IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received 9 December 2022, accepted 30 December 2022, date of publication 3 January 2023, date of current version 12 January 2023.

Digital Object Identifier 10.1109/ACCESS.2023.3233946

RESEARCH ARTICLE

Robustness Analysis of Fuzzy Cellular Neural Network With Deviating Argument and Stochastic Disturbances

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This work was supported by the Natural Science Foundation of China under Grant 62072164 and Grant 12074111.

ABSTRACT Robustness analysis of fuzzy cellular neural networks with deviating arguments and stochastic disturbances is the main topic of discussion in this paper. The issue at hand is what the upper bounds of the disturbances and deviating intervals for the fuzzy cellular neural network can withstand before losing its stability. We solve these problems by using Gronwall-Bellman lemma and some inequality techniques. The theoretical results point that for an exponentially stable fuzzy cellular neural network, the perturbed fuzzy cellular neural network still keep its globally exponential stability if the upper bound of the length of deviating intervals or the intensity of stochastic disturbances is less than the upper bound derived in this paper. A number of numerical cases are offered to support the availability of conjectural values.

INDEX TERMS Fuzzy cellular neural network, robustness analysis, deviating argument, stochastic disturbances.

I. INTRODUCTION

The appearance of artificial neural network (ANN) aims to simulate biological neurons. Based on ANN, many extensions of ANN have been proposed and be widely used [1], [2], for example, cellular neural network (CNN). In [3] and [4], CNN as a branch of ANN, was first proposed by Chua and Yang, it overcomes the drawbacks of ANN well, this model reduces the numbers of interconnections and keeps the advantages of parallel processing of ANN. Besides, neurons in a CNN only connect to other neurons in a specific area. Based on these properties of CNN, CNN and its extension are widely used in image encryption technology [5], parallel signal processing [6] and so on [7], [8], and [9].

Fuzzy cellular neural network (FCNN) as one of the most important extensions of CNN, was proposed by Yang and Yang et al. [10], [11] in 1996. Due to the existence of fuzzy logic, FCNN model can better describe the uncertain behaviors in practical applications than CNN, so it is very necessary to study the various properties of FCNN. Therefore, FCNN has received extensive attentions in recent decades, there are many classic methods to explore the properties of FCNN, such as Lyapunov theory, Razumikhin-type method, Linear Matrix Inequality method (LMI), etc. References [12], [13], [14], [15], [16], [17], [18], [19], [20], and [21]. Furthermore, in [22], Zhang and Xiang investigated the existence, uniqueness and global asymptotic stability of delayed FCNN by using the properties of M-matrix and the topological degree theory. And in [23], Aravind and Balasubramaniam use the fractional Barbalat's lemma and some inequalities to design a new Lyapunov-Krasovskii functional method to discussed the global asymptotic stability of fractional order complex valued FCNN with impulsive interference attentively. At the same time, the available methods and techniques to explore the robustness of stability of FCNN with disturbances are very limited.

Deviating argument, as one of the disturbance factors, which first appeared at the end of the 18 th century, was initially established to solve geometric problems. In the 1950s, with the repeated analysis of the theory and the combination with various applications, the theory of differential equation system with deviating argument was formed. For this type system, it is a mixture of continuous and discrete systems, which has the properties of difference and differential

The associate editor coordinating the review of this manuscript and approving it for publication was Qi Zhou.

systems. Due to these property of system with deviating argument, it is very interesting to study its dynamic behaviors. For this kind hybrid equations, the traditional definition of the solution of differential equation is no longer applicable. For this problem, Akhmet et al. gave a groundbreaking definition of the solutions of differential equations or neural networks with deviating arguments in [24], [25], and [26]. In addition, stability as a prerequisite for the application of the neural networks with deviating arguments, it has been widely studied, see, [24], [25], [26], [27], [28], [29]. Furthermore, the length of the deviating interval also affects the stability of a dynamical system, in [30] and [31], Zhang and Si further studied the robustness of exponential stability of nonlinear system with deviating argument on the basis of the result of stability. And they estimate the upper bound of the interval length of deviating function. This provides a theoretical basis for the design and application of general neural networks and differential equations with deviating arguments. However, the length of deviating interval that can make FCNN with deviating argument (FCNNDA) to be stable is rarely explored.

On the other hand, in practical applications, there are usually random disturbances that can make the neural networks lose its stability and convergence, just as Haykin pointed in [1], thus, the influence of random disturbances on the system cannot be ignored. In real human neural networks, random disturbances are usually noisy processes caused by random fluctuations and other probabilistic when synapses transmit neurotransmitters. Due to the influence of perturbations, the locus of differential systems will become a stochastic motion [32]. With the development of the theory of stochastic analysis, and its increasingly wide applications in the field of control, many notions are proposed to depict the stability of stochastic processes, see [32]. In [14], Long et al. give new conditions to stabilize delayed SFCNN by presenting a new L-operator inequality with time-varying delays. And Zhang et al. applied stochastic theory to fractional-order SFCNN in [16], and not only the proof of the existence and uniqueness of solutions of fractional SFCNN is given, but also a criterion to guarantee the uniform stability of fractional-order SFCNN. In addition, in [33], Qi et al. designed a sliding mode control law based on stochastic fuzzy model, and proposed a new stochastic stability criterion related to sojourn time for the corresponding sliding mode equation using Lyapunov method. Whereas, the maximum noise intensity that can stabilize SFCNN is rarely estimated.

Based on the above discussions, the purpose of this paper is to investigate the robustness of the exponential stability of FCNN with deviating argument and random disturbances. Roughly speaking, the main works and contributions of this paper are as follows.

- The robustness of the stability of the SFCNN is investigated, and the upper bound of noise to keep the stability of the SFCNN is estimated.
- The property of the deviating function is explored, and the length of the interval of the deviating argument that can maintain the stability of the FCNNDA is estimated.
- The robustness of the stability of FCNN with both deviating argument and random disturbances (SFCNNDA) is investigated, and the upper bounds of the two disturbances are estimated. And the mutual restriction between the two disturbances is pointed out.
- The works of this paper provide a theoretical basis for the design of FCNNs that meets the performance requirements via using Gronwall-Bellman Lemma and inequality techniques.

Finally, we give the rest organizational structures of this paper. We first consider SFCNN without deviating argument, and for an exponentially stable FCNN, we estimate the upper bound of the noise intensity that can make SFCNN remains globally exponentially stable in Section II. In Section III, the model we considered and some assumptions we use are given, concurrently, we explore the impacts of deviating argument on system stability, and give the maximum length of deviating argument intervals. Besides, in Section IV, based on the model given in Section III, we add random perturbations and discuss the influences of both random perturbations and deviating argument on the stability of SFCNNDA. At the same time, we give the upper bound of the noise intensities and the max length of the intervals of deviating argument by solving transcendental equations to maintain its stability. In addition, three instances are given to support our academic consequences in Section V.

Notation: Denote $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N} = \{1, 2, \cdots\}$, and $\mathbb{R}^m$ denotes the space which is made up of all m-dimensional vectors. For a vector $\chi = (\chi_1, \chi_2, \ldots, \chi_m)^T$, we denote $||\chi|| = \sum_{\varsigma=1}^{m} |\chi_\varsigma|$, $\varsigma \in \mathbb{N}$ where $\chi_\varsigma \in \mathbb{R}$. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is a complete filtered probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous filtration and satisfies the usual conditions, that means the space embraces all P-null sets. $\mho(t)$, a scalar Brownian movement, which is defined at $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. And E stands an operator which is used to compute the mathematical expectation for the given probability measure P. Fuzzy AND and fuzzy OR operations are represented by $\bigwedge$ and $\bigvee$, respectively.

II. NOISE EFFECT ON STABILITY

We first consider the following FCNN.

$$\begin{cases} \dot{\Upsilon}_\rho(t) = -\varrho_\rho \Upsilon_\rho(t) + \bigwedge_{\varsigma=1}^{m} \check{\varphi}_{\rho\varsigma} f_\varsigma(\Upsilon_\varsigma(t)) \\ \qquad\qquad + \bigvee_{\varsigma=1}^{m} \check{\omega}_{\rho\varsigma} f_\varsigma(\Upsilon_\varsigma(t)) + \bigwedge_{\varsigma=1}^{m} \mathfrak{I}_{\rho\varsigma} \mathfrak{L}_\varsigma \\ \qquad\qquad + \bigvee_{\varsigma=1}^{m} \mathfrak{I}_{\rho\varsigma} \mathfrak{L}_\varsigma + \mathfrak{Z}_\rho, \\ \Upsilon_\rho(t_0) = \Upsilon_\rho^0, \end{cases} \tag{1}$$

where $\rho, \varsigma \in \mathbb{N}$, $\Upsilon_\rho^0 \in \mathbb{R}$ is the initial value of FCNN (1). $\Upsilon_\rho(t)$, $\Upsilon_\varsigma(t)$, $\mathfrak{L}_\varsigma$ and $\mathfrak{Z}_\rho$ denote the states of FCNN (1) and external inputs, respectively. $\check{\varphi}_{\rho\varsigma}$ and $\mathfrak{I}_{\rho\varsigma}$ are the elements of fuzzy feedback MIN template and fuzzy feed-forward MIN template, respectively. $\check{\omega}_{\rho\varsigma}$, $\mathfrak{I}_{\rho\varsigma}$ are the elements of fuzzy feedback MAX template and fuzzy feed-forward MAX template, respectively. $f_\varsigma(\cdot)$ is the activation function.

Assume $\Upsilon^\star$ is the equilibrium point of FCNN (1), where $\Upsilon^\star = \{\Upsilon_1^\star, \cdots, \Upsilon_m^\star\}^T$, then let $\tilde{\Lambda}(t) = \Upsilon(t) - \Upsilon^\star$ and $\Gamma_\varsigma(\tilde{\Lambda}_\varsigma(t)) = f(\tilde{\Lambda}_\varsigma(t)+\Upsilon_\varsigma^\star) - f(\Upsilon^\star)$, $\tilde{\Lambda}_\rho^0 = \Upsilon_\rho^0 - \Upsilon_\rho^\star$, then, FCNN (1) can be rewritten as

$$\begin{cases} \dot{\tilde{\Lambda}}_\rho(t) = -\varrho_\rho \tilde{\Lambda}_\rho(t) \\ \qquad + \bigwedge_{\varsigma=1}^m \breve{\varphi}_{\rho\varsigma} \Gamma_\varsigma(\tilde{\Lambda}_\varsigma(t)) + \bigvee_{\varsigma=1}^m \breve{\omega}_{\rho\varsigma} \Gamma_\varsigma(\tilde{\Lambda}_\varsigma(t)), \\ \tilde{\Lambda}_\rho(t_0) = \tilde{\Lambda}_\rho^0. \end{cases} \tag{2}$$

Here are the assumptions the activation function $\Gamma_\varsigma(\cdot)$ and deviating function need to satisfy.

$\mathcal{I}(1)$. Assume activation function $\Gamma_\varsigma(t)$ satisfies the following inequality,

$$|\Gamma_\varsigma(u) - \Gamma_\varsigma(v)| \le \mu_\varsigma |u - v|, \quad \varsigma = 1, 2, \cdots$$

and $\Gamma_\varsigma(0) = 0$.

Remark 1: Assumption $\mathcal{I}(1)$ means that the function $\Gamma_\varsigma(\cdot)$ is forced up and down by a linear function. The slope of the linear function does not exceed the lipschitz constant μ_ς. This assumption holds for activation functions of most neural networks, such as tanh(·), sin(·) and so on. Furthermore, if assumption $\mathcal{I}(1)$ holds, then, the following lemma we need to point.

Lemma 1 [16]: If assumption $\mathcal{I}(1)$ holds, the solution $\tilde{\Lambda}(t) = (\tilde{\Lambda}_1(t), \ldots, \tilde{\Lambda}_m)^T$ of FCNN (2) meets the initial condition is unique.

Proof: We can think of FCNN (2) as a particular case of the system in [16], thus, the proof of this lemma is similar to the Theorem 1 in [16]. So we omit it here. □

For simplicity, we call globally exponentially stable, mean square exponentially stable and almost surely globally exponentially stable as GES, MSES and ASGES respectively. Then, we give the definition of GES of FCNN (2).

Definition 1: FCNN (2) is said to be GES if there exist $\nu > 0$, $\vartheta > 0$ such that

$$||\tilde{\Lambda}(t)|| \le \nu ||\tilde{\Lambda}^0|| \exp(-\vartheta(t - t_0)) \tag{3}$$

holds, where $\tilde{\Lambda}(t) = \{\tilde{\Lambda}_1(t), \ldots, \tilde{\Lambda}_m(t)\}^T$ and $\tilde{\Lambda}^0 = \{\tilde{\Lambda}_1(t_0), \cdots, \tilde{\Lambda}_m(t_0)\}^T$ are the state and initial value of FCNN (2) respectively.

Then, based on FCNN (2), we consider the effect of disturbances on FCNN. We consider the following SFCNN.

$$\begin{cases} d\Lambda_\rho(t) = \Big[-\varrho_\rho \Lambda_\rho(t) + \bigwedge_{\varsigma=1}^m \breve{\varphi}_{\rho\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t)) \\ \qquad + \bigvee_{\varsigma=1}^m \breve{\omega}_{\rho\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t))\Big] dt \\ \qquad + \sum_{\varsigma=1}^m \sigma_{\rho\varsigma} \Lambda_\varsigma(t) d\mho(t), \\ \Lambda_\rho(t_0) = \Lambda_\rho^0 = \tilde{\Lambda}_\rho^0, \end{cases} \tag{4}$$

where $\rho, \sigma \in \mathbb{N}$, Λ_ρ^0 is the initial value of SFCNN (4). $\Gamma_\varsigma(\cdot)$ is the ςth activation function.

Then, we can get the following two definitions from [14].

Definition 2 [14]: SFCNN (4) is said to be MSES if $\forall t \ge t_0 \ge 0$, $\Lambda^0 \in \mathbb{R}^n$, the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t} \ln E||\Lambda(t; t_0, \Lambda^0)||^2 < 0 \tag{5}$$

or exist $\zeta > 0$, $\varpi > 0$ such that

$$E||\tilde{\Lambda}(t; t_0, \tilde{\Lambda}^0)||^2 \le \zeta E||\tilde{\Lambda}^0||^2 \exp(-\varpi(t - t_0)). \tag{6}$$

Definition 3 [14]: SFCNN (4) is said to be ASGES, if for all $t \ge t_0 \ge 0$, $\Lambda^0 \in \mathbb{R}^n$, the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t} \ln |\Lambda(t; t_0, \Lambda^0)| < 0 \tag{7}$$

almost sure.

Furthermore, we give another lemma that we need for this paper.

Lemma 2 [10]: Assume $\hat{\Lambda}(t) = \{\hat{\Lambda}_1(t), \ldots, \hat{\Lambda}_m(t)\}^T$ and $\check{\Lambda}(t) = \{\check{\Lambda}_1(t), \ldots, \check{\Lambda}_m(t)\}^T$ are two states of FCNN (2), then

$$\left|\bigwedge_{\varsigma=1}^m \breve{\varphi}_{\rho\varsigma} \Gamma_\varsigma(\hat{\Lambda}_\varsigma(t)) - \bigwedge_{\varsigma=1}^m \breve{\varphi}_{\rho\varsigma} \Gamma_\varsigma(\check{\Lambda}_\varsigma(t))\right| \le \sum_{\varsigma=1}^m \mu_\varsigma |\breve{\varphi}_{\rho\varsigma}| |\hat{\Lambda}_\varsigma(t) - \check{\Lambda}_\varsigma(t)|,$$

$$\left|\bigvee_{\varsigma=1}^m \breve{\omega}_{\rho\varsigma} \Gamma_\varsigma(\hat{\Lambda}_\varsigma(t)) - \bigvee_{\varsigma=1}^m \breve{\omega}_{\rho\varsigma} \Gamma_\varsigma(\check{\Lambda}_\varsigma(t))\right| \le \sum_{\varsigma=1}^m \mu_\varsigma |\breve{\omega}_{\rho\varsigma}| |\hat{\Lambda}_\varsigma(t) - \check{\Lambda}_\varsigma(t)|$$

hold.

From (5), (7), we can easily see that MSES can be derived from ASGES, but vice versa is not true. However, if assumption $\mathcal{I}(1)$ and Lemma 2 hold, the MSES of SFCNN (4) implies the ASGES of SFCNN (4) [34].

Next, let us explore the effect of stochastic perturbations on stability of FCNN firstly.

Theorem 1: If assumption $\mathcal{I}(1)$ and Lemma 2 hold, FCNN (2) is GES, then SFCNN (4) is ASGES if $|\sigma_*| \le \breve{\sigma}_{\rho\varsigma}$, where $\breve{\sigma}_{\rho\varsigma}$ is the unique positive solution of the following transcendental equation.

$$2\nu^2 \exp(-2\vartheta\Delta) + 2\sigma_*^2 \nu^2/\vartheta \exp\{8\Delta[\Delta(\kappa_1 + \kappa_2)^2 + \sigma_*^2]\} = 1, \tag{8}$$

where

$$\Delta > \ln(2\nu^2)/(2\vartheta), \quad \kappa_1 = \max_{1\le\rho\le m} |\varrho_\rho|,$$

$$\kappa_2 = \max_{1\le\rho\le m} \mu_\rho \sum_{\varsigma=1}^m (|\breve{\varphi}_{\varsigma\rho}| + |\breve{\omega}_{\varsigma\rho}|), \quad \sigma_* = \max_{1\le\rho\le m} \sum_{\varsigma=1}^m |\sigma_{\varsigma\rho}|.$$

Proof: From FCNN (2), SFCNN (4) and assumption $\mathcal{I}(2)$, we have

$$\begin{aligned}
&|\varLambda_\rho(t)-\tilde{\varLambda}_\rho(t)| \\
&\leq \int_{t_0}^{t}\Big\{|\varrho_\rho||\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)| \\
&\quad + \bigwedge_{\varsigma=1}^{m}|\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)| \\
&\quad + \bigvee_{\varsigma=1}^{m}|\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)|\Big\}ds \\
&\quad + \sum_{\varsigma=1}^{m}|\sigma_{\rho\varsigma}||\varLambda_\varsigma(s)|d\mho(s).
\end{aligned} \tag{9}$$

Let $\kappa_1=\max_{1\leq\rho\leq m}|\varrho_\rho|$, $\kappa_2=\max_{1\leq\rho\leq m}\mu_\rho\sum_{\varsigma=1}^{m}(|\breve{\varphi}_{\varsigma\rho}|+|\breve{\omega}_{\varsigma\rho}|)$ and $\sigma_*=\max_{1\leq\rho\leq m}\sum_{\varsigma=1}^{m}|\sigma_{\varsigma\rho}|$, thus,

$$\begin{aligned}
&||\varLambda(t)-\tilde{\varLambda}(t)|| \\
&=\sum_{\rho=1}^{m}|\varLambda_\rho(t)-\tilde{\varLambda}_\rho(t)| \\
&\leq \int_{t_0}^{t}\Big\{\sum_{\rho=1}^{m}|\varrho_\rho||\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)|\Big\}ds \\
&\quad + \int_{t_0}^{t}\sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\sigma_{\rho\varsigma}||\varLambda_\varsigma(s)|d\mho(s) \\
&\leq \int_{t_0}^{t}\Big\{\sum_{\rho=1}^{m}|\varrho_\rho||\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)| \\
&\quad + \sum_{\varsigma=1}^{m}\sum_{\rho=1}^{m}|\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)| \\
&\quad + \sum_{\varsigma=1}^{m}\sum_{\rho=1}^{m}|\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\varLambda_\varsigma(s)-\tilde{\varLambda}_\varsigma(s)|\Big\}ds \\
&\quad + \int_{t_0}^{t}\sum_{\varsigma=1}^{m}\sum_{\rho=1}^{m}|\sigma_{\rho\varsigma}||\varLambda_\varsigma(s)|d\mho(s) \\
&\leq \int_{t_0}^{t}\Big\{\sum_{\rho=1}^{m}|\varrho_\rho||\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\breve{\varphi}_{\varsigma\rho}|\mu_\rho|\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\breve{\omega}_{\varsigma\rho}|\mu_\rho|\varLambda_\rho(s)-\tilde{\varLambda}_\rho(s)|\Big\}ds \\
&\quad + \int_{t_0}^{t}\sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m}|\sigma_{\varsigma\rho}||\varLambda_\rho(s)|d\mho(s) \\
&\leq \int_{t_0}^{t}\kappa_1||\varLambda(s)-\tilde{\varLambda}(s)||+\kappa_2||\varLambda(s)-\tilde{\varLambda}(s)||ds \\
&\quad + \sigma_*\int_{t_0}^{t}||\varLambda(s)||d\mho(s).
\end{aligned} \tag{10}$$

Therefore, by *Itô* formula,

$$\begin{aligned}
&||\varLambda(t)-\tilde{\varLambda}(t)||^2 \\
&\leq 2\Big\{\int_{t_0}^{t}\kappa_1||\varLambda(s)-\tilde{\varLambda}(s)||+\kappa_2||\varLambda(s)-\tilde{\varLambda}(s)||ds\Big\}^2 \\
&\quad + 2\sigma_*^2\int_{t_0}^{t}||\varLambda(s)||^2ds \\
&\leq 2(t-t_0)\int_{t_0}^{t}(\kappa_1+\kappa_2)^2||\varLambda(s)-\tilde{\varLambda}(s)||^2ds \\
&\quad + 2\sigma_*^2\int_{t_0}^{t}||\varLambda(s)||^2ds.
\end{aligned} \tag{11}$$

Then, for $t\geq t_0+2\Delta$,

$$\begin{aligned}
&E||\varLambda(t)-\tilde{\varLambda}(t)||^2 \\
&\leq [4\Delta(\kappa_1+\kappa_2)^2+4\sigma_*^2]\int_{t_0}^{t}E||\varLambda(s)-\tilde{\varLambda}(s)||^2ds \\
&\quad + 2\sigma_*^2\nu^2/\vartheta E||\tilde{\varLambda}^0||^2.
\end{aligned} \tag{12}$$

Then for $t_0+\Delta\leq t\leq t_0+2\Delta$, by Gronwall-Bellman inequality,

$$\begin{aligned}
E||\varLambda(t)-\tilde{\varLambda}(t)||^2 &\leq 2\sigma_*^2\nu^2/\vartheta E||\tilde{\varLambda}^0||^2 \\
&\quad \times \exp\{8\Delta[\Delta(\kappa_1+\kappa_2)^2+\sigma_*^2]\}.
\end{aligned} \tag{13}$$

Thus, for $t_0+\Delta\leq t\leq t_0+2\Delta$,

$$\begin{aligned}
&E||\varLambda(t)||^2 \\
&\leq 2E||\tilde{\varLambda}(t)||^2+2E||\varLambda(t)-\tilde{\varLambda}(t)||^2 \\
&\leq \Big\{2\nu^2\exp(-2\vartheta\Delta)+2\sigma_*^2\nu^2/\vartheta \\
&\quad \times \exp\{8\Delta[\Delta(\kappa_1+\kappa_2)^2+\sigma_*^2]\}\Big\}E||\tilde{\varLambda}^0||^2.
\end{aligned} \tag{14}$$

Select

$$\begin{aligned}
\mathfrak{H}(\sigma_*) &= 2\nu^2\exp(-2\vartheta\Delta) \\
&\quad + 2\sigma_*^2\nu^2/\vartheta\exp\{8\Delta[\Delta(\kappa_1+\kappa_2)^2+\sigma_*^2]\}.
\end{aligned}$$

We can easily observe that $\mathfrak{H}(\sigma_*)$ is strictly increasing for σ_*. And noting that $\Delta > \ln(2\nu^2)/(2\vartheta)$, thus, $\mathfrak{H}(0) < 1$. Therefore, there must be a $\breve{\sigma}_{\rho\varsigma}>0$, such that $\mathfrak{H}(\breve{\sigma}_*)=1$.

Then, let $P=-\ln\mathfrak{H}(\sigma_*)/\Delta$, we can see $P>0$, from (14), for $t_0+\Delta\leq t\leq t_0+2\Delta$, we can obtain that

$$E||\varLambda(t)||^2\leq\exp(-P\Delta)E||\tilde{\varLambda}^0||^2. \tag{15}$$

Therefore, from Lemma 1, we can get

$$\begin{aligned}
\varLambda(t;t_0,\varLambda^0) &= \varLambda(t;t_0+(r-1)\Delta, \\
&\qquad \varLambda(t_0+(r-1)\Delta;t_0,\varLambda^0)).
\end{aligned}$$

where r is positive constant. Thus, for $t \geq t_0 + m\Delta$,

$$\begin{aligned} E||\Lambda(t)||^2 &= E||\Lambda(t; t_0 + (q-1)\Delta, \Lambda(t_0 + (q-1)\Delta; t_0, \Lambda^0))||^2 \\ &\leq \exp(-P\Delta)E||\Lambda(t; t_0 + (q-1)\Delta; t_0, \Lambda^0)||^2 \\ &= \exp(-P\Delta)E||\Lambda(t; t_0 + (q-2)\Delta, \\ &\quad \Lambda(t_0 + (q-2)\Delta; t_0, \Lambda^0))||^2 \\ &\quad \cdots \\ &\leq \exp(-qP\Delta)E||\Lambda^0||^2. \end{aligned} \tag{16}$$

Therefore, for any $t > t_0 + \Delta$, there must be a positive constant u such that

$$E||\Lambda(t; t_0, \Lambda^0)||^2 \leq \exp(-P(t-t_0))\exp(P\Delta)E||\Lambda^0||^2 \tag{17}$$

for $t_0 + (u-1)\Delta \leq t \leq t_0 + u\Delta$ holds. And it is obviously that (17) is also hold for $t_0 \leq t \leq t_0 + \Delta$. Thus, SFCNN (4) is ASGES. □

III. DEVIATING ARGUMENT EFFECT ON STABILITY

In this part, we mainly explore the effect of deviating argument, and consider following FCNNDA.

$$\begin{cases} \dot{\Lambda}_\rho(t) = -\varrho_\rho \Lambda_\rho(t) + \bigwedge_{\varsigma=1}^m \breve{\varphi}_{\rho\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))) \\ \qquad\qquad + \bigvee_{\varsigma=1}^m \breve{\omega}_{\rho\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))), \\ \Lambda_\rho(t_0) = \Lambda_\rho^0 = \tilde{\Lambda}_\rho^0, \end{cases} \tag{18}$$

where $\mathfrak{H}(t)$ is deviating function, and $\mathfrak{H}(t) = \mathfrak{H}_k^*$, when $t \in [\aleph_k, \aleph_{k+1})$. $\Gamma_\varsigma(\cdot)$ is the ςth activation function. And exist two sequences $\{\aleph_k\}$, $\{\mathfrak{H}_k^*\}$, $k \in \mathbb{N}$ such that $\aleph_k \leq \mathfrak{H}_k^* < \aleph_{k+1}$, $\forall k \in \mathbb{N}$, and $\aleph_k \to \infty$, $\mathfrak{H}_k^* \to \infty$. Thus, if $\aleph_k \leq t \leq \mathfrak{H}_k^* < \aleph_{k+1}$, the FCNNDA (18) is an advanced system. On the contrary, if $\aleph_k \leq \mathfrak{H}_k^* \leq t < \aleph_{k+1}$, (18) is a delayed system. Therefore, FCNNDA (18) is a mix type of advanced and delayed system that means the dynamic behavior of FCNNDA (18) depends not only on its past values, but also on its advanced values. This type of system (18) can be used to simulate the behaviours of pathodynamical system [27].

Here are some assumptions for $\mathfrak{H}(t)$ need to satisfy.

$\mathcal{I}(2)$. There is a $\aleph > 0$ such that $\aleph_{k+1} - \aleph_k < \aleph$, $k \in \mathbb{N}$.

$\mathcal{I}(3)$. $\aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2] < 1$.

$\mathcal{I}(4)$. $\nu\exp(-\vartheta\Delta) + 2\kappa_2\nu/\vartheta \exp[2\Delta(\kappa_1 + 3\kappa_2)] < 1$.

Remark 2: As we all know, for deviating argument, the deviating interval affects the stability of the perturbed neural network. Therefore, assumption $\mathcal{I}(2)$ guarantees that the deviating interval of the perturbed neural network is bounded. Besides, the relationship between the states $\Lambda(\mathfrak{H}(t))$ and $\Lambda(t)$ is guaranteed by assumption $\mathcal{I}(3)$.

Firstly, we explore the relationship between the state $\Lambda(t)$ and $\Lambda(\mathfrak{H}(t))$.

Lemma 3: Let $\mathcal{I}(1)$-$\mathcal{I}(3)$ hold, then the following inequality

$$||\Lambda(\mathfrak{H}(t))|| \leq \Xi||\Lambda(t)||, \quad \forall t \in \mathbb{R}^+ \tag{19}$$

holds, where $\Lambda(t)$ is a solution of FCNNDA (18) and

$$\Xi = \left\{1 - \aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2]\right\}^{-1},$$

$$\kappa_1 = \max_{1\leq\rho\leq m} |\varrho_\rho|, \quad \kappa_2 = \max_{1\leq\rho\leq m} \mu_\rho \sum_{\varsigma=1}^m (|\breve{\varphi}_{\varsigma\rho}| + |\breve{\omega}_{\varsigma\rho}|).$$

Proof: Since $k \in \mathbb{N}$, for any $t \in [\aleph_k, \aleph_{k+1})$, from assumptions $\mathcal{I}(1)$ and Lemma 2, we have

$$\begin{aligned} \sum_{\rho=1}^m |\Lambda_\rho(t)| = \sum_{\rho=1}^m |\Lambda_\rho(\mathfrak{H}_k^*)| + \sum_{\rho=1}^m \int_{\mathfrak{H}_k^*}^t \Big[|\varrho_\rho||\Lambda_\rho(t)| \\ + \sum_{\varsigma=1}^m |\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}_k^*)| \\ + \sum_{\varsigma=1}^m |\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}_k^*)| \Big] ds. \end{aligned} \tag{20}$$

Further, we can get

$$\begin{aligned} ||\Lambda(t)|| &\leq ||\Lambda(\mathfrak{H}_k^*)|| + \int_{\mathfrak{H}_k^*}^t \Big[\kappa_1||\Lambda(t)|| + \kappa_2||\Lambda(\mathfrak{H}_k^*)||\Big] ds \\ &\leq (1 + \kappa_2\aleph)||\Lambda(\mathfrak{H}_k^*)|| + \int_{\mathfrak{H}_k^*}^t \kappa_1||\Lambda(t)||ds. \end{aligned} \tag{21}$$

Using Gronwall-Bellman inequality, we have

$$||\Lambda(t)|| \leq (1 + \kappa_2\aleph)||\Lambda(\mathfrak{H}_k^*)|| \exp(\kappa_1\aleph). \tag{22}$$

Similarly,

$$\begin{aligned} &||\Lambda(\mathfrak{H}_k^*)|| \\ &\leq ||\Lambda(t)|| + \int_{\mathfrak{H}_k^*}^t \Big[\kappa_1||\Lambda(t)|| + \kappa_2||\Lambda(\mathfrak{H}_k^*)||\Big] dt \\ &\leq ||\Lambda(t)|| + \aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2]||\Lambda(\mathfrak{H}_k^*)|| \\ &\leq \Xi||\Lambda(t)||. \end{aligned} \tag{23}$$

where $\Xi = \left\{1 - \aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2]\right\}^{-1}$.

Since t and k are random constant, thus, (23) holds for $t \in [\aleph_k, \aleph_{k+1})$. □

Based on Lemma 3, we investigated the effect on stability of FCNN disturbed by deviating argument.

Theorem 2: Let assumptions $\mathcal{I}(1)$-$\mathcal{I}(4)$ hold, and FCNN (2) is GES, then the FCNNDA is also GES if $\aleph < \min\{\Delta/2, \bar{\bar{\aleph}}\}$ and $\bar{\bar{\aleph}}$ is the unique solution of the following transcendental equation.

$$\begin{aligned} &\nu\exp(-\vartheta(\Delta - \aleph)) + \nu/\vartheta\kappa_2\Big\{1 + (1 - \aleph[\kappa_1(1 \\ &\quad + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2])^{-1}\Big\}\exp\Big\{2\Delta\Big(\kappa_1 + 2\kappa_2 \\ &\quad + \kappa_2\Big[1 + (1 - \aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) \\ &\quad + \kappa_2])^{-1}\Big]\Big)\Big\} = 1, \end{aligned} \tag{24}$$

where $\Delta > \ln\upsilon/\vartheta$.

Proof: For any $t \geq t_0 \geq 0$, from (2), (4), thus,

$$\begin{aligned}
&\sum_{\rho=1}^{m} |\Lambda_\rho(t) - \tilde{\Lambda}_\rho(t)| \\
&\leq \int_{t_0}^{t} \Bigg[\sum_{\rho=1}^{m} |\varrho_\rho||\Lambda_\rho(s) - \tilde{\Lambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m} |\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}(s)) - \tilde{\Lambda}_\varsigma(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m} |\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}(s)) - \tilde{\Lambda}_\varsigma(s)| \Bigg] ds \\
&\leq \int_{t_0}^{t} \Bigg[\sum_{\rho=1}^{m} |\varrho_\rho||\Lambda_\rho(s) - \tilde{\Lambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m} |\breve{\varphi}_{\varsigma\rho}|\mu_\rho|\Lambda_\rho(\mathfrak{H}(s)) - \tilde{\Lambda}_\rho(s)| \\
&\quad + \sum_{\rho=1}^{m}\sum_{\varsigma=1}^{m} |\breve{\omega}_{\varsigma\rho}|\mu_\rho|\Lambda_\rho(\mathfrak{H}(s)) - \tilde{\Lambda}_\rho(s)| \Bigg] ds. \qquad (25)
\end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned}
&||\Lambda(t) - \tilde{\Lambda}(t)|| \\
&\leq \int_{t_0}^{t} \Big[(\kappa_1 + \kappa_2)||\Lambda(s) - \tilde{\Lambda}(s)|| \\
&\quad + \kappa_2||\Lambda(\mathfrak{H}(s)) - \Lambda(s)|| \Big] ds \\
&\leq (\kappa_1 + 2\kappa_2 + \kappa_2\Xi) \int_{t_0}^{t} ||\Lambda(s) - \tilde{\Lambda}(s)||ds \\
&\quad + \kappa_2(1 + \Xi)\nu/\vartheta||\tilde{\Lambda}^0||. \qquad (26)
\end{aligned}$$

Thus, by applying Gronwall-Bellman lemma, for $t_0 + \aleph \leq t \leq t_0 + 2\Delta$, we have

$$\begin{aligned}
&||\Lambda(t) - \tilde{\Lambda}(t)|| \\
&\leq \kappa_2(1 + \Xi)\nu/\vartheta||\tilde{\Lambda}^0|| \\
&\quad \times \exp[2\Delta(\kappa_1 + 2\kappa_2 + \kappa_2\Xi)]. \qquad (27)
\end{aligned}$$

Noting that $\aleph \leq \Delta/2$, for $t_0 - \aleph + \Delta \leq t \leq t_0 - \aleph + 2\Delta$,

$$\begin{aligned}
||\Lambda(t)|| &\leq ||\Lambda(t) - \tilde{\Lambda}(t)|| + ||\tilde{\Lambda}(t)|| \\
&\leq \Big\{ \nu\exp(-\vartheta(\Delta - \aleph)) + \kappa_2(1 + \Xi)\nu/\vartheta \\
&\quad \times \exp[2\Delta(\kappa_1 + 2\kappa_2 + \kappa_2\Xi)] \Big\} ||\tilde{\Lambda}^0||. \qquad (28)
\end{aligned}$$

Let $\mathfrak{F}(\aleph) = \nu\exp(-\vartheta(\Delta - \aleph)) + \kappa_2(1 + \Xi)\nu/\vartheta \exp[2\Delta(\kappa_1 + 2\kappa_2 + \kappa_2\Xi)]$, $\mathfrak{E}(\aleph) = \aleph[\kappa_1(1 + \kappa_2\aleph)\exp(\kappa_1\aleph) + \kappa_2]$, then, we can easily observe that $\mathfrak{E}(\aleph)$ is strictly increasing for $\aleph$, thus, there must be a $\bar{\aleph}$ such that $\mathfrak{E}(\bar{\aleph}) = 1$. On the other hand, $\mathfrak{F}(\aleph)$ is also strictly increasing for $\aleph$ and from $\mathcal{I}(4)$ we have $\mathfrak{F}(0) < 1$, thus, there must be another positive constant $\bar{\bar{\aleph}} \in (0, \bar{\aleph})$ such that $\mathfrak{F}(\bar{\bar{\aleph}}) = 1$. Since $\mathfrak{F}(\aleph)$ is also increasing for $\aleph$ on interval $(0, \bar{\aleph})$, therefore, we can know that $\mathfrak{F}(\aleph) < 1$, when $\aleph < \bar{\bar{\aleph}}$. Then, from what has been discussed above, we know that $\mathfrak{F}(\aleph) < 1$, when $\aleph < \min\{\Delta/2, \bar{\bar{\aleph}}\}$.

Setting $K = -\ln\mathfrak{F}/\Delta$, then for $t_0 - \aleph + \Delta \leq t \leq t_0 - \aleph + 2\Delta$ we have

$$||\Lambda(t)|| \leq \exp(-\Delta K)||\Lambda^0||. \qquad (29)$$

From Lemma 1, we have

$$\begin{aligned}
\Lambda(t; t_0, \Lambda^0) = \Lambda(t; t_0 + (m-1)\Delta, \\
\Lambda(t_0 + (m-1)\Delta; t_0, \Lambda^0)).
\end{aligned}$$

where m is positive constant. Thus, for $t \geq t_0 - \aleph + m\Delta$,

$$\begin{aligned}
&||\Lambda(t)|| \\
&= ||\Lambda(t; t_0 + (m-1)\Delta, \Lambda(t_0 + (m-1)\Delta; t_0, \Lambda^0))|| \\
&\leq \exp(-\Delta K)||\Lambda(t; t_0 + (m-1)\Delta; t_0, \Lambda^0)|| \\
&= \exp(-\Delta K)||\Lambda(t; t_0 + (m-2)\Delta, \\
&\quad \Lambda(t_0 + (m-2)\Delta; t_0, \Lambda^0))|| \\
&\quad \ldots \\
&\leq \exp(-m\Delta K)||\Lambda^0||. \qquad (30)
\end{aligned}$$

Therefore, for any $t > t_0 - \aleph + \Delta$, there must be an $l > 0$ such that

$$\begin{aligned}
||\Lambda(t; t_0, \Lambda^0)|| \leq \exp(-K(t - t_0)) \\
\times \exp(K(\Delta - \aleph))||\Lambda^0||, \qquad (31)
\end{aligned}$$

for $t_0 - \aleph + (l-1)\Delta \leq t \leq t_0 - \aleph + l\Delta$ holds. And it is obviously that (31) is also hold for $t_0 \leq t \leq t_0 - \aleph + \Delta$. Thus, FCNNDA (18) is also GES. □

Remark 3: Assumption $\mathcal{I}(4)$ guarantees that the derived transcendental equation (24) has real solutions on the interval $(0, +\infty)$.

IV. THE EFFECT OF DEVIATING ARGUMENT AND STOCHASTIC DISTURBANCES

In this part, we will consider random perturbations on the basis of FCNNDA (18) and explore the influence of both random perturbation and deviating argument on the stability of FCNNDA (18). Then, the model of SFCNNDA is as follows.

$$\begin{cases}
d\Lambda_\rho(t) = \Big[-\varrho_\rho\Lambda_\rho(t) + \bigwedge_{\varsigma=1}^{m} \breve{\varphi}_{\rho\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t)) \\
\qquad + \bigvee_{\varsigma=1}^{m} \breve{\omega}_{\rho\varsigma}\Gamma_\varsigma(\Lambda_j(\mathfrak{H}(t)) \Big] dt \\
\qquad + \sum_{\varsigma=1}^{m} \sigma_{\rho\varsigma}\Lambda_\varsigma(t)d\mho(t), \\
\Lambda_\rho(t_0) = \Lambda_\rho^0 = \tilde{\Lambda}_\rho^0,
\end{cases} \qquad (32)$$

where $\sigma = (\sigma_{\rho\varsigma})_{m\times m}$ is a matrix of diffusion coefficients.

For the purpose of investigating the impact of deviating argument and random disturbances on stability, we give two assumptions we will use in the follows.

$\mathcal{I}(5)$.

$$\begin{aligned}
\phi = 6\aleph^2\kappa_2^2 + 9\aleph(2\aleph\kappa_1^2 + \sigma_*^2) \\
\times (1 + 2\aleph^2\kappa_2^2)\exp[\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)] < 1.
\end{aligned}$$

$\mathcal{I}(6)$.

$$2\nu^2 \exp(-2\vartheta\Delta) + 256\kappa_2^2\Delta\nu^2/\vartheta \times \exp\{16\Delta^2(\kappa_1^2 + 34\kappa_2^2)\} < 1.$$

And the relation between $E||\Lambda(t)||^2$ and $E||\Lambda(\mathfrak{H}(t))||^2$ is unmasked as follows.

Lemma 4: Let assumptions $\mathcal{I}(1)$, $\mathcal{I}(2)$ and $\mathcal{I}(5)$ hold, then $\forall t \in \mathbb{R}^+$, such that

$$E||\Lambda(\mathfrak{H}(t))||^2 \leq \xi E||\Lambda(t)||^2 \tag{33}$$

holds, where

$$\begin{aligned}\xi &= 3(1-\phi)^{-1},\\ \phi &= 6\aleph^2\kappa_2^2 + 9\aleph(2\aleph\kappa_1^2 + \sigma_*^2)\\ &\quad \times (1 + 2\aleph^2\kappa_2^2)\exp[\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)],\end{aligned}$$

and $\Lambda(t)$ is the solution of SFCNNDA (32).

Proof: Since $t \in \mathbb{R}^+$, and $k \in \mathbb{N}$, thus, there exists $k \in \mathbb{R}^+$, for all $t \in [\aleph_k, \aleph_{k+1})$, such that $\mathfrak{H}(t) = \mathfrak{H}_k^*$. Then by assumption $\mathcal{I}(1)$ and Lemma 2, we can obtain that

$$\begin{aligned}&\sum_{\rho=1}^{m}|\Lambda_\rho(t)|\\ &= ||\Lambda(t)||\\ &\leq \sum_{\rho=1}^{m}|\Lambda_\rho(\mathfrak{H}_k^*)| + \sum_{\rho=1}^{m}\int_{\mathfrak{H}_k^*}^{t}\Big[|\varrho_\rho||\Lambda_\rho(s)|\\ &\quad + \sum_{\varsigma=1}^{m}\mu_\varsigma|\breve{\varphi}_{\rho\varsigma}||\Lambda_\varsigma(\mathfrak{H}_k^*)| + \sum_{\varsigma=1}^{m}\mu_\varsigma|\breve{\omega}_{\rho\varsigma}||\Lambda_\varsigma(\mathfrak{H}_k^*)|\Big]ds\\ &\quad + \sum_{\rho=1}^{m}\int_{\mathfrak{H}_k^*}^{t}\sum_{\varsigma=1}^{m}|\sigma_{\rho\varsigma}||\Lambda_\varsigma(s)|d\mho(s)\\ &\leq ||\Lambda(\mathfrak{H}_k^*)|| + \int_{\mathfrak{H}_k^*}^{t}\Big[\kappa_1||\Lambda(s)|| + \kappa_2||\Lambda(\mathfrak{H}_k^*)||\Big]ds\\ &\quad + \int_{\mathfrak{H}_k^*}^{t}\sigma_*||\Lambda(s)||d\mho(s).\end{aligned} \tag{34}$$

Thus,

$$\begin{aligned}&E||\Lambda(t)||^2\\ &\leq 3E||\Lambda(\mathfrak{H}_k^*)||^2 + 6\aleph\int_{\mathfrak{H}_k^*}^{t}\kappa_1^2E||\Lambda(s)||^2\\ &\quad + \kappa_2^2E||\Lambda(\mathfrak{H}_k^*)||^2ds + 3\sigma_*^2\int_{\mathfrak{H}_k^*}^{t}E||\Lambda(s)||^2ds\\ &\leq (6\aleph\kappa_1^2 + 3\sigma_*^2)\int_{\mathfrak{H}_k^*}^{t}E||\Lambda(s)||^2ds\\ &\quad + (3 + 6\aleph^2\kappa_2^2)E||\Lambda(\mathfrak{H}_k^*)||^2.\end{aligned} \tag{35}$$

Then, by Gronwall-Bellman lemma,

$$E||\Lambda(t)||^2 \leq (3 + 6\aleph^2\kappa_2^2)E||\Lambda(\mathfrak{H}_k^*)||^2 \times \exp\{\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)\}. \tag{36}$$

Similarly,

$$\begin{aligned}&E||\Lambda(\mathfrak{H}_k^*)||^2\\ &\leq 3E||\Lambda(t)||^2 + 6\aleph^2\kappa_2^2E||\Lambda(\mathfrak{H}_k^*)||^2\\ &\quad + \Big\{\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)(3 + 6\aleph^2\kappa_2^2)\\ &\quad \times \exp[\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)]\Big\}E||\Lambda(\mathfrak{H}_k^*)||^2\\ &\leq 3E||\Lambda(t)||^2 + \phi E||\Lambda(\mathfrak{H}_k^*)||^2\\ &\leq 3(1-\phi)^{-1}E||\Lambda(t)||^2,\end{aligned} \tag{37}$$

where

$$\phi = 6\aleph^2\kappa_2^2 + 9\aleph(2\aleph\kappa_1^2 + \sigma_*^2) \times (1 + 2\aleph^2\kappa_2^2)\exp[\aleph(6\aleph\kappa_1^2 + 3\sigma_*^2)].$$

Therefore,

$$E||\Lambda(\mathfrak{H}_k^*)||^2 \leq \xi E||\Lambda(t)||^2 \tag{38}$$

holds for all $t \geq t_0$, where $\xi = 3(1-\phi)^{-1}$.

Since t and k are random, thus, $\forall t \in \mathbb{R}^+$ (38) holds. □

Remark 4: Through the proof process, we can find that if assumption $\mathcal{I}(5)$ not hold, then $\xi < 0$, this is contrary to the fact that $E||\Lambda(\mathfrak{H})_k^*||^2 \geq 0$.

Theorem 3: Let assumptions $\mathcal{I}(1)$, $\mathcal{I}(2)$, $\mathcal{I}(5)$ and $\mathcal{I}(6)$ hold, and FCNN (2) *is GES, then SFCNNDA* (32) *is MSES if $|\sigma_*| < \bar{\sigma}_*/\sqrt{2}$ and $\aleph < \min\{\Delta/2, \hat{\aleph}\}$, where $\bar{\sigma}_*$ and $\hat{\aleph}$ are the solutions of following two transcendental equations.*

$$2\nu^2\exp(-2\vartheta\Delta) + 4[64k_2^2\Delta + \bar{\sigma}_*^2]\nu^2/\vartheta \times \exp\{16\Delta^2(k_1^2 + 34k_2^2) + 8\Delta\bar{\sigma}_*\} = 1 \tag{39}$$

and

$$2\nu^2\exp(-2\vartheta(\Delta - \hat{\aleph})) + 2\tilde{\mathfrak{S}}\exp(2\tilde{\mathfrak{Q}}\Delta) = 1. \tag{40}$$

where κ_1 and κ_2 are as same as we defined in Theorem 1,

$$\begin{aligned}\Delta &> \ln(2\nu^2)/(2\vartheta),\\ \tilde{\mathfrak{S}} &= \{32k_2^2\Delta[1 + 3(1 - 6\hat{\aleph}^2k_2^2 + 9\hat{\aleph}(2\hat{\aleph}k_1^2 + \sigma_*^2/2)\\ &\quad \times (1 + 2\hat{\aleph}^2k_2^2)\exp[\hat{\aleph}(6\hat{\aleph}k_1^2 + 1.5\bar{\sigma}_*^2)])^{-1}]\}\nu^2/\vartheta,\\ \tilde{\mathfrak{Q}} &= 8\Delta(k_1^2 + 2k_2^2) + 64k_2^2\Delta[1 + 3(1 - 6\hat{\aleph}^2k_2^2\\ &\quad + 9\hat{\aleph}(2\hat{\aleph}k_1^2 + \sigma_*^2/2)(1 + 2\hat{\aleph}^2k_2^2)\\ &\quad \times \exp[\hat{\aleph}(6\hat{\aleph}k_1^2 + 1.5\bar{\sigma}_*^2)])^{-1}] + 2\sigma_*^2.\end{aligned}$$

Proof: From (2), (32), we have

$$\begin{aligned}&\sum_{\rho=1}^{m}|\Lambda_\rho(t) - \tilde{\Lambda}_\rho(t)|\\ &\leq \sum_{\rho=1}^{m}\int_{t_0}^{t}\Big[|\varrho_\rho||\Lambda_\rho(s) - \tilde{\Lambda}_\rho(s)|\\ &\quad + \sum_{\varsigma=1}^{m}|\breve{\varphi}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}(s)) - \tilde{\Lambda}_\varsigma(s)|\end{aligned}$$

$$+\sum_{\varsigma=1}^{m}|\breve{\omega}_{\rho\varsigma}|\mu_\varsigma|\Lambda_\varsigma(\mathfrak{H}(s))-\tilde{\Lambda}_\varsigma(s)|\Bigg]ds$$
$$+\sum_{\rho=1}^{m}\int_{t_0}^{t}\left|\sum_{\varsigma=1}^{m}\sigma_{\rho\varsigma}\Lambda_\varsigma(s)\right|d\mho(s). \quad (41)$$

Then, we have

$$\begin{aligned}&E||\Lambda(t)-\tilde{\Lambda}(t)||^2\\&\le 4(t-t_0)\int_{t_0}^{t}\Big[\kappa_1^2E||\Lambda(s)-\tilde{\Lambda}(s)||^2\\&\quad+\kappa_2^2E||\Lambda(\mathfrak{H}(s))-\tilde{\Lambda}(s)||^2\Big]ds\\&\quad+2\sigma_*^2\int_{t_0}^{t}E||\Lambda(s)||^2d\mho(s)\\&\le\Big\{4(t-t_0)(\kappa_1^2+2\kappa_2^2)+32\kappa_2^2(t-t_0)(1+\xi)\\&\quad+4\sigma_*^2\Big\}\int_{t_0}^{t}E||\Lambda(s)-\tilde{\Lambda}(s)||^2ds\\&\quad+[16\kappa_2^2(t-t_0)(1+\xi)+2\sigma_*^2]\nu^2/\vartheta E||\tilde{\Lambda}^0||^2. \quad (42)\end{aligned}$$

Let $\mathfrak{Q}=8\Delta(\kappa_1^2+2\kappa_2^2)+64\kappa_2^2\Delta(1+\xi)+4\sigma_*^2$, $\mathfrak{S}=[32\kappa_2^2\Delta(1+\xi)+2\sigma_*^2]\nu^2/\vartheta$. Using Gronwall-Bellman lemma for (42), then for $t_0+\aleph\le t\le t_0+2\Delta$

$$E||\Lambda(t)-\tilde{\Lambda}(t)||^2\le\mathfrak{S}E||\Lambda^0||^2\exp(2\mathfrak{Q}\Delta). \quad (43)$$

Noting that $\aleph<\Delta/2$, therefore, for $t_0-\aleph+\Delta\le t\le t_0-\aleph+2\Delta$,

$$E||\Lambda(t)||^2\le\Big\{2\nu^2\exp(-2\vartheta(\Delta-\aleph))+2\mathfrak{S}\exp(2\mathfrak{Q}\Delta)\Big\}E||\Lambda^0||^2. \quad (44)$$

Select $\mathfrak{Y}(\sigma_*,\aleph)=2\nu^2\exp(-2\vartheta(\Delta-\aleph))+2\mathfrak{S}\exp(2\mathfrak{Q}\Delta)$, from assumption $\mathcal{I}(6)$, we know that $\mathfrak{Y}(0,0)<1$. On the other hand, we can easily observe that $\mathfrak{Y}(\sigma_*,0)$ is strictly increasing for σ_*. Thus, we can find a $\bar{\sigma}_*>0$ that meets $\mathfrak{Y}(\bar{\sigma}_*,0)=1$. And we also see that $\mathfrak{Y}(\sigma_*,\aleph)$ is increasing monotonically with respect to $\aleph$, thus there must be a $\hat{\aleph}$ such that $\mathfrak{Y}(\sigma_*,\aleph)<1$, when $|\sigma_*|<\bar{\sigma}_*/\sqrt{2}$, $\aleph<\min\{\Delta/2,\hat{\aleph}\}$.

Setting $\Omega=-\ln(\mathfrak{Y}(\sigma_*,\aleph))/\Delta$, the rest of the proof is as same as Theorem 1, so we omit it here.

Therefore, $\forall t\ge t_0$, $E||\Lambda(t)||^2\le\exp(-\Omega(t-t_0))\exp(\Omega(\Delta-\aleph))E||\Lambda^0||^2$ holds. Then, SFCNNDA (32) is ASGES. □

Remark 5: The proof in this part is not a simple combination of the previous two parts. It can be found that the Theorem 3 reveals the constraint relationship between the upper bounds of the two perturbations to a certain extent. Furthermore, if assumption $\mathcal{I}(6)$ is not hold, then equation (39) and (40) have no roots on interval $(0,+\infty)$.

Remark 6: Table 1 shows a brief comparison between a part of existing literature and this paper. The elements of comparison include the following parts: fuzzy logic, deviating argument, stochastic disturbance, exponentially stable, asymptotic stability, robust stability. Because the neural network introduced in this paper contains fuzzy logic, its application will be more extensive compared with the general systems. For example, template learning [35], digital image restoration [27] and so on.

V. EXAMPLES

There are some instances to prove the validity of theoretical results in this part.

Example 1: We consider the following SFCNN.

$$\begin{cases}d\Lambda_1(t)=\Big[-\varrho_1\Lambda_1(t)+\bigwedge_{\varsigma=1}^{2}\breve{\varphi}_{1\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\\\qquad+\bigvee_{\varsigma=1}^{2}\breve{\omega}_{1\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\Big]dt\\\qquad+\sum_{\varsigma=1}^{2}\sigma_{1\varsigma}\Lambda_\varsigma(t)d\mho(t),\\d\Lambda_2(t)=\Big[-\varrho_2\Lambda_2(t)+\bigwedge_{\varsigma=1}^{2}\breve{\varphi}_{2\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\\\qquad+\bigvee_{\varsigma=1}^{2}\breve{\omega}_{2\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\Big]dt\\\qquad+\sum_{\varsigma=1}^{2}\sigma_{2\varsigma}\Lambda_\varsigma(t)d\mho(t),\end{cases} \quad (45)$$

where

$$\varrho=(\varrho_{\rho\varsigma})_{2\times2}=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \varphi=(\breve{\varphi}_{\rho\varsigma})_{2\times2}=\begin{bmatrix}0.1&0.2\\0.2&0.1\end{bmatrix},\quad \omega=(\breve{\omega}_{\rho\varsigma})_{2\times2}=\begin{bmatrix}0.2&0.1\\0.1&0.2\end{bmatrix}.$$

From the known conditions, and according to the principle of comparison, the FCNN without stochastic disturbances

$$\begin{cases}\dot{\Lambda}_1(t)=-\varrho_1\Lambda_1(t)+\bigwedge_{\varsigma=1}^{2}\breve{\varphi}_{1\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\\\qquad+\bigvee_{\varsigma=1}^{2}\breve{\omega}_{1\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t)),\\\dot{\Lambda}_2(t)=-\varrho_2\Lambda_2(t)+\bigwedge_{\varsigma=1}^{2}\breve{\varphi}_{2\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t))\\\qquad+\bigvee_{\varsigma=1}^{2}\breve{\omega}_{2\varsigma}\Gamma_\varsigma(\Lambda_\varsigma(t)),\end{cases} \quad (46)$$

is GES with $\nu=1$, $\vartheta=0.9$.

We select $\Delta=0.5$ and $\Gamma_\varsigma(\cdot)=\tanh(\cdot)$, then $|\Gamma_\varsigma(u)-\Gamma_\varsigma(v)|\le|u-v|$ holds, that means $\mu_\varsigma=1$. Therefore, we can get that $\kappa_1=1$, $\kappa_2=0.6$.

And from (8), the following transcendental equation can be obtained.

$$2\exp(-0.9)+2.2222\sigma_*^2\exp(5.12+4\sigma_*^2)=1. \quad (47)$$

We solved (47) by MATLAB, then we can obtain $\breve{\sigma}_*=0.0224$, thus, SFCNN (45) is still GES when the column sum of the matrix σ is less than $\breve{\sigma}_*$. Fig. 1 shows the states of (45) with σ in Table 1, we can know that $\sigma_*=0.02$. Fig. 2 shows the states of (45) with σ in Table 2, then $\sigma_*=0.04$, thus, SFCNN (45) is GES.

TABLE 1. Differences between other literature and our paper.

	Fuzzy logic	Deviating argument	Stochastic disturbance	Exponential stability	Asymptotic stability	Robust stability
Akhmet(2010)[36]	-	✓	-	✓	✓	-
Zhu(2011)[22]	✓	-	-	-	✓	-
Zhu(2012)[37]	✓	-	✓	-	-	✓
Wan(2017)[29]	✓	✓	✓	-	-	-
Zhang(2017)[31]	-	-	✓	✓	-	✓
Si(2021)[30]	-	✓	✓	✓	-	✓
This paper	✓	✓	✓	✓	-	✓

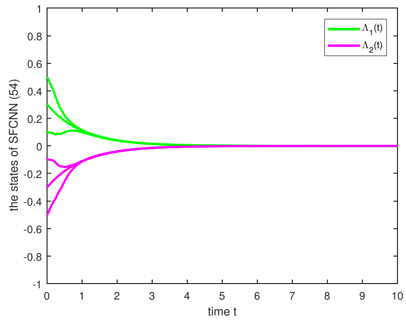

FIGURE 1. States of SFCNN (45) with $\sigma_* = 0.02$.

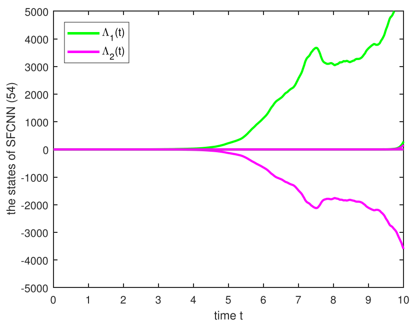

FIGURE 2. States of SFCNN (45) with $\sigma_* = 0.04$.

TABLE 2. Values of σ. (✓ and ✗ represent whether σ_* is lower than $\breve{\sigma}_*$).

Fig \ σ	σ_{11}	σ_{12}	σ_{21}	σ_{22}	σ_*
Fig. 1	0.01	0.01	0.01	0.01	0.02✓
Fig. 2	0.01	0.03	0.01	0.01	0.04✗

Example 2: We consider the following FCNNDA.

$$\begin{cases} \Lambda_1(t) = -\varrho_1 \Lambda_1(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{1\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{1\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))), \\ \Lambda_2(t) = -\varrho_2 \Lambda_2(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{2\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{2\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))), \end{cases} \tag{48}$$

where

$$\varrho = (\varrho_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\varphi = (\breve{\varphi}_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} 0.003 & 0.004 \\ 0.004 & 0.003 \end{bmatrix},$$

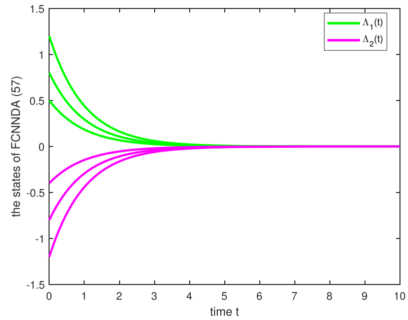

FIGURE 3. States of FCNNDA (48) with $\{\aleph_k\} = k/50$ and $\{\mathfrak{H}_k^*\} = (2k+1)/100$.

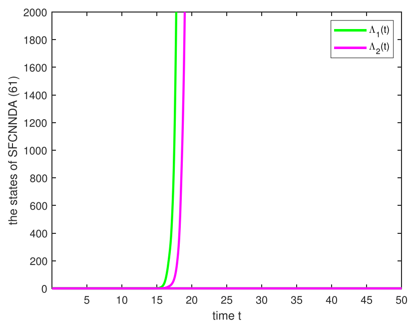

FIGURE 4. States of SFCNNDA (52) with $\{\aleph_k\} = k/2700$, $\{\mathfrak{H}_k\} = (2k+1)/5400$ and $\sigma_* = 0.00032$.

$$\omega = (\breve{\omega}_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} 0.004 & 0.003 \\ 0.003 & 0.004 \end{bmatrix}.$$

And in this instance, we set $\{\aleph_k\} = \frac{k}{50}$, $\{\mathfrak{H}_k^*\} = \frac{2k+1}{100}$. By computing the parameters we gave above, we can obtain that $\kappa_1 = 1$, $\kappa_2 = 0.014$.

On the other hand, the system (48) without deviating argument is as shown below.

$$\begin{cases} \Lambda_1(t) = -\Lambda_1(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{1\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t)) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{1\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t)), \\ \Lambda_2(t) = -\Lambda_2(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{2\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t)) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{2\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(t)). \end{cases} \tag{49}$$

It follows from the comparison principle that it is GES with $\nu = 1$ and $\vartheta = 0.8$.

Then we select $\Delta = 0.1$, $\Gamma_\varsigma = \tanh(\cdot)$ and $\mu_\varsigma = 1$, $\varsigma = 1, 2$. Thus, we can obtain

$$\exp(-0.8 \times 0.1) + 0.035 \exp(0.2084) = 0.9662 < 1. \tag{50}$$

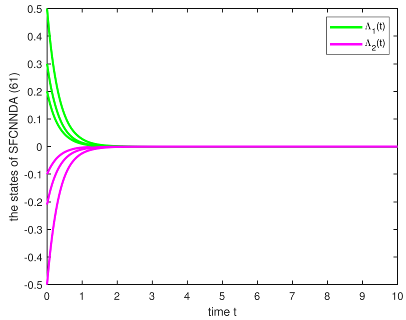

FIGURE 5. **States of SFCNNDA** (52) **with** $\{\aleph_k\} = k/2700$, $\{\mathfrak{H}_k\} = (2k+1)/5400$ **and** $\sigma_* = 0.00011$.

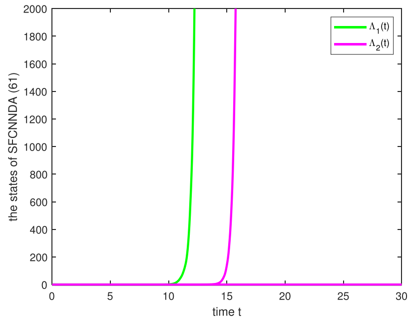

FIGURE 6. **States of SFCNNDA (52) with** $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (2k+1)/20000$ **and** $\sigma_* = 0.0004$.

By solving the following transcendental equation

$$\exp(-0.8 \times (0.1 - \aleph)) + 0.0175\{1 + (1 - \aleph[(1 + 0.014\aleph) \times \exp \aleph + 0.014])^{-1}\} \exp\left\{0.2(1.028 + 0.014[1 + (1 - \aleph[(1 + 0.014\aleph) \exp \aleph + 0.014])^{-1}])\right\} = 1. \tag{51}$$

We can easily obtain $\bar{\bar{\aleph}} = 0.0435$. Therefore, the FCNNDA is said to be GES if $\aleph < 0.0435$.

Fig. 3 shows the states of FCNNDA (48) with $\{\aleph_k\} = \frac{k}{50}$, and $\{\mathfrak{H}_k^*\} = \frac{2k+1}{100}$. It meets the conditions of Theorem 1, thus, it is GES.

Example 3: We consider the following SFCNNDA.

$$\begin{cases} d\Lambda_1(t) = \Big[-\varrho_1 \Lambda_1(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{1\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{1\varsigma} \Gamma_\varsigma(\Lambda_j(\mathfrak{H}(t)))\Big]dt \\ \qquad + \sum_{\varsigma=1}^{2} \sigma_{1\varsigma} \Lambda_\varsigma(t) d\mho(t), \\ d\Lambda_2(t) = \Big[-\varrho_2 \Lambda_2(t) + \bigwedge_{\varsigma=1}^{2} \breve{\varphi}_{2\varsigma} \Gamma_\varsigma(\Lambda_\varsigma(\mathfrak{H}(t))) \\ \qquad + \bigvee_{\varsigma=1}^{2} \breve{\omega}_{2\varsigma} \Gamma_\varsigma(\Lambda_j(\mathfrak{H}(t)))\Big]dt \\ \qquad + \sum_{\varsigma=1}^{2} \sigma_{2\varsigma} \Lambda_\varsigma(t) d\mho(t), \end{cases} \tag{52}$$

where

$$\varrho = (\varrho_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix},$$
$$\varphi = (\breve{\varphi}_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} 0.00008 & 0.00007 \\ 0.00007 & 0.00008 \end{bmatrix},$$
$$\omega = (\breve{\omega}_{\rho\varsigma})_{2\times 2} = \begin{bmatrix} 0.0004 & 0.0006 \\ 0.0006 & 0.0004 \end{bmatrix}.$$

Then, the system (50) without deviating argument and stochastic disturbances is as same as the system (49). Similarly, by comparison principle [34], it is GES with $\nu = 1$ and $\vartheta = 2.8$. We select $\Delta = 0.26$. Thus, we can easily get $\kappa_1 = 3$, $\kappa_2 = 0.0002$. Let $\Gamma_\varsigma(\cdot) = \tanh(\cdot)$, then, similar to Example 1, we take $\mu_\varsigma = 1$, $\varsigma = 1, 2$. Therefore, we can get the following equations that by plugging in the above parameters,

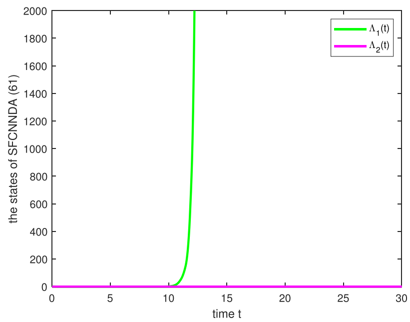

FIGURE 7. **States of SFCNNDA (52) with** $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (2k+1)/20000$ **and** $\sigma_* = 0.0004$.

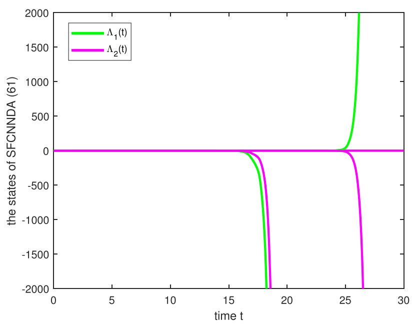

FIGURE 8. **States of SFCNNDA (52) with** $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (2k+1)/20000$ **and** $\sigma_* = 0.0008$.

$$2\exp(-1.456) + 3.1438 \times 10^{-5} \times \exp(9.7344) = 0.9973 < 1. \tag{53}$$

On the other hand, we plug the parameters into (39), (40), then, by solving equations (39) and (40), we can get $\bar{\sigma}_* = 0.00033438$, and $\hat{\aleph} = 0.00050941$. Therefore, when $|\sigma_*| < \bar{\sigma}_*/\sqrt{2} = \bar{\bar{\sigma}}_* = 0.00023644$ and $\aleph < 0.00050941$, SFCNNDA (52) is said to be GES. Since $\sigma_* = \max_{1\le\rho\le m} \sigma_{\rho\varsigma}$, that means the system is ASGES when the max sum of the columns of σ is less than $\bar{\bar{\sigma}}_*$ we derived.

From Table 3, we can easily observe that the max sum of columns of σ is 0.0003 in Fig. 4, it is larger than $\bar{\bar{\sigma}}_*$, thus, the system is not ASGES.

TABLE 3. Parameters of Fig. 4-Fig. 10.

Fig \ Parameters	$\aleph_k$	$\mathfrak{H}_k$	σ_{11}	σ_{12}	σ_{21}	σ_{22}	σ_*
Fig. 4	$k/2700$	$(2k+1)/5400$	0.0003	0.00001	0.00002	0.00001	0.00031
Fig. 5	$k/2700$	$(2k+1)/5400$	0.00006	0.00004	0.00004	0.00007	0.00011
Fig. 6	$3k/10000$	$(6k+1)/20000$	0.0004	0	0	0.0003	0.0004
Fig. 7	$3k/10000$	$(6k+1)/20000$	0.0004	0	0	0.0001	0.0004
Fig. 8	$3k/10000$	$(6k+1)/20000$	0	0.0008	0.00009	0	0.0008
Fig. 9	$3k/10000$	$(6k+1)/20000$	0	0.0004	0.0006	0	0.0006
Fig. 10	$3k/10000$	$(6k+1)/20000$	0.0005	0.0008	0.0006	0.0003	0.0011

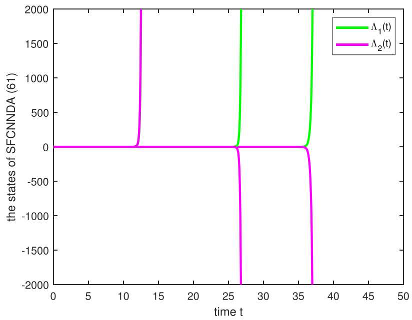

FIGURE 9. States of SFCNNDA (52) with $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (2k+1)/20000$ and $\sigma_* = 0.0006$.

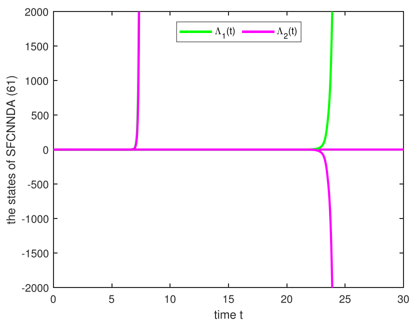

FIGURE 10. States of SFCNNDA (52) with $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (2k+1)/20000$ and $\sigma_* = 0.0011$.

In Fig. 5, we choose the same $\{\aleph_k\}$ and $\{\mathfrak{H}_k\}$ as in Fig. 4, and σ is shown in Table 3. Since the max sum of columns of σ is lower than $\bar{\bar{\sigma}}_*$, thus, SFCNNDA (52) is ASGES.

Fig. 6 - Fig. 9 show the states of SFCNNDA (52) with $\{\aleph_k\} = 3k/10000$, $\{\mathfrak{H}_k\} = (6k+1)/20000$ and the value of σ is shown in Table 3 respectively. We can easily see that both of the sum of columns of σ of Fig. 6 - Fig. 9 are lager than $\bar{\bar{\sigma}}_*$, thus, it is not ASGES.

According to the parameters in Table 3, we can see that Fig. 10 is also not ASGES.

VI. CONCLUSION

The robustness of the stability of FCNN with deviating arguments and stochastic noises are examined in this article. In order to make the system with external disturbances we suggested in this paper to stay globally exponential stable, the upper bounds of noise and the deviating interval have to be determined. By resolving the transcendental equations, we can estimate the upper boundaries of the interference that we considered. The conclusions we reached here offer bedrock for applications and designs of FCNN. Improvements of upper limits may be the focus of future research to provide bigger stability margins that can survive random disturbances and deviating arguments.

DATA AVAILABILITY

No data were used to support this study.

CONFLICT OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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