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RESEARCH ARTICLE

Priority Degrees and Distance Measures of Complex Hesitant Fuzzy Sets With Application to Multi-Criteria Decision Making

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ABSTRACT The notion of a complex hesitant fuzzy set (CHFS) is one of the better tools in order to deal with complex information. Since distance plays a crucial role in order to differentiate between two things or sets, in this paper, we first develop a priority degree for the comparison between complex hesitant fuzzy elements (HFEs). Then a variety of distance measures are developed, namely, Complex hesitant normalized Hamming-Hausdorff distance (CHNHHD), Complex hesitant normalized Euclidean-Hausdorff distance (CHNEHD), Generalized complex hesitant normalized Hausdorff distance (GCHNHD), Complex hesitant hybrid normalized Hamming distance (CHHNHD), Complex hesitant hybrid normalized Euclidean distance (CHHNED), Generalized complex hesitant hybrid normalized distance (GCHHND) and their weighted forms. Moreover, the continuous form of the proposed distances is also developed. Further, the proposed distances are applied to medical diagnosis problems for their effectiveness and application. Furthermore, a multi-criteria decision making (MCDM) approach is developed based on the TOPSIS method and proposed distances. Finally, a practical example related to the effectiveness of COVID-19 tests is presented for the application and validity of the proposed method. A comparison study was also done with the method that was already in place to see how well the new method worked.

INDEX TERMS Complex fuzzy sets, priority degree, distance measure, closeness coefficient, TOPSIS method.

LIST OF ABBREVIATIONS

Symbols	*Description*
FS	Fuzzy Set.
HFS	Hesitant Fuzzy Set.
HD	Hesitancy degree.
IFS	Intuitionistic fuzzy set.
MCDM	Multi criteria decision making.
CHFS	Complex Hesitant Fuzzy Set.
PD	Priority degree.
DM	Distance measure.
CF	Closeness coefficient.
TM	TOPSIS method.

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I. INTRODUCTION

The fuzzy set theory initiated by Zadeh [1] is a useful tool for dealing with real-world decision making (DM) problems that involve uncertainty. In practical problems, we have a variety of uncertain and unclear information. Therefore, to deal with a such types of issues, many researchers have investigated research and extend the idea of fuzzy sets by developing intuitionistic fuzzy sets (IFS) [2], type-1 fuzzy sets [3] type-2 fuzzy sets [4] fuzzy multi-set [5], hesitant fuzzy sets (HFS) [6], etc. Atanassov [2] initiated the notion of IFS, which is characterized by a membership function, nonmembership function, and hesitancy degree (HD).

Due to the lack of knowledge, limited time and other factors in practical application, people are not always able to agree on specific components of complex decision making it extremely difficult to achieve an agreement. For example, a pair of decision makers may debate the degree to which an element X belongs to set A, and one should allocate 0.4 to it, while the other wishes to allocate 0.8 to the elements. The difficulty in generating a standard membership value stems from the presence of a large number of values rather than the presence of a range of errors or certain apparent distribution values [6]. So, Torra [7], Torra and Narukava [8] introduced the notion of HFS, which allowed membership to be based on an infinite set of values. HFS has been extensively studied since its beginning [9], [10], [11] and has been successfully used in a variety of uncertain decision-making situations. In addition, there has been research conducted on the application of HFS algorithms based on distance and similarity measurement [12], [13] have also been applied to multi-criteria decision making (MCDM) problems [14], [15], [16]. Many scientific domains, including healthcare [17], medical diagnosis [18], economics and sociology [19], [20], have used MCDM approaches. In general, aggregation and utilization are the two processes of MCDM. Decision making techniques, in which experts use of HFS to convey their opinions, have developed considerable aggregation approaches for this purpose. Hesitant fuzzy weighted averaging (HFWA) operator and Hesitant fuzzy weighted geometric (HFWG) operator were introduced by Xia and Xu [21], with various extensions and generalizations, including GHFWA operator, a GHFWG operator, a HFWA operator and a HFWG operator. To aggregate the hesitant fuzzy information, Senapati et al. [22] introduced the concept of Aczel-Alsina t-norm and t-conorm based aggregation operators. Many scholars are concerned about what happens if we extend the range of FS to include real numbers instead of complex numbers in the unit disc in the complex plane, and the impact of this change. Ramot et al. [23] introduced the concept of the complex fuzzy set to solve this issue. CFS has gotten more attention among researchers in the last few years. Moreover, Talafha et al. [24] extended the notion of CFS and introduced the concept of complex hesitant fuzzy (CHFS). CHFS is the combination of HFS and CFS. The degree of membership is complex-valued and is given in polar coordinates. Distance measurement is an important technique for determining the difference between two things. Distance measure is an important topic in fuzzy set theory and is widely used in many research areas, including machine learning, pattern recognition, and decision-making problems [25], [26], [27]. In view of the significance of similarity measures, several researchers have recently developed different similarity measures. Xu and Xi [12] introduced the distances and similarity measures for HFSs and also proposed some novel measures under the Hamming and Euclidean distances between HFSs. Furthermore, Li et al. [13] developed some new distance measures and proposed a MCDM method based on these distances. Moreover, Cheng and Li [28] generalized the distance measures proposed in [12] by adding hesitance degree and applied the concept to pattern recognition. Peng et al. [29] introduced a distance and similarity for HFSs and it is widely used to deal with MCDM [30], [31] problems. Some HFS ordering relations have been developed, and play a vital role in decision making. Rodriguez et al. [19] defined order relations amongst HFSs. Lan [32] discussed the priority degree and studied some properties. Farhadinia [33] also created two HFS ordering algorithms. In a hesitant fuzzy environment, Yang and Hussain [34] introduced the hesitant Hausdorff distance and similarity measure. Modified the distance measure proposed in [12], Singha et al. [35] defined modified distance measure on HFSs. Some improved distance measures have been developed by Rezaei and Rezaei [36], and applied to hierarchical clustering. Li et al. [37] developed parameterized distance measures on HFS with credibility degree and applied it to pattern recantation. Moreover, Lv et al. [38] developed a hesitant fuzzy distance and similarity measure-based hesitant fuzzy network clustering algorithm. Bi et al. [39] proposed the distance-and-entropy measure for CFS. Rani and Garg [40] defined the distance and similarity measures for CIFSs. Garg et al. [41] also introduced the distance and similarity measures for CHFSs.

In order to make the best decisions possible in real-world settings, there are numerous instances where we must quantify the uncertainty in the data. For managing the ambiguous information that is present in our day-to-day problems, information measures are crucial tools. The ambiguous information is processed by various information measures, including similarity, distance, entropy, and inclusion, which allows us to draw some conclusions. Due to their numerous applications in a variety of domains, including pattern recognition, medical diagnosis, clustering analysis, and image segmentation, these measures have recently attracted the attention of many authors. All currently used information-based decision-making methods in HF deal with membership functions, that have real values. In CHFS theory, membership degrees are complex-valued and are represented in polar coordinates. The phase term linked with membership degree delivers the additional information, typically related to periodicity, while the amplitude term connected with membership degree gives the amount of an object's belongingness in a CHFS. Motivated by the priority degree proposed by Li et al. [37] and distance measure proposed by Xia and Xu [12], and keeping the advantages of the CHFS in this paper the key contributions are investigated:

1) To define priority degree for CHFSs.
2) To propose Complex hesitant normalized Hamming-Hausdorff distance (CHNHHD), Complex hesitant normalized Euclidean-Hausdorff distance (CHNEHD) and their weighted forms.
3) To develop Generalized complex hesitant normalized Hausdorff distance (GCHNHD), Complex hesitant Hybrid normalized Hamming distance (CHHNHD), Complex hesitant Hybrid normalized Euclidean distance (CHHNED), Generalized complex hesitant Hybrid normalized distance (GCHHND) and their weighted forms.
4) To investigate the continuous form of the proposed distances.
5) To propose a MCDM method based on TOPSIS and developed distances.

Remainder of the paper is organize as: Section II of this paper describes some basic ideas of hesitant fuzzy sets. Section III investigates order relations of complex hesitant fuzzy set (CHFS), while in Section IV, we define some generalized distances between two CHFSs, namely the Hamming distance, Euclidean distance, Hausdorff distance and Hybrid distance between CHFSs. We also define their weighted form and continuous weighted forms. In Section V, a new technique for complex hesitant fuzzy multiple attribute decision making is created based on the suggested formula for the priority degree. In Section VI, we give an example to illustrate the applicability of the new method. Section VII compares the proposed technique to other pertinent techniques in order to determine its suitability. In Section VIII, we end the paper with a conclusion and a future plan.

II. PRELIMINARIES

A. HESITANT FUZZY SET

To deal with MCDM problems, that have some possible value, Torra [7], proposed the idea of HFS. HFS is characterized by membership degree (MD) of the set of possible values belonging to [0,1]. It can be defined as:

Definition 1: Let X be a universe of discourse and $x \in X$. Then a hesitant fuzzy set $\breve{E}$ defined on X may be written as a collection of ordered pairs,

$$\breve{E} = \{(x, p_{\breve{E}}(x)) : x \in X\} \tag{1}$$

where $p_{\breve{E}}(x)$ is a collection of some values in $[0, 1]$, that represent the membership value of the element $x \in X$. For convenience, $p = p_{\breve{E}}(x)$ is called a hesitant fuzzy element (*HFE*) and H be the set of all *HFEs*.

Xia and Xu [21] defined the comparison procedure to compare two HFEs: For a *HFE* p in H, the score $q(p) = \frac{1}{lp}\sum_{\gamma\in p}\gamma$ is known as score function of p, where l_p is the total number of elements in p. Some properties of score function.

Let any two *HFEs*, p_1 and p_2, then
if $q(p_1) > q(p_2)$, then $p_1 > p_2$;
if $q(p_1) < q(p_2)$, then $p_1 < p_2$;
if $q(p_1) = q(p_2)$, then $p_1 = p_2$.

For any three *HFEs* p, p_1, and p_2, Xia and Xu [21] and Torra [7] defined the following operations:

1) $p^c = \cup_{\gamma\in p}\{1-\gamma\}$,
2) $p_1 \cup p_2 = \cup_{\gamma\in p_1,\gamma\in p_2} max\{\gamma_1, \gamma_2\}$,
3) $p_1 \cap p_2 = \cap_{\gamma\in p_1,\gamma\in p_2} min\{\gamma_1, \gamma_2\}$,
4) $p_1 \oplus p_2 = \cup_{\gamma\in p_1,\gamma\in p_2}\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$,
5) $p_1 \otimes p_2 = \cup_{\gamma\in p_1,\gamma\in p_2}\{\gamma_1\gamma_2\}$,
6) $\alpha p = \cup_{\gamma\in p}\{1-(1-\gamma)^\alpha\}$,
7) $p^\alpha = \cup_{\gamma\in p}\{\gamma^\alpha\}$.

To deal the ordering relation between *HFSs*, Lan et al. [32] proposed the priority degree formula and can be defined as;

Definition 2: Given a set $X = \{x_1, x_2, \ldots, x_n\}$. Let $\breve{E}_1 = \{< x_i, \hat{h}_{\breve{E}_1}(x_i) > : x_i \in X\}$, and $\breve{E}_2 = \{< x_i, \hat{h}_{\breve{E}_2}(x_i) >: x_i \in X\}$ be two HFSs. Then the priority degree for $\breve{E}_1 \succ \breve{E}_2$ is defined below;

$$P\left(\breve{E}_1 \succ \breve{E}_2\right) = \begin{cases} \dfrac{\sum_{i=1}^{n} S_1(x_i)}{\sum_{i=1}^{n} S_1(x_i) + \sum_{i=1}^{n} S_2(x_i)}, \text{ when} \\ \sum_{i=1}^{n} S_1(x_i) + \sum_{i=1}^{n} S_2(x_i) \neq 0 \\ 0.5, \sum_{i=1}^{n} S_1(x_i) + \sum_{i=1}^{n} S_2(x_i) = 0 \end{cases} \tag{2}$$

where

$$\begin{aligned}
\hat{h}_{\breve{E}_1}(x_i) &= \{u_m(x_i) \,|u_m(x_i) \in [0,1], m = 1, 2, \ldots, j_i\},\\
\hat{h}_{\breve{E}_2}(x_i) &= \{v_n(x_i) \,|v_n(x_i) \in [0,1], n = 1, 2, \ldots, k_i\},\\
A_1(x_i) &= \{(u_m(x_i), v_n(x_i))|u_m(x_i) - v_n(x_i) > 0,\\
&\qquad (u_m(x_i), v_n(x_i)) \in \hat{h}_{\breve{E}_1}(x_i) \times \hat{h}_{\breve{E}_2}(x_i)\Big\},\\
A_2(x_i) &= \{(u_m(x_i), v_n(x_i))|u_m(x_i) - v_n(x_i) < 0,\\
&\qquad (u_m(x_i), v_n(x_i)) \in \hat{h}_{\breve{E}_1}(x_i) \times \hat{h}_{\breve{E}_2}(x_i)\Big\},\\
A_3(x_i) &= \{(u_m(x_i), v_n(x_i))|u_m(x_i) - v_n(x_i) = 0,\\
&\qquad (u_m(x_i), v_n(x_i)) \in \hat{h}_{\breve{E}_1}(x_i) \times \hat{h}_{\breve{E}_2}(x_i)\Big\}
\end{aligned}$$

and

$$S_1(x_i) = \begin{cases} \dfrac{1}{j_i k_i} \displaystyle\sum_{(u_m(c_i), v_n(c_i))\in A_1(c_i)} u_m(c_i) - v_n(c_i), \\ \text{when } A_1(x_i) \neq \phi; \\ 0, \text{ when } A_1(x_i) = \phi \end{cases}$$

$$S_2(x_i) = \begin{cases} \dfrac{1}{j_i k_i} \displaystyle\sum_{(u_m(c_i), v_n(c_i))\in A_1(c_i)} v_n(c_i) - u_m(c_i), \\ \text{when } A_2(x_i) \neq \phi; \\ 0, \text{ when } A_2(x_i) = \phi. \end{cases}$$

B. COMPLEX HESITANT FUZZY SETS

In the DM problem, the complexity of the expert opinion changes for the same object but at different times, and

HFS [7], [8] cannot deal with such information effectively. Therefore, to deal with such information, Ramot et al. [23] introduced the concept of CFS as a generalization of FS and is characterized by complex valued membership degree. Talafha et al. [24] extended the notion of CFS and introduced the concept of complex hesitant fuzzy set and a generalization of HFS with CFS. The CHFS [24] concept makes it one of the most effective strategies for dealing with unfavorable and complex data on practical decision-making problems. CHFS [24], membership value has a complex value and is reflected in polar coordinates. The amplitude term associated with the membership level gives the object size to the CHFS, and the phase time associated with the membership degrees provides additional details, which are often related to the passage of time. It can be defined as:

Definition 3 ([28]): Let X be a universe of discourse. Then a complex hesitant fuzzy set (CHFS) Ch is:

$$Ch = \{(x, h_{Ch}(x)) : x \in X\} \tag{3}$$

where $h_{Ch}(x) = \gamma_{Ch}(x).e^{i2\pi(\theta_{Ch}(x))}$ indicate the complex valued in polar coordinate, where $\gamma_{Ch}(x) \in \eta(x), \theta_{Ch}(x) \in \psi(x)$ and where $\gamma_{Ch}(x)$ and $\theta_{Ch}(x) \in (0, 2\pi]$. Furthermore, when x is the singleton set then the *CHFE* denoted by $Ch = (x, \eta(x).e^{i2\pi(\psi(x))})$. We denote a complex hesitant fuzzy element (CHFE) by $Ch = \eta e^{i2\pi(\psi)}$ and $\gamma_{Ch}(x)$ and $\theta_{Ch}(x) \in [0, 1]$ and $(0, 2\pi]$ respectively.

To compare CHFE's Talafha et al. [24] developed score function as: Let Ch be a *CHFE* on X,

$$s(Ch) = \frac{1}{2}\left(\frac{1}{l_{Ch}}\sum_{\gamma\in\eta}\gamma + \frac{1}{l_{Ch}}\sum_{\theta\in\psi}\theta\right) \tag{4}$$

is known as score function of Ch, where l_{Ch} is the number of elements.

For any two *HFEs* Ch_1 and Ch_2, if $s(Ch_1)$ *greater than* $s(Ch_2)$, then Ch_1 *greater than* Ch_2; if $s(Ch_1)$ *equal to* (Ch_2), then Ch_1 *equal to* Ch_2;

if $s(Ch_1)$ less than $s(Ch_2)$, then Ch_1 less than Ch_2.

Moreover, Talafha et al. [24] developed some operational laws for *CHFEs*

For any three *CHFN* Ch, Ch_1, and Ch_2, then the properties of *CHFN* as follows:

1. $(Ch)^c = \cup_{\substack{\gamma\in\eta\\ \theta\in\psi}}\{(1-\gamma).e^{i2\pi(1-\theta)}\}$
2. $Ch_1 \cup Ch_2 = \cup_{\substack{\gamma\in\eta_1,\gamma\in\eta_2\\ \theta\in\psi_1,\theta\in\psi_2}}\{(\max\{\gamma_1,\gamma_2\})e^{i2\pi(\max\{\theta_1,\theta_2\})}\}$
3. $Ch_1 \cap Ch_2 = \cap_{\substack{\gamma\in\eta_1,\gamma\in\eta_2\\ \theta\in\psi_1,\theta\in\psi_2}}\{(\min\{\gamma_1,\gamma_2\})e^{i2\pi(\min\{\theta_1,\theta_2\})}\}$
4. $Ch_1 \oplus Ch_2 = \cup_{\substack{\gamma\in\eta_1,\gamma\in\eta_2\\ \theta\in\psi_1,\theta\in\psi_2}}\{(\gamma_1+\gamma_2-\gamma_1\gamma_2) e^{i2\pi(\theta_1+\theta_2-\theta_1\theta_2)}\}$
5. $Ch_1 \otimes Ch_2 = \cup_{\substack{\gamma\in\eta_1,\gamma\in\eta_2\\ \theta\in\psi_1,\theta\in\psi_2}}\{(\gamma_1\gamma_2) e^{i2\pi(\theta_1\theta_2)}\}$
6. $\lambda Ch = \cup_{\substack{\gamma\in\eta\\ \theta\in\psi}}\{\left(1-(1-\gamma)^{\lambda}\right) e^{i2\pi\left(1-(1-\theta)^{\lambda}\right)}\}$
7. $(Ch)^{\lambda} = \cup_{\substack{\gamma\in\eta\\ \theta\in\psi}}\{(\gamma^{\lambda})e^{i2\pi(\theta^{\lambda})}\}$

Garg at el. [41] introduced the distances measures for CHFS as;

Let $Ch_1 = (x_k, \delta_{Ch}(x_k).e^{i2\pi(\theta_{Ch}(x_k))})$ and $\acute{C}h_2 = (x_k, \delta_{Ch}(x_k).e^{i2\pi(\theta_{Ch}(x_k))})$ be two CHFSs on the set $X = \{x_1, x_2, \ldots, x_k\},$. Then the distance between Ch_1 and $\acute{C}h_2$ is denoted by $d(Ch_1, \acute{C}h_2)$, satisfy the following properties.

(1) $0 \le d(Ch_1, \acute{C}h_2) \le 1$ OR $d(Ch_1, \acute{C}h_2) \in [0, 1]$

(2) $d(Ch_1, \acute{C}h_2) = 0$ if and only if $Ch_1 = \acute{C}h_2$

(3) $d(Ch_1, \acute{C}h_2) = d(\acute{C}h_2, Ch_1)$.

III. PRIORITY DEGREE FOR COMPLEX HESITANT FUZZY SETS

Motivated by the priority degree developed by Lan et al. [32] for HFS, in this section, we propose a priority degree formula for CHFS in order to deal with the ordering relationship among CHFESs.

Definition 4: Given a fixed set $C = \{c_1, c_2, \ldots c_n\}$, suppose that two given complex hesitant fuzzy sets are, $E_1 = \left\{\left\langle c_i, \hbar_{E_1}(c_i)\, e^{i2\pi w_{E_1}(c_i)}\right\rangle | c_i \in C\right\}$, and $E_2 = \left\{\left\langle c_i, \hbar_{E_2}(c_i)\, e^{i2\pi w_{E_2}(c_i)}\right\rangle | c_i \in C\right\}$.

The priority degree for E_1 and E_2, denoted by $E_1 \succ E_2$ is defined by;

$$P(E_1 \succ E_2) = \begin{cases} \frac{\sum_{i=1}^{n} S_1(c_i)}{\sum_{i=1}^{n} S_1(c_i)+\sum_{i=1}^{n} S_2(c_i)}, \text{ when} \\ \quad \sum_{i=1}^{n} S_1(c_i) + \sum_{i=1}^{n} S_2(c_i) \neq 0 \\ 0.5, \ \sum_{i=1}^{n} S_1(c_i) + \sum_{i=1}^{n} S_2(c_i) = 0 \end{cases} \tag{5}$$

where,

$$\begin{aligned}
\acute{C}\hbar_{E_1}(c_i) &= \Big\{u_r(c_i)\, e^{i2\pi w_{E_1}(c_i)} | u_r(c_i)\, e^{i2\pi w_{E_1}(c_i)} \\
&\quad \in [0,1], r = 1, 2, 3 \ldots\ldots m_i\Big\}, \\
\acute{C}\hbar_{E_2}(c_i) &= \Big\{v_t(c_i)\, e^{i2\pi w_{E_1}(c_i)} | v_t(c_i)\, e^{i2\pi w_{E_2}(c_i)} \\
&\quad \in [0,1], t = 1, 2, 3 \ldots\ldots n_i\Big\}, \\
A_1(c_i) &= \{(u_r(c_i), v_t(c_i)) | u_r(c_i) - v_t(c_i) > 0, \\
&\quad (u_r(c_i), v_t(c_i)) \in \acute{C}\hbar_{E_1}(c_{i\phi}) \times \acute{C}\hbar_{E_2}(c_i)\Big\}, \\
\theta_1(c_i) &= \{(w_r(c_i), \overline{w_t}(c_i)) | w_r(c_i) - \overline{w_t}(c_i) > 0, \\
&\quad (w_r(c_i), \overline{w_t}(c_i)) \in \acute{C}\hbar_{E_1}(c_i) \times \acute{C}\hbar_{E_2}(c_i)\Big\}, \\
A_2(c_i) &= \{(u_r(c_i), v_t(c_i)) | u_r(c_i) - v_t(c_i) < 0, \\
&\quad (u_r(c_i), v_t(c_i)) \in \acute{C}\hbar_{E_1}(c_i) \times \acute{C}\hbar_{E_2}(c_i)\Big\}, \\
\theta_2(c_i) &= \{(w_r(c_i), \overline{w_t}(c_i)) | w_r(c_i) - \overline{w_t}(c_i) < 0, \\
&\quad (w_r(c_i), \overline{w_t}(c_i)) \in \acute{C}\hbar_{E_1}(c_i) \times \acute{C}\hbar_{E_2}(c_i)\Big\}, \\
A_3(c_i) &= \{(u_r(c_i), v_t(c_i)) | u_r(c_i) - v_t(c_i) = 0, \\
&\quad (u_r(c_i), v_t(c_i)) \in \acute{C}\hbar_{E_1}(c_i) \times \acute{C}\hbar_{E_2}(c_i)\Big\}, \\
\theta_3(c_i) &= \{(w_r(c_i), \overline{w_t}(c_i)) | w_r(c_i) - \overline{w_t}(c_i) = 0, \\
&\quad (w_r(c_i), \overline{w_t}(c_i)) \in \acute{C}\hbar_{E_1}(c_i) \times \acute{C}\hbar_{E_2}(c_i)\Big\},
\end{aligned}$$

$$S_1(c_i) = \begin{cases} \frac{1}{2}\left\{ \begin{array}{l} \frac{1}{m_i n_i} \sum\limits_{(u_r(c_i),v_t(c_i))\in A_1(c_i)} u_r(c_i) - v_t(c_i) + \\ \sum\limits_{(w_r(c_i),\overline{w}_t(c_i))\in \theta_1(c_i)} w_r(c_i) - \overline{w}_t(c_i) \end{array} \right\}, \\ \quad \text{where } A_1(c_i) \neq \phi;\ \theta_1(c_i) \neq \phi \\ \frac{1}{2}\left\{ \frac{1}{m_i n_i} \sum\limits_{(u_r(c_i),v_t(c_i))\in A_1(c_i)} u_r(c_i) - v_t(c_i) \right\}, \\ \quad \text{where } A_1(c_i) \neq \phi;\ \theta_1(c_i) = \phi \\ \frac{1}{2}\left\{ \sum\limits_{(w_r(c_i),\overline{w}_t(c_i))\in \theta_1(c_i)} w_r(c_i) - \overline{w}_t(c_i) \right\}, \\ \quad \text{where } A_1(c_i) = \phi;\ \theta_1(c_i) \neq \phi, \\ 0,\ \text{where } A_1(c_i) = \phi;\ \theta_1(c_i) = \phi \end{cases}$$

$$S_2(c_i) = \begin{cases} \frac{1}{2}\left\{ \begin{array}{l} \frac{1}{m_i n_i} \sum\limits_{(u_r(c_i),v_t(c_i))\in A_1(c_i)} v_t(c_i) - u_r(c_i) + \\ \sum\limits_{(w_r(c_i),\overline{w}_t(c_i))\in \theta_1(c_i)} \overline{w}_t(c_i) - w_r(c_i) \end{array} \right\}, \\ \quad \text{where } A_2(c_i) \neq \phi;\ \theta_2(c_i) \neq \phi \\ \frac{1}{2}\left\{ \frac{1}{m_i n_i} \sum\limits_{(u_r(c_i),v_t(c_i))\in A_1(c_i)} v_t(c_i) - u_r(c_i) \right\}, \\ \quad \text{where } A_2(c_i) \neq \phi;\ \theta_2(c_i) = \phi \\ \frac{1}{2}\left\{ \sum\limits_{(w_r(c_i),\overline{w}_t(c_i))\in \theta_1(c_i)} \overline{w}_t(c_i) - w_r(c_i) \right\}, \\ \quad \text{where } A_2(c_i) = \phi;\ \theta_2(c_i) \neq \phi, \\ 0,\ \text{where } A_2(c_i) = \phi;\ \theta_2(c_i) = \phi \end{cases}$$

Remark 1: $S_1(c_i)$ represents the average residual amount for $\acute{C}\hbar_{E_1}(c_i)$ over $\acute{C}\hbar_{E_2}(c_i)$. And $S_2(c_i)$ represents the average residual amount for $\acute{C}\hbar_{E_2}(c_i)$ over $\acute{C}\hbar_{E_1}(c_i)$. $P(E_1 \succ E_2)$ represents the priority degree for $E_1 \succ E_2$. And $P(E_2 \succ E_1)$ represents the priority degree for $E_2 \succ E_1$.

For example, $P(E_1 \succ E_2) = 0.25$ indicates that the priority degree for $E_1 \succ E_2$ is 0.25, and is written as: $E_1 \succ_{(0.25)} E_2$.

The notation $E_1 \succ_{P(E_1 \succ E_2)} E_2$ doesn't mean that E_1 is absolutely superior to E_2; its just shows that the priority degree of $E_1 \succ E_2$ which is denoted by $P(E_1 \succ E_2)$. In fact, the priority degree of $P(E_2 \succ E_1) = 1 - P(E_1 \succ E_2)$.

Example 1: Suppose that $C = \{c_1, c_2, c_3\}$,

$$E_1 = \left\{ \begin{array}{c} \langle c_1, \{0.5e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.8)}\}\rangle, \\ \langle c_2, \{0.1e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.5)}\}\rangle, \\ \langle c_3, \{0.4e^{i2\pi(0.1)}, 0.7e^{i2\pi(0.2)}\}\rangle \end{array} \right\}$$

and

$$E_2 = \left\{ \begin{array}{c} \langle c_1, \{0.2e^{i2\pi(0.4)}, 0.6e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.8)}\}\rangle, \\ \langle c_2, \{0.8e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}\}\rangle, \\ \langle c_3, \{0.9e^{i2\pi(0.1)}, 0.7e^{i2\pi(0.3)}\}\rangle \end{array} \right\}$$

are two CHFSs. Then by above Definition (3.1) we have,

$$\begin{aligned} A_1(c_1) &= \{(0.5, 0.2), (0.5, 0.4)\}, \\ A_1(c_2) &= \{(0.3, 0.1)\}, A_1(c_3) = \phi. \\ \theta_1(c_1) &= \{(0.6, 0.4), (0.8, 0.4), (0.8, 0.6)\}, \\ \theta_1(c_2) &= \{(0.5, 0.3), (0.5, 0.2)\}, \\ \theta_1(c_3) &= \{(0.2, 0.1)\}. \\ A_2(c_1) &= \{(0.5, 0.6), (0.2, 0.6), (0.2, 0.4), (0.3, 0.6), \\ &\qquad (0.3, 0.4)\}, \\ A_2(c_2) &= \{(0.1, 0.8), (0.1, 0.3), (0.3, 0.8)\}, \\ A_2(c_3) &= \{(0.4, 0.9), (0.4, 0.7), (0.7, 0.9)\}. \\ \theta_2(c_1) &= \{(0.4, 0.6), (0.4, 0.8), (0.6, 0.8)\}, \\ \theta_2(c_2) &= \{(0.2, 0.3), (0.2, 0.5)\}, \\ \theta_2(c_3) &= \{(0.1, 0.3), (0.2, 0.3)\}. \\ S_1(c_1) &= 0.0667, S_1(c_2) = 0.0583, S_1(c_3) = 0.0125. \\ S_2(c_1) &= 0.1055, S_2(c_2) = 0.1500, S_2(c_3) = 0.1625. \end{aligned}$$

$$P(E_1 \succ E_2) = 0.2475, P(E_2 \succ E_1) = 0.7525.$$
$$E_1 \succ_{(0.2475)} E_2, E_2 \succ_{(0.7525)} E_1.$$

The above results only shows that the priority degree for $E_1 \succ E_2$ is 0.2475 and the priority degree for $E_2 \succ E_1$ is 0.7525. However, this does not imply that E_1 is inherently superior to E_2. We may deduce the following characteristics concerning priority degree.

Property 1 (Normalization): $0 \leq P(E_1 \succ E_2) \leq 1$, $0 \leq P(E_2 \succ E_1) \leq 1$.

Proof: In above example, $P(E_1 \succ E_2) = 0.2475 \leq 1$, $P(E_2 \succ E_1) = 0.7525 \leq 1$ hold. □

Property 2:(Complementarity): $P(E_1 \succ E_2) + P(E_2 \succ E_1) = 1$.

Proof: Let

$$P(E_1 \succ E_2) = \frac{\sum_{i=1}^n S_1(c_i)}{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)}$$

and

$$P(E_2 \succ E_1) = \frac{\sum_{i=1}^n S_2(c_i)}{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)}.$$

Then

$$\begin{aligned} &P(E_1 \succ E_2) + P(E_2 \succ E_1) \\ &= \frac{\sum_{i=1}^n S_1(c_i)}{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)} \\ &\quad + \frac{\sum_{i=1}^n S_2(c_i)}{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)} \\ &= \frac{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)}{\sum_{i=1}^n S_1(c_i) + \sum_{i=1}^n S_2(c_i)} = 1. \end{aligned}$$

□

In above example, $P(E_1 \succ E_2) = 0.2475$ and $P(E_2 \succ E_1) = 0.7525$, $P(E_1 \succ E_2) + P(E_2 \succ E_1) = 0.2475 + 0.7525 = 1$ hold.

IV. DISTANCE MEASURES BETWEEN CHFSs

In this section, we define some generalized distances between two CHFSs, namely the Hamming distance, Euclidean distance, Hausdorff distance and Hybrid distance between CHFSs. We also define their weighted form and continuous weighted forms.

Let C be the universal set,

$$X = \{\langle c, \gamma_{X_j}(c)\, e^{i2\pi\left(w_{\gamma X_k}(c)\right)}\rangle | c \in C\}$$

and

$$Y = \{\langle c, \gamma_{Y_j}(c)\, e^{i2\pi\left(w_{\gamma Y_k}(c)\right)}\rangle | c \in C\}$$

be two *CHFSs*, where $k = \{1, 2, \ldots n\}$ and $j = \{1, 2, \ldots n\}$ then the distance measures between X and Y can be presented as follows:

Complex hesitant normalized Hamming distance (CHNHD) [41]:

$$d_{CHNHD}(X, Y) = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\begin{array}{l}\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| \\ +\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|\end{array}\right\}\right] \quad (6)$$

Complex hesitant normalized Euclidean distance (CHNED) [41]:

$$d_{CHNED}(X, Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}\left(\begin{array}{l}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right|^2 \\ +\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|^2\end{array}\right)\right]^{\frac{1}{2}} \quad (7)$$

where $\gamma_{X_k}(c_j), \gamma_{Y_k}(c_i), w_{\gamma X_k}(c_j)$ and $w_{\gamma Y_k}(c_j)$ are the *kth* largest value in X and Y, $lc_j = \max\{l(X), l(Y)\}$ and respectively. Eq. (6) and Eq. (7) may be used to form a Generalized complex hesitant normalized distance:

Generalized complex hesitant normalized distance (GCHND) [41]:

$$d_{GCHND}(X, Y) = \frac{1}{2n}\left[\sum_{i=1}^{n}\left\{\begin{array}{l}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left|\gamma_X(c_i) - \gamma_Y(c_i)\right|^{\lambda} \\ +\left|w_{\gamma X}(c_i) - w_{\gamma Y}(c_i)\right|^{\lambda}\end{array}\right\}\right]^{\frac{1}{\lambda}} \quad (8)$$

where $\lambda > 0$.

- If we take $\lambda = 1$,then the Generalized complex hesitant normalized distance is reduce to complex hesitant normalized Hamming distance.
- If we take $\lambda = 2$, then the Generalized complex hesitant normalized distance reduce to complex hesitant normalized Euclidean distance.

Now we follow the Hausdorff metric and generalize the distance measures proposed in [12] to CHFS environment and develop complex hesitant normalized Hamming-Hausdorff and Euclidean-Hausdorff distance as follows;

Complex hesitant normalized Hamming-Hausdorff distance (CHNHHD):

$$d_{CHNHHD}(X, Y) = \frac{1}{2n}\left[\sum_{i=1}^{n}\left\{\max_i\left(\begin{array}{l}\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| + \\ \max_i\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|\end{array}\right)\right\}\right] \quad (9)$$

Complex hesitant normalized Euclidean-Hausdorff distance (CHNEHD):

$$d_{CHNEHD}(X, Y) = \left[\frac{1}{2n}\left\{\sum_{i=1}^{n}\left(\begin{array}{l}\max_i\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right|^2 + \\ \max_i\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|^2\end{array}\right)\right\}\right]^{\frac{1}{2}} \quad (10)$$

Generalized complex hesitant normalized Hausdorff distance (GCHNHD):

$$d_{GCHNHD}(X, Y) = \left[\frac{1}{2n}\left\{\sum_{i=1}^{n}\left(\begin{array}{l}\max_i\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right|^{\lambda} + \\ \max_i\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right)\right\}\right]^{\frac{1}{\lambda}} \quad (11)$$

where $\lambda > 0$, if $\lambda = 1$ then the Generalized complex hesitant normalized Hausdorff distance reduce to a Complex hesitant normalized Hamming-Hausdorff distance, and if $\lambda = 2$ then the Generalized complex hesitant normalized Hausdorff distance reduce to a Generalized complex hesitant normalized Hausdorff distance.

Now we define complex hesitant hybrid normalized hamming distance, complex hesitant hybrid normalized Euclidean distance and generalized complex hesitant hybrid normalized distance measure by combining the above distances respectively.

Complex hesitant Hybrid normalized Hamming distance (CHHNHD): Please see (12).

$$d_{CHHNHD}(X, Y) = \frac{1}{4n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\begin{array}{l}\left(\begin{array}{l}\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| + \\ \left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|\end{array}\right) \\ +\max_i\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| + \\ \max_i\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|\end{array}\right\}\right] \quad (12)$$

Complex hesitant Hybrid normalized Euclidean distance (CHHNED): Please see (13).

$$d_{CHHNED}(X, Y) = \left[\frac{1}{4n}\sum_{i=1}^{n}\left\{\begin{array}{l}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{l}\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right|^2 + \\ \left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|^2\end{array}\right) \\ +\max_i\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right|^2 + \\ \max_i\left|w_{\gamma X_k}(c_i) - w_{\gamma Y_k}(c_i)\right|^2\end{array}\right\}\right]^{\frac{1}{2}} \quad (13)$$

Generalized complex hesitant Hybrid normalized distance (GCHHND): Please see (14),

$$d_{GCHHND}(X,Y) = \left[\frac{1}{4n}\sum_{i=1}^{n}\left\{\begin{array}{c}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right)\\ +\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right\}\right]^{\frac{1}{\lambda}} \tag{14}$$

where $\lambda > 0$.

Normally, the weighted vector W_j of every element $c_i \in C$, such that $W_j(j = 1, 2, \ldots, n)$ with $W_j \in [0, 1]$ and $\sum_{j=1}^{n} W_j = 1$, then the Generalized complex hesitant weighted distance are define as follows;

Generalized complex hesitant weighted normalized distance (GCHWND) [41]: Please see (15),

$$d_{GCHWND}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}W_j\left(\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right)\right)\right]^{\frac{1}{\lambda}} \tag{15}$$

where $\lambda > 0$. If we put $\lambda = 1$ and $\lambda = 2$, then we get complex hesitant normalized hamming distance [41], and complex hesitant normalized Euclidean distance [41] respectively as follows:

Complex hesitant weighted normalized Hamming distance (CHWNHD) [41]: Please see (16).

$$d_{CHWNHD}(X,Y) = \frac{1}{2n}\sum_{i=1}^{n}W_j\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\end{array}\right)\right] \tag{16}$$

Complex hesitant weighted normalized Euclidean distance (CHWNED) [41]: Please see (17).

$$d_{CHWNED}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}W_j\left(\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{2}+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{2}\end{array}\right)\right)\right]^{\frac{1}{2}} \tag{17}$$

Next we define generalized complex hesitant weighted Housdorff distance as:

Generalized complex hesitant weighted normalized Hausdorff distance (GCHWNHD): Please see (18),

$$d_{GCHWNHD}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}W_j\left(\begin{array}{c}\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right)\right]^{\frac{1}{\lambda}} \tag{18}$$

where $\lambda \geq 1$. When $\lambda = 1$, then we get a complex hesitant weighted normalized Hamming-Hausdorff distance (CHWNHHD). When $\lambda = 2$, then we get complex hesitant weighted normalized Euclidean-Hausdorff distance (CHWNEHD) respectively as follows;

Complex hesitant weighted normalized Hamming-Hausdorff distance (CHWNHHD): Please see (19).

$$d_{CHWNHHD}(X,Y) = \frac{1}{2n}\sum_{i=1}^{n}W_j\left[\begin{array}{c}\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\end{array}\right]. \tag{19}$$

Complex hesitant weighted normalized Euclidean-Hausdorff distance (CHWNEHD): Please see (20).

$$d_{CHWNEHD}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}W_j\left(\begin{array}{c}\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{2}+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{2}\end{array}\right)\right]^{\frac{1}{2}} \tag{20}$$

Furthermore, by combining the GCHWWND and GCHWNHD we define the Generalized complex hesitant weighted Hybrid normalized distance as follows:

Generalized complex hesitant weighted Hybrid normalized distance (GCHWHND): Please see (21),

$$d_{GCHWHND}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n}W_j\left\{\begin{array}{c}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right)\\ +\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^{\lambda}+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^{\lambda}\end{array}\right\}\right]^{\frac{1}{\lambda}} \tag{21}$$

where $\lambda > 0$.

In particular, if we put $\lambda = 1$ and $\lambda = 2$ in Eq. (21), then a complex hesitant weighted Hybrid normalized Hamming distance and complex hesitant weighted Hybrid normalized Euclidean distance are obtained respectively.

Complex hesitant weighted Hybrid normalized Hamming distance (CHWHNHD): Please see (22).

$$d_{CHWHNHD}(X,Y) = \frac{1}{2n}\sum_{i=1}^{n}W_j\left[\begin{array}{c}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\end{array}\right\}\\ +\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\end{array}\right] \tag{22}$$

Complex hesitant weighted Hybrid normalized Euclidean distance (CHWHNED): Please see (23).

$$d_{CHWHNED}(X,Y) = \left[\frac{1}{2n}\sum_{i=1}^{n} W_j\left\{\begin{array}{c}\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left(\begin{array}{c}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^2+\\ \left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^2\end{array}\right)\\ +\max_i\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|^2+\\ \max_i\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|^2\end{array}\right\}\right]^{\frac{1}{2}}. \tag{23}$$

Example 2: Let $C=\{c_1,c_2,c_3\}$ be a universal set and

$$Ch_1=\left\{\begin{array}{l}\left\langle c_1,\left\{0.1e^{i2\pi(0.3)},0.3e^{i2\pi(0.4)}\right\}\right\rangle,\\ \left\langle c_2,\left\{0.8e^{i2\pi(0.5)},0.5e^{i2\pi(0.1)}\right\}\right\rangle,\\ \left\langle c_3,\left\{0.6e^{i2\pi(0.9)},0.9e^{i2\pi(0.5)}\right\}\right\rangle.\end{array}\right\}$$

and

$$Ch_2=\left\{\begin{array}{l}\left\langle c_1,\left\{0.9e^{i2\pi(0.1)},0.7e^{i2\pi(0.5)}\right\}\right\rangle,\\ \left\langle c_2,\left\{0.3e^{i2\pi(0.1)},0.4e^{i2\pi(0.4)}\right\}\right\rangle,\\ \left\langle c_3,\left\{0.1e^{i2\pi(0.6)},0.2e^{i2\pi(0.9)}\right\}\right\rangle.\end{array}\right\}.$$

be two CHFEs. Then applying complex hesitant normalized Hamming distance (*CHNHD*) we get

$d_{CHNHD}(Ch_1,Ch_2)=0.3917$ (Please see (24), as shown at the bottom of the page),

$d_{CHNED}(Ch_1,Ch_2)=0.4425$ (Please see (25), as shown at the bottom of the page),

$d_{GCHNED}(Ch_1,Ch_2)=0.4841$ (Please see (24)).

Also by applying the complex hesitant normalized Hausdorff distance (CHNHD), we get:

$$d_{CHNHHD}(Ch_1,Ch_2)=0.5000,$$
$$d_{CHNEHD}(Ch_1,Ch_2)=0.5385,$$
$$d_{GCHNHD}(Ch_1,Ch_2)=0.5708.$$

Similarly, utilizing the Complex hesitant Hybrid normalized distances we get:

$$d_{CHHNHD}(Ch_1,Ch_2)=0.3208,$$
$$d_{CHHNED}(Ch_1,Ch_2)=0.4128,$$
$$d_{GCHHND}(Ch_1,Ch_2)=0.4691.$$

Let $\{w_1,w_2,w_3\}$, such that, $w_1=0.3$, $w_2=0.2$ and $w_3=0.5$, be the weighted vector. Then utilizing the weighted distances we get:

$$d_{CHWNHD}(Ch_1,Ch_2)=0.3719,$$
$$d_{CHWNED}(Ch_1,Ch_2)=0.2088$$
$$d_{GCHWND}(Ch_1,Ch_2)=0.2100,$$
$$d_{CHWNHHD}(Ch_1,Ch_2)=0.1717$$
$$d_{CHWNEHD}(Ch_1,Ch_2)=0.3191,$$
$$d_{GCHWNHD}(Ch_1,Ch_2)=0.4046$$
$$d_{CHWHNHD}(Ch_1,Ch_2)=0.3100,$$
$$d_{CHWHNED}(Ch_1,Ch_2)=0.4157$$
$$d_{GCHWHND}(Ch_1,Ch_2)=0.4759.$$

Now, we determined that Each of the following distance measures is discrete if the universe of discourse and the weight of components are both continuous and the weighted vector of $c\in C=[a,b]$ is $w(c)$, where as $w(c)\in[0,1]$, and $\int_a^b w(c)dc=1$. Then a continuous complex hesitant weighted distance measures are defined.

Continuous complex hesitant weighted Hamming distance (CCHWHD): Please see (27).

$$d(X,Y) = \frac{1}{2}\int_a^b W(c)\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{c}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{array}\right\}\right)dc \tag{27}$$

Continuous complex hesitant weighted Euclidean distance (CCHWED): Please see (28).

$$d(X,Y) = \left[\frac{1}{2}\int_a^b W(c)\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{c}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^2+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^2\end{array}\right\}\right)dc\right]^{\frac{1}{2}} \tag{28}$$

$$d_{CHNHD}(Ch_1,Ch_2)=\frac{1}{2\times 3}\left(\begin{array}{l}\frac{1}{2}(|0.1-0.9|+|0.3-0.7|+|0.3-0.1|+|0.4-0.5|)\\ +\frac{1}{2}(|0.8-0.3|+|0.5-0.4|+|0.5-0.1|+|0.1-0.4|)\\ +\frac{1}{2}(|0.6-0.1|+|0.9-0.2|+|0.9-0.6|+|0.5-0.9|)\end{array}\right)=0.3917 \tag{24}$$

$$d_{CHNED}(Ch_1,Ch_2)=\left[\frac{1}{2\times 3}\left(\begin{array}{l}\frac{1}{2}(|0.1-0.9|^2+|0.3-0.7|^2+|0.3-0.1|^2+|0.4-0.5|^2)\\ +\frac{1}{2}(|0.8-0.3|^2+|0.5-0.4|^2+|0.5-0.1|^2+|0.1-0.4|^2)\\ +\frac{1}{2}(|0.6-0.1|^2+|0.9-0.2|^2+|0.9-0.6|^2+|0.5-0.9|^2)\end{array}\right)\right]^{\frac{1}{2}}=0.4425 \tag{25}$$

$$d_{GCHNED}(Ch_1,Ch_2)=\left[\frac{1}{2\times 3}\left(\begin{array}{l}\frac{1}{2}(|0.1-0.9|^3+|0.3-0.7|^3+|0.3-0.1|^3+|0.4-0.5|^3)\\ +\frac{1}{2}(|0.8-0.3|^3+|0.5-0.4|^3+|0.5-0.1|^3+|0.1-0.4|^3)\\ +\frac{1}{2}(|0.6-0.1|^3+|0.9-0.2|^3+|0.9-0.6|^3+|0.5-0.9|^3)\end{array}\right)\right]^{\frac{1}{3}}=0.4841 \tag{26}$$

Generalized continuous complex hesitant weighted distance (GCCHWD): Please see (29),

$$d(X,Y) = \left[\frac{1}{2}\int_a^b W(c)\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{l}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{array}\right\}\right)dc\right]^{\frac{1}{\lambda}} \tag{29}$$

where $\lambda > 0$, if $W(c) = \frac{1}{(b-a)}$, for any $c \in [a,b]$, then the Generalized continuous complex hesitant weighted distance is reduced to a Generalized continuous complex hesitant normalized distance, and the continuous complex hesitant weighted Hamming distance is reduced to a continuous complex hesitant normalized Hamming distance, and the continuous complex hesitant weighted Euclidean distance reduced to a continuous complex hesitant normalized Euclidean distance respectively.

Generalized continuous complex hesitant normalized distance (GCCHND): Please see (30),

$$d(X,Y) = \left[\frac{1}{2(b-a)}\int_a^b\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{l}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{array}\right\}\right)dc\right]^{\frac{1}{\lambda}} \tag{30}$$

where $\lambda > 0$. if $\lambda = 1$ then the continuous complex hesitant normalized Hamming distance is obtained;

Continuous complex hesitant normalized Hamming distance (CCHNHD): Please see (31),

$$d(X,Y) = \frac{1}{2(b-a)} \times \int_a^b\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{l}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{array}\right\}\right)dc \tag{31}$$

if $\lambda = 2$ then the continuous complex hesitant normalized Euclidean distance is obtained.

Continuous complex hesitant normalized Euclidean distance (CCHNED): Please see (32).

$$d(X,Y) = \left[\frac{1}{2(b-a)}\int_a^b\left(\frac{1}{lc}\sum_{k=1}^{lc}\left\{\begin{array}{l}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{array}\right\}\right)dc\right]^{\frac{1}{2}} \tag{32}$$

Using traditional Hausdorff metric,we proposed the generalized continuous complex hesitant weighted distance are follow as:

Generalized continuous complex hesitant weighted Hausdorff distance (GCCHWHD): Please see (33).

$$d(X,Y) = \left[\int_a^b W(c)\left(\begin{array}{l}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{array}\right)\right]^{\frac{1}{\lambda}} \tag{33}$$

In particular cases, if we take $\lambda = 1, 2$ in above Eq. (32) and (33), the Continuous complex hesitant weighted Hamming-Hausdorff distance and a continuous complex hesitant weighted Euclidean-Hausdorff distance are obtained respectively.

Continuous complex hesitant weighted Hamming-Hausdorff distance (CCHWHHD): Please see (34).

$$d(X,Y) = \left[\frac{1}{2}\int_a^b W(c)\left(\begin{array}{l}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c_i)\right|+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{array}\right)\right] \tag{34}$$

Continuous complex hesitant weighted Euclidean - Hausdorff distance (CCHWEHD): Please see (35).

$$d(X,Y) = \left[\frac{1}{2}\int_a^b W(c)\left(\begin{array}{l}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{array}\right)\right]^{\frac{1}{2}} \tag{35}$$

If $W(c) = \frac{1}{(b-a)}$ for any $c \in [a,b]$, then the continuous complex hesitant weighted Hamming-Hausdorff distance becomes reduced to a continuous complex hesitant normalized Hamming-Hausdorff distance.

Continuous complex hesitant normalized Hamming-Hausdorff distance (CCHNHHD): Please see (36).

$$d(X,Y) = \left[\frac{1}{2(b-a)}\int_a^b\left(\begin{array}{l}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{array}\right)\right] \tag{36}$$

Similarly, for any $c \in [a,b]$, if $W(c) = \frac{1}{(b-a)}$ then the continuous complex hesitant weighted Euclidean-Hausdorff distance is reduced to a continuous complex hesitant normalized Euclidean-Hausdorff distance, and a Generalized continuous complex hesitant weighted Hausdorff distance is reduced to a Generalized continuous complex hesitant normalized Hausdorff distance.

Continuous complex hesitant normalized Euclidean-Hausdorff distance (CCHNEHD): Please see (37).

$$d(X,Y) = \left[\frac{1}{2(b-a)}\int_a^b\left(\begin{array}{l}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{array}\right)\right]^{\frac{1}{2}} \tag{37}$$

Generalized continuous complex hesitant normalized Hausdorff distance (GCCHNHD): Please see (38),

$$d(X,Y)=\left[\frac{1}{2(b-a)}\int_a^b\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{pmatrix}\right]^{\frac{1}{\lambda}} \tag{38}$$

where $\lambda > 0$. Combining the above distances, we can obtained the Generalized continuous complex hesitant weighted Hybrid distance, which is defined as;

Generalized continuous complex hesitant weighted Hybrid distance (GCCHWHD): Please see (39),

$$d(X,Y)=\left[\frac{1}{2}\int_a^b W(c)\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{pmatrix}\end{pmatrix}dc\right]^{\frac{1}{\lambda}} \tag{39}$$

where $\lambda > 0$.

In special cases, if we take $\lambda = 1$ then the GCCHWHD reduced to CCHWHHD:

Continuous complex hesitant weighted Hybrid Hamming distance (CCHWHHD): Please see (40).

$$d(X,Y)=\frac{1}{2}\int_a^b W(c)\times\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{pmatrix}\end{pmatrix}dc. \tag{40}$$

If take $\lambda = 2$ then the GCCHWHD reduced to CCHWHED:

Continuous complex hesitant weighted Hybrid Euclidean distance (CCHWHED): Please see (41).

$$d(X,Y)=\left[\frac{1}{2}\int_a^b W(c)\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{pmatrix}\end{pmatrix}dc\right]^{\frac{1}{2}} \tag{41}$$

If $W(c)=\frac{1}{(b-a)}$, then (32),(33) and (34) are reduced to a Generalized continuous complex hesitant Hybrid normalized distance, Continuous complex hesitant Hybrid normalized Hamming distance and a continuous complex hesitant Hybrid normalized Euclidean distance respectively, for every $c \in [a,b]$.

Generalized continuous complex hesitant Hybrid normalized distance (GCCHHND): Please see (42),

$$d(X,Y)=\left[\frac{1}{2(b-a)}\int_a^b\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{\lambda}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{\lambda}\end{pmatrix}\end{pmatrix}dc\right]^{\frac{1}{\lambda}} \tag{42}$$

where $\lambda > 0$.

Continuous complex hesitant Hybrid normalized Hamming distance (CCHHNHD): Please see (43).

$$d(X,Y)=\frac{1}{2(b-a)}\int_a^b\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|\end{pmatrix}\end{pmatrix}dc \tag{43}$$

Continuous complex hesitant Hybrid normalized Euclidean distance (CCHHNED): Please see (44).

$$d(X,Y)=\left[\frac{1}{2(b-a)}\int_a^b\begin{pmatrix}\frac{1}{2lc}\sum_{k=1}^{lc}\begin{Bmatrix}\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{Bmatrix}\\ +\frac{1}{2}\begin{pmatrix}\max_k\left|\gamma_{X_k}(c)-\gamma_{Y_k}(c)\right|^{2}+\\ \max_k\left|w_{\gamma X_k}(c)-w_{\gamma Y_k}(c)\right|^{2}\end{pmatrix}\end{pmatrix}dc\right]^{\frac{1}{2}} \tag{44}$$

A. ANALYSIS OF COMPLEX HESITANT FUZZY SET CHFS ORDERING RELATIONS

In this section,we examine the CHFS ordering relation using example.

Example 3: Suppose that $C = \{c\}$, $Ch_1(c) = \{0.2e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.8)}\}$ and $Ch_2(c) = \{0.1e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.7)}\}$ are two CHFSs on C. When we use the score function we get, $S(Ch_1) = 0.60$, $S(Ch_2) = 0.433$ $S(Ch_1) > S(Ch_2)$ From the above definition,we have $Ch_1(c) > Ch_2(c)$. Aggregation operators, in general, are monotonous increasing functions. Using the score function to handle the ordering of $Ch_1(c)$ and $Ch_2(c)$. TOPSIS can be used to deal with the ordering relationship between $Ch_1(c)$ and $Ch_2(c)$. Now the same result can be derived using the closeness coefficient formula. Let us take the CHF-negative ideal and CHF-positive Ideal i.e.,, $Ch^{-}(c) = \{0e^{i2\pi(0)}, 0e^{i2\pi(0)}, 0e^{i2\pi(0)}\}$ and $Ch^{+}(c) = \{1e^{i2\pi(1)}, 1e^{i2\pi(1)}, 1e^{i2\pi(1)}\}$. Now we find the closeness coefficient using the distance formulas which

TABLE 1. Ranking order of Closeness coefficient by utilizing different distance measures.

Distances	$C_{Ch_1(c)}$	$C_{Ch_2(c)}$	$Ranking$
Closeness coefficient by CHNHD [41]	0.6	0.433	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by CHNED [41]	0.581	0.446	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by GCHND [41]	0.568	0.531	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by CHNHHD	0.548	0.484	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by CHNEHD	0.546	0.482	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by GCHNHD	0.544	0.481	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by CHHNHD	0.569	0.464	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by CHHNED	0.558	0.470	$C_{Ch_1(c)} > C_{Ch_2(c)}$
Closeness coefficient by GCHHND	0.551	0.473	$C_{Ch_1(c)} > C_{Ch_2(c)}$

are defined above. We apply a complex hesitant normalized hamming distance (CHNHD): $d_{ch}(Ch_1(c), Ch^-(c)) = 0.6$, $d_{ch}(Ch_2(c), Ch^-(c)) = 0.433$ apply complex hesitant normalized Euclidean distance (CHNED): $d_{ch}(Ch_1(c), Ch^+(c)) = 0.4$, $d_{ch}(Ch_2(c), Ch^+(c)) = 0.567$ The closeness coefficient of $Ch_1(c)$ and $Ch_2(c)$, denoted by $C_{Ch_1(c)}$ and $C_{Ch_2(c)}$ are denoted as,

$$\frac{d_{ch}(Ch_1(c), Ch^-(c))}{d_{ch}(Ch_1(c), Ch^-(c)) + d_{ch}(Ch_1(c), Ch^+(c))} = 0.60$$
$$\frac{d_{ch}(Ch_2(c), Ch^-(c))}{d_{ch}(Ch_2(c), Ch^-(c)) + d_{ch}(Ch_2(c), Ch^+(c))} = 0.433$$

Since, $C_{Ch_1(c)} > C_{Ch_2(c)}$ Therefore $0.60 > 0.434$ i-e,

$$\frac{d_{ch}(Ch_1(c), Ch^-(c))}{d_{ch}(Ch_1(c), Ch^-(c)) + d_{ch}(Ch_1(c), Ch^+(c))} > \frac{d_{ch}(Ch_2(c), Ch^-(c))}{d_{ch}(Ch_1(c), Ch^-(c)) + d_{ch}(Ch_1(c), Ch^+(c))}.$$

from which, $Ch_1(c) > Ch_2(c)$.

From Table 1, we see that the ranking order by utilizing the proposed distance measures are the same as compared with the existing distances. However the distance measures proposed in [12] and [41] are the special cases of the distances developed in this paper. Therefore, the proposed distance measures are the generalization of the existing distance measures.

Example 4: To make a proper diagnosis, $C = C_1$ (Viral fever), C_2 (Malaria), C_3 (Typhoid), C_4 (Stomach problem), C_5 (Chest problem) for an affected person (patient) with the given values of the signs and symptoms, $S = S_1$ (Temperature), S_2(Headache), S_3(Cough), S_4(Stomach pain) and S_5 (Chest pain), taking into consideration all possible diagnosis Szmidt [18] considered all possible diagnoses and symptoms and signs and symptoms as *HFEs*. Utilizing *CHFSs* can take an a lot of the more data taken into consideration; the more values, we achieve from affected person, the greater epistemic reality we have. So, in this paper, we use *CHFEs* to deal with such cases. Every symptom is defined through a *CHFE*, which is defined by sets $(\gamma_{X_i}(c)\, e^{i2\pi(w_{\gamma X_k}(c))})$ to indicate the degree that signs and symptoms characteristic S_i satisfy the considered diagnosis C_i. The data is given in Table 2. The set of patients is P = Nida, Tania, Dania, Faryal, Wajiha. The signs and symptoms which may be additionally defined through *CHFEs* are given in Table 3. We are seeking a diagnosis for every affected person.

We applied the existing distances [41] and the proposed distance measure in order to calculate a diagnosis for each patient. Tables 4-12 show a list of all the consequences (results) for the patient under consideration. In Tables 4, 5, and 6, by utilizing the complex hesitant normalized Hamming distance [41], we found that Nida suffers from stomach problems, which is the same by using complex hesitant normalized Hamming-Hausdorff distance (Table 7), complex hesitant normalized Euclidean-Hausdorff distance (Table 8), complex hesitant normalized generalized-Hausdorff (Table 9), hybrid complex hesitant normalized Hamming distance (Table 10), hybrid complex hesitant normalized Euclidean distance (Table 11), and hybrid complex hesitant normalized generalized distance (Table 12). The proposed distances, on the other hand, are a generalisation of the distance measure developed in [41] and are spatial cases of the developed distances. Also, from Table 4, we see that Tania suffers from stomach problems, but Table 5 and Table 6 show that Tania suffers form malaria. However, by using complex hesitant normalized Hamming-Hausdorff distance (Table 7), complex hesitant normalized Euclidean-Hausdorff distance (Table 8), complex hesitant normalized generalized-Hausdorff (Table 9), hybrid complex hesitant normalized Hamming distance (Table 10), hybrid complex hesitant normalized Euclidean distance (Table 11), hybrid complex hesitant normalized generalized distance (Table 12), Tania suffers from malaria. Which is the same by utilizing distance measure proposed in [41] (Table 5 and Table 6). This shows that the proposed distance measure presents a better result as compared to the existing distances [41]. Furthermore, in Table 4, we see that Dania suffers from malaria. The same result can be seen in Table 5 and Table 6. However, in Tables 7-12, we found that Dania suffers from malaria, which is the same as seen in Table 4-6. Moreover, from Table 4, we see that Faryal suffers from stomach problems, which is the same result as presented in Tables 7-12. However, from Table 5 and Table 6, by utilizing distances developed in [41], it is found that Faryal suffers from malaria. Furthermore, from Table 4, we see that Wajiha suffers from stomach problem, but Table 5 and Table 6 show that Wajiha suffers from typhoid.

TABLE 2. Symptoms characteristic of the considered diagnosis.

	Temperature	Headache	Cough	Stomach pain	Chest pain
Viral fever	$\left\{ 0.1e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.3)} \right\}$	$\left\{ 0.3e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.5)} \right\}$	$\left\{ 0.4e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.2)} \right\}$	$\left\{ 0.2e^{i2\pi(0.4)}, 0.9e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.1)} \right\}$	$\left\{ 0.4e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.7)} \right\}$
Malaria	$\left\{ 0.6e^{i2\pi(0.2)}, 0.9e^{i2\pi(0.3)} \right\}$	$\left\{ 0.2e^{i2\pi(0.9)}, 0.1e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)} \right\}$	$\left\{ 0.3e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.2)}, 0.6e^{i2\pi(0.1)} \right\}$	$\left\{ 0.2e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.4)}, 0.1e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)} \right\}$	$\left\{ 0.2e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.1)} \right\}$
Typhoid	$\left\{ 0.4e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.7)} \right\}$	$\left\{ 0.9e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.2)} \right\}$	$\left\{ 0.4e^{i2\pi(0.2)}, 0.5e^{i2\pi(0.5)}, 0.7e^{i2\pi(0.3)} \right\}$	$\left\{ 0.3e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.4)} \right\}$	$\left\{ 0.8e^{i2\pi(0.2)}, 0.9e^{i2\pi(0.4)} \right\}$
Stomach problem	$\left\{ 0.9e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.5)} \right\}$	$\left\{ 0.4e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.4)} \right\}$	$\left\{ 0.8e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.4)} \right\}$	$\left\{ 0.5e^{i2\pi(0.3)}, 0.6e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.9)}, 0.8e^{i2\pi(0.1)} \right\}$	$\left\{ 0.1e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.1)} \right\}$
Chest problem	$\left\{ 0.5e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.2)} \right\}$	$\left\{ 0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.1)} \right\}$	$\left\{ 0.9e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.6)} \right\}$	$\left\{ 0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.1)}, 0.6e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.3)} \right\}$	$\left\{ 0.6e^{i2\pi(0.9)}, 0.9e^{i2\pi(0.3)} \right\}$

TABLE 3. Symptoms characters of the considered affected person(patient).

	Temperature	Headache	Cough	Stomach pain	Chest pain
Nida	$\left\{ 0.3e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.1)} \right\}$	$\left\{ 0.5e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.6)} \right\}$	$\left\{ 0.2e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.5)} \right\}$	$\left\{ 0.1e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.8)}, 0.8e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.5)} \right\}$	$\left\{ 0.9e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)} \right\}$
Tania	$\left\{ 0.4e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.8)} \right\}$	$\left\{ 0.1e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)} \right\}$	$\left\{ 0.1e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.9)}, 0.3e^{i2\pi(0.8)} \right\}$	$\left\{ 0.6e^{i2\pi(0.2)}, 0.6e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.5)} \right\}$	$\left\{ 0.9e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.8)} \right\}$
Dania	$\left\{ 0.9e^{i2\pi(0.5)}, 0.2e^{i2\pi(0.4)} \right\}$	$\left\{ 0.9e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.2)} \right\}$	$\left\{ 0.5e^{i2\pi(0.9)}, 0.9e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.2)} \right\}$	$\left\{ 0.6e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.3)} \right\}$	$\left\{ 0.7e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.2)} \right\}$
Faryal	$\left\{ 0.1e^{i2\pi(0.4)}, 0.2e^{i2\pi(0.9)} \right\}$	$\left\{ 0.2e^{i2\pi(0.1)}, 0.9e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.5)} \right\}$	$\left\{ 0.1e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)} \right\}$	$\left\{ 0.2e^{i2\pi(0.5)}, 0.1e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.9)} \right\}$	$\left\{ 0.4e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.6)} \right\}$
wajiha	$\left\{ 0.5e^{i2\pi(0.7)}, 0.8e^{i2\pi(0.1)} \right\}$	$\left\{ 0.1e^{i2\pi(0.5)}, 0.8e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)} \right\}$	$\left\{ 0.8e^{i2\pi(0.5)}, 0.6e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.4)} \right\}$	$\left\{ 0.6e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.3)} \right\}$	$\left\{ 0.6e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.4)} \right\}$

TABLE 4. Result obtained by complex hesitant normalized Hamming distance [41].

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.2883	0.3083	0.2925	0.3575	0.3033
Tania	0.2658	0.3775	0.2450	0.3800	0.3258
Dania	0.2967	0.3450	0.2492	0.2908	0.2850
Faryal	0.2417	0.3533	0.2925	0.3625	0.3333
Wajiha	0.2517	0.3033	0.3242	0.3258	0.2033

TABLE 5. Result obtained by complex hesitant normalized Euclidean distance [41].

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.3227	0.3663	0.3544	0.4130	0.3568
Tania	0.30997	0.4462	0.3154	0.4365	0.4049
Dania	0.3651	0.4147	0.3116	0.3504	0.3391
Faryal	0.3467	0.4229	0.3417	0.4182	0.4078
Wajiha	0.2855	0.3514	0.3902	0.3807	0.2705

TABLE 6. Result obtained by complex hesitant normalized Generalized distance [41].

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.3559	0.4169	0.40098	0.4567	0.3994
Tania	0.3433	0.4960	0.3752	0.4804	0.4656
Dania	0.4176	0.4599	0.3619	0.3915	0.3930
Faryal	0.4123	0.4730	0.3823	0.4646	0.4565
Wajiha	0.3152	0.3867	0.4404	0.4312	0.3252

TABLE 7. Result obtained by complex hesitant normalized Hamming-Hausdorff distance:.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.420	0.510	0.490	0.520	0.490
Tania	0.40	0.560	0.390	0.530	0.530
Dania	0.440	0.590	0.420	0.480	0.430
Faryal	0.450	0.560	0.500	0.600	0.520
Wajiha	0.350	0.470	0.530	0.500	0.360

However, from Tables 7-12, we also see that Wajiha suffers from typhoid. From the above discussion, we see that the proposed distances gives better results as compared to the distances developed in [41]. The main reason is that Garg et al. [41] distances give different results, while the proposed distances give the same results.

V. TOPSIS METHOD FOR MULTI-ATTRIBUTE/CRITERIA DECISION MAKING APPROACH UNDER COMPLEX HESITANT FUZZY SETTING

In this section, we described a TOPSIS-based MCDM model for solving a faculty selection decision-making issue using (CHFs). The MCDM problem may be stated as a decision

TABLE 8. Result obtained by complex hesitant normalized Euclidean-Hausdorff distance.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.44272	0.53198	0.50695	0.54589	0.51865
Tania	0.42426	0.58481	0.45277	0.54863	0.57009
Dania	0.48785	0.60249	0.45607	0.50398	0.45497
Faryal	0.51672	0.61725	0.49901	0.62929	0.54955
Wajiha	0.38079	0.47117	0.55767	0.51962	0.40000

TABLE 9. Result obtained by complex hesitant normalized Generalized-Hausdorff.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.46354	0.55924	0.52208	0.56629	0.53844
Tania	0.44072	0.60623	0.49853	0.56639	0.60157
Dania	0.52293	0.61243	0.48573	0.52632	0.47812
Faryal	0.56242	0.63471	0.55656	0.65108	0.57307
wajiha	0.40514	0.49434	0.58098	0.53924	0.43480

TABLE 10. Hybrid complex hesitant normalized Hamming distance.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.35417	0.40917	0.39125	0.43875	0.39667
Tania	0.33292	0.46875	0.31750	0.45500	0.42792
Dania	0.36833	0.46750	0.33458	0.38541	0.35750
Faryal	0.34583	0.45667	0.39625	0.48125	0.42667
Wajiha	0.30083	0.38667	0.42708	0.41292	0.28167

TABLE 11. Hybrid complex hesitant normalized Euclidean distance.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.38740	0.45670	0.43737	0.48404	0.44516
Tania	0.37154	0.52014	0.39019	0.49574	0.49443
Dania	0.43089	0.51720	0.39057	0.43402	0.40125
Faryal	0.50365	0.52907	0.50769	0.53428	0.48390
Wajiha	0.33653	0.41563	0.48127	0.45548	0.34144

TABLE 12. Hybrid complex hesitant normalized Generalized distance.

	Viral fever	Malaria	Typhiod	Stomach problem	Chest problem
Nida	0.41667	0.49825	0.46934	0.51731	0.47902
Tania	0.39797	0.55660	0.44538	0.52689	0.54212
Dania	0.47612	0.54679	0.43267	0.46861	0.43968
Faryal	0.49864	0.56542	0.48509	0.57301	0.52130
Wajiha	0.36570	0.44700	0.52020	0.49115	0.38774

matrix (DM), with columns representing the collection of attributes/criteria and rows representing alternatives. Thus, for $DMZ_{m\times n}$, consider the following set of m choices and n criteria. The k is the DMs' unknown weight vector is denoted by $W = (W_1, W_2, \ldots, W_j)^T$, such that when weight is completely unknown and when weight is partially known, with $W_j \in [0, 1]$, $\sum_{t=1}^{k} W_j = 1$. The complex hesitant fuzzy decision making (CHF-DMs) provided by the DMs is denoted by $\check{Z}^{(k)} = ([\rho_{ij}]^{(k)})_{m\times n}$. Since criteria in the decision making process are of two types (i) cost and (ii) benefit, if the criteria are of the cost type, we will change the criteria to the benefit type by normalising the CHF-DMs $\check{Z}^{*(k)} = ([\rho^*_{ij}]^{(k)})_{m\times n}$ as follows:

$$[\rho^*_{ij}]^{(k)} = \begin{cases} [\rho_{ij}]^{(k)} \text{ for benefit criteria,} \\ ([\rho_{ij}]^{(k)})^c \text{ for cost criteria,} \end{cases}$$

where $([\rho_{ij}]^{(k)})^c$ is the complement of $[\rho_{ij}]^{(k)}$ $(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$.

The steps of the developed approach are as under:

Step 1. Construct CHFs decision matrices.

Step 2. When the criteria's weights are supplied, use them. If not, then determine the weight using the optimization problem approach. The complicated hesitant fuzzy decision matrix is stated as follows: Because the criteria have varying degrees of relevance, the weight vector of all the criteria, as determined by the DMs, is specified as by $W = \{W_1, W_2, \ldots, W_j\}^T$ where $0 \leq W_j \leq 1$, $\sum_{t=1}^{k} W_j = 1$. And W_j is the important degree for all attribute/criterias. In general, the significant degrees of the qualities must be established by the decision-makers. As a result of the complexity and ambiguity of real decision-making difficulties, as well as the inherent subjectivity of human thought, information on criteria weights is frequently inadequate. We believe that the criteria weight information supplied by the DMs might be presented in the following ways [42], for $i \neq j$:

1. A weak ranking:$\{W_i \geq W_j\}$;
2. A strick ranking:$\{W_i - W_j \geq \sigma_i (> 0)\}$;
3. A ranking with multiples: $\{W_i \geq \sigma_i W_j\}$, $0 \leq \sigma_i \leq 1$;
4. An interval form: $\{\lambda_i \leq W_i \leq \lambda_i + \sigma_i\}$, $0 \leq \lambda_i \leq \lambda_i + \sigma_i \leq 1$;

5. A ranking of differences: $\{W_i - W_j \geq W_k - W_l\}$, for $j \neq k \neq l$.

In MCDM, it is important to evaluate the weight of the criteria. According to Wang [43], criteria with a large deviation value should be given a high weight in the MCDM problem, whereas criteria with a small deviation value should be provided a low weight. In a complicated and hesitant fuzzy environment, we created an optimization model based on deviation maximisation approaches to discover the ideal weights for a set of criteria. For the criteria $T_i \in T$, the deviation of the criteria can be determined as:

$$D_{ij}(W_j) = \sum_{k=1}^{m} W_j d\left(X_{ij}, Y_{kj}\right),$$

$$d\left(X, Y\right) = \frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\{\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| + \left|w_{\gamma X_k}\left(c_i\right) - w_{\gamma Y_k}\left(c_i\right)\right|\}\right],$$

denotes the complex hesitant fuzzy normalizied Hamming distance between the CHFs X and Y. Let

$$D_j(W) = \sum_{k=1}^{m} D_{ij}(W) = \sum_{i=1}^{m}\sum_{k=1}^{m} W_j\left(\frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\{\left|\gamma_{X_k}(c_i) - \gamma_{Y_k}(c_i)\right| + \left|w_{\gamma X_k}\left(c_i\right) - w_{\gamma Y_k}\left(c_i\right)\right|\}\right]\right)$$

$j = 1, 2, \ldots, n$ then $D_j(W)$ represents the deviation value of all alternatives.

$$M_1 = \begin{cases} \max D(W) = \sum_{i=1}^{m}\sum_{J=1}^{m}\sum_{K=1}^{m} W_j d\left(X_{ij}, Y_{kj}\right) \\ s.t \quad W_j \geq 0, t = 1, 2, \ldots, n, \sum_{t=1}^{m} W_j = 1 \end{cases}$$

To solve the above model, we let

$$L(W, \xi) = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{k=1}^{m} W_j d\left(X_{ij}, Y_{kj}\right) + \frac{\xi}{2}\left(\sum_{t=1}^{m} W_j - 1\right) = 0.$$

It depicts a constrained optimization's Lagrange function $(M - 1)$. Where is a real number that indicates a variable for the Lagrange multiplier. Next, the partial derivative of L are,

$$\frac{\partial L}{\partial W_t} = \sum_{i=1}^{m}\sum_{k=1}^{m} W_j d\left(X_{ij}, Y_{kj}\right) + \xi W_j = 0 \quad (a)$$
$$\frac{\partial L}{\partial \xi} = \frac{1}{2}\left(\sum_{t=1}^{m} W_j - 1\right) = 0 \tag{45}$$

It follows from above (45) that

$$W_j = \frac{-\sum_{i=1}^{m}\sum_{j=1}^{n} W_j d\left(X_{ij}, Y_{kj}\right)}{\xi}, j = 1, 2, \ldots, n, \tag{46}$$

$$\xi = -\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(X_{ij}, Y_{kj}\right)\right)^2} \tag{47}$$

Clearly $\xi < 0, \sum_{i=1}^{m}\sum_{j=1}^{n} d\left(X_{ij}, Y_{kj}\right)$ means the sum of deviation of all the alternatives with respect to the jth attribute/criteria, and $\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(X_{ij}, Y_{kj}\right)\right)^2}$ means the sum of deviation of all alternatives with respect to all criteria. Then combining Equation (a) and (b), we can get

$$W_j = \frac{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{K=1}^{m} W_j d\left(X_{ij}, Y_{kj}\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} d\left(X_{ij}, Y_{kj}\right)\right)^2}}.$$

By normalizing $W_j (j = 1, 2, \ldots, n)$, we can convert their sum into a unit and obtain Eq. (48), as shown at the bottom of the next page. However, there are times when the information regarding the weight vector is partially known rather than wholly unknown. For these cases, we create the following constrained optimization model (M_2) (49), as shown at the bottom of the next page, based on the set of known weight information.

The (M_2) model is a linear programming model that can be run using the LINGO 11.0 math software package. The solution of this model gives the optimal solution $W = (W_1, W_2, \ldots, W_n)$, which can be used as a criteria weight vector.

Step 3. Calculate the complex hesitant fuzzy positive ideal solution (CHF-PIS), denoted

$$\widetilde{C}^+ = (\widetilde{C}_1^+, \widetilde{C}_2^+, \ldots, \widetilde{C}_3^+) \tag{50}$$

and the complex hesitant fuzzy negative ideal solution (CHF-NIS), denoted by:

$$\widetilde{C}^- = (\widetilde{C}_1^-, \widetilde{C}_2^-, \ldots, \widetilde{C}_3^-) \tag{51}$$

where

$$\widetilde{C}^+ = [\widetilde{\rho}\ \widehat{\upsilon}_{\max j}] = [\vee_1 \leq i \leq m\widetilde{\rho}\ \widehat{\upsilon}_{ij}, \wedge_1 \leq i \leq m\widetilde{\rho}\ \widehat{\phi}_{ij}],$$

and

$$\widetilde{C}^- = [\widetilde{\rho}\ \widehat{\upsilon}_{\min j}] = [\wedge_1 \leq i \leq m\rho\ _{\upsilon ij}, \vee_1 \leq i \leq m\rho\ _{\phi ij}].$$

Step 4. Compute the distance measured from each alternative to a CHF-PIS and a CHP-NIS by utilizing the proposed distances.

Step 5. Calculate the closeness coefficient of each alternative by using the distances of alternatives from CHF-PIS and CHF-NIS based on the following equations:

$$C_{A_i} = \frac{d_j^+(\tilde{C}_j, \tilde{C}_j^+)}{d_j^+(\tilde{C}_j, \tilde{C}_j^+) + d_j^-(\tilde{C}_j, \tilde{C}_j^-)} \tag{52}$$

Step 6. Ranking the alternatives $A_i (i = 1, 2, \ldots, m)$ according to the closeness coefficients $C_{A_i} (i = 1, 2, \ldots, m)$.

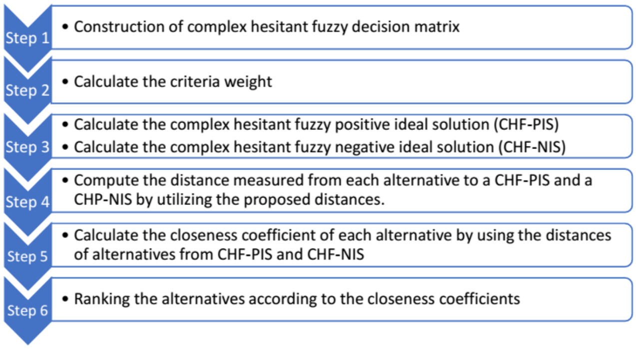

FIGURE 1. Flowchart of the proposed method.

VI. NUMERICAL EXAMPLES

This section presents an MCDM problem regarding the ranking effectiveness of COVID-19 tests in order to show the applicability and application of the developed procedure.

1) RANKING EFFECTIVENESS OF COVID-19 TESTS

The 2019 Coronavirus (COVID-19) disease is a highly infectious disease caused by the SARS-CoV-2 (Severe Acute Respiratory Syndrome Coronavirus-2) [44]. This viral sickness has been the most serious concern for humanity since it was first reported to the WHO (World Health Organization) [45] on December 31, 2019, when multiple cases of an unknown pneumonia-like disease were recorded in the Chinese city of Wuhan. Fever, cough, headache, malaise, dyspnea, and loss of smell and taste are only a few of the symptoms of COVID 19 [46]. Symptoms might emerge anywhere from 1 to 14 days after being exposed to the virus [47]. At least one-third of those afflicted do not show any signs or symptoms. The majority (81%) of individuals who acquire severe symptoms and are categorised as patients have mild to moderate symptoms (up to mild pneumonia) and 14 have severe symptoms (dyspnea). Hypoxia, or lung involvement in imaging (50% or more), causes significant symptoms in 5% of people (dyspnea, shock, or multi-organ dysfunction). The chance of having severe symptoms is higher in older adults. Long-term organ damage has been seen in some people who continue to have varied consequences for several months after recovery (long COVID) [48]. Years of research are being done to find out more about what this disease will mean in the long run [49].

COVID-19 is spread through the air when humans inhale droplets containing viruses or tiny airborne particles. Inhaling them poses the most risk when individuals are close by, but they can also be breathed across greater distances, especially indoors. Infection can also be spread by squirting or spraying infected fluids into the eyes, nose, or mouth, as well as on contaminated surfaces in rare cases. People can be infectious for up to 20 days and spread the virus without showing any signs of symptoms [50]. Several test techniques have been established to diagnose the condition.

On March 11, 2020, the WHO Director-General proclaimed the COVID-19 outbreak a pandemic due to a significant increase in cases outside China worldwide. The WHO has proclaimed the outbreak to be a global health emergency. This unique and very fatal illness has successfully spread in all weather conditions and across all health standards (though a bit slower in certain situations). People who already have chronic conditions like diabetes, heart disease, or respiratory difficulties have been shown to be more susceptible to the virus. This illness affects people of all ages and genders, leaving no one immune. Children who are not particularly susceptible to the disease are disproportionately affected by socioeconomic changes and the dread engendered by the epidemic. Strict lockdowns, the shutdown of educational institutions, a scarcity of supplies, and other issues brought on by the pandemic are also wreaking havoc on businesses and society.Since its breakout, this illness has proven to be a game changer, affecting economies, restricting travel, isolating nations, dramatically increasing casualties, and causing large demographic shifts. COVID-19 is caused by a rapidly mutating virus and is therefore unpredictable. COVID-19's high transmissibility is one of the causes of its rapid spread, and there is presently no vaccine or pharmacological therapy available to protect people against the virus [51]. Although less deadly than SARS (Severe Acute Respiratory Syndrome) or MERS-COV (Middle East Respiratory Syndrome Coronavirus) [52], COVID-19 has produced a huge worldwide

$$W_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m}\left[\frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\right\}\right]\right]}{\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right|+\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\right\}\right]\right)\right]} \tag{48}$$

$$(M_2)\left\{\begin{array}{c} MaxD(W) \\ =\sum_{j=1}^{n}\left[\sum_{i=1}^{m}\sum_{k=1}^{m}\left(\frac{1}{2n}\sum_{i=1}^{n}\left[\frac{1}{lc_i}\sum_{k=1}^{lc_i}\left\{\begin{array}{l}\left|\gamma_{X_k}(c_i)-\gamma_{Y_k}(c_i)\right| \\ +\left|w_{\gamma X_k}(c_i)-w_{\gamma Y_k}(c_i)\right|\end{array}\right\}\right]\right)\right] \\ s.t\ \ W_j \geq 0,\quad j=1,2,\ldots,n,\quad \sum_{t=1}^{m} W_j = 1 \end{array}\right. \tag{49}$$

TABLE 13. Complex hesitant fuzzy decision matrix.

	$\tilde{C}_1$	$\tilde{C}_2$	$\tilde{C}_3$	$\tilde{C}_4$	$\tilde{C}_5$
A_1	$\left\{\begin{array}{c}0.5e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.6)},\\ 0.3e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.6e^{i2\pi(0.5)},\\ 0.9e^{i2\pi(0.4)}\end{array}\right\}$	$\left\{\begin{array}{c}0.1e^{i2\pi(0.8)}, 0.3e^{i2\pi(0.3)},\\ 0.5e^{i2\pi(0.1)}, 0.1e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.5e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.6)},\\ 0.3e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.1e^{i2\pi(0.8)},\\ 0.2e^{i2\pi(0.5)}\end{array}\right\}$
A_2	$\left\{\begin{array}{c}0.8e^{i2\pi(0.2)}, 0.7e^{i2\pi(0.3)},\\ 0.1e^{i2\pi(0.5)}\end{array}\right\}$	$\left\{\begin{array}{c}0.7e^{i2\pi(0.2)},\\ 0.9e^{i2\pi(0.7)}\end{array}\right\}$	$\left\{\begin{array}{c}0.8e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.1)},\\ 0.7e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.1e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.8)},\\ 0.2e^{i2\pi(0.6)}\end{array}\right\}$	$\left\{\begin{array}{c}0.3e^{i2\pi(0.5)},\\ 0.4e^{i2\pi(0.1)}\end{array}\right\}$
A_3	$\left\{\begin{array}{c}0.2e^{i2\pi(0.5)}, 0.3e^{i2\pi(0.7)},\\ 0.4e^{i2\pi(0.3)}\end{array}\right\}$	$\left\{\begin{array}{c}0.5e^{i2\pi(0.8)},\\ 0.7e^{i2\pi(0.5)}\end{array}\right\}$	$\left\{\begin{array}{c}0.7e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.5)},\\ 0.6e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.5)}\end{array}\right\}$	$\left\{\begin{array}{c}0.5e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.5)},\\ 0.6e^{i2\pi(0.1)}\end{array}\right\}$	$\left\{\begin{array}{c}0.9e^{i2\pi(0.2)},\\ 0.7e^{i2\pi(0.8)}\end{array}\right\}$
A_4	$\left\{\begin{array}{c}0.6e^{i2\pi(0.8)}, 0.1e^{i2\pi(0.5)},\\ 0.7e^{i2\pi(0.5)}\end{array}\right\}$	$\left\{\begin{array}{c}0.2e^{i2\pi(0.1)},\\ 0.5e^{i2\pi(0.7)},\end{array}\right\}$	$\left\{\begin{array}{c}0.1e^{i2\pi(0.3)}, 0.2e^{i2\pi(0.4)},\\ 0.6e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.3)}\end{array}\right\}$	$\left\{\begin{array}{c}0.3e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.4)},\\ 0.3e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.5e^{i2\pi(0.1)},\\ 0.8e^{i2\pi(0.6)}\end{array}\right\}$
A_5	$\left\{\begin{array}{c}0.9e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.4)},\\ 0.7e^{i2\pi(0.2)}\end{array}\right\}$	$\left\{\begin{array}{c}0.1e^{i2\pi(0.3)},\\ 0.2e^{i2\pi(0.4)}\end{array}\right\}$	$\left\{\begin{array}{c}0.5e^{i2\pi(0.4)}, 0.8e^{i2\pi(0.2)},\\ 0.7e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.3)}\end{array}\right\}$	$\left\{\begin{array}{c}0.7e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.1)},\\ 0.5e^{i2\pi(0.3)}\end{array}\right\}$	$\left\{\begin{array}{c}0.3e^{i2\pi(0.4)},\\ 0.2e^{i2\pi(0.7)}\end{array}\right\}$

health catastrophe because of the overcrowding of hospitals and healthcare systems. The hospital's ability to treat the disabled may be exceeded by the number of active cases. There have been a large number of people who have needed to be hospitalised for more than a week and up to two months [53]. According to the most recent WHO statistics as of March 20, 2021 [54], there have been 121,969,223 confirmed COVID-19 cases worldwide, with 2,694,094 fatalities documented, and the numbers are continually rising. We do not have a drug to treat this sickness because it is a unique and mutating condition. Different nations and institutes claimed that various drugs were effective in treating COVID-19, but this was later proven to be false. Individual vaccines are being produced in several nations to prevent individuals from being sick, but providing vaccines to each and every person will take a long time. Furthermore, these vaccinations may not function with the virus's new strands and only have an 80% to 90% effectiveness rate. As a result, it's critical to keep a close eye on the progress of the disease so that it doesn't spread to unaffected or unaffected places. For COVID-19 spread analysis, test effectiveness is ranked. Currently, a variety of COVID-19 tests are available, each of which plays an important role in evaluating the disease's progress and assisting decision-makers and governments in formulating policies. Let the four arttibutives is $\{T_1, T_2, T_3, T_4, T_5\}$ be the set of available tests for COVID-19, where

(i): T_1 rRT-PCR (Real-time Reverse Transcription Polymerase Chain Reaction) is a highly specific and sensitive COVID-19 diagnostic test. Swabs from the nose or mouth are used in the test.

(ii): T_2 CBNAAT (cartridge-based nucleic acid amplification test) is a diagnostic test that was formerly used to diagnose tuberculosis but is currently being used in certain countries to diagnose COVID-19. It is more rapid than RT-PCR.

(iii): T_3 To detect coronavirus as a foreign substance, antigen assays target the spike proteins of the virus. Nasal swabs are used to collect samples.

(iv): T_4 COVID-19 antibody tests, commonly known as serological testing, identify antibodies produced in the body against COVID-19. By identifying IgG and IgM antibodies, this test diagnoses previous and continuing illnesses. This test requires a blood sample.

(V): T_5 The reverse transcription loop-modified isothermal amplification test (RT-LAMP) is a type of reverse transcription loop-modified isothermal amplification. We must calculate the weight using an unknown weight information approach as described in model. And then let the show begin (w_i). And let the set $A_i\{i = 1, 2, 3, 4, 5\}$ be the alternatives. The evaluation values provided by the experts in the form of complex hesitant fuzzy elements are presented in Table 13.

Step 1. Since all criteria are of the benefit type, there is no need for normalization.

Case I: When the attribute weight is completely unknown:

Step 2. Since the attribute weight is completely unknown. Therefore, utilizing Equation 42, we get the following attribute weight $w_1 = 0.192$, $w_2 = 0.205$, $w_3 = 0.218$, $w_4 = 0.190$, $w_5 = 0.195$.

Step 3. Taking $CHF - PIS = \{1e^{i2\pi(1)}, \ldots\}$ and $CHF - NIS = \{0e^{i2\pi(0)}, \ldots\}$, the CHF-PIS and CHF-NIS respectively.

Step 4. Calculate the distance of each alternatives from $CHF - PIS$ and $CHF - NIS$ by utilizing the proposed distance measures, we get:

$$
\begin{aligned}
&d^+_{CHWNEHD1} = 0.0541,\ d^+_{CHWNEHD2} = 0.0572,\\
&d^+_{CHWNEHD3} = 0.0487,\ d^+_{CHWNEHD4} = 0.0461,\\
&d^+_{CHWNEHD5} = 0.0556,\\
&d^-_{CHWNEHD1} = 0.0452,\ d^-_{CHWNEHD2} = 0.0469,\\
&d^-_{CHWNEHD3} = 0.0566,\ d^-_{CHWNEHD4} = 0.0588,\\
&d^-_{CHWNEHD5} = 0.0388,
\end{aligned}
$$

Step 5. Calculate the closeness coefficient of each alternative from CHF-PIS and CHF-NIS by utilizing Equation 45 we get: $C_{A_1} = 0.4552$, $C_{A_2} = 0.4504$, $C_{A_3} = 0.5378$, $C_{A_4} = 0.4393$, $C_{A_5} = 0.4112$.

Step 6. According to the closeness coefficient, the ranking of alternatives is: $A_3 > A_2 > A_5 > A_4 > A_1$. Hence, the best alternative is A_3.

Case II: When the criteria weight is partially known:

Step 2'. Suppose that the information about the criteria weight is partially known i.e. $w_1 \in [0.15, 0.2]$, $w_2 \in [0.2, 0.3]$, $w_3 \in [0.24, 0.31]$, $w_4 \in [0.2, 0.32]$, $w_5 \in [0.2, 0.25]$.

$$
\begin{aligned}
max(D) = 1.0815w_1 + 1.1600w_2 + 1.2333w_3\\
+1.0733w_4 + 1.1058w_5.
\end{aligned}
$$

such that, $C_{A_1} = 0.4552$, $C_{A_2} = 0.4504$, $C_{A_3} = 0.5378$, $C_{A_4} = 0.4393$, $C_{A_5} = 0.4112$.

TABLE 14. Result obtained by Generalized complex hesitant weighted normalized Hausdorff distance.

	C_{A_1}	C_{A_2}	C_{A_3}	C_{A_4}	C_{A_5}	$Ranking$
$\lambda = 1$	0.4552	0.4504	0.5378	0.4393	0.4112	$C_{A_3} > C_{A_1} > C_{A_2} > C_{A_4} > C_{A_5}$
$\lambda = 2$	0.4614	0.4562	0.5296	0.4416	0.4289	$C_{A_3} > C_{A_1} > C_{A_2} > C_{A_4} > C_{A_5}$
$\lambda = 3$	0.4654	0.4615	0.5230	0.4440	0.4420	$C_{A_3} > C_{A_1} > C_{A_2} > C_{A_5} > C_{A_4}$

TABLE 15. Result obtained by Generalized complex hesitant weighted normalized Hybrid distance.

	C_{A_1}	C_{A_2}	C_{A_3}	C_{A_4}	C_{A_5}	$Ranking$
$\lambda = 1$	0.4183	0.4377	0.5352	0.4168	0.4085	$C_{A_3} > C_{A_2} > C_{A_1} > C_{A_4} > C_{A_5}$
$\lambda = 2$	0.4344	0.4496	0.5286	0.4288	0.4241	$C_{A_3} > C_{A_2} > C_{A_1} > C_{A_4} > C_{A_5}$
$\lambda = 3$	0.4452	0.4579	0.5224	0.4360	0.4368	$C_{A_3} > C_{A_2} > C_{A_1} > C_{A_5} > C_{A_4}$

TABLE 16. Comparison with other CHFs distances.

$Operators$	A_1	A_2	A_3	A_4	A_5	$RANKING$
$CHFWA$ [24]	0.4600	0.4916	0.5858	0.4461	0.4511	$A_3 > A_2 > A_1 > A_5 > A_4$
$CHFWG$ [24]	0.3310	0.3606	0.4781	0.3474	0.3534	$A_3 > A_2 > A_5 > A_4 > A_1$
$CHWNHD$ [41]	0.4001	0.4311	0.5338	0.4051	0.4072	$A_3 > A_2 > A_5 > A_4 > A_1$
$CHWNED$ [41]	0.4188	0.4456	0.5280	0.4207	0.4215	$A_3 > A_2 > A_5 > A_4 > A_1$
$GCHWND$ [41]	0.4322	0.4555	0.5220	0.4304	0.4336	$A_3 > A_2 > A_5 > A_1 > A_4$
$CHWNHHD$	0.4552	0.4504	0.5378	0.4393	0.4112	$A_3 > A_1 > A_2 > A_4 > A_5$
$CHWNEHD$	0.4614	0.4562	0.5296	0.4416	0.4289	$A_3 > A_1 > A_2 > A_4 > A_5$
$GCHWNHD$	0.4654	0.4615	0.5230	0.4440	0.4420	$A_3 > A_1 > A_2 > A_5 > A_4$
$CHWHNHD$	0.4183	0.4377	0.5352	0.4068	0.4175	$A_3 > A_2 > A_1 > A_5 > A_4$
$GHWHND$	0.4252	0.4579	0.5224	0.4360	0.4468	$A_3 > A_2 > A_5 > A_4 > A_1$

By solving this model we get $w_1 = 0.15$, $w_2 = 0.20$, $w_3 = 0.25$, $w_4 = 0.2$, $w_5 = 0.2$.

Step 3'. This step is similar as Step 3 above.

Step 4'. Calculate the distance of each alternative from $CHF-PIS$ and $CHF-NIS$ by utilizing the proposed distance measures, we get:

$$\begin{aligned}
&d^{+}_{CHWNEHD1} = 0.1205, d^{+}_{CHWNEHD2} = 0.1143,\\
&d^{+}_{CHWNEHD3} = 0.0925, d^{+}_{CHWNEHD4} = 0.0793,\\
&d^{+}_{CHWNEHD5} = 0.1190,\\
&d^{-}_{CHWNEHD1} = 0.0795, d^{-}_{CHWNEHD2} = 0.0856,\\
&d^{-}_{CHWNEHD3} = 0.1057, d^{-}_{CHWNEHD4} = 0.1207,\\
&d^{-}_{CHWNEHD5} = 0.0810,
\end{aligned}$$

Step 5. Calculate the closeness coefficient of each alternative from CHF-PIS and CHF-NIS by utilizing Equation 45 we get: $C_{A_1} = 0.3975$, $C_{A_2} = 0.4281$, $C_{A_3} = 0.5333$, $C_{A_4} = 0.3965$, $C_{A_5} = 0.4050$.

Step 6. According to the closeness coefficient, the ranking of alternatives is: $A_3 > A_2 > A_5 > A_4 > A_1$. Hence, the best alternative is A_3.

Hence, in both cases, the best alternative is A_3.

A. IMPACT OF PARAMETER AND RANKING OF ALTERNATIVES BY UTILIZING THE PROPOSED DISTANCES

As the above result is based on the parameter λ. Tables 14 and 15 shows a more detailed overview of the effect of the various parameters by adjusting the parameter λ. It is seen that corresponding to $\lambda = 1, 2, 3$, the order of the alternatives is $A_3 > A_2 > A_5 > A_4 > A_1$.

However, from Tables 14 and 15, we see that if we increase the value of λ, then the values of the closeness coefficient tend to increase. But the ranking of alternatives remains the same. Thus, for different values of parameters, the DMs can pick the nature of the problem.

VII. COMPARISON ANALYSIS WITH EXISTING METHODS

A. COMPARISON WITH COMPLEX HESITANT FUZZY SETS

In this section, the proposed method is compared with the existing methods in the CHFS environment [24]. Table 16 shows how the proposed complex hesitant fuzzy TOPSIS method and the methods from [24] and [41] were used to rank the options. From Table 16, we observe that the best alternative by using the proposed method and existing methods is the same. The ranking of alternatives by using the CHFWA operator [24] is $A_3 > A_2 > A_1 > A_5 > A_4$, while using the GCHWNHD we have $A_3 > A_1 > A_2 > A_5 > A_4$. The best alternative in both approaches is the same. However, based on the CHFWA operator, the ranking order of A_1 and A_2 is $A_2 > A_1$, while in the proposed approach it is $A_1 > A_2$. The main reason is that while using the existing CHFWA operator, we have a loss of information. In the same way, by utilizing the CHFWG operator [24], the ranking result is $A_3 > A_2 > A_5 > A_4 > A_1$. The best alternative is to achieve the same result by using GCHWNHD. However, the ranking order of other alternatives by using the CHFWG operator [24] is different as compared to the proposed approach, which is again due to the loss of information. This shows that the developed approach is more effective in dealing with the MCDM problem. Moreover, by utilizing CHWNHD [41], the ranking of alternative is $A_3 > A_2 > A_5 > A_4 > A_1$. The best alternative is the same as what we get by using GCHWNHD. However, the ranking order of other alternatives is $A_2 > A_5 > A_4 > A_1$ which is different from the

TABLE 17. Comparison with other HFs distances.

$distance$	A_1	A_2	A_3	A_4	A_5	$Ranking$
$HFWA$ [21]	0.45016	0.55265	0.59881	0.46926	0.50065	$A_3 > A_2 > A_5 > A_4 > A_1$
$HFWG$ [21]	0.31286	0.39540	0.50156	0.37301	0.36387	$A_3 > A_2 > A_4 > A_5 > A_1$
$HHWNHD$ [12]	0.3971	0.4810	0.5585	0.4374	0.4212	$A_3 > A_2 > A_5 > A_4 > A_1$
$HHWNED$ [12]	0.4187	0.4859	0.5491	0.4463	0.4443	$A_3 > A_2 > A_5 > A_4 > A_1$
NHD [13]	0.3903	0.4861	0.5433	0.4223	0.4451	$A_3 > A_5 > A_2 > A_4 > A_1$
NED [13]	0.4109	0.4895	0.5364	0.4345	0.4559	$A_3 > A_2 > A_5 > A_4 > A_1$
$MNHD$ [35]	0.6766	0.7178	0.7550	0.6951	0.7022	$A_3 > A_2 > A_5 > A_4 > A_1$
$MNED$ [35]	0.6521	0.6797	0.7229	0.6710	0.6726	$A_3 > A_2 > A_5 > A_4 > A_1$
NHD [17]	0.3851	0.4827	0.5344	0.40478	0.4591	$A_3 > A_5 > A_2 > A_4 > A_1$
NED [17]	0.4060	0.4868	0.5288	0.43042	0.4670	$A_3 > A_2 > A_5 > A_4 > A_1$
$GCHWNHD$	0.4654	0.4615	0.5230	0.4440	0.4420	$A_3 > A_1 > A_2 > A_5 > A_4$

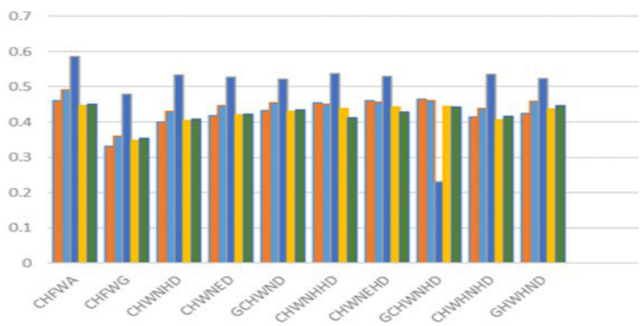

FIGURE 2. Ranking results obtained by the comparison with CHF operators.

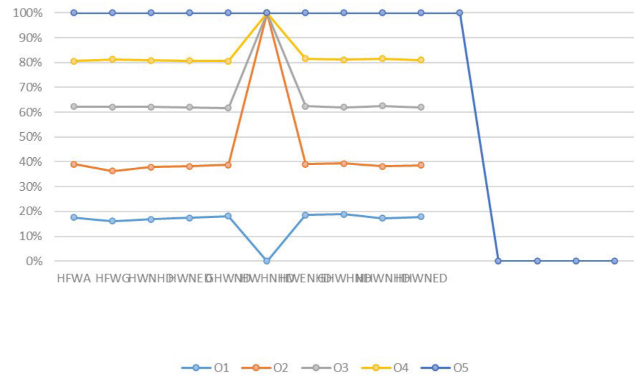

FIGURE 3. Ranking results obtained by the comparison with HF operators.

proposed method. The developed GCHWNHD is the generalization of the existing CHWNHD measure [41]. Similarly, the ranking of alternatives by utilizing CHWNED [41] is $A_3 > A_2 > A_5 > A_4 > A_1$. The best option is the same as using the proposed GCHWNHD, but the ranking order of the other options is different. A similar debate exists for GCHWND [41] and GCHWNHD. From the discussion, we conclude that the developed approach is more effective and applicable as compared with existing methods [24], [41]. In our approach, we use the maximize deviation method in order to find the criteria weight. Further, we utilize the GCHWNHD measure, which is the generalization of the existing distances developed in [41]. It should also be noted that the existing distances are the special cases of the proposed GCHWNHD.

B. COMPARISON WITH HESITANT FUZZY SETS

HFS can be considered as a special case of CHFS when DMs only consider one-dimensional information [7], [8]. For comparison, CHFEs can be converted to HFEs [21] by removing the phase term from the evaluation values. The ranking result of the proposed method and existing methods is presented in Table 17.

According to Table 17, the ranking of alternatives by using the HFWA operator and the HFWG operator [21] are $A_3 > A_2 > A_5 > A_4 > A_1$ and $A_3 > A_2 > A_4 > A_5 > A_1$ respectively. The best alternative is the same as that obtained by the proposed *GCHWNHD*. However, the proposed *GCHWNHD* deals with the CHF information while the methods used in [21] only deal with the HF information and not two-phase or periodic information. Therefore, the proposed *GCHWNHD* gives more accurate results as compared with the HFWA operator and HFWG operator [21]. Also, the ranking of alternatives by utilizing *HHWNHD* and *HHWNED* [12] are $A_3 > A_2 > A_5 > A_4 > A_1$ and $A_3 > A_2 > A_5 > A_4 > A_1$ respectively. The best alternative is found in both the *HHWNHD* and *HHWNED* [12] as our approach. However, *HHWNHD* and *HHWNED* [12] can be utilize only in HFS environment and cannot deal with two phase information. Moreover, the ranking of alternatives by utilizing the methods developed in [13], [17], and [35] is the same as that obtained by proposed method. The existing methods only deal with the HF information and not two-phase information. The main advantage of the proposed approach is that it not only generalizes the existing approaches but also deals with periodic or two-phase information. Consequently, we get more accurate results by utilizing the proposed method as compared with the existing approaches.

VIII. CONCLUDING REMARKS

In a range of scientific disciplines, including clustering analysis, pattern recognition, and decision-making, distance and similarity measurements are fundamentally significant. To deal with complex two phase information CHFS is one of the effective tool and is characterized by complex hesitant fuzzy membership degree, and complex hesitant fuzzy non-membership degree. In order to make the best decisions possible in real-world settings, there are multiple cases where we must quantify the uncertainty in the data. For handling the uncertain information that is present in our day-to-day problems, information measures are crucial tools. Since in literature no such distances measure were defined for CHFSs. Therefore, in this paper, we developed priority degree and distance measures in a CHF setting based on traditional distance measures such as Hamming distance, Euclidean distance, and Hausdorff distance. First, a priority degree is developed and elaborated in order to rank the CHF information. Then a variety of distances, namely CHNHHD, CHNEHD, GCHNHD, CHHNHD, CHHNED and GCHHND are proposed to be examined. Further, the developed distances are applied to a medical diagnosis problem. Moreover, the TOPSIS method was developed based on the proposed distance measures with unknown criteria weight information. Furthermore, a numerical example related to COVID-19 is presented for the application and effectiveness of the developed approach. We also compared the proposed method with the existing methods. In future, we will extend the proposed work to complex intuitionistic hesitant fuzzy sets, complex Pythagorean hesitant fuzzy sets, complex fermatean hesitant fuzzy sets.

DECLARATION OF INTERESTS

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article. The authors also declare that there is no conflict of interests regarding the publication of this article.

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