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RESEARCH ARTICLE

Synchronization of Stochastic Multilayer Networks by Finite-Time Controller

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ABSTRACT It is well known that complex networks are inevitably affected by random disturbances. In the study of complex network stability, finite-time control has always been a research hot spot. This paper discuss finite-time synchronization (FTSY) of stochastic multilayer networks (MLNs). First, novel finite-time stability (FTST) theorems of deterministic complex dynamical system are given, and the expression of finite stability time is also estimated. Second, sufficient criteria are obtained for FTSY of stochastic MLNs. Compared with some the existing research results of FTSY in stochastic MLNs, FTSY of stochastic MLNs is realized by using FTST of deterministic complex dynamical systems, and the relationship between control intensity and network layers is also discussed under the condition of minimum convergence time, which will be meaningful to help how to select appropriate in actual complex system control. Lastly, the availability of the method is checked by numerical simulation base on the theory proposed in this paper.

INDEX TERMS Multilayer networks, finite-time control, synchronization.

I. INTRODUCTION

In the actual system, the influence of random factors is inevitable, which makes the system show uncertainty. The existence of random factors makes the system more complex. Therefore, it is very difficult to make quantitative analysis of this kind of system. People often have paied attention to its qualitative characteristics and made qualitative research on it. In the analysis of stochastic systems, stability has been an important dynamic characteristic and one of the main objectives of engineering design. In order to ensure the stability of deterministic systems, pole placement constraints were widely used. This technology had been deeply studied [1], [2], [3], [4], [6]. However, in stochastic systems, the poles of the system can not be determined, and the method of pole assignment is impossible. Therefore, this method could not be used to study stochastic stability. The stability of stochastic systems has always been a difficult problem to be solved. In recent years, researchers have done a lot of work to study the stability of stochastic systems. Some new research methods have been introduced and many new achievements have been made. For example, in [7], Sheng studied output-feedback control of stochastic systems. In [8], Zhao discussed state estimation of networks under stochastic protocol. In [9], Yang discussed synchronization of stochastic networks using a novel hybrid controller. In [10], Wu synchronization of stochastic network by periodically intermittent discrete observation control. In [11], Liu discussed stochastic nonlinear systems by tracking feedback control. However, these research results were asymptotically stable systems in infinite time domain. In fact, in addition to being interested in the asymptotic stability of stochastic systems, people have been more care for stochastic systems meeting certain transient performance requirements. Moreover, due

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to the fractional power term in the finite-time controller, the finite-time closed-loop control system has better robustness and anti-interference performance compared with the non-finite time closed-loop system. The finite-time control method is applied in many fields, such as robot control, satellite attitude control, motor control, etc [2]. The concept of FTST for the sake of discussing the transient performance of the system was proposed earlier in [12], and the sufficient conditions for the existence of solutions of the problem were proposed by using Lyapunov function method. Subsequently, many new achievements have been made in the FTST of stochastic systems. For example, in [13], Zhang discussed FTSY of stochastic networks by adaptive control. In [14], Yuan discussed FTSY of stochastic networks via impulsive control. In [15], Wu discussed FTST of stochastic networks via feedback control. In [16], Ren discussed control of stochastic systems via finite-time sliding control strategy.

In addition, the structure and behavior of the single network have been deeply studied in the past decade, but in the context of complex networks, the composition of networks usually can not explain multiple interactions in different time and space. For example, transportation networks between cities (including railway network layer, highway network layer, aviation network layer, water transport network layer, etc.), commodity trade networks between countries and regions, metabolic networks between different cells, etc. A variety of multi-layer networks have been seen everywhere. With the development of network science, many expressions related to multi-layer networks have appeared in the existing research, such as interconnected systems, network of network [17], etc. The emerging research direction "multi-layer network" provides a natural framework for analyzing the complexity of MLNs, which makes more and more scholars take care the structure and properties of MLNs [18], especially synchronization of stochastic MLNs [19].

In the existing research, the FTST theorem of stochastic system is generally used to judge FTSY of stochastic network. For example, in [20], for $d\mu(t) = f_1(\mu)dt + f_2(\mu)d\omega(t)$, if the differential operator $\mathbf{L}\upsilon(\mu) \leq -a_1\upsilon^p(\mu)$, then the zero of stochastic ODE was FTST. In [21], if $\mathbf{L}\upsilon(\mu) \leq -a_1\upsilon^p(\mu) - a_2\upsilon^r(\mu)$, then the zero of stochastic ODE was FTST, where, $a_{1,2} > 0, 0 < p < 1, r > 1$. In [22], if $\mathbf{L}\upsilon(\mu) \leq -a_1\upsilon^p(\mu) - a_2\upsilon(\mu)$, then the zero of stochastic ODE was FTST, where $a_{1,2} > 0, p > 0$. In addition to the above methods, is there any other method to realize FTST of stochastic nonlinear systems?

Encouraged by the above discussion, the main contribution of this paper attempts to focus on the following two aspects:

(i) New FTST theorems for deterministic dynamical systems are proposed, and the FTSY criterion for stochastic MLNs is given by using new FTST theorems, and the expressions of finite settling time are estimated respectively.

(ii) Under the condition of minimum convergence time, the relationship between control intensity and network layers is gained.

The structural arrangement of this paper is as below: the second part describes the research model, some necessary assumptions, definitions and lemmas. In Section 3, some FTSY conditions for MLNs are given. An example of numerical simulation is given in Section 4. The conclusions are discussed in Section 5.

II. SOME PRELIMINARIES

The stochastic MLNs is considered

$$\dot{\mu}_i(t) = h(\mu_i(t)) + \sum\nolimits_{k=1}^{m} \sum\nolimits_{j=1}^{N} d_{ij}^{(k)} \Xi_k \mu_j(t) + \sigma_i(t, \mu_i(t))\, \dot{\omega}_i(t) \tag{1}$$

where $\mu_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{in})^T \in R^n$, $h(\mu_i(t)) = (h_1(\mu_{i1}(t)), \ldots, h_n(\mu_{in}(t)))^T . D^{(k)} = (d_{ij}^{(k)})_{N\times N}, (k = 1, 2, \ldots, m)$ is irreducible and satisfies $d_{ii}^{(k)} = -\sum_{j=1, j\neq i}^{N} d_{ij}^{(k)}$, $i = 1, 2 \ldots N$. Ξ_k means the kth layer's inner coupling matrix. $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T$ is the n-dimensional Brown moment, $\omega_i = (\omega_{i1}, \omega_{i2}, \ldots, \omega_{in})^T \epsilon R^n$, and $\sigma = (\sigma_{ij})_{n\times n} : R^+ \times R^n \times R^n \to R^{n\times n}$ is the noise intensity matrix.

The response stochastic MLNs is constructed from the drive systems (1) as follows:

$$\dot{\vartheta}_i(t) = h(\vartheta_i(t)) + \sum\nolimits_{k=1}^{m} \sum\nolimits_{j=1}^{N} d_{ij}^{(k)} \Xi_k \vartheta_j(t) + \sigma_i(t, \vartheta_i(t))\, \dot{\omega}_i(t) + \pi_i(t) \tag{2}$$

where $\vartheta_i = (\vartheta_{i1}, \vartheta_{i2}, \ldots, \vartheta_{in})^T \in R^n$, the controller is $\pi_i(t) = (\pi_{i1}(t), \pi_{i2}(t), \ldots, \pi_{in}(t))^T \in R^n$.

Let $\varpi_i = \vartheta_i(t) - \mu_i(t)$, $\varpi(t) = ((\varpi_1(t))^T, \ldots, \varpi_N(t))^T)^T \in R^{nN}$, $\mathrm{H}(\mu(t)) = (\mathrm{h}(\mu_1(t))^T, \ldots, \mathrm{h}(\mu_N(t))^T)^T$, $\pi(t) = (\pi_1(t), \ldots, \pi_N(t))^T$, the stochastic error system from MLNs (1)-(2) can be gained:

$$\mathrm{d}\bar{\omega}(t) = \Big[H(\vartheta(t)) - H(\mu(t)) + \sum\nolimits_{k=1}^{m} (D^{(k)} \otimes \Xi_k)\varpi(t) + \pi(t)\Big]dt + \varphi(t)\, d\varphi(t) \tag{3}$$

Definition 1 [23]: Considering the stochastic system as follows

$$\mathrm{d}\mu(t) = \mathrm{g}_1(\mu)\,\mathrm{dt} + \mathrm{g}_2(\mu)\,\mathrm{d}\omega(t) \tag{4}$$

where $\mu(t) \in R^n$, $g_1 : R^n \to R^n$ and $g_2 : R^n \to R^{n\times m}$, $\omega(\cdot)$ means the m-dimensional Brown moment. $g_1(0) = 0$, $g_2(0) = 0$, the system (4) has a unique global solution, expressed as $\mu(t, \mu_0)$, $0 \leq t < \infty$, where μ_0 is the initial state.

For each $\upsilon \in C^{2,1}(R^n \times R^+, R^+)$, the operator $\mathbf{L}\upsilon$ relative to Eq. (4) is

$$\mathbf{L}\upsilon = \frac{\partial \upsilon}{\partial \mu} \cdot g_1 + \frac{1}{2} trace\left(g_2^T \cdot \frac{\partial^2 u}{\partial \mu^2} \cdot g_2\right),$$

where $\frac{\partial \upsilon}{\partial \mu} = (\frac{\partial \upsilon}{\partial \mu_1}, \frac{\partial \upsilon}{\partial \mu_2}, \ldots, \frac{\partial \upsilon}{\partial \mu_n})$, $\frac{\partial^2 u}{\partial \mu^2} = (\frac{\partial^2 u}{\partial u_j u_k})_{n\times n} (j, k = 1, 2, \ldots, n)$.

Definition 2: If

$$\lim_{t\to T(\mu 0)} E|\mu(t)|^p \leq \ell, \forall \mu_0 \in R^n, \ell > 0, \mathrm{p} > 0,$$

then the equilibrium point of system (4) is quasi-stable with finite time pth moment, where $T(\mu_0)$ is interrelated to the starting value.

Remark 1: If $p = 2$, definition 2 means quasi-stable with finite time mean square moment. In definition 2, when the convergence time T is independent of the initial value, we call it fixed-time quasi-stable.

Hypothesis 1 (H1): Suppose $h(\cdot)$ satisfies

$$|h(\vartheta_i(t)) - h(\mu_i(t))| \le \gamma_1 \varpi_i(t), \gamma_1 > 0.$$

Hypothesis 2 (H2): If σ (t) satisfies

$$\text{trace}\left(\sigma^T(t,\varpi)\sigma(t,\varpi)\right) \le \gamma_2 \sum\nolimits_{i=1}^{N} |\varpi_i|^2, \gamma_2 > 0.$$

Lemma 1[24]: Let $\alpha \in R^n$, $\beta \in R^n$, then

$$\alpha^T\beta + \beta^T\alpha \le \gamma\alpha^T\alpha + \gamma^{-1}\beta^T\beta, \gamma > 0.$$

Lemma 2 [25]: For $\lambda_i \ge 0, i = 1, \ldots, n, \chi > 1$ then

$$\sum\nolimits_{i=1}^{n} \lambda_i^{\chi} \ge n^{1-\chi}(\sum\nolimits_{i=1}^{n} \lambda_i)^{\chi}$$

Lemma 3 [26]: If $\kappa(t)$ is a continuous function, and $\int_0^t \kappa(x)dx \le 0$, then $\kappa(t) \le 0$ for $t > 0$.

Proof: According to the [26], if there is a $t_0 > 0$, s.t. $\kappa(t_0) > 0$.

As $\kappa(t)$ is a continuous function, $\exists c > 0$, making $\kappa(t) > 0$ for $\forall t \in (t_0-c, t_0+c)$. From the mean value integral theorem,

$$\int_{t_0-c}^{t_0+c} \kappa(s)ds = 2c\kappa(h) > 0, h \in [t_0-c, t_0+c],$$

which is contradiction with $\int_0^t \kappa(x)dx \le 0$, So $\kappa(t) \le 0$ for $t > 0$.

Lemma 4 [27]: For

$$\dot{\varsigma} = G(\varsigma(t)), \varsigma(0) = \varsigma_0, \tag{5}$$

If $\Theta(\varsigma(t))$ is the continuous, positive-definite function, also

$$\dot{\Theta}(\varsigma) \le -\lambda_1 - \lambda_2\Theta(\varsigma) - \lambda_3\Theta^q(\varsigma), \tag{6}$$

where $q > 1, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$. Then, the zero of system (5) is fixed-time stable,

$$T \le \frac{1}{\lambda_2}\ln\frac{\lambda_1+\lambda_2}{\lambda_1} + \frac{1}{\lambda_3(q-1)}.$$

Lemma 5: For

$$\dot{\varsigma} = G(\varsigma(t)), \varsigma(0) = \varsigma_0 \tag{7}$$

If $\Theta(\varsigma(t))$ is the continuous, positive-definite function, also

$$\Theta(\varsigma(t)) \le \Theta(\varsigma(0)) - \lambda_1 t - \lambda_2\int_0^t \Theta(\varsigma(s))ds - \lambda_3\int_0^t \Theta^q(\varsigma(s))ds, \tag{8}$$

where $q > 1, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$. Then, the zero of system (7) is fixed-time stable,

$$T \le \frac{1}{\lambda_2}\ln\frac{\lambda_1+\lambda_2}{\lambda_1} + \frac{1}{\lambda_3(q-1)}.$$

Proof: By inequality (8), there are

$$\Theta(\varsigma(t)) - \Theta(\varsigma(0) \le \int_0^t \left(-\lambda_1 - \lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s))\right)ds,$$

thus

$$\int_0^t \dot{\Theta}(\varsigma(s))ds \le \int_0^t (-\lambda_1 - \lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s)))ds,$$

that is

$$\int_0^t \left(\dot{\Theta}(\varsigma(s)) - (-\lambda_1 - \lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s)))\right)ds \le 0.$$

From Lemma 3,

$$\dot{\Theta}(\varsigma(t)) \le -\lambda_1 - \lambda_2\Theta(\varsigma(t)) - \lambda_3\Theta^q(\varsigma(t)).$$

According to Lemma 4, the zero of system (7) is FTST, also

$$T \le \frac{1}{\lambda_2}\ln\frac{\lambda_1+\lambda_2}{\lambda_1} + \frac{1}{\lambda_3(q-1)}.$$

Lemma 6. If $\Theta(\varsigma(t))$ satisfies:

$$\Theta(\varsigma(t)) \le -\lambda_2\int_0^t \Theta(\varsigma(s))ds - \lambda_3\int_0^t \Theta^q(\varsigma(s))ds. \tag{9}$$

where $q > 1, \lambda_2 > 0, \lambda_3 > 0$. Then, the zero of system (7) is fixed-time stable, also

(i) $\Theta(\varsigma(0)) \ge \frac{\lambda_2}{e^{\lambda_2\left(1-\frac{1}{\lambda_3(q-1)}\right)}-1}$,

$$T \le \frac{1}{\lambda_2}\ln\frac{\Theta(\varsigma(0))+\lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)} \le 1.$$

(ii) $\Theta(\varsigma(0)) < -\lambda_2$,

$$1 \le T \le \frac{1}{\lambda_2}\ln\frac{\Theta(\varsigma(0))+\lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)}.$$

Proof: By inequality (9), there are

$$\Theta(\varsigma(t)) - \Theta(\varsigma(0)) \le -\Theta(\varsigma(0)) + \int_0^t \left(-\lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s))\right)ds.$$

(i) If $\Theta(\varsigma(0)) > 0, t \le 1$,

$$\Theta(\varsigma(t)) - \Theta(\varsigma(0)) \le -\Theta(\varsigma(0))t + \int_0^t \left(-\lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s))\right)ds.$$

thus

$$\int_0^t (\dot{\Theta}(\varsigma(s))ds \le \int_0^t \left(-\Theta(\varsigma(0)) - \lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s))\right)ds$$

that is

$$\int_0^t \left(\dot{\Theta}(\varsigma(s)) - (-\Theta(\varsigma(0)) - \lambda_2\Theta(\varsigma(s)) - \lambda_3\Theta^q(\varsigma(s)))\right) ds \le 0.$$

From Lemma 3,

$$\dot{\Theta}(\varsigma(t)) \le -\Theta(\varsigma(0)) - \lambda_2\Theta(\varsigma(t)) - \lambda_3\Theta^q(\varsigma(t)).$$

According to Lemma 4, the zero of system (7) is FTST, also

$$T \le \frac{1}{\lambda_2} ln \frac{\Theta(\varsigma(0)) + \lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)}, \tag{10}$$

Since inequality (10) holds under the condition of $\Theta(\varsigma(0)) > 0, t \le 1$, so

$$T \le \frac{1}{\lambda_2} ln \frac{\Theta(\varsigma(0)) + \lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)} \le 1,$$

that is, $\Theta(\varsigma(0)) \ge \frac{\lambda_2}{e^{\lambda_2\left(1-\frac{1}{\lambda_3(q-1)}\right)}-1}$, which meet $\Theta(\varsigma(0)) > 0$.
So,

$$\Theta(\varsigma(0)) \ge \frac{\lambda_2}{e^{\lambda_2\left(1-\frac{1}{\lambda_3(q-1)}\right)}-1}, T \le \frac{1}{\lambda_2} ln \frac{\Theta(\varsigma(0)) + \lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)} \le 1.$$

(ii) If $\Theta(\varsigma(0)) > 0, t \le 1$, similar to (i), the origin of system (7) is FTST, also

$$\Theta(\varsigma(0)) < -\lambda_2, 1 \le T \le \frac{1}{\lambda_2} \ln \frac{\Theta(\varsigma(0)) + \lambda_2}{\Theta(\varsigma(0))} + \frac{1}{\lambda_3(q-1)}.$$

III. MAIN RESULTS

In the section, the FTSY of stochastic MLNs is discussed by using FTST theorem of deterministic systems.

Theorem 1: Under H1-H2 and the controller (11), the stochastic error system (3) is quasi-stable under finite-time pth moment

$$\pi_i(t) = -k_1\varpi_i - k_2\varpi_i^q, \tag{11}$$

and

$$\lim_{t\to T} E\left[|\varpi_i(t)|^p\right] \le 2^{\frac{q+1}{2}} \frac{k_0}{k_4},$$
$$E[T] \le \frac{1}{k_3} \ln \frac{k_0 + k_3}{k_0} + \frac{1}{\left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_1\right)(q-1)},$$

where $k_3 = 2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi > 0$, $\upsilon(0) > \sqrt[\frac{q+1}{2}]{k_0/k_1}$, $\Phi = \left\{\lambda_{max}\left(D^{(k)} \otimes \Xi_k\right)\left(D^{(k)} \otimes \Xi_k\right)^T\right\}$, $p = q+1$, $q > 1$, $k_0 > 0, k_1 > 0, k_2 > 0, 0 < k_4 < 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}}$, $k_1 < 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}}$.

Proof: Choosing the following function

$$\upsilon(t) = \frac{1}{2}\sum_{i=1}^N \varpi_i^T \varpi_i.$$

Based on the Itô's formula,

$$d\upsilon(t) = \mathbf{L}\upsilon(t)\,dt + \varpi^T \sigma(t, \varpi)\,d\omega(t),$$

and the differential operator

$$\mathbf{L}\nu(t) = \sum_{i=1}^N \varpi_i \Big[h(\vartheta_i(t)) - h(\mu_i(t)) + \sum_{k=1}^m \sum_{j=1}^N d_{ij}^{(k)} \Xi_k \varpi_j(t) + \pi_i(t)\Big] + \frac{1}{2}\text{trace}\left((\varphi_i(t))^T \varphi_i(t)\right), \tag{12}$$

By using (H1) and (H2), yields

$$\begin{aligned}
\mathbf{L}\nu(t) \le{} & \sum_{i=1}^N \Big[\gamma_1 \varpi_i^T(t)\varpi_i(t) + \sum_{k=1}^m \sum_{j=1}^N d_{ij}^{(k)} \Xi_k \varpi_i^T(t)\varpi_j(t) - k_1\varpi_i^T(t)\varpi_i(t) - k_2\varpi_i^T(t)\varpi_i^q(t)\Big] + \frac{1}{2}\gamma_2 \sum_{i=1}^N \varpi_i^T(t)\varpi_i(t) \\
\le{} & \left(\gamma_1 + \frac{1}{2}\gamma_2\right)\varpi^T\varpi + \frac{1}{2}\varpi^T\varpi + \sum_{k=1}^m \left(\frac{1}{2}\varpi^T\left(D^{(k)} \otimes \Xi_k\right)\left(D^{(k)} \otimes \Xi_k\right)^T\varpi\right) - k_1\varpi^T\varpi - k_2\varpi^T\varpi^q \\
\le{} & \left(\gamma_1 + \frac{1}{2}\gamma_2 + \frac{1}{2}\right)\varpi^T\varpi + m\Phi\frac{1}{2}\varpi^T\varpi - k_1\varpi^T\varpi - k_2\varpi^T\varpi^q \\
={} & -(2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi)\,\upsilon - k_2\varpi^T\varpi^q.
\end{aligned} \tag{13}$$

According to Lemma 2, when $q > 1$, $\varpi^T(t)\varpi^q(t) \ge 2^{\frac{q+1}{2}}(N_n)^{\frac{1-q}{2}}(\upsilon(t))^{\frac{q+1}{2}}$, one has

$$\mathbf{L}\upsilon(t) \le -k_3\upsilon(t) - 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} \upsilon(t)^{\frac{q+1}{2}},$$

where $k_3 = 2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi$.

From the Itô's formula, we have

$$\mathrm{E}\upsilon(t) = \mathrm{E}\upsilon(0) + \mathrm{E}\int_0^t \mathbf{L}\upsilon(\theta)\,d\theta = \mathrm{E}\upsilon(0) + \int_0^t \mathrm{E}[\mathbf{L}\upsilon(\theta)]\,d\theta,$$

which

$$\begin{aligned}
E[\mathbf{L}\upsilon(t)] &\le -k_3E\upsilon(t) - 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} E\upsilon(t)^{\frac{q+1}{2}} \\
&= -k_3E\upsilon(t) - \left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_4\right)E\upsilon(t)^{\frac{q+1}{2}} - k_4E\upsilon(t)^{\frac{q+1}{2}} + k_0 - k_0.
\end{aligned}$$

If $E\upsilon(t)^{\frac{q+1}{2}} > \frac{k_0}{k_4}$, then

$$E[\mathbf{L}\upsilon(t)] \leq -k_3 E\upsilon(t) - \left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_4\right) E\upsilon(t)^{\frac{q+1}{2}} - k_0$$

When q > 1, $E\upsilon(t)^{\frac{q+1}{2}} \geq (E\upsilon(t))^{\frac{q+1}{2}}$, so

$$E[\mathbf{L}\upsilon(t)] \leq -k_3 E\upsilon(t) - \left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_4\right) E\upsilon(t)^{\frac{q+1}{2}} - k_0.$$

So,

$$E\upsilon(t) \leq E\upsilon(0) + \int_0^t \left[-k_3 E\upsilon(\theta) - \left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_4\right) E\upsilon(\theta)^{\frac{q+1}{2}} - k_0\right] d\theta,$$

thus,

$$E\upsilon(t) - E\upsilon(0) \leq -k_3 \int_0^t E[\upsilon(\theta)] d\theta - \left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_4\right) \int_0^t (E\upsilon(\theta))^{\frac{q+1}{2}} d\theta - k_0 t.$$

Let $\Theta(\varsigma(t)) = E\upsilon(t)$, according to Lemma 5, we can estimate the following setting time

$$E[T] \leq \frac{1}{k_3} \ln \frac{k_0 + k_3}{k_0} + \frac{1}{\left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_1\right)(q-1)}.$$

So, when $E\upsilon(t)^{\frac{q+1}{2}} > \frac{k_0}{k_4}$, the decrease of $\upsilon(\mu)$ in finite-time stochastic drives the trajectories of the closed-loop system into $E\upsilon(t)^{\frac{q+1}{2}} \leq \frac{k_0}{k_4}$, that is,

$$E \parallel \varpi(t) \parallel^p = E \parallel \varpi(t) \parallel^{q+1} = 2^{\frac{q+1}{2}} E\upsilon(t)^{\frac{q+1}{2}} \leq 2^{\frac{q+1}{2}} \frac{k_0}{k_4},$$

where p = q + 1.

Remark 2: Let the function $T(m) \leq \frac{1}{k_3} \ln \frac{k_0+k_3}{k_0} + \frac{1}{\left(2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} - k_1\right)(q-1)}$, $k_3 = 2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi$. If $\frac{dT(m)}{dm} = 0$, then the range of layer of network $\frac{2k_1 - 2\gamma_1 - \gamma_2 - 1 - (e-1)k_0}{\Phi} < m < \frac{2k_1 - 2\gamma_1 - \gamma_2 - 1}{\Phi}$ can be gained for getting the minimum value $T(m)$.

Remark 3: From $2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi > 0$ and $k_1 < 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}}$, we can get that the relationship among the control intensity k_1, k_2 and the number of network layers m is $\gamma_1 + (\gamma_2 + 1 + m\Phi) < k_1 < 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}}$ under the condition of minimum convergence time $T(m)$, which is different from the [14].

Remark 4: Although the stochastic disturbance of the system affects the stability of the system, the control system is still robust under the action of the controller.

Remark 5: Compared with some the existing research results of FTSY in stochastic network in [13], [15], [19], and [20], FTSY of stochastic network is realized by using FTST theorems of deterministic complex dynamical systems, which provides a new analysis method for the stability analysis of stochastic systems.

Therorem 2: Under H1-H2 and the controller (14), the stochastic error system (3) is stable under finite-time mean moment

$$\pi_i(t) = -k_1 \varpi_i - k_2 \varpi_i^q, \tag{14}$$

and

$$\upsilon(\varpi(0)) < -k_3,$$
$$1 \leq T \leq \frac{1}{k_3} \ln \frac{\upsilon(\varpi(0)) + k_3}{\upsilon(\varpi(0))} + \frac{1}{2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-\frac{q+1}{2}}{2}} \left(\frac{q+1}{2} - 1\right)}.$$

where $k_3 = 2k_1 - 2\gamma_1 - \gamma_2 - 1 - m\Phi > 0$, $\Phi = \max_{k=1,\ldots,N} \left\{\lambda_{max}\left(D^{(k)} \otimes \Theta_k\right)\left(D^{(k)} \otimes \Theta_k\right)^T\right\}$, $k_1 > 0$, $k_2 > 0$.

Proof: Selecting the following function

$$\upsilon(t) = \frac{1}{2} \sum_{i=1}^{N} \varpi_i^T \varpi_i.$$

Similar to that of Theorem 1, one have

$$E\upsilon(t, \varpi(t)) \leq E\upsilon(t, \varpi(0)) + \int_0^t \left[-k_3 E\upsilon(t) - 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} (E\upsilon(t))^{\frac{q+1}{2}}\right] d\theta$$

When $\upsilon(\varpi(0)) < -k_3$, we have

$$E\upsilon(t, \varpi(t)) \leq -k_3 \int_0^t E[\upsilon(t)] d\theta - 2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-q}{2}} \int_0^t (E\upsilon(t))^{\frac{q+1}{2}} d\theta.$$

Furthermore, according to Lemma 6, let $\Theta(\varsigma(t)) = E\upsilon(t)$, we have

$$1 \leq T \leq \frac{1}{k_3} \ln \frac{\upsilon(\varpi(0)) + k_3}{\upsilon(\varpi(0))} + \frac{1}{2^{\frac{q+1}{2}} k_2 (Nn)^{\frac{1-\frac{q+1}{2}}{2}} \left(\frac{q+1}{2} - 1\right)}.$$

Remark 6: Although controllers (11) and (14) are the same equation, different conclusions are obtained due to different conditions that the theorem satisfies. Compared with theorem 2, theorem 1 needs to satisfy weaker conditions to achieve finite time pth moment quasi-stability of the error system, while theorem 2 is finite time mean square stability of the error system.

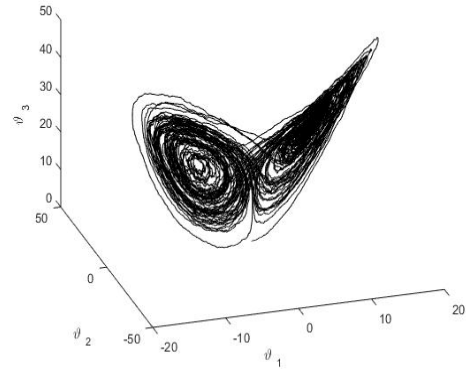

FIGURE 1. The stochastic Lorenz system.

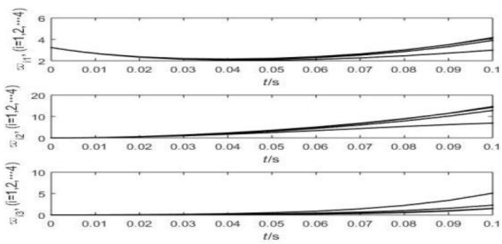

FIGURE 2. Synchronous evolution curve without the controller.

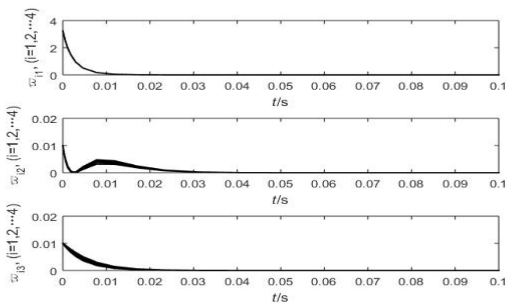

FIGURE 3. Synchronous evolution curve with q=1.5.

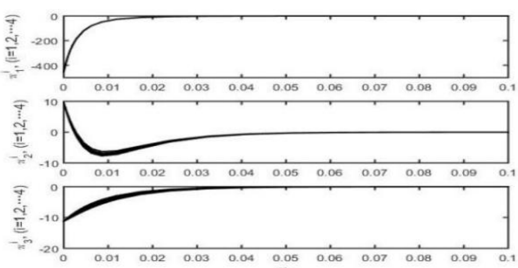

FIGURE 4. The controller evolution curve with q=1.5.

IV. ILLUSTRATIVE EXAMPLE

It is assumed that the node of the network is Lorenz system, i.e,

$$\begin{cases} \dot{s}_{i1} = 10\left(s_{i1} - s_{i2}\right) \\ \dot{s}_{i2} = 28 s_{i1} - s_{i2} - s_{i1} s_{i3} \\ \dot{s}_{i3} = -8 s_{i3}/3 + s_{i1} s_{i2}, \end{cases}$$

By simple calculation [28],

$$|\mathrm{h}(\vartheta_i) - \mathrm{h}(\mu_i)| \leq \gamma_1 |\varpi_i| \approx 100.571 |\varpi_i|$$

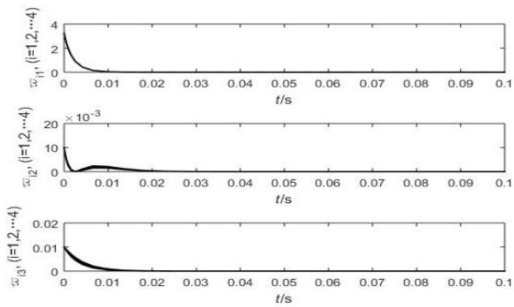

FIGURE 5. Synchronous evolution curve.

If $\sigma(t, \vartheta) = 0.1 diag\{\vartheta_1, \vartheta_2, \vartheta_3\}$, N = 4, n = 3, $\Theta_k = I$, m = 2, and

$$D^{(1)} = \begin{pmatrix} -1.5 & -0.5 & 1 & 1 \\ -0.5 & 1 & 0.5 & -1 \\ 1 & 0.5 & -0.5 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

$$D^{(2)} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 \\ 1 & 2 & -2 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

then $\Phi = 2.5206$. According to Theorem 1, by simple calculation, $k_1 = 104$, $k_2 = 82$, $k_0 = 1$, $k_4 = 100$, $k_3 = 0.7446$. If the initial values of the systems state are assumed to be rand [0, 5]. Figure 1 shows the stochastical Lorenz system. Figure 2 means the error systems evolution curve without the controller. Fig. 3 means the synchronization of the networks for q = 1.5. Figure 4 shows the controller evolution curve with q = 1.5. Fig. 5 means the synchronization of the networks for q=2.5 and $\sigma(t, \vartheta) = 0.01 diag\{\vartheta_1, \vartheta_2, \vartheta_3\}$. Fig. 5 shows that the smaller the random disturbance and the larger the q, the faster the error systems converges. The effectiveness of Theorem 1 is tested by numerical simulation.

V. CONCLUSION

The paper had discussed FTSY of MLNs. Novel FTST theorems have been gained for deterministic dynamical system, and novel sufficient criteria are also gained for FTSY of stochastic MLNs. In particularly, under the condition of minimum convergence time, the relationship between control intensity and network layers has been given. Lastly, an example is used to check the validity of the theoretical results. Since the fixed-time stability of the network is independent of the initial value of the system, it may be better than the FTST method of the network. Future work will further discuss the fixed-time stability of random MLNs based on the deterministic system stability theory.

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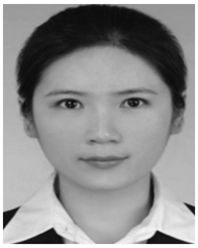

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