{a}_{N-1}^T \boldsymbol{x}_{N-1} + \boldsymbol{b}_{N-1}^T \boldsymbol{u}(N-1) \\ &\quad + E[\boldsymbol{r}_N^T((\boldsymbol{F}_{N-1} + \alpha_{N-1}\boldsymbol{F}_{N-1}\eta_{N-1} \\ &\quad + \beta_{N-1}\boldsymbol{F}_{N-1}\tau_j)\boldsymbol{x}(N-1)(\boldsymbol{G}_{N-1} + \alpha_{N-1}\boldsymbol{G}_{N-1}\eta_{N-1} \\ &\quad + \beta_{N-1}\boldsymbol{G}_{N-1}\tau_{N-1})\boldsymbol{u}(N-1))]\}. \end{aligned} \tag{16}$$

We obtain

$$\begin{aligned} &J(\boldsymbol{x}_{N-1}, N-1) \\ &\quad = \max_{\boldsymbol{u}(N-1)\in[a,b]^r} \{(\boldsymbol{a}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\qquad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{F}_{N-1}^T \boldsymbol{r}_N)^T \boldsymbol{x}_{N-1} \\ &\qquad + (\boldsymbol{b}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\qquad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{G}_{N-1}^T \boldsymbol{r}_N)^T \boldsymbol{u}(N-1)\}. \end{aligned} \tag{17}$$

Let $\boldsymbol{u}^*(N-1)$ be the solution of the former equation. Denote $\boldsymbol{\alpha}_{N-1} = \boldsymbol{b}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{G}_{N-1}^T \boldsymbol{r}_N$. Then

$$\max_{\boldsymbol{u}(N-1)\in[a,b]^r} \{\boldsymbol{\alpha}_{N-1}^T \boldsymbol{u}(N-1)\} = \boldsymbol{\alpha}_{N-1}^T \boldsymbol{u}^*(N-1). \tag{18}$$

So we can get the optimal control $\boldsymbol{u}^*(N-1)$ as

$$u_j^*(N-1) = \begin{cases} b, & \text{if } (\boldsymbol{\alpha}_{N-1})_j > 0, \\ a, & \text{if } (\boldsymbol{\alpha}_{N-1})_j < 0, \\ \textit{undetermined}, & \text{otherwise}, \end{cases} \tag{19}$$

where $(\boldsymbol{\alpha}_{N-1})_j$ represents the j-th element in vector $\boldsymbol{\alpha}_{N-1}$. Hence, we can get

$$\begin{aligned} &J(\boldsymbol{x}_{N-1}, N-1) \\ &\quad = (\boldsymbol{a}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\qquad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{F}_{N-1}^T \boldsymbol{r}_N)^T \boldsymbol{x}_{N-1} \\ &\qquad + (\boldsymbol{b}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\qquad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{G}_{N-1}^T \boldsymbol{r}_N \boldsymbol{u}^*(N-1). \end{aligned} \tag{20}$$

Denote

$$\begin{aligned} \boldsymbol{r}_{N-1} &= \boldsymbol{a}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\quad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{F}_{N-1}^T \boldsymbol{r}_N, \end{aligned} \tag{21}$$

$$\begin{aligned} t_{N-1} &= (\boldsymbol{b}_{N-1} + (1 + \alpha_{N-1}E[\eta_{N-1}] \\ &\quad + \beta_{N-1}E[\tau_{N-1}])\boldsymbol{G}_{N-1}^T \boldsymbol{r}_N)^T \boldsymbol{u}^*(N-1). \end{aligned} \tag{22}$$

Then

$$J(\boldsymbol{x}_{N-1}, N-1) = \boldsymbol{r}_{N-1}^T \boldsymbol{x}_{N-1} + t_{N-1}. \tag{23}$$

For $k = N-2$, by using the recurrence equation (12), we have

$$\begin{aligned} J(\boldsymbol{x}_{N-2}, N-2) &= \max_{\boldsymbol{u}(N-2)\in[a,b]^r} \{(\boldsymbol{a}_{N-2} + (1+\alpha_{N-2}E[\eta_{N-2}] \\ &\quad + \beta_{N-2}E[\tau_{N-2}])\boldsymbol{F}_{N-2}^T \boldsymbol{r}_{N-1})^T \boldsymbol{x}_{N-2} \\ &\quad + (\boldsymbol{b}_{N-2} + (1+\alpha_{N-2}E[\eta_{N-2}] + \beta_{N-2} \\ &\quad \cdot E[\tau_{N-2}])\boldsymbol{G}_{N-2}^T \boldsymbol{r}_{N-1})^T \boldsymbol{u}(N-2)\}. \end{aligned} \tag{24}$$

Similar to the calculation of $k = N-1$, denote

$$\begin{aligned} \boldsymbol{r}_{N-2} &= \boldsymbol{a}_{N-2} + (1+\alpha_{N-2}E[\eta_{N-2}] \\ &\quad + \beta_{N-2}E[\tau_{N-2}])\boldsymbol{F}_{N-2}^T \boldsymbol{r}_{N-1}, \end{aligned} \tag{25}$$

$$\begin{aligned} t_{N-2} &= (\boldsymbol{b}_{N-2} + (1+\alpha_{N-2}E[\eta_{N-2}] + \beta_{N-2} \\ &\quad \cdot E[\tau_{N-2}])\boldsymbol{G}_{N-2}^T \boldsymbol{r}_{N-1})^T \boldsymbol{u}^*(N-2). \end{aligned} \tag{26}$$

Then

$$J(\boldsymbol{x}_{N-2}, N-2) = \boldsymbol{r}_{N-2}^T \boldsymbol{x}_{N-2} + \sum_{i=N-2}^{N-1} t_i, \tag{27}$$

and the optimal control $\boldsymbol{u}^*(N-2)$ is

$$u_j^*(N-2) = \begin{cases} b, & \text{if } (\boldsymbol{\alpha}_{N-2})_j > 0, \\ a, & \text{if } (\boldsymbol{\alpha}_{N-2})_j < 0, \\ \textit{undetermined}, & \text{otherwise}, \end{cases} \tag{28}$$

with $\boldsymbol{\alpha}_{N-2} = \boldsymbol{b}_{N-2} + (1+\alpha_{N-2}E[\eta_{N-2}] + \beta_{N-2}E[\tau_{N-2}])$ $\boldsymbol{G}_{N-2}^T \boldsymbol{r}_{N-1}$, and $(\boldsymbol{\alpha}_{N-2})_j$ represents the j-th element in the vector $\boldsymbol{\alpha}_{N-2}$. By induction, the theorem is proved.

Compared with the general optimal control problem, we obtain the analytical expression for the optimal control (14) of the optimal control problem (13). This type of optimal control (14) is easy to implement compared to some optimal controls and has a wide range of applications [25], [27].

Different from [27], we study an optimal control model whose system matrices and control matrices are multiplied by random sequences and uncertain sequences. The optimal results of the optimal control model whose systems involve both multiplicative noises and additive noises are easily deduced based on Theorem 3.2 in [27] and Theorem 2. The model is presented in the form of

$$\begin{cases} J(\boldsymbol{x}_0, 0) \\ = \min\limits_{\substack{\boldsymbol{u}(j)\in[a,b]^r \\ 0\le j\le N-1}} E\left[\sum\limits_{j=0}^{N-1} (\boldsymbol{a}_j \boldsymbol{x}(j) + \boldsymbol{b}_j \boldsymbol{u}(j)) + \boldsymbol{a}_N \boldsymbol{x}(N)\right] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j \boldsymbol{F}_j \eta_j + \beta_j \boldsymbol{F}_j \tau_j)\boldsymbol{x}(j) + (\boldsymbol{G}_j \\ \qquad\qquad + \alpha_j \boldsymbol{G}_j \eta_j + \beta_j \boldsymbol{G}_j \tau_j)\boldsymbol{u}(j) + \boldsymbol{m}_j \xi_j, \\ \quad j = 0, 1, \cdots, N-1, \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{29}$$

where $\boldsymbol{m}_j \in \mathbb{R}^n$ are constant matrices, the uncertain random variables $\xi_j = f_j(\tilde{\eta}_j, \tilde{\tau}_j)$ where f_j are measurable functions, $\tilde{\eta}_j$ are random variables and $\tilde{\tau}_j$ are uncertain variables, for $j = 0, 1, \cdots, N-1$. Moreover, the uncertain variables $\tau_j, \tilde{\tau}_j, j = 0, 1, \cdots, N-1$ are independent.

Theorem 3: The optimal controls $\boldsymbol{u}^*(k)$ of (29) are provided by

$$u_j^*(k) = \begin{cases} b, & \text{if } (\boldsymbol{\alpha}_k)_j > 0, \\ a, & \text{if } (\boldsymbol{\alpha}_k)_j < 0, \\ \textit{undetermined}, & \text{otherwise}, \end{cases} \tag{30}$$

with $\boldsymbol{\alpha}_k = \boldsymbol{b}_k + (1+\alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1}$, for $k = 0, 1, 2, \cdots, N-1, j = 1, 2, \cdots, r$. The optimal values are

$$J(\boldsymbol{x}_N, N) = \boldsymbol{r}_N^T \boldsymbol{x}_N, \tag{31}$$

$$J(\boldsymbol{x}_k, k) = \boldsymbol{r}_k^T \boldsymbol{x}_k + \sum_{i=k}^{N-1} t_i + \sum_{i=k}^{N-1} \boldsymbol{r}_i^T \boldsymbol{m}_i E[\xi_i], \tag{32}$$

where

$$\begin{aligned} \boldsymbol{r}_N = \boldsymbol{a}_N, \boldsymbol{r}_k &= \boldsymbol{a}_k + (1+\alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{F}_k^T \boldsymbol{r}_{k+1}, \\ t_k &= (\boldsymbol{b}_k + (1+\alpha_k E[\eta_k] \\ &\quad + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1})^T \boldsymbol{u}^*(k), \end{aligned}$$

and $u_j^*(k)$ and $(\boldsymbol{b}_k + (1+\alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1})_j$ mean the j-th elements in vectors $\boldsymbol{u}^*(k)$ and $\boldsymbol{b}_k + (1+\alpha_k E[\eta_k] + \beta_k E[\tau_k])\boldsymbol{G}_k^T \boldsymbol{r}_{k+1}$, respectively, for $k = 0, 1, 2, \cdots, N-1$. The proof of Theorem 3 is similar to Theorem 2, so we omit it.

Example 1: Based on Theorem 2, we consider the following optimal control problems. First, an optimal control problem is presented as below

$$\begin{cases} J(\boldsymbol{x}_0, 0) \\ = \max\limits_{\substack{\boldsymbol{u}(i)\in[-1,1] \\ 0\le i\le 10}} E\left[\sum\limits_{j=0}^{9} (\boldsymbol{a}_j^T \boldsymbol{x}(j) + \boldsymbol{b}_j^T \boldsymbol{u}(j)) + \boldsymbol{a}_{10}^T \boldsymbol{x}(10)\right] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j \boldsymbol{F}_j \eta_j + \beta_j \boldsymbol{F}_j \tau_j)\boldsymbol{x}(j) \\ \qquad\qquad + (\boldsymbol{G}_j + \alpha_j \boldsymbol{G}_j \eta_j + \beta_j \boldsymbol{G}_j \tau_j)\boldsymbol{u}(j), \\ \quad j = 0, 1, 2, \cdots, 9, \ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{33}$$

where

$$\boldsymbol{x}(j) = (x_1(j), x_2(j))^T, \boldsymbol{F} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{a}_0^T \\ \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \\ \boldsymbol{a}_4^T \\ \boldsymbol{a}_5^T \\ \boldsymbol{a}_6^T \\ \boldsymbol{a}_7^T \\ \boldsymbol{a}_8^T \\ \boldsymbol{a}_9^T \\ \boldsymbol{a}_{10}^T \end{bmatrix} = \begin{bmatrix} 0.0736 & 0.8530 \\ 0.5290 & 0.7858 \\ -0.3651 & 0.4269 \\ 0.5669 & 0.0014 \\ 0.1618 & -0.2906 \\ -0.5123 & 0.1088 \\ -0.6075 & 0.5787 \\ 0.7344 & 0.9610 \\ 0.7875 & 0.7975 \\ 0.8244 & 0.2787 \\ -0.2119 & -0.8214 \end{bmatrix},$$

TABLE 1. The coefficients of F_j.

F_j	f_{11}	f_{12}	f_{21}	f_{22}
F_0	0.8908	−0.9876	0.3471	0.4755
F_1	0.4754	0.2421	−0.0064	0.1606
F_2	0.1606	0.2992	0.9096	0.7502
F_3	0.3339	−0.0003	0.8607	0.2635
F_4	0.2635	0.5780	0.4007	0.1541
F_5	0.5138	−0.1391	0.9235	0.8985
F_6	0.6511	0.2007	0.3404	0.5410
F_7	0.1509	0.1791	0.5096	0.5075
F_8	−0.0008	0.1409	0.7393	0.8639
F_9	0.8639	-0.3085	0.9512	0.3120

TABLE 2. The coefficients of G_j.

G_j	g_{11}	g_{12}	g_{21}	g_{22}
G_0	0.0358	0.2747	0.2704	0.2562
G_1	−0.7429	1.3979	0.4593	0.4623
G_2	0.3891	0.8152	0.5701	0.5823
G_3	−0.5464	0.4010	0.0098	0.0731
G_4	0.2718	−0.0143	−0.4143	0.9982
G_5	0.4711	0.2951	0.8801	0.8590
G_6	0.1914	0.0347	0.6688	0.4786
G_7	0.1778	0.5363	0.4796	0.8297
G_8	−0.8297	0.3430	0.4691	0.5812
G_9	0.2045	0.9136	0.4306	0.6193

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix} = \begin{bmatrix} 0.5474 & -0.1692 \\ 0.8020 & -0.8615 \\ 0.0362 & -0.7086 \\ 0.2732 & 0.4704 \\ 0.2294 & 0.0845 \\ 0.1767 & -0.0487 \\ 0.9664 & 0.8507 \\ -0.4864 & -0.8967 \\ 0.1181 & 0.3162 \\ -0.9045 & 0.1447 \end{bmatrix},$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix} = \begin{bmatrix} 0.0971 \\ 0.0554 \\ 0.0522 \\ -0.0314 \\ -0.0025 \\ -0.0244 \\ 0.0637 \\ 0.0094 \\ 0.0477 \\ -0.0402 \end{bmatrix}, \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \end{bmatrix} = \begin{bmatrix} -0.0494 \\ 0.0607 \\ 0.0913 \\ 0.0965 \\ 0.0935 \\ 0.0724 \\ -0.0143 \\ -0.0726 \\ -0.0254 \\ 0.0071 \end{bmatrix}.$$

The coefficients of F_j and G_j are shown in Tables 1 and 2, respectively.

The $\eta_j \sim U(-1, 1)$ are independent uniform random variables and $\tau_j \sim \mathcal{L}(-1, 1)$ are independent linear uncertain variables, and we can get $E[\eta_j] = 0, E[\tau_j] = 0$, for $j = 0, 1, 2, \cdots, 9$. Suppose that we have the initial state $x_0 = (0.8143, 0.2435)^T$. Then the optimal controls $u^*(j)$ and optimal objective values $J(x_j, j)$ of problem (33) are obtained by Theorem 2 and listed in Table 3.

Remark 8: In columns 2 and 3 of Table 3, the corresponding states $x(j+1) = (F_j + \alpha_j F_j c_j + \beta_j F_j d_j)x(j) + (G_j + \alpha_j G_j c_j + \beta_j G_j d_j)u(j)$ with initial state $x_0 = (0.8143, 0.2435)^T$, where c_j and d_j are the realization of random variables η_j and uncertain variables τ_j, and generated by $0 < \frac{c_j+1}{2} < 1, 0 < \frac{d_j+1}{2} < 1$ for $j = 0, 1, 2, \cdots, 9$.

TABLE 3. The optimal results of problem (33).

Stage	c_j	d_j	$x(j)$	u_j^*	$J(x_j, j)$
0	0.8162	0.1044	$(0.8143, 0.2435)^T$	$(1, 1)^T$	18.2798
1	−0.9341	−0.8923	$(0.8543, 0.9936)^T$	$(1, -1)^T$	17.7337
2	0.6101	−0.0973	$(-0.4418, 0.1351)^T$	$(1, 1)^T$	14.8247
3	−0.2347	0.5793	$(1.2007, 0.8715)^T$	$(-1, 1)^T$	15.3254
4	−0.2714	0.0647	$(1.4334, 1.4103)^T$	$(-1, 1)^T$	14.6082
5	0.4233	0.7430	$(0.9129, 2.2190)^T$	$(1, 1)^T$	14.9621
6	−0.3426	0.3002	$(0.9668, 4.7748)^T$	$(1, 1)^T$	15.4244
7	0.9497	−0.8481	$(1.7665, 3.9536)^T$	$(1, 1)^T$	11.2060
8	0.1740	−0.1722	$(1.8078, 4.5132)^T$	$(1, 1)^T$	7.8475
9	−0.3817	−0.4723	$(0.1496, 6.3655)^T$	$(-1, -1)^T$	2.3970
10			$(-2.9880, 1.0914)^T$		−0.2633

TABLE 4. The optimal results of problem (34).

Stage	$x(j)$	u_j^*	$J(x_j, j)$
0	$(0.8143, 0.2435)^T$	$(1, 1)^T$	18.2798
1	$(0.7954, 0.9250)^T$	$(1, -1)^T$	17.6339
2	$(-0.5387, 0.1405)^T$	$(1, 1)^T$	14.8228
3	$(1.1598, 0.7678)^T$	$(-1, 1)^T$	15.2385
4	$(1.3344, 1.2639)^T$	$(-1, 1)^T$	14.3828
5	$(0.7960, 2.1420)^T$	$(1, 1)^T$	14.6790
6	$(0.8773, 4.3988)^T$	$(1, 1)^T$	14.7258
7	$(1.6801, 3.8258)^T$	$(1, 1)^T$	10.8960
8	$(1.6528, 4.1071)^T$	$(1, 1)^T$	7.3687
9	$(0.0907, 5.8203)^T$	$(-1, -1)^T$	2.3574
10	$(-2.8353, 0.8523)^T$		−0.0993

Next, we investigate an optimal control model subject to a multistage system without randomness and uncertainty compared with problem (33).

$$\begin{cases} J(x_0, 0) \\ = \max\limits_{\substack{u(i)\in[-1,1] \\ 0\le i\le 10}} E\left[\sum_{j=0}^{9}(a_j^T x(j) + b_j^T u(j)) + a_{10}^T x(10)\right] \\ \text{subject to} \\ \quad x(j+1) = F_j x(j) + G_j u(j), \\ \quad j = 0, 1, 2, \cdots, 9, \ x(0) = x_0, \end{cases} \tag{34}$$

where the matrices F_j, G_j and vectors a_j, b_j, $x(j)$, $u(j)$ are the same mean as problem (33).

The optimal controls $u^*(j)$ and optimal objective values $J(x_j, j)$ of problem (34) are obtained by Theorem 2 and listed in Table 4.

From Tables 3 and 4, the optimal controls for problem (33) and problem (34) are completely the same, they are illustrated in Figure 1. But the state vectors of two problems are distinctive, the state vectors of the problem (33) and problem (34) are displayed in Figures 2 and 3, respectively. The reason for the same optimal controls is that the expected values of the uncertain variables and uncertain variables are zero. More preciously, with the same initial state, the optimal values $J(x_0, 0)$ of the problem (33) and problem (34) are identical.

Remark 9: Comparing uncertain random optimal control problem (33) with optimal control problem (34) without randomness and uncertainty, we see that problem (34) is just a special case of problem (33). Similarly, we can do a comparison between the uncertain random optimal control

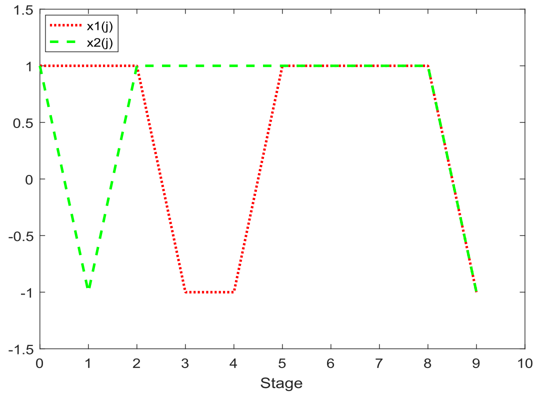

FIGURE 1. Trajectories concerning components of $u(j)$.

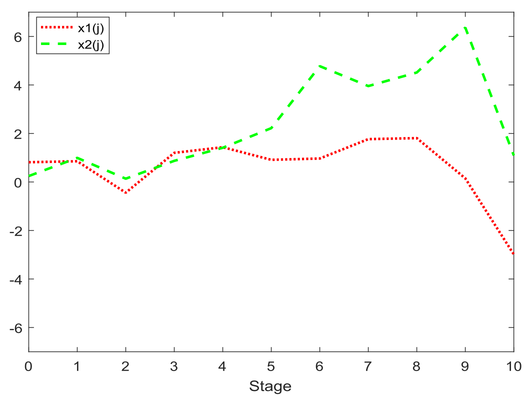

FIGURE 2. Trajectories concerning components of $x(j)$ for (33).

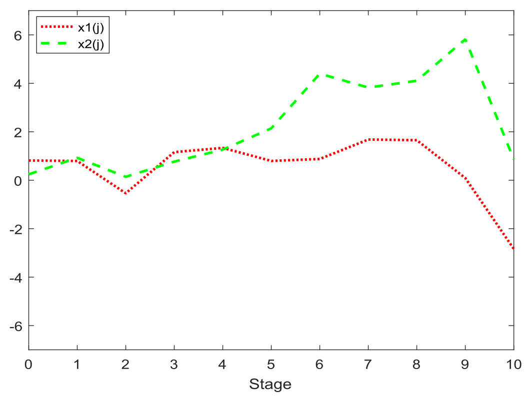

FIGURE 3. Trajectories concerning components of $x(j)$ for (34).

problem (33) and the optimal control problem with randomness or uncertainty.

V. LINEAR QUADRATIC PROBLEM OF OPTIMAL CONTROL MODEL

The linear quadratic optimal control problem is one of the most classic optimal control problems and has been playing a central role in modern control theory. In this section, we study an uncertain random LQ optimal control model where the weighting matrices in the objective function are allowed to be indefinite. To begin with, a linear quadratic optimal control model is introduced as follows

$$\begin{cases} J(\boldsymbol{x}_0, 0) = \min\limits_{\substack{\boldsymbol{u}(j)\in U(j)\\ 0\le j\le N-1}} E\Bigg[\sum\limits_{j=0}^{N-1}(\boldsymbol{x}^T(j)\boldsymbol{A}_j\boldsymbol{x}(j) \\ \qquad\qquad +\boldsymbol{u}^T(j)\boldsymbol{B}_j\boldsymbol{u}(j)) + \boldsymbol{x}^T(N)\boldsymbol{A}_N\boldsymbol{x}(N)\Bigg] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j\boldsymbol{F}_j\eta_j + \beta_j\boldsymbol{F}_j\tau_j)\boldsymbol{x}(j) \\ \qquad\qquad +(\boldsymbol{G}_j + \alpha_j\boldsymbol{G}_j\eta_j + \beta_j\boldsymbol{G}_j\tau_j)\boldsymbol{u}(j), \\ \quad j = 0, 1, \cdots, N-1,\ \boldsymbol{x}(0) = \boldsymbol{x}_0, \end{cases} \tag{35}$$

where $\boldsymbol{A}_j \in \mathbb{R}^{n\times n}$, $\boldsymbol{B}_j \in \mathbb{R}^{r\times r}$ are symmetric matrices, the noises $\eta_j \sim U(-1, 1)$ are independent random variables and $\tau_j \in \mathcal{L}(-1, 1)$ are independent uncertain variables, for $j = 0, 1, 2, \cdots, N-1$.

Since the weighting matrices $\boldsymbol{A}_0, \boldsymbol{A}_1, \cdots, \boldsymbol{A}_N, \boldsymbol{B}_0, \boldsymbol{B}_1, \cdots, \boldsymbol{B}_{N-1}$ are indefinite, the model (35) maybe ill-posed. Therefore, we have the following definitions.

Definition 6: The LQ problem (35) is called well-posed if $J(\boldsymbol{x}_0, 0) > -\infty$, for any $\boldsymbol{x}_0 \in \mathbb{R}^n$.

Definition 7: A well-posed problem is called attainable if there exists a control sequence $(\boldsymbol{u}^*(0), \boldsymbol{u}^*(1), \boldsymbol{u}^*(2), \cdots, \boldsymbol{u}^*(N-1))$ such that achieves $J(\boldsymbol{x}_0, 0)$ for any $\boldsymbol{x}_0 \in \mathbb{R}^n$. In this case $(\boldsymbol{u}^*(0), \boldsymbol{u}^*(1), \boldsymbol{u}^*(2), \cdots, \boldsymbol{u}^*(N-1))$ is called an optimal control sequence.

Before giving the main results of the LQ problem (35), some useful lemmas are presented.

Lemma 4 (Penrose [28]): Let a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ be given. Then there exists a unique matrix $\boldsymbol{A}^+ \in \mathbb{R}^{n\times m}$, which is called the Moore-Penrose pseudo inverse of $\boldsymbol{A}$, such that $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}, \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+, (\boldsymbol{A}\boldsymbol{A}^+)^T = \boldsymbol{A}\boldsymbol{A}^+, (\boldsymbol{A}^+\boldsymbol{A})^T = \boldsymbol{A}^+\boldsymbol{A}$.

Lemma 5 (Penrose [28]): Let a symmetric matrix $\boldsymbol{S}$ be given. Then (*i*) $\boldsymbol{S}^+ = (\boldsymbol{S}^+)^T$; (*ii*) $\boldsymbol{S} \geq \boldsymbol{0}$ if and only if $\boldsymbol{S}^+ \geq \boldsymbol{0}$; (*iii*) $\boldsymbol{S}\boldsymbol{S}^+ = \boldsymbol{S}^+\boldsymbol{S}$.

Lemma 6 (Penrose [28]): Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be given matrices with appropriate sizes, then the matrix equation $\boldsymbol{A}\boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$ have a solution $\boldsymbol{X}$ if and only if $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{C}\boldsymbol{B}^+\boldsymbol{B} = \boldsymbol{C}$. Moreover, any solution to $\boldsymbol{A}\boldsymbol{X}\boldsymbol{B} = \boldsymbol{C}$ is represented by $\boldsymbol{X} = \boldsymbol{A}^+\boldsymbol{C}\boldsymbol{B}^+ + \boldsymbol{Y} - \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{Y}\boldsymbol{B}\boldsymbol{B}^+$, where $\boldsymbol{Y}$ is any matrix with appropriate size.

The main results of the LQ problem (35) are given as follows.

Theorem 4: Suppose the LQ problem (35) is attainable by a feedback control law $\boldsymbol{u}(j) = \boldsymbol{L}_j\boldsymbol{x}_j$ where $\boldsymbol{L}_j \in \mathbb{R}^{r\times n}$ are constant matrices, then there exist symmetric matrices $\boldsymbol{S}_j \in \mathbb{R}^{n\times n}$ satisfy the following constrained difference equation (CDE)

$$\begin{cases} \boldsymbol{S}_j = \boldsymbol{A}_j + (1 + \frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{F}_j^T\boldsymbol{S}_{j+1}\boldsymbol{F}_j - \boldsymbol{M}_j^T\boldsymbol{P}_j^+\boldsymbol{M}_j, \\ \boldsymbol{S}_N = \boldsymbol{A}_N, \\ \boldsymbol{P}_j\boldsymbol{P}_j^+\boldsymbol{M}_j = \boldsymbol{M}_j, \\ \boldsymbol{P}_j = \boldsymbol{B}_j + (1 + \frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{G}_j^T\boldsymbol{S}_{j+1}\boldsymbol{G}_j \geq \boldsymbol{0}, \\ \boldsymbol{M}_j = (1 + \frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{G}_j^T\boldsymbol{S}_{j+1}\boldsymbol{F}_j, \end{cases} \tag{36}$$

where P_j^+ mean the Moore-Penrose inverse of P_j, for $j = 0, 1, 2, \cdots, N-1$. Moreover, the optimal feedback control gain matrices are provided by

$$L_j = -P_j^+ M_j + Y_j - P_j^+ P_j Y_j, \tag{37}$$

with $Y_j \in \mathbb{R}^{r\times n}$ being any given real matrices, for $j = 0, 1, 2, \cdots, N-1$. The optimal values are

$$J(x_j, j) = x_j^T S_j x_j, \tag{38}$$

for $j = 0, 1, 2, \cdots, N$.

Proof: Denote the optimal controls of the LQ problem (35) as $u^*(0), u^*(1), u^*(2), \cdots, u^*(N-1)$. By recurrence equation (11), we have $J(x_N, N) = x_N^T S_N x_N$, where $S_N = A_N$. For $j = N-1$, by recurrence equation (12), we have

$$\begin{aligned} J(x_{N-1}, N-1) &= \min_{u(N-1)\in U(N-1)} E\{x_{N-1}^T A_{N-1} x_{N-1} \\ &+ u^T(N-1) B_{N-1} u(N-1) \\ &+ [F_{N-1}x_{N-1} + G_{N-1}u(N-1)]^T S_N \\ &\cdot [F_{N-1}x_{N-1} + G_{N-1}u(N-1)] \\ &\cdot (1 + \alpha_{N-1}^2\eta_{N-1}^2 + \beta_{N-1}^2\tau_{N-1}^2 + 2\alpha_{N-1}\eta_{N-1} \\ &+ 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1})\}. \end{aligned} \tag{39}$$

Using the linear property of expected value with chance measure, it holds that

$$\begin{aligned} J(x_{N-1}, N-1) &= \min_{u(N-1)\in U(N-1)} \Big\{x_{N-1}^T A_{N-1} x_{N-1} \\ &+ u^T(N-1) B_{N-1} u(N-1) \\ &+ [F_{N-1}x_{N-1} + G_{N-1}u(N-1)]^T S_N \\ &\cdot [F_{N-1}x_{N-1} + G_{N-1}u(N-1)] \cdot \\ &\{1 + \alpha_{N-1}^2 E[\eta_{N-1}^2] + 2\alpha_{N-1}E[\eta_{N-1}] \\ &+ E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} \\ &+ 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}]\}\Big\}. \end{aligned} \tag{40}$$

Since $\eta_{N-1} \sim U(-1, 1)$, it obtains $E[\eta_{N-1}] = 0, E[\eta_{N-1}^2] = \frac{1}{3}$. Next, let us discuss the expected value of uncertain random variable $\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}$.

(i) If $\alpha_{N-1} = \beta_{N-1} = 0$, it's easy to know

$$[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] = 0. \tag{41}$$

(ii) If $\alpha_{N-1} = 0, \beta_{N-1} \neq 0$, we find

$$\begin{aligned} &E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] \\ &= \beta_{N-1}^2 E[\tau_{N-1}^2 + \frac{2}{\beta_{N-1}}\tau_{N-1}]. \end{aligned} \tag{42}$$

Because of $0 <| \beta_{N-1} |\leq 1$, we have $| \frac{2}{\beta_{N-1}} |\geq 2$. By Lemma 1, $E[\tau_{N-1}^2 + \frac{2}{\beta_{N-1}}\tau_{N-1}] = \frac{1}{3}$. Therefore, we obtain

$$\begin{aligned} &E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] \\ &= \frac{1}{3}\beta_{N-1}^2. \end{aligned} \tag{43}$$

(iii) If $\alpha_{N-1} \neq 0, \beta_{N-1} \neq 0$, let Φ_{N-1} be the probability distribution of random variable η_{N-1}. Using Lemma 2, we have

$$\begin{aligned} &E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] \\ &= \int_{-1}^{1} \beta_{N-1}^2 E\left[\tau_{N-1}^2 + \frac{2+2\alpha_{N-1}y_{N-1}}{\beta_{N-1}}\tau_{N-1}\right] \\ &\times d\Phi_{N-1}(y_{N-1}). \end{aligned} \tag{44}$$

We get $| \frac{2+2\alpha_{N-1}y_{N-1}}{\beta_{N-1}} |\geq \frac{2-|2\alpha_{N-1}|}{|\beta_{N-1}|} \geq 2$ with conditions $| \alpha_{N-1} | + | \alpha_{N-1} |\leq 1$ and $y_{N-1} \in [-1, 1]$. By Lemma 1, $E[\tau_{N-1}^2 + \frac{2+2\alpha_{N-1}y_{N-1}}{\beta_{N-1}}\tau_{N-1}] = \frac{1}{3}$. Thus

$$\begin{aligned} &E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] \\ &= \int_{-1}^{1} \frac{1}{3}\beta_{N-1}^2 d\Phi_{N-1}(y_{N-1}) \\ &= \frac{1}{3}\beta_{N-1}^2. \end{aligned} \tag{45}$$

Combining the above three cases, we have

$$\begin{aligned} &E[\beta_{N-1}^2\tau_{N-1}^2 + 2\beta_{N-1}\tau_{N-1} + 2\alpha_{N-1}\beta_{N-1}\eta_{N-1}\tau_{N-1}] \\ &= \frac{1}{3}\beta_{N-1}^2. \end{aligned} \tag{46}$$

Furthermore, we obtain

$$\begin{aligned} J(x_{N-1}, N-1) &= \min_{u(N-1)\in U(N-1)} \{x_{N-1}^T A_{N-1} x_{N-1} \\ &+ u^T(N-1) B_{N-1} u(N-1) \\ &+ (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)[F_{N-1}x_{N-1} \\ &+ G_{N-1}u(N-1)]^T S_N [F_{N-1}x_{N-1} \\ &+ G_{N-1}u(N-1)]\}. \end{aligned} \tag{47}$$

It's assumed that LQ problem (35) is attainable by a feedback control $u(N-1) = L_{N-1}x_{N-1}$, thus

$$\begin{aligned} J(x_{N-1}, N-1) &= \min_{L_{N-1}} \{x_{N-1}^T [A_{N-1} + L_{N-1}^T B_{N-1} L_{N-1} \\ &+ (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)(F_{N-1} + G_{N-1}L_{N-1})^T \\ &\cdot S_N (F_{N-1} + G_{N-1}L_{N-1})] x_{N-1}\}. \end{aligned} \tag{48}$$

Denote

$$\begin{aligned} S_{N-1} &= A_{N-1} + L_{N-1}^T B_{N-1} L_{N-1} \\ &+ (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)(F_{N-1} + G_{N-1}L_{N-1})^T \\ &\cdot S_N (F_{N-1} + G_{N-1}L_{N-1}). \end{aligned} \tag{49}$$

Then it follows from the first-order necessary conditions for optimality that

$$\frac{\partial S_{N-1}}{L_{N-1}}$$

$$= 2(\boldsymbol{B}_{N-1} + (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)\boldsymbol{G}_{N-1}^T\boldsymbol{S}_N\boldsymbol{G}_{N-1}) \cdot \boldsymbol{L}_{N-1} + 2(1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)\boldsymbol{G}_N^T\boldsymbol{S}_N\boldsymbol{F}_{N-1} = \boldsymbol{0}. \tag{50}$$

Let

$$\boldsymbol{P}_{N-1} = \boldsymbol{B}_{N-1} + (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2) \cdot \boldsymbol{G}_{N-1}^T\boldsymbol{S}_N\boldsymbol{G}_{N-1}, \tag{51}$$

$$\boldsymbol{M}_{N-1} = (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)\boldsymbol{G}_N^T\boldsymbol{S}_N\boldsymbol{F}_{N-1}. \tag{52}$$

Then we have $\boldsymbol{P}_{N-1}\boldsymbol{L}_{N-1} + \boldsymbol{M}_{N-1} = \boldsymbol{0}$. By Lemma 6, the equation (50) has a solution if and only if $\boldsymbol{P}_{N-1}\boldsymbol{P}_{N-1}^+ \boldsymbol{M}_{N-1} = \boldsymbol{M}_{N-1}$. Thus, we can get

$$\begin{cases} \boldsymbol{L}_{N-1} = -\boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1} + \boldsymbol{Y}_{N-1} - \boldsymbol{P}_{N-1}^+\boldsymbol{P}_{N-1}\boldsymbol{Y}_{N-1}, \\ \forall \boldsymbol{Y}_{N-1} \in \mathbb{R}^{r\times n}. \end{cases} \tag{53}$$

Substituting (53) into (49), it holds that

$$\boldsymbol{S}_{N-1} = \boldsymbol{A}_{N-1} + (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)\boldsymbol{F}_{N-1}^T \cdot \boldsymbol{S}_N\boldsymbol{F}_{N-1} - \boldsymbol{M}_{N-1}^T\boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}. \tag{54}$$

Therefore, we have

$$J(\boldsymbol{x}_{N-1}, N-1) = \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1}. \tag{55}$$

In the following, we assert that $\boldsymbol{S}_{N-1}$ must satisfy

$$\boldsymbol{P}_{N-1} = \boldsymbol{B}_{N-1} + (1 + \frac{1}{3}\alpha_{N-1}^2 + \frac{1}{3}\beta_{N-1}^2)\boldsymbol{G}_{N-1}^T\boldsymbol{S}_N\boldsymbol{G}_{N-1} \geq \boldsymbol{0}. \tag{56}$$

If not, there is a negative eigenvalue $\mu < 0$ for $\boldsymbol{P}_{N-1}$. Denote the unitary eigenvector with respect to μ by $\boldsymbol{v}_\mu$ (i.e., $\boldsymbol{v}_\mu^T\boldsymbol{v}_\mu = 1$ and $\boldsymbol{P}_{N-1}\boldsymbol{v}_\mu = \mu\boldsymbol{v}_\mu$). Let $\sigma \neq 0$ be an arbitrary scalar and construct a control $\tilde{\boldsymbol{u}}(N-1)$ as follow

$$\tilde{\boldsymbol{u}}(N-1) = \sigma \mid \mu \mid^{\frac{1}{2}} \boldsymbol{v}_\mu - \boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_k. \tag{57}$$

According to lemmas 4, 5, by equations (47), (51), (52), (54), the associated objective function becomes

$$\begin{aligned} J(\boldsymbol{x}_{N-1}, N-1) &= \min_{\boldsymbol{u}(N-1)\in \boldsymbol{U}(N-1)} \{\boldsymbol{x}_{N-1}^T\boldsymbol{M}_{N-1}^T\boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1} \\ &\quad + \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1} \\ &\quad + 2\boldsymbol{u}^T(N-1)\boldsymbol{P}_{N-1}\boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1} \\ &\quad + \boldsymbol{u}^T(N-1)\boldsymbol{P}_{N-1}\boldsymbol{u}(N-1)\} \\ &= \min_{\boldsymbol{u}(N-1)\in \boldsymbol{U}(N-1)} \{[\boldsymbol{u}(N-1) + \boldsymbol{P}_{N-1}^+ \cdot \boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1}]^T\boldsymbol{P}_{N-1} \\ &\quad \cdot [\boldsymbol{u}(N-1) + \boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1}] \\ &\quad + \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1}\} \\ &\leq [\tilde{\boldsymbol{u}}(N-1) + \boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1}]^T\boldsymbol{P}_{N-1}[\tilde{\boldsymbol{u}}(N-1) \\ &\quad + \boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1}\boldsymbol{x}_{N-1}] + \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1} \\ &= [(\sigma \mid \mu \mid^{-\frac{1}{2}} \boldsymbol{v}_\mu)^T\boldsymbol{P}_{N-1}(\sigma \mid \mu \mid^{-\frac{1}{2}} \boldsymbol{v}_\mu) \\ &\quad + \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1} \\ &= -\sigma^2 + \boldsymbol{x}_{N-1}^T\boldsymbol{S}_{N-1}\boldsymbol{x}_{N-1}. \end{aligned} \tag{58}$$

Let $\sigma \to \infty$. Then $J(\boldsymbol{x}_0, \boldsymbol{u}) \to -\infty$, which contradicts the assumption of the theorem.

For $j = N-2$, similar to the calculation of $j = N-1$, we have

$$\begin{aligned} J(\boldsymbol{x}_{N-2}, N-2) &= \min_{\boldsymbol{u}(N-2)\in \boldsymbol{U}(N-2)} \{\boldsymbol{x}_{N-2}^T\boldsymbol{A}_{N-2}\boldsymbol{x}_{N-2} \\ &\quad + \boldsymbol{u}^T(N-2)\boldsymbol{B}_{N-2}\boldsymbol{u}(N-2) \\ &\quad + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)[\boldsymbol{F}_{N-2}\boldsymbol{x}_{N-2} + \boldsymbol{G}_{N-2} \\ &\quad \cdot \boldsymbol{u}(N-2)]^T\boldsymbol{S}_{N-1}[\boldsymbol{F}_{N-2}\boldsymbol{x}_{N-2} + \boldsymbol{G}_{N-2}\boldsymbol{u}(N-2)]\} \\ &= \min_{\boldsymbol{L}_{N-2}} \{\boldsymbol{x}_{N-2}^T[\boldsymbol{A}_{N-2} + \boldsymbol{L}_{N-2}^T\boldsymbol{B}_{N-2}\boldsymbol{L}_{N-2} \\ &\quad + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)(\boldsymbol{F}_{N-2} + \boldsymbol{G}_{N-2}\boldsymbol{L}_{N-2})^T \\ &\quad \cdot \boldsymbol{S}_{N-1}(\boldsymbol{F}_{N-2} + \boldsymbol{G}_{N-2}\boldsymbol{L}_{N-2})]\boldsymbol{x}_{N-2}\}. \end{aligned} \tag{59}$$

Denote

$$\boldsymbol{S}_{N-2} = \boldsymbol{A}_{N-2} + \boldsymbol{L}_{N-2}^T\boldsymbol{B}_{N-2}\boldsymbol{L}_{N-2} + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)(\boldsymbol{F}_{N-2} + \boldsymbol{G}_{N-2}\boldsymbol{L}_{N-2})^T \cdot \boldsymbol{S}_{N-1}(\boldsymbol{F}_{N-2} + \boldsymbol{G}_{N-2}\boldsymbol{L}_{N-2}). \tag{60}$$

According to the necessary conditions for first order optimality, we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{S}_{N-2}}{\boldsymbol{L}_{N-2}} &= 2(\boldsymbol{B}_{N-2} + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)\boldsymbol{G}_{N-2}^T\boldsymbol{S}_{N-1} \\ &\quad \cdot \boldsymbol{G}_{N-2})\boldsymbol{L}_{N-2} + 2(1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2) \\ &\quad \cdot \boldsymbol{G}_{N-2}^T\boldsymbol{S}_{N-1}\boldsymbol{F}_{N-2} \\ &= \boldsymbol{0}. \end{aligned} \tag{61}$$

Then $\boldsymbol{P}_{N-2}\boldsymbol{L}_{N-2} + \boldsymbol{M}_{N-2} = \boldsymbol{0}$ has a solution if and only if $\boldsymbol{P}_{N-1}\boldsymbol{P}_{N-1}^+\boldsymbol{M}_{N-1} = \boldsymbol{M}_{N-1}$ where $\boldsymbol{P}_{N-2} = \boldsymbol{B}_{N-2} + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)\boldsymbol{G}_{N-2}^T\boldsymbol{S}_{N-1}\boldsymbol{G}_{N-2}, \boldsymbol{M}_{N-2}\ (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)\boldsymbol{G}_{N-2}^T\boldsymbol{S}_{N-1}\boldsymbol{F}_{N-2}$. Furthermore, we obtain

$$\begin{cases} \boldsymbol{L}_{N-2} = -\boldsymbol{P}_{N-2}^+\boldsymbol{M}_{N-2} + \boldsymbol{Y}_{N-2} - \boldsymbol{P}_{N-2}^+\boldsymbol{P}_{N-2}\boldsymbol{Y}_{N-2}, \\ \forall \boldsymbol{Y}_{N-2} \in \mathbb{R}^{r\times n}, \end{cases} \tag{62}$$

and

$$\boldsymbol{S}_{N-2} = \boldsymbol{A}_{N-2} + (1 + \frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)\boldsymbol{F}_{N-2}^T\boldsymbol{S}_{N-1} \cdot \boldsymbol{F}_{N-2} - \boldsymbol{M}_{N-2}^T\boldsymbol{P}_{N-2}^+\boldsymbol{M}_{N-2}. \tag{63}$$

So the optimal value is

$$J(\boldsymbol{x}_{N-2}, N-2) = \boldsymbol{x}_{N-2}^T\boldsymbol{S}_{N-2}\boldsymbol{x}_{N-2}. \tag{64}$$

With the similar method to $N-1$, it holds that $\boldsymbol{P}_{N-2} = \boldsymbol{B}_{N-2} + (1+\frac{1}{3}\alpha_{N-2}^2 + \frac{1}{3}\beta_{N-2}^2)\boldsymbol{G}_{N-2}^T \boldsymbol{S}_{N-1}\boldsymbol{G}_{N-2} \geq \boldsymbol{0}$. By induction, the theorem is proved.

Different from [22], we study an LQ optimal model (35) where the weighting matrices in the objective function are allowed to be indefinite. Compared with [3] and [14], we study an optimal control whose system matrices and control matrices are multiplied by uniform random sequences and linear uncertain sequences, and the effects of randomness and uncertainty are displayed by $\frac{1}{3}\alpha_j^2$ and $\frac{1}{3}\beta_j^2$, respectively. From Theorem 1 in [14] and Theorem 4, we can see that both of them are similar in form. Then Theorems 5 and 6 can be easily proved by the same procedures as in Theorems 2 and 3 of [14].

Theorem 5: The LQ problem (35) is well-posed if there exist symmetric matrices $\boldsymbol{S}_j$ satisfying the following LMI condition

$$\begin{bmatrix} \boldsymbol{L}_{11} & \boldsymbol{L}_{12} \\ \boldsymbol{L}_{21} & \boldsymbol{L}_{22} \end{bmatrix} \geq \boldsymbol{0}, \tag{65}$$

with

$$\boldsymbol{L}_{11} = \boldsymbol{A}_j + (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{F}_j^T \boldsymbol{S}_{j+1}\boldsymbol{F}_j - \boldsymbol{S}_j, \tag{66}$$

$$\boldsymbol{L}_{12} = (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{F}_j^T \boldsymbol{S}_{j+1}\boldsymbol{G}_j, \tag{67}$$

$$\boldsymbol{L}_{21} = (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{G}_j^T \boldsymbol{S}_{j+1}\boldsymbol{F}_j, \tag{68}$$

$$\boldsymbol{L}_{22} = \boldsymbol{B}_j + (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{G}_j^T \boldsymbol{S}_{j+1}\boldsymbol{G}_j, \tag{69}$$

for $j = 0, 1, 2, \cdots, N-1$, and $\boldsymbol{S}_N \leq \boldsymbol{A}_N$.

Theorem 6: The following are equivalent

(i) The LQ problem (35) is well-posed.
(ii) The LQ problem (35) is attainable.
(iii) The LMI condition (65) is feasible.
(iv) The CDE (36) is solvable.

Next, we provide a complete characterization of all optimal controls. More preciously, we show that any optimal control can be expressed in terms of the solution to the CDE (36) with two degrees of freedom.

Theorem 7: Suppose $\boldsymbol{S}_j (j = 0, 1, 2, \cdots, N-1)$ solve the CDE (36). Then the set of all optimal controls $\boldsymbol{u}(j)$ for the LQ problem (35) is provided by

$$\begin{cases} \boldsymbol{u}(j) = -(\boldsymbol{P}_j^+\boldsymbol{M}_j + \boldsymbol{Y}_j - \boldsymbol{P}_j^+\boldsymbol{P}_j\boldsymbol{Y}_j)\boldsymbol{x}_j + \boldsymbol{Z}_j - \boldsymbol{P}_j^+\boldsymbol{P}_j\boldsymbol{Z}_j, \\ j = 0, 1, 2, \cdots, N-1, \end{cases} \tag{70}$$

where $\boldsymbol{Y}_j \in \mathbb{R}^{r\times n}$ and $\boldsymbol{Z}_j \in \mathbb{R}^r$ are any given real matrices and real vectors, respectively. Furthermore, the optimal values are

$$J(\boldsymbol{x}_j, j) = \boldsymbol{x}_j^T \boldsymbol{S}_j \boldsymbol{x}_j, \tag{71}$$

for $j = 0, 1, 2, \cdots, N-1, N$.

Proof: Sufficiency. Let $\boldsymbol{S}_j (j = 0, 1, 2, \cdots, N-1)$ solves the CDE (36). Similar proof process as Theorem 4, we have

$$\begin{aligned} J(\boldsymbol{x}_j, j) &= \min_{\boldsymbol{u}(j)\in \boldsymbol{U}(j)} \{\boldsymbol{x}_j^T[\boldsymbol{A}_j + (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{F}_j^T\boldsymbol{S}_{j+1}\boldsymbol{F}_j]\boldsymbol{x}_j \\ &\quad + 2(1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{u}^T(j)\boldsymbol{G}_j^T\boldsymbol{S}_{j+1}\boldsymbol{F}_j\boldsymbol{x}_j \\ &\quad + \boldsymbol{u}^T(j)[\boldsymbol{B}_j + (1+\frac{1}{3}\alpha_j^2 + \frac{1}{3}\beta_j^2)\boldsymbol{G}_j^T\boldsymbol{S}_{j+1}\boldsymbol{G}_j]\boldsymbol{u}(j) \\ &= \min_{\boldsymbol{u}(j)\in \boldsymbol{U}(j)} \{\boldsymbol{x}_j^T\boldsymbol{M}_j^T\boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j + 2\boldsymbol{u}^T(j)\boldsymbol{M}_j\boldsymbol{x}_j \\ &\quad + \boldsymbol{u}^T(j)\boldsymbol{P}_j\boldsymbol{u}(j) + \boldsymbol{x}_j^T\boldsymbol{S}_j\boldsymbol{x}_j\}. \end{aligned} \tag{72}$$

Let $\boldsymbol{T}_j = -(\boldsymbol{Y}_j - \boldsymbol{P}_j^+\boldsymbol{P}_j\boldsymbol{Y}_j)$ and $\tilde{\boldsymbol{T}}_j = -(\boldsymbol{Z}_j - \boldsymbol{P}_j^+\boldsymbol{P}_j\boldsymbol{Z}_j)$. Then we have $\boldsymbol{P}_j\boldsymbol{T}_j = \boldsymbol{0}, \boldsymbol{P}_j\tilde{\boldsymbol{T}}_j = \boldsymbol{0}$. A completion of square implies

$$\begin{aligned} J(\boldsymbol{x}_j, j) &= \min_{\boldsymbol{u}(j)\in \boldsymbol{U}(j)} \{[\boldsymbol{u}(j) + (\boldsymbol{P}_j^+\boldsymbol{M}_j + \boldsymbol{T}_j)\boldsymbol{x}_j + \tilde{\boldsymbol{T}}_j]^T\boldsymbol{P}_j[\boldsymbol{u}(j) \\ &\quad + (\boldsymbol{P}_j^+\boldsymbol{M}_j + T_j^1)\boldsymbol{x}_j + T_j^2]\} + \boldsymbol{x}_j^T\boldsymbol{S}_j\boldsymbol{x}_j. \end{aligned} \tag{73}$$

Since $\boldsymbol{P}_j \geq \boldsymbol{0}$, we know that $\boldsymbol{u}(j) = -[(\boldsymbol{P}_j^+\boldsymbol{M}_j + \boldsymbol{T}_j)\boldsymbol{x}_j + \tilde{\boldsymbol{T}}_j]$ which minimizes the objective function $J(\boldsymbol{x}_j, j)$. Furthermore, the optimal values are $J(\boldsymbol{x}_j, j) = \boldsymbol{x}_j^T\boldsymbol{S}_j\boldsymbol{x}_j$, for $j = 0, 1, 2, \cdots, N-1, N$.

Necessity. If any control sequence $\tilde{\boldsymbol{u}}(j)$ which minimizes the objective function $J(\boldsymbol{x}_j, j)$, thus

$$\begin{aligned} J(\boldsymbol{x}_j, j) &= \min_{\boldsymbol{u}(j)\in \boldsymbol{U}(j)} \{[\boldsymbol{u}(j) + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j]^T\boldsymbol{P}_j[\boldsymbol{u}(j) \\ &\quad + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j]\} + \boldsymbol{x}_j^T\boldsymbol{S}_j\boldsymbol{x}_j \\ &= \boldsymbol{x}_j^T\boldsymbol{S}_j\boldsymbol{x}_j. \end{aligned} \tag{74}$$

The above equality implies that

$$\begin{cases} [\tilde{\boldsymbol{u}}(j) + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j]^T\boldsymbol{P}_j[\tilde{\boldsymbol{u}}(j) + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j] = \boldsymbol{0}, \\ j = 0, 1, 2, \cdots, N-1. \end{cases} \tag{75}$$

As $\boldsymbol{P}_j \geq 0$, we know $\boldsymbol{P}_j = \boldsymbol{C}_j^T\boldsymbol{C}_j$ where $\boldsymbol{C}_j \in \mathbb{R}^{r\times r}$ is a constant matrix. Then, we obtain

$$\boldsymbol{C}_j[\tilde{\boldsymbol{u}}(j) + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j] = \boldsymbol{0}, \tag{76}$$

which means that

$$\boldsymbol{P}_j[\tilde{\boldsymbol{u}}(j) + \boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j] = \boldsymbol{0}. \tag{77}$$

Hence $\tilde{\boldsymbol{u}}(j)$ solves the following equation

$$\boldsymbol{P}_j\tilde{\boldsymbol{u}}(j) + \boldsymbol{P}_j\boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j = \boldsymbol{0}. \tag{78}$$

By Lemma 6.3 with $\boldsymbol{A} = \boldsymbol{P}_j, \boldsymbol{B} = \boldsymbol{I}, \boldsymbol{C} = -\boldsymbol{P}_j\boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j$, we have the following solution of equation (78) with

$$\begin{cases} \tilde{\boldsymbol{u}}(j) = -\boldsymbol{P}_j^+\boldsymbol{M}_j\boldsymbol{x}_j + \boldsymbol{Z}_j - \boldsymbol{P}_j^+\boldsymbol{P}_j\boldsymbol{Z}_j, \\ \forall \boldsymbol{Z}_j \in \mathbb{R}^r, j = 0, 1, 2, \cdots, N-1. \end{cases} \tag{79}$$

Thus the optimal control can be represented by (70).

Example 2: Based on Theorem 4, we consider the following LQ optimal control problems. First, an LQ optimal the control problem is presented below

$$\begin{cases} J(\boldsymbol{x}_0, 0) = \min_{\substack{\boldsymbol{u}(j)\in U(j) \\ 0\le j\le 3}} E\Bigg[\sum_{j=0}^{2}(\boldsymbol{x}^T(j)\boldsymbol{A}_j\boldsymbol{x}(j) \\ \qquad\qquad +\boldsymbol{u}^T(j)\boldsymbol{B}_j\boldsymbol{u}(j)) + \boldsymbol{x}^T(3)\boldsymbol{A}_3\boldsymbol{x}(3)\Bigg] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = (\boldsymbol{F}_j + \alpha_j\boldsymbol{F}_j\eta_j + \beta_j\boldsymbol{F}_j\tau_{1j})\boldsymbol{x}(j) \\ \qquad\qquad +(\boldsymbol{G}_j + \beta_j\boldsymbol{G}_j\eta_j + \beta_j\boldsymbol{G}_j\tau_{1j})\boldsymbol{u}(j), \\ \quad j = 0, 1, 2, \ \boldsymbol{x}(0) = \boldsymbol{x}_0. \end{cases} \tag{80}$$

We study a three-stage system (80) with initial state $\boldsymbol{x}_0 = (0.3163, -0.2279)^T$. The coefficients of the dynamic system are as follows

$$\boldsymbol{F}_0 = \begin{bmatrix} 0.4315 & 0.5201 \\ -0.3349 & 0.4768 \end{bmatrix}, \boldsymbol{F}_1 = \begin{bmatrix} 0.7820 & 0.3403 \\ -0.6957 & 0.1354 \end{bmatrix},$$
$$\boldsymbol{F}_2 = \begin{bmatrix} 0.7178 & 0.5037 \\ 0.3369 & 0.6848 \end{bmatrix}, \boldsymbol{G}_0 = \begin{bmatrix} 0.7813 & 0.5816 \\ 0.4792 & 0.7321 \end{bmatrix},$$
$$\boldsymbol{G}_1 = \begin{bmatrix} 0.6987 & -0.7249 \\ -0.5871 & 0.5679 \end{bmatrix}, \boldsymbol{G}_2 = \begin{bmatrix} 0.3897 & 0.5288 \\ 0.4697 & 0.5721 \end{bmatrix}.$$

And $\alpha_0 = 0.3, \alpha_1 = -0.1, \alpha_2 = 0.2, \beta_0 = 0.2, \beta_1 = -0.1, \beta_2 = 0.3$. The $\eta_j \sim U(-1, 1)$ are independent uniformly distributed random variables and $\tau_j \sim \mathcal{L}(-1, 1)$ are independent linear uncertain variables for $j = 0, 1, 2$. Finally, the state and control weights are

$$\boldsymbol{A}_0 = \begin{bmatrix} 1.3521 & 0 \\ 0 & -0.3847 \end{bmatrix}, \boldsymbol{A}_1 = \begin{bmatrix} 1.5326 & 0 \\ 0 & 0.7712 \end{bmatrix},$$
$$\boldsymbol{A}_2 = \begin{bmatrix} -1.5879 & 0 \\ 0 & -0.6423 \end{bmatrix}, \boldsymbol{A}_3 = \begin{bmatrix} 1.3789 & 0 \\ 0 & 0.7824 \end{bmatrix},$$
$$\boldsymbol{B}_0 = \begin{bmatrix} 0.8429 & 0 \\ 0 & 1.2437 \end{bmatrix}, \boldsymbol{B}_1 = \begin{bmatrix} -0.3417 & 0 \\ 0 & -1.1022 \end{bmatrix},$$
$$\boldsymbol{B}_2 = \begin{bmatrix} 0.5920 & 0 \\ 0 & 1.4328 \end{bmatrix}.$$

We solve the corresponding CDE of the problem (80) stage by stage and construct the optimal feedback control law $\boldsymbol{L}_j$. Finally, we can calculate the optimal results.

Specially, for the CDE (36), the terminal condition is $\boldsymbol{S}_3 = \boldsymbol{A}_3$.

Stage 3: for $j = 2$, we have

$$\boldsymbol{P}_2 = \boldsymbol{B}_2 + (1 + \frac{1}{3}\alpha_2 + \frac{1}{3}\beta_2)\boldsymbol{G}_2^T\boldsymbol{S}_3\boldsymbol{G}_2 = \begin{bmatrix} 0.9906 & 0.5158 \\ 0.5158 & 2.1023 \end{bmatrix} \ge \boldsymbol{0}, \tag{81}$$

$$\boldsymbol{P}_2^+ = \boldsymbol{P}_2^{-1} = \begin{bmatrix} 1.1574 & -0.2840 \\ -0.2840 & 0.5454 \end{bmatrix}, \tag{82}$$

$$\boldsymbol{M}_2 = (1 + \frac{1}{3}\alpha_2 + \frac{1}{3}\beta_2)\boldsymbol{G}_2^T\boldsymbol{S}_3\boldsymbol{F}_2 = \begin{bmatrix} 0.5316 & 0.5450 \\ 0.7034 & 0.7030 \end{bmatrix}, \tag{83}$$

$$\boldsymbol{S}_2 = \boldsymbol{A}_2 + (1 + \frac{1}{3}\alpha_2 + \frac{1}{3}\beta_2)\boldsymbol{F}_2^T\boldsymbol{S}_3\boldsymbol{F}_2 - \boldsymbol{M}_2^T\boldsymbol{P}_2^+\boldsymbol{M}_2 = \begin{bmatrix} -1.1385 & 0.3185 \\ 0.3185 & -0.2901 \end{bmatrix}. \tag{84}$$

The optimal feedback control gain is

$$\boldsymbol{L}_2 = -\boldsymbol{P}_2^+\boldsymbol{M}_2 = \begin{bmatrix} -0.4155 & -0.4311 \\ -0.2326 & -0.2286 \end{bmatrix}. \tag{85}$$

Stage 2: for $j = 1$, we have

$$\boldsymbol{P}_1 = \boldsymbol{B}_1 + (1 + \frac{1}{3}\alpha_1 + \frac{1}{3}\beta_1)\boldsymbol{G}_1^T\boldsymbol{S}_2\boldsymbol{G}_1 = \begin{bmatrix} -1.2649 & 0.9415 \\ 0.9415 & -2.0626 \end{bmatrix} \ge \boldsymbol{0}, \tag{86}$$

$$\boldsymbol{P}_1^+ = \boldsymbol{P}_1^{-1} = \begin{bmatrix} -1.1974 & -0.5466 \\ -0.5466 & -0.7343 \end{bmatrix}, \tag{87}$$

$$\boldsymbol{M}_1 = (1 + \frac{1}{3}\alpha_1 + \frac{1}{3}\beta_1)\boldsymbol{G}_1^T\boldsymbol{S}_2\boldsymbol{F}_1 = \begin{bmatrix} -1.0486 & -0.2830 \\ 1.0692 & 0.2908 \end{bmatrix}, \tag{88}$$

$$\boldsymbol{S}_1 = \boldsymbol{A}_1 + (1 + \frac{1}{3}\alpha_1 + \frac{1}{3}\beta_1)\boldsymbol{F}_1^T\boldsymbol{S}_2\boldsymbol{F}_1 - \boldsymbol{M}_1^T\boldsymbol{P}_1^+\boldsymbol{M}_1 = \begin{bmatrix} 1.2719 & -0.0679 \\ -0.0679 & 0.7307 \end{bmatrix}. \tag{89}$$

The optimal feedback control gain is

$$\boldsymbol{L}_1 = -\boldsymbol{P}_1^+\boldsymbol{M}_1 = \begin{bmatrix} -0.6712 & -0.1800 \\ 0.2120 & 0.0588 \end{bmatrix}. \tag{90}$$

Stage 1: for $j = 0$, we have

$$\boldsymbol{P}_0 = \boldsymbol{B}_0 + (1 + \frac{1}{3}\alpha_0 + \frac{1}{3}\beta_0)\boldsymbol{G}_0^T\boldsymbol{S}_1\boldsymbol{G}_0 = \begin{bmatrix} 1.7750 & 0.8102 \\ 0.8102 & 2.0409 \end{bmatrix} \ge \boldsymbol{0}, \tag{91}$$

$$\boldsymbol{P}_0^+ = \boldsymbol{P}_0^{-1} = \begin{bmatrix} 0.6881 & -0.2732 \\ -0.2732 & 0.5984 \end{bmatrix}, \tag{92}$$

$$\boldsymbol{M}_0 = (1 + \frac{1}{3}\alpha_0 + \frac{1}{3}\beta_0)\boldsymbol{G}_0^T\boldsymbol{S}_2\boldsymbol{F}_0 = \begin{bmatrix} 0.3289 & 0.6694 \\ 0.1375 & 0.6209 \end{bmatrix}, \tag{93}$$

$$\boldsymbol{S}_0 = \boldsymbol{A}_0 + (1 + \frac{1}{3}\alpha_0 + \frac{1}{3}\beta_0)\boldsymbol{F}_0^T\boldsymbol{S}_1\boldsymbol{F}_0 - \boldsymbol{M}_0^T\boldsymbol{P}_0^+\boldsymbol{M}_0 = \begin{bmatrix} 1.6441 & 0.0522 \\ 0.0522 & -0.1995 \end{bmatrix}. \tag{94}$$

The optimal feedback control gain is

$$\boldsymbol{L}_0 = -\boldsymbol{P}_0^+\boldsymbol{M}_0 = \begin{bmatrix} -0.1887 & -0.2910 \\ 0.0075 & -0.1887 \end{bmatrix}. \tag{95}$$

Suppose that we have the initial state $\boldsymbol{x}_0 = (0.3163, -0.2279)^T$. Then the optimal controls $\boldsymbol{u}^*(j) = \boldsymbol{L}_j\boldsymbol{x}(j)$ and optimal objective values $J(\boldsymbol{x}_j, j)$ of problem (80) are obtained by Theorem 4 and listed in Table 5.

Remark 10: In columns 2 and 3 of Table 5, the corresponding states $\boldsymbol{x}(j+1) = (\boldsymbol{F}_j+\alpha_j\boldsymbol{F}_jc_j+\beta_j\boldsymbol{F}_jd_j)\boldsymbol{x}(j)+(\boldsymbol{G}_j+\alpha_j\boldsymbol{G}_jc_j+\beta_j\boldsymbol{G}_jd_j)\boldsymbol{u}(j)$ *with initial state* $\boldsymbol{x}_0 = (0.3163, -0.2279)^T$, *where* c_j *and* d_j *are the realization of random variables* η_j

TABLE 5. The optimal results of problem (80).

Stage	$\boldsymbol{x}(j)$	$\boldsymbol{u}^*(j)$	$J(\boldsymbol{x}(j), j)$
0	$(0.3163, -0.2279)^T$	$(0.0066, 0.0454)^T$	0.1466
1	$(0.0668, -0.2402)^T$	$(-0.0016, 0.0000)^T$	0.0500
2	$(-0.0229, -0.0761)^T$	$(0.0452, 0.0244)^T$	-0.0012
3	$(-0.0285, -0.0263)^T$		0.0017

TABLE 6. The optimal results of problem (96).

Stage	$\boldsymbol{x}(j)$	$\boldsymbol{u}^*(j)$	$J(\boldsymbol{x}(j), j)$
0	$(0.3163, -0.2279)^T$	$(0.0071, 0.0440)^T$	0.1454
1	$(0.0490, -0.1790)^T$	$(-0.0006, -0.0002)^T$	0.0276
2	$(-0.0229, -0.0581)^T$	$(0.0339, 0.0182)^T$	-0.0008
3	$(-0.0228, -0.0211)^T$		0.0011

and uncertain variables τ_j, and generated by $0 < \frac{c_j+1}{2} < 1, 0 < \frac{d_j+1}{2} < 1$ *for* $j = 0, 1, 2, \cdots, 9$.

Next, we investigate an LQ optimal control model subject to a multistage system without randomness and uncertainty compared with problem (80).

$$\begin{cases} J(\boldsymbol{x}_0, 0) = \min\limits_{\substack{\boldsymbol{u}(j)\in U(j)\\ 0\le j\le 3}} E\Bigg[\sum_{j=0}^{2}(\boldsymbol{x}^T(j)\boldsymbol{A}_j\boldsymbol{x}(j) \\ \qquad\qquad +\boldsymbol{u}^T(j)\boldsymbol{B}_j\boldsymbol{u}(j)) + \boldsymbol{x}^T(3)\boldsymbol{A}_3\boldsymbol{x}(3)\Bigg] \\ \text{subject to} \\ \quad \boldsymbol{x}(j+1) = \boldsymbol{F}_j\boldsymbol{x}(j) + \boldsymbol{G}_j\boldsymbol{u}(j), \\ \quad j = 0, 1, 2,\ \boldsymbol{x}(0) = \boldsymbol{x}_0. \end{cases} \tag{96}$$

where the matrices $\boldsymbol{A}_j, \boldsymbol{B}_j, \boldsymbol{F}_j, \boldsymbol{G}_j$ and vectors $\boldsymbol{x}(j), \boldsymbol{u}(j)$ are the same mean as problem (80).

Specially, for the CDE (36), the terminal condition is $\tilde{\boldsymbol{S}}_3 = \boldsymbol{A}_3$.

Stage 3: for $j = 2$, we have

$$\tilde{\boldsymbol{P}}_2 = \boldsymbol{B}_2 + \boldsymbol{G}_2^T\tilde{\boldsymbol{S}}_3\boldsymbol{G}_2 = \begin{bmatrix} 0.9740 & 0.4944 \\ 0.4944 & 2.0745 \end{bmatrix} \ge \boldsymbol{0}, \tag{97}$$

$$\tilde{\boldsymbol{P}}_2^+ = \tilde{\boldsymbol{P}}_2^{-1} = \begin{bmatrix} 1.1680 & -0.2784 \\ -0.2784 & 0.5484 \end{bmatrix}, \tag{98}$$

$$\tilde{\boldsymbol{M}}_2 = \boldsymbol{G}_2^T\tilde{\boldsymbol{S}}_3\boldsymbol{F}_2 = \begin{bmatrix} 0.5095 & 0.5223 \\ 0.6742 & 0.6738 \end{bmatrix}, \tag{99}$$

$$\begin{aligned}\tilde{\boldsymbol{S}}_2 &= \boldsymbol{A}_2 + \boldsymbol{F}_2^T\tilde{\boldsymbol{S}}_3\boldsymbol{F}_2 - \tilde{\boldsymbol{M}}_2^T\tilde{\boldsymbol{P}}_2^+\tilde{\boldsymbol{M}}_2 \\ &= \begin{bmatrix} -1.1499 & 0.3127 \\ 0.3127 & -0.2972 \end{bmatrix}.\end{aligned} \tag{100}$$

The optimal feedback control gain is

$$\tilde{\boldsymbol{L}}_2 = -\tilde{\boldsymbol{P}}_2^+\tilde{\boldsymbol{M}}_2 = \begin{bmatrix} -0.4074 & -0.4225 \\ -0.2279 & -0.2241 \end{bmatrix}. \tag{101}$$

Stage 2: for $j = 1$, we have

$$\begin{aligned}\tilde{\boldsymbol{P}}_1 &= \boldsymbol{B}_1 + \boldsymbol{G}_1^T\tilde{\boldsymbol{S}}_2\boldsymbol{G}_1 \\ &= \begin{bmatrix} -1.2620 & 0.9386 \\ 0.9386 & -2.0597 \end{bmatrix} \ge \boldsymbol{0},\end{aligned} \tag{102}$$

$$\tilde{\boldsymbol{P}}_1^+ = \tilde{\boldsymbol{P}}_1^{-1} = \begin{bmatrix} -1.1986 & -0.5462 \\ -0.5462 & -0.7344 \end{bmatrix}, \tag{103}$$

$$\tilde{\boldsymbol{M}}_1 = \boldsymbol{G}_1^T\tilde{\boldsymbol{S}}_2\boldsymbol{F}_1 = \begin{bmatrix} -1.0452 & -0.2827 \\ 1.0658 & 0.2905 \end{bmatrix}, \tag{104}$$

$$\begin{aligned}\tilde{\boldsymbol{S}}_1 &= \boldsymbol{A}_1 + \boldsymbol{F}_1^T\tilde{\boldsymbol{S}}_2\boldsymbol{F}_1 - \tilde{\boldsymbol{M}}_1^T\tilde{\boldsymbol{P}}_1^+\tilde{\boldsymbol{M}}_1 \\ &= \begin{bmatrix} 1.2721 & -0.0678 \\ -0.0678 & 0.7295 \end{bmatrix}.\end{aligned} \tag{105}$$

The optimal feedback control gain is

$$\tilde{\boldsymbol{L}}_1 = -\tilde{\boldsymbol{P}}_1^+\tilde{\boldsymbol{M}}_1 = \begin{bmatrix} -0.6707 & -0.1801 \\ 0.2118 & 0.0590 \end{bmatrix}. \tag{106}$$

Stage 1: for $j = 0$, we have

$$\tilde{\boldsymbol{P}}_0 = \boldsymbol{B}_0 + \boldsymbol{G}_0^T\tilde{\boldsymbol{S}}_1\boldsymbol{G}_0 = \begin{bmatrix} 1.7362 & 0.7763 \\ 0.7763 & 2.0072 \end{bmatrix} \ge \boldsymbol{0}, \tag{107}$$

$$\tilde{\boldsymbol{P}}_0^+ = \tilde{\boldsymbol{P}}_0^{-1} = \begin{bmatrix} 0.6964 & -0.2693 \\ -0.2693 & 0.6024 \end{bmatrix}, \tag{108}$$

$$\tilde{\boldsymbol{M}}_0 = \boldsymbol{G}_0^T\tilde{\boldsymbol{S}}_2\boldsymbol{F}_0 = \begin{bmatrix} 0.3155 & 0.6414 \\ 0.1322 & 0.5948 \end{bmatrix}, \tag{109}$$

$$\begin{aligned}\tilde{\boldsymbol{S}}_0 &= \boldsymbol{A}_0 + \boldsymbol{F}_0^T\tilde{\boldsymbol{S}}_1\boldsymbol{F}_0 - \tilde{\boldsymbol{M}}_0^T\tilde{\boldsymbol{P}}_0^+\tilde{\boldsymbol{M}}_0 \\ &= \begin{bmatrix} 1.6330 & 0.0519 \\ 0.0519 & -0.2025 \end{bmatrix}.\end{aligned} \tag{110}$$

The optimal feedback control gain is

$$\tilde{\boldsymbol{L}}_0 = -\boldsymbol{P}_0^+\boldsymbol{M}_0 = \begin{bmatrix} -0.1841 & -0.2865 \\ 0.0054 & -0.1855 \end{bmatrix}. \tag{111}$$

Then the optimal controls $\boldsymbol{u}^*(j) = \boldsymbol{L}_j\boldsymbol{x}(j)$ and optimal objective values $J(\boldsymbol{x}_j, j)$ of problem (96) are obtained by Theorem 4 and listed in Table 6.

From Tables 5 and 6, the optimal controls for the problem (80) and problem (96) are distinctive due to the interference of uncertain variables and random variables. The optimal value $J(\boldsymbol{x}_0, 0)$ of problem (80) is greater than that of problem (96). Comparing LQ optimal control problem (96) without randomness and uncertainty, we need to pay a higher price to meet the optimal value of the problem (80). This phenomenon is different from that in Example 1.

Remark 11: Comparing uncertain random LQ optimal control problem (80) with LQ optimal control problem (96) without randomness and uncertainty, we see that problem (96) is just a special case of problem (80). Similarly, we can do the comparison between the uncertain random LQ optimal control problem (80) and LQ optimal control problem with randomness or uncertainty.

VI. CONCLUSION

Different from the separate indeterministic environment such as the stochastic or uncertain situation, this paper considered an optimal control whose system matrices and control matrices are multiplied by random sequence and uncertain sequence. To solve such a model, recurrence equations were presented based on Bellman's principle and chance theory. With the help of recurrence equations, the bang-bang optimal controls for two types of multistage uncertain random systems were obtained. Then we investigated an LQ optimal control problem, allowing the weighting matrices in the objective function to be indefinite. Moreover, numerical examples were given to show the effectiveness of the results obtained.

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XIN CHEN received the B.S. degree in applied mathematics from the College of Information Engineering, Fuyang Normal University, Fuyang, China, in 2016, and the Ph.D. degree in mathematics from the Nanjing University of Science and Technology, Nanjing, China, in 2022. He is currently a Lecturer with the School of Science, Nanjing Forestry University, Nanjing. His major research interests include optimal control and chance theory.

TING JIN received the Ph.D. degree in mathematics from the Nanjing University of Science and Technology, Nanjing, China. She is currently an Associate Professor with the School of Science, Nanjing Forestry University, Nanjing. She has authored or coauthored on several journals, including *Knowledge-Based Systems*, *Applied Soft Computing*, *Energy Conversion and Management*, *Physica A: Statistical Mechanics and its Applications*, *Applied Mathematics and Computation*, and *Chaos, Solitons & Fractals*. Her current research interests include reliability analysis, decision making, and fractional-order differential equation and their applications.