The Designated Convergence Rate Problems of Consensus or Flocking of Double-Integrator Agents With General Nonequal Velocity and Position Couplings: Further Results and Patterns of Convergence Rate Contours

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***Abstract*—This paper considers the designated convergence rate (DCR) (or the designated convergence margin) problems of consensus or flocking of coupled double-integrator agents. The DCR problems are more valuable for systems design than just convergence or stability conditions. The system setting in this paper is general, i.e., the velocity coupling and position coupling (VCPC) between agents, respectively, are set to be general and nonequal (up to rescaling), together with distinct damping and stiffness gains for the VCPC, respectively. This paper has two primary contributions on consensus: 1) further necessary and sufficient conditions are established to guarantee the DCR problems of the system, which have enriched the previous results and 2) the patterns of the convergence rate contours for the DCR are characterized, in terms of the damping and stiffness gains, which are closely related to the characteristics of the spectra of the two Laplacian matrices of the VCPC. Additionally, this paper has a contribution on matrix theory, i.e., the sufficient conditions for the simultaneous upper-triangularization of two independent Laplacian matrices, particularly from an easily verifiable topological perspective on the corresponding digraphs of these Laplacian matrices.**

***Index Terms*—Consensus margin, convergence margin, convergence rate contour, cooperative control, designated convergence margin, designated convergence rate (DCR), flocking, formation, rendezvous, stability margin.**

I. INTRODUCTION

COOPERATIVE control of multiple agents has attracted a great deal of attention recently, e.g., consensus [2], [6]–[9], [15], [19]–[22], [25], [30], [32], coverage [16], deployment [18], flocking [1], [3]–[5], [26], [27], rendezvous [12], [24], swarming [11], [13], [14], [18], formation in the Euclidean space [17], or on the spherical manifolds [18], etc. For a first-order (single-integrator) model, there is only position coupling between agents, for which many results have been obtained. Flocking generally refers to the scenario when the agents (typically the double-integrator agents) will move at a same velocity with an explicitly [1], [3] or implicitly [26] described formation; such a motion is called (second-order) consensus when the agents will converge to a zero formation [1], [28], [34].

Flocking of double-integrator agents provides a framework, within which one can introduce more complex behaviors of agents into the networked system. For example, one can introduce a leader or an external reference [23], coupling delays [19], [20], [31], input constraints [28], optimal control [29], into flocking [22].

For second-order (double-integrator) models of flocking or consensus, the state of a single agent includes both velocity and position. Thus, generally, there exist both velocity coupling and position coupling (VCPC) among agents, together with distinct damping and stiffness gains for the VCPC, respectively.

The consideration of nonequal VCPC (up to rescaling) has not only theoretical merits but also application implications. For example, physical agents can possibly carry range detection sensors or proximity sensors (e.g., laser range-finders or infrared proximity sensors), as well as velocity detection sensors (e.g., a Doppler radar is a specialized radar that uses the Doppler effect to produce velocity data about objects at a distance). Just due to the measurements (on relative velocity and position) using different physical sensors, the nonequal VCPC setting has its very physical meaning.

However, the VCPC setting in most reports is merely assumed to be identical; it is known that, the VCPC play distinct roles in systems dynamics, thus, should be considered *nonequal* in general [1], [3], [4], with the *equal* setting as its special case. There is also insufficient attention on a general setting of the damping/stiffness gains for the VCPC: usually in literature, the stiffness gain was set to be unity, with only the damping gain being a possible variable parameter, or the damping and the stiffness gains were set to be just identical (except for a few works, e.g., [1], [3], and [4], to the author's knowledge).

Manuscript received August 6, 2016; revised November 25, 2016 and February 4, 2017; accepted February 9, 2017. Date of publication March 14, 2017; date of current version April 13, 2017. This paper was recommended by Associate Editor Q. Liu.

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Digital Object Identifier 10.1109/TCYB.2017.2668458

Moreover, another major concern is what conditions can ensure the designated convergence rate (DCR) or the designated convergence margin of a system, which extends many categories of cooperative control problems (e.g., flocking, consensus, swarming, formation, etc., which generally concern merely about the convergence or stability) in the very large literature. Convergence or stability conditions alone are insufficient for understanding the problems for at least two reasons: 1) a system should be designed more robust than having merely the theoretical convergence or stability (in applications, noise and inaccurate measurements may make a theoretically convergent or stable system unstable) and 2) it is often important to design a system with a DCR (for consensus problems and the consensus margin) [1]. Deriving conditions for a system with a DCR has important implications, which unfortunately was rarely concerned in most literature.

Recently, Li and Chen [1] have investigated one of the DCR problems of consensus or flocking of double-integrator agents with nonequal VCPC (as well as the *distinct* damping and stiffness gains for the VCPC, respectively), and established the necessary and sufficient conditions to guarantee the DCR, which generalized the existing results in the field, and which are valuable for systems design than merely the convergence or stability conditions; refer to [1, Fig. 1] for the coverage of the convergence and the DCR problems. Even for a linear system, solving a DCR problem for *unknown* damping/stiffness gains is still difficult, particular with a *general nonequal setting* of the VCPC.

In this paper, we consider more forms of the DCR problems. For convenience, we list the rationales for the consideration of the different DCR forms in this paper in Section II-D, as compared with the DCR form in [1].

The main contributions in this paper are listed as follows.

1) We provide further necessary and sufficient conditions to guarantee the DCR problems, which have enriched the previous results in [1]. Refer to Sections V and VI.
2) Particularly, we characterize different patterns of the convergence rate contours for the DCR, which are the functions of the damping and stiffness gains, and which are closely related to the characteristics of the spectra of the two Laplacian matrices of the VCPC, refer to Section V-C. The convergence rate contours also show that, increasing the damping gain does not necessarily increase the convergence margin of consensus (or consensus margin), which may be unexpected to one's intuition.
3) Additionally, this paper has another contribution on matrix theory, i.e., the sufficient conditions for simultaneous upper-triangularization of two independent Laplacian matrices, particularly from an easily verifiable topological perspective on the corresponding digraphs of these two Laplacian matrices. Refer to Section IV.

The rest of this paper is arranged as follows. Section II is the problem. Section III is the preparations. Section IV is the sufficient conditions for simultaneous upper-triangularization of two independent Laplacian matrices. Sections V and VI are the main results on the DCR problems. Finally, Section VII is the conclusions.

II. Problem Description

A. Model

Consider n agents in the N-dimensional Euclidean space. Denote $x_i \in \mathbb{R}^N$ as the position of agent i, $i = 1, 2, \ldots, n$, the dynamics of the agents can be described by the double-integrators [1], [4], [25], [34]

$$\dot{x}_i = v_i$$
$$\dot{v}_i = u_i$$

where the control input [1], [4]

$$u_i = \sum_{j\in\mathcal{M}_i} -bw_{ij}\big(\dot{x}_i - \dot{x}_j\big) + \sum_{j\in\mathcal{N}_i} -kv_{ij}\big(x_i - x_j - a_{ij}\big) \tag{1}$$

where $b, k > 0$ are the damping and stiffness gains, respectively; $w_{ij} \geq 0$ ($v_{ij} \geq 0$) is the velocity (position) coupling weight on agent i from agent j; $\mathcal{M}_i$ and $\mathcal{N}_i$ are the neighbor sets of agent i; $w_{ij} > 0$ if $j \in \mathcal{M}_i$, and $v_{ij} > 0$ if $j \in \mathcal{N}_i$; vectors $a_{ij} \in \mathbb{R}^N$, $i, j = 1, 2, \ldots, n$, are compatible.

System (1) is convergent *iff* $\dot{x}_i \to \dot{x}_j$ and $x_i - x_j \to a_{ij}$ for all i, j. System (1) is called second-order consensus if the agents will achieve the zero formation (i.e., $a_{ij} = \mathbf{0}$ for all i, j).

The Laplacian matrices $L = [L_{ij}] \in \mathbb{R}^{n\times n}$ and $H = [H_{ij}] \in \mathbb{R}^{n\times n}$, which model the VCPC structures, respectively, are defined as

$$L_{ij} = \begin{cases} -w_{ij} & i \neq j \\ \sum_{j=1, j\neq i}^{n} w_{ij} & i = j \end{cases}, \quad H_{ij} = \begin{cases} -v_{ij} & i \neq j \\ \sum_{j=1, j\neq i}^{n} v_{ij} & i = j. \end{cases}$$

Define the weighted digraph of L as $\mathcal{G}_L = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$; $e_{ij} = (i, j) \in \mathcal{E}$ *iff* there is a directed link from i to j, with w_{ji} being the weight of e_{ij}; $\mathcal{A} = [\mathcal{A}_{ij}] = [w_{ji}] \in \mathbb{R}^{n\times n}$ is the weighted adjacency matrix (note the sequence of i, j). Define the weighted digraph $\mathcal{G}_H$ of H similarly.

Denote I as the identity matrix, $\mathbf{1} = [1, 1, \ldots, 1]^T$, $\mathbf{0} = [0, 0, \ldots, 0]^T$, with the dimension determined in the subscript or the context. $\lambda(\cdot)$ denotes the spectrum of a matrix. Define

$$A := \begin{pmatrix} -bL & -kH \\ I_n & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

which has at least two zero eigenvalues [1, Proposition 1]; thus, without loss of generality, denote

$$\lambda(A) = \{\Lambda_j \in \mathbb{C},\ j = 1, 2, \ldots, 2(n-1),\ \Lambda_{2n-1} = \Lambda_{2n} = 0\}.$$

Define the maximum of the real parts of the $2(n-1)$ eigenvalues as

$$r_{\max} := \max_{j=1,\ldots,2(n-1)} \big\{\mathrm{Re}\big(\Lambda_j\big)\big\}$$

where $r_{\max}$ is a function of L, H, b, and k.

B. DCR

If L, H, b, and k are all known, then calculation of $r_{\max}$ is trivial.

Here, we consider an *inverse problem*—the DCR problem: the gains b and k are two *unknown* variables to be designed, then what values of b and k can ensure a designated condition of $r_{\max}$?

Although system (1) is linear, the DCR problem for *unknown* gains is *nonlinear*, particular with respect to general *nonequal* VCPC [1].

C. Main Concerns in This Paper

Denote constant $r_0 > 0$. A DCR problem considered in [1] is for

$$r_{\max} < -r_0. \tag{2}$$

If $r_0 = 0$, conditions for (2) to hold reduce to the consensus condition.

Different from [1], this paper considers the DCR problems for

$$r_{\max} = -r_0 \tag{3}$$
$$r_{\max} \leq -r_0. \tag{4}$$

D. Rationale for Different Concerns of the DCR Problems (2)–(4)

One may wonder, the DCR problems (2)–(4) can be merged as one DCR problem, e.g., (4). Here, they are separated for many reasons.

Reasons for Separating the DCR Problem (2):

a) For some cases, there is a solution for (2), but no solution for (3) [i.e., (4) with the equal sign], e.g., as shown in Proposition 3 (the first two items).

b) The reduction of (2) with $r_0 = 0$ is just the pure consensus problem; thus, separating (2) is convenient to compare the generalization/reduction.

Reasons for Separating the DCR Problem (3): The convergence rate contours of b and k for (3) have important properties, which are closely related to different types of the spectra of the two Laplacian matrices of the VCPC; refer to, e.g., Fig. 1, Proposition 2, Corollary 2, and Proposition 3 (the last two items).

The DCR problem (3) is the main focus in this paper.

III. Preparations

A. Lemmas

Lemma 1: For a Laplacian matrix, there is at least one zero eigenvalue, and all *nonzero eigenvalues* have positive real parts (which can be derived from Geršgorin Disks Theorem [33]).

Without loss of generality, denote the n eigenvalues of the Laplacian matrix L as $\lambda_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$, and denote the n eigenvalues of the Laplacian matrix H as $\mu_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$

$$\mathrm{Re}(\lambda_j) \geq 0,\ j = 1, 2, \ldots, n-1;\ \lambda_n = 0$$
$$\mathrm{Re}(\mu_j) \geq 0,\ j = 1, 2, \ldots, n-1;\ \mu_n = 0.$$

Further, $\mathrm{Re}(\lambda_j) > 0$ for all $j = 1, 2, \ldots, n-1$, *iff* $\mathcal{G}_L$ has a directed spanning tree [34]. $\lambda(H)$ has the same properties as $\lambda(L)$.

Furthermore, we write λ_j and μ_j, respectively, as

$$\lambda_j = u_j + v_j \mathrm{i}$$
$$\mu_j = \tilde{u}_j + \tilde{v}_j \mathrm{i} \tag{5}$$

where $u_j, v_j, \tilde{u}_j, \tilde{v}_j \in \mathbb{R}$, $u_j, \tilde{u}_j \geq 0$, $\mathrm{i} = \sqrt{-1}$ (the imaginary unit).

Lemma 2 (Schur Decomposition): For matrix L, there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ (i.e., $U^* = U^{-1}$, where U^* is the complex conjugate transpose of U) such that $U^*LU = V_1$ as

$$U^*LU = U^{-1}LU = V_1 := \begin{bmatrix} \lambda_1 & * & \ldots & * & * \\ 0 & \lambda_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_{n-1} & * \\ 0 & 0 & \ldots & 0 & \lambda_n \end{bmatrix}.$$

If $\lambda(L) \in \mathbb{R}$, then U reduces to be an orthogonal matrix $U \in \mathbb{R}^{n\times n}$.

B. $\mathcal{L}$-Assumption for Two Independent Laplacian Matrices

In this paper, assume that the Laplacian matrix H can be also transformed upper triangular by the same matrix U in Lemma 2, i.e.,

$$U^*HU = U^{-1}HU = V_2 := \begin{bmatrix} \mu_1 & * & \ldots & * & * \\ 0 & \mu_2 & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \mu_{n-1} & * \\ 0 & 0 & \ldots & 0 & \mu_n \end{bmatrix}.$$

Definition 1: The $\mathcal{L}$-assumption for two Laplacian matrices is defined as follows: the two Laplacian matrices L and H can be made upper triangular simultaneously by a same unitary matrix.

Remark 1: Section IV provides one solution of the $\mathcal{L}$-assumption from an easily verifiable topological perspective. Note that if the $\mathcal{L}$-assumption holds, then the two zero eigenvalues λ_n and μ_n of L and H can be made simultaneously appeared as the last diagonal entries in V_1 and V_2, respectively, (for the reason, refer Remark 12 in the Appendix).

C. On Relaxation Concern of the $\mathcal{L}$-Assumption

The $\mathcal{L}$-assumption is certainly not a necessary condition for consensus or the DCR problems. But it is still unknown that, to what extent, the $\mathcal{L}$-assumption could be relaxed for solving the DCR problems (for the pure consensus problem with nonequal VCPC, refer to [1], [3], and [4]).

An interesting yet unexpected simple example would provide some challenges on this relaxation question, please refer to [1, Example 1], in which the VCPC of the agents are modeled by two independent Laplacian matrices, and each of the corresponding digraphs $\mathcal{G}_H$ and $\mathcal{G}_L$ is *just a directed spanning tree* with the unit weight on each edge, but the directions of the edges in $\mathcal{G}_H$ and $\mathcal{G}_L$ are opposite, refer to the topologies illustrated in [1, Fig. 2] (here the two Laplacian matrices of the topologies do not satisfy the $\mathcal{L}$-assumption).

This example is very simple and satisfies the *cliché* of the connectivity condition (that the digraphs have spanning directed trees). However, the system with the very simple topologies (the two different Laplacian matrices) will never be stable for any positive gains b, k, not to mention consensus.

IV. Solving the $\mathcal{L}$-Assumption From Easily Verifiable Topological Perspective

Generally, it is very difficult to verify the $\mathcal{L}$-assumption for two *arbitrary* Laplacian matrices L and H, particularly if some entries of L or H are either variables or unknown constants.

Here, we provide some sufficient conditions to ensure the $\mathcal{L}$-assumption, particularly from an easily verifiable topological perspective, even for the Laplacian matrices L, H in which some entries can be either variables or unknown constants.

Remark 2: This is another contribution in this paper on matrix theory (i.e., the conditions for simultaneous upper-triangularization of two independent Laplacian matrices).

A. Definition

Definition 2: The topology of a Laplacian matrix $C \in \mathbb{R}^{n\times n}$ belongs to the concatenated-directed-star (CDS) topology, if C has the structure

$$C = \varepsilon \mathbf{1} I_n - \mathbf{1}\varepsilon \in \mathbb{R}^{n\times n} \tag{6}$$

where

$$\varepsilon := (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^{1\times n}, \quad \varepsilon \geq 0$$

is the *arbitrary non-negative row vector*, $\varepsilon \geq 0$ means that all $\varepsilon_i \geq 0$ with at least one $\varepsilon_i > 0$. Note that $\varepsilon_i > 0$ can be any positive value, not necessarily the unity.

From Definition 2, an all-to-all topology is a special type of the CDS topology with $\varepsilon > 0$; a directed-star topology is a special CDS topology with only one $\varepsilon_i > 0$ in ε, which has only $n-1$ directed edges from one agent i to agents $\{1, 2, \ldots, n\}\backslash i$.

The *geometric interpretation* of the general topology is that: for each index i, which index satisfying $\varepsilon_i > 0$, then there are $(n-1)$ directed edges from agent i to all other agents $\{1, 2, \ldots, n\} \setminus i$, with the same edge weight ε_i. Each $\varepsilon_i > 0$ just corresponds to a directed-star topology, this is why the word "concatenation" is used.

B. Property of the CDS Topology

Denote $\gamma := (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbb{R}^{1\times n}$ as an *arbitrary* vector with constraint $\gamma \mathbf{1} = 1$, denote $\gamma_{(1)} := [\gamma_2, \ldots, \gamma_n]$, and define matrix $T_1 \in \mathbb{R}^{n\times n}$ as [3], [4] (T_1 in [3] is denoted as T)

$$T_1 := \begin{bmatrix} \gamma_1 & \gamma_{(1)} \\ -\mathbf{1}_{n-1} & I_{n-1} \end{bmatrix}.$$

Consider the similarity transformation on an *arbitrary* Laplacian matrix L, then

$$T_1 L T_1^{-1} = \begin{bmatrix} 0 & * \\ \mathbf{0} & L_1 \end{bmatrix}$$

where L_1 is independent of γ [4] (one may use any other $T_i, i = 1, 2, \ldots, n$, in [4] to perform the similarity transformation).

Lemma 3: Consider the Laplacian matrix with the CDS topology

$$L = \varepsilon \mathbf{1} I_n - \mathbf{1}\varepsilon \in \mathbb{R}^{n\times n}$$

then

$$L_1 = \varepsilon \mathbf{1} I_{n-1}.$$

Proof: Refer to the unique and elegant structure of L_1 in [4]. ■

C. Results

Theorem 1: Consider two digraphs $\mathcal{G}_L$ and $\mathcal{G}_H$. If any one ($\mathcal{G}_L$ or $\mathcal{G}_H$) belongs to the CDS topology, while another topology ($\mathcal{G}_H$ or $\mathcal{G}_L$) can be *arbitrary*, with *arbitrary* edge weights, then the corresponding Laplacian matrices L, H of the digraphs satisfy the $\mathcal{L}$-assumption.

Proof: Refer to Proof of Theorem 1 in Appendix C. ■

Corollary 1: To ensure the $\mathcal{L}$-assumption, the digraphs $\mathcal{G}_L$ and $\mathcal{G}_H$ are *not* required to contain a same directed spanning tree or to have a same numbered root-agent in their respective directed spanning trees.

Remark 3: For the DCR of the system, each of $\mathcal{G}_H$ and $\mathcal{G}_L$ should have a directed spanning tree as a prerequisite [1, Proposition 2].

V. Results for the DCR Problem (3)

This section provides the solutions of the DCR problem (3) and the algorithm in a general case, and then provides the analytical results for different types of nonequal VCPC, finally, the interesting patterns of the convergence rate contours of b and k for the DCR problem (3).

For system (1) with nonequal VCPC, always assume that: the $\mathcal{L}$-assumption holds, and each of $\mathcal{G}_L$ and $\mathcal{G}_H$ has a directed spanning tree.

A. General Case

For $j = 1, 2, \ldots, n-1$, define

$$\begin{aligned}\Gamma_j(b,k) &:= -\tilde{v}_j^2 k^2 + f_j(b)k + g_j(b)\\ C_j(b,k) &:= 4\tilde{u}_j k + \left(u_j^2 + v_j^2\right)b^2 - 8u_j r_0 b + 8r_0^2\end{aligned}$$

where

$$\begin{aligned}f_j(b) &:= \tilde{u}_j(u_j b - 2r_0)^2 + u_j v_j \tilde{v}_j b^2\\ g_j(b) &:= -r_0(u_j b - r_0)\left(\left(u_j b - 2r_0\right)^2 + v_j^2 b^2\right)\end{aligned}$$

and

$$\begin{aligned}u_{\min} &:= \min_{i=1,2,\ldots,n-1} u_i\\ b_0 &:= \max_{j=1,2,\ldots,n-1}\left\{\frac{2r_0}{u_j}\right\} = \frac{2r_0}{u_{\min}}.\end{aligned} \tag{7}$$

Proposition 1: For the DCR (3) of system (1), a necessary and sufficient condition on the gains is that, there exists an index set $\mathcal{I}(b,k) \subset \{1, 2, \ldots, n-1\}$ (as a function of b and k) such that

$$\begin{cases} \Gamma_j(b,k) = 0, & j \in \mathcal{I}(b,k)\\ C_j(b,k) \geq 0, & j \in \mathcal{I}(b,k)\\ \Gamma_j(b,k) > 0, & j = 1, 2, \ldots, n-1, \quad j \notin \mathcal{I}(b,k)\\ C_j(b,k) > 0, & j = 1, 2, \ldots, n-1, \quad j \notin \mathcal{I}(b,k)\\ b \geq b_0\\ k > 0. \end{cases} \tag{8}$$

Algorithm 1 Numerical Solutions of b, k for Inequalities (8)

given L, H, r_0, and thus b_0 [refer to (7) for its calculation]
given an upper limit $b_{\max}$ of b, i.e., $b \in (b_0, b_{\max})$
$\triangleright$ *i.e., calculate a valid solution for the designated interval*
for $b = b_0$ **do**
  calculate a solution of variable k for (8) with current value b:
  - κ_0 is a possible solution of (8) for all j with $\tilde{v}_j = 0$
  - κ_1, κ_2 are possible solutions of (8) for all j with $\tilde{v}_j \neq 0$

  denote $\kappa_0 = \kappa_1 = 0$, $\kappa_2 = \kappa$ (κ is a large enough constant)
  for $j = 1$ **do**
    if $\tilde{v}_j = 0$ **then**
      calculate k_0 from $\Gamma_j(b, k_0) = 0$, and k_c from $C_j(b, k_c) = 0$
      if $k_0 \geq k_c$ and $k_0 > \kappa_0$ **then**
        $\kappa_0 \leftarrow k_0$
      else
        return, no valid solution
      end if
    else if $\tilde{v}_j \neq 0$ **then**
      calculate solutions k_1, k_2 for k from $\Gamma_j(b, k) = 0$
      if $k_1, k_2 > 0$ **then** $\triangleright$ assume $k_1 \leq k_2$
        if $k_1 \in [\kappa_1, \kappa_2]$ or $k_2 \in [\kappa_1, \kappa_2]$ **then**
          $\kappa_1 \leftarrow \max\{\kappa_1, k_1\}$, $\kappa_2 \leftarrow \min\{\kappa_2, k_2\}$
        else
          return, no valid solution
        end if
      else
        return, no valid solution
      end if
    end if
    $j \leftarrow j + 1$
  end for
  reassign $k_c = \max\{k_j\}$, where $C_j(b, k_j) = 0$ for all j with $\tilde{v}_j \neq 0$. Only when $\kappa_2 \geq \max\{\kappa_0, k_c\}$, inequalities (8) have solutions:
  - κ_2 is one solution;
  - κ_1 is a solution if $\kappa_1 \geq \max\{\kappa_0, k_c\}$;
  - κ_0 is a solution if $\kappa_0 \geq \max\{\kappa_1, k_c\}$.

  $b \leftarrow b + \epsilon$, until $b > b_{\max}$ $\triangleright$ $\epsilon > 0$ *is an incremental constant*
end for

Proof: Refer to Proof of Proposition 1 in Appendix C. ■

Remark 4: One sufficient condition is that: $\mathcal{I}(b, k)$ is a constant set that is independent of b and k (thus, denoted as $\mathcal{I}$).

Remark 5: Note that b_0 may not be the infimum or the "greatest lower bound" of b in (8), since $\Gamma_j(b, k) = 0$ may have no solution for $b = b_0$ or $b \to b_0$ [which requires at least $f_j^2(b) + 4\tilde{v}_j^2 g_j(b) \geq 0$].

For given L, H and r_0, Algorithm 1 provides numerical solutions of k (as a function of b) for inequalities (8) for a designated interval $b \in (b_0, b_{\max})$, where b_0 is defined in (7) and $b_{\max}$ is the user-defined value. For $b = b_0$, one needs to consider if $f_j(b) = 0$ when $\tilde{v}_j = 0$.

B. Analytical Results for Different Types of Nonequal VCPC

Proposition 2: If the eigenvalues of H are all real. Then, system (1) has the DCR (3), if $b > b_0$ and k is a function of b

$$k = \max_{j=1,2,\ldots,n-1} \left\{ \frac{r_0}{\tilde{u}_j}(u_j b - r_0)\left(1 + \frac{v_j^2 b^2}{(u_j b - 2r_0)^2}\right) \right\}. \tag{9}$$

Proof: Refer to Proof of Proposition 2 in Appendix C. ■

Definition 3: Denote ℓ as the index set such that

$$u_j = u_{\min}, \ \forall j \in \ell.$$

Denote $\ell = \{1, 2, \ldots, n-1\}$ when $u_1 = u_2 = \cdots = u_{n-1}$.

Corollary 2: Trajectory (9) as the function of b and k has the following properties:

1) For $b \to b_0$, by definition in (7), then:
   a) if there is one $v_j \neq 0$ with $j \in \ell$, then $k \to \infty$;
   b) if $v_j = 0$ for all $j \in \ell$, then

$$k = \max\left\{\hat{k}_0(b), \tilde{k}_0(b)\right\} < \infty \tag{10}$$

where

$$\hat{k}_0(b) := \max_{j \notin \ell} \left\{ \frac{r_0}{\tilde{u}_j}(u_j b - r_0)\left(1 + \frac{v_j^2 b^2}{(u_j b - 2r_0)^2}\right) \right\}$$

$$\tilde{k}_0(b) := \max_{j \in \ell} \left\{ \frac{r_0}{\tilde{u}_j}(u_j b - r_0) \right\} = \max_{j \in \ell} \left\{ \frac{r_0^2}{\tilde{u}_j} \right\}.$$

2) If b is sufficiently large, then

$$k \approx \max_{j=1,2,\ldots,n-1} \left\{ \frac{r_0}{\tilde{u}_j}(u_j b - r_0)\left(1 + \frac{v_j^2}{u_j^2}\right) \right\}$$

which is approximately a linear function of b.

Proposition 3 in the following is the result on whether $b = b_0$ is the infimum to ensure (3).

Proposition 3: Assume that the eigenvalues of H are all real. Then, for $b = b_0$:

1) if the eigenvalues of L are all complex (except $\lambda_n = 0$), then the system will never have the DCR (3) for any $k > 0$;
2) if $v_j \neq 0$ for one $j \in \ell$, then the system will never have the DCR (3) for any $k > 0$;
3) for $\ell \neq \{1, 2, \ldots, n-1\}$, a necessary and sufficient condition for the DCR (3) is that

$$\begin{cases} v_j = 0, \ j \in \ell \\ k \geq \max_{j \notin \ell} \left\{ \frac{r_0}{\tilde{u}_j}(u_j b_0 - r_0)\left(1 + \frac{v_j^2 b_0^2}{(u_j b_0 - 2r_0)^2}\right) \right\} \\ k \geq \max_{j \in \ell} \left\{ \frac{r_0^2}{\tilde{u}_j} \right\} \end{cases} \tag{11}$$

4) for $\ell = \{1, 2, \ldots, n-1\}$, a necessary and sufficient condition for the DCR (3) is that

$$\begin{cases} v_j = 0, \ j = 1, 2, \ldots, n-1 \\ k \geq \max_{j=1,2,\ldots,n-1} \left\{ \frac{r_0^2}{\tilde{u}_j} \right\}. \end{cases} \tag{12}$$

Proof: Refer to Proof of Proposition 3 in Appendix C. ■

Remark 6: In Propositions 2 and 3, the values of b and k for the DCR (3) constitute a piece-wise *continuous* trajectory. For (9), we have:

1) if one $v_j \neq 0$ with $j \in \ell$, then $k \to \infty$ (Corollary 2) as $b \to b_0$, which is consistent with the nonexistence of k in items 1) and 2) in Proposition 3;
2) if $v_j = 0$ for all $j \in \ell$, then (9) [refer to (10)], with $b \to b_0$, is the lower bound of k in items 3) and 4) in Proposition 3.

Remark 7: Note that $u_j b_0 > 2r_0$ for $j \notin \ell$.

Corollary 3: If the eigenvalues of L and H are all real. Then, to ensure the DCR (3), a necessary and sufficient condition on the gains is that

$$\begin{aligned} &b > b_0,\ k = \max_{j=1,2,\ldots,n-1}\left\{\frac{r_0}{\tilde{u}_j}\left(u_j b - r_0\right)\right\} \\ &b = b_0,\ k \geq \max_{j=1,2,\ldots,n-1}\left\{\frac{r_0}{\tilde{u}_j}\left(u_j b_0 - r_0\right)\right\}. \end{aligned} \tag{13}$$

Proof: For $b = b_0$, inequalities (11) reduce to be

$$\begin{cases} k \geq \max\limits_{j \notin \ell}\left\{\frac{r_0}{\tilde{u}_j}(u_j b_0 - r_0)\right\} \\ k \geq \max\limits_{j \in \ell}\left\{\frac{r_0^2}{\tilde{u}_j}\right\} \end{cases}$$

which are equivalent to (13), since

$$\frac{r_0}{\tilde{u}_j}(u_j b_0 - r_0) = \frac{r_0^2}{\tilde{u}_j},\ \text{ for } j \in \ell.$$

Condition $b > b_0$ is derived from Proposition 2. The results hold. ■

Remark 8: Trajectory of b, k for (3) is piece-wise continuous. If $\ell = \{1, 2, \ldots, n-1\}$, then (13) becomes

$$k \geq \max_{j \in \ell} \frac{r_0^2}{\tilde{u}_j}$$

which is consistent with (12).

C. Patterns of the Convergence Rate Contours for the DCR (3)

There are four types of the patterns of the convergence rate contours of b and k for the DCR (3), which are closely related with different types of the spectra of the two Laplacian matrices L and H.

1) *Condition:* If H has complex eigenvalues.
   Pattern: This type of the pattern of b and k is illustrated in Fig. 1(a), and note that b_0 may not be the infimum of b (Remark 5).
   Example: Refer to $\Gamma_1(b, k) = 0$ in Example 1.
2) *Condition:* If the eigenvalues of H are all real; while L has complex eigenvalues, and there is one $v_j \neq 0$ with $j \in \ell$.
   Pattern: This type of the pattern of b and k is described by (9), with $k \to \infty$ as $b \to b_0$, as illustrated in Fig. 1(b).
   Example: Refer to the first part of Item 1 in Corollary 2.
3) *Condition:* If the eigenvalues of H are all real; while L has complex eigenvalues, and $v_j = 0$ for all $j \in \ell$.
   Pattern: This type of the pattern of b and k has two parts, as illustrated in Fig. 1(c), which are described by (11) [corresponding to the dotted line in Fig. 1(c)] and (9) [corresponding to the solid line in Fig. 1(c)]; the two lines are continuous, and b_0 is the infimum of b.
   Example: Refer to the second part of item 1 in Corollary 2.
4) *Condition:* If the eigenvalues of L and H are all real.
   Pattern: This type of the pattern of b and k is given in Corollary 3, and illustrated in Fig. 1(d). b_0 is the infimum of b.

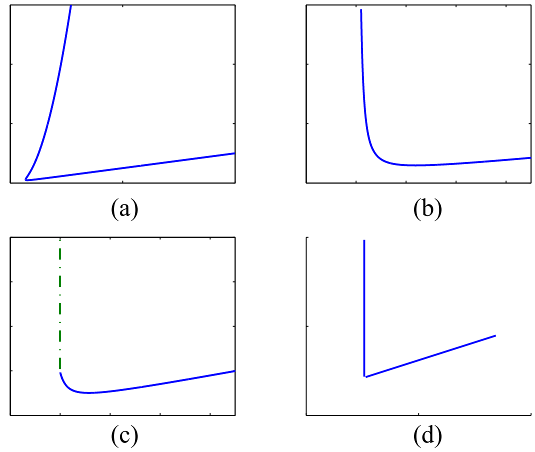

Fig. 1. Illustration of four types of patterns of b and k for ensuring the DCR (3). The x-axis represents the gain b, with $b \geq b_0$; the y-axis represents the gain k. For Fig. 1(a), b_0 may not be the infimum of b. For Fig. 1(b)–(d), b_0 is the infimum of b.

VI. Results for the DCR Problem (4)

A. General Case

Proposition 4: A necessary and sufficient condition for (4) is that

$$\begin{cases} \Gamma_j(b, k) \geq 0, & j = 1, 2, \ldots, n-1 \\ C_j(b, k) \geq 0, & j = 1, 2, \ldots, n-1 \\ b \geq b_0 \\ k > 0. \end{cases} \tag{14}$$

Proof: Refer to Proof of Proposition 4 in Appendix C. ■

Remark 9: Note that b_0 may not be the infimum of b in (14), since $\Gamma_j(b, k) \geq 0$ may have no solution for $b = b_0$ or $b \to b_0$.

For solutions of inequalities (14): if H has complex eigenvalues, deriving analytic solutions for b, k is complex; numerical solutions provide practical alternatives (similar to [1, Algorithm 1]). If the eigenvalues of H are all real, analytic solutions can be derived in an elegant way, which refers to the following results.

Example 1: For system (1) given in [1, Example 2], i.e., $n = 4$

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},\ H = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

To determine gains b and k to ensure the DCR (4) with $r_0 = 1$. Here $g_j(b) = -b^3+5b^2-8b+4$, $j = 1, 2, 3$. $f_j(b) = b^2-4b+4$, and $j = 1, 2$. $f_3(b) = 2b^2 - 8b + 8$. Thus, $b \geq 2r_0 = 2$, and

$$\begin{aligned} \Gamma_j(b, k) &= -k^2 + \left(b^2 - 4b + 4\right)k - b^3 \\ &\quad + 5b^2 - 8b + 4,\ j = 1, 2 \\ \Gamma_3(b, k) &= \left(2b^2 - 8b + 8\right)k - b^3 + 5b^2 - 8b + 4 \\ C_j(b, k) &= 4k + b^2 - 8b + 8,\ j = 1, 2 \\ C_3(b, k) &= 8k + b^2 - 8b + 8. \end{aligned}$$

From [1, Algorithm 1] with the additional solutions for $\Gamma_j(b, k) = 0$, one gets the results of b, k (Fig. 2), which

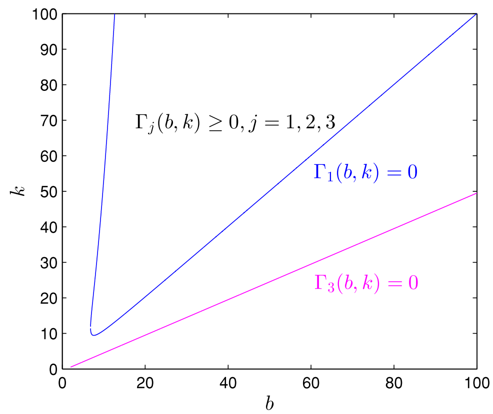

Fig. 2. Illustration of $\Gamma_j(b,k) \geq 0$. The two curves $C_1(b,k) = 0$ and $C_3(b,k) = 0$ are below the curve $\Gamma_1(b,k) = 0$.

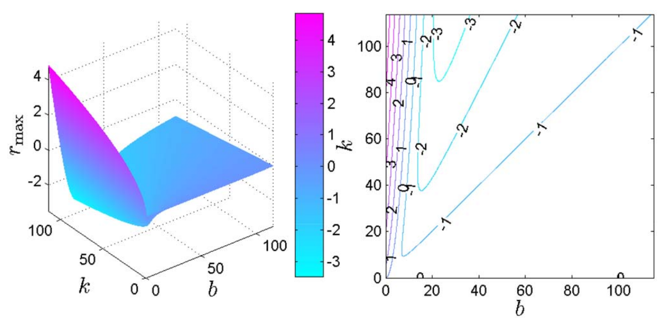

Fig. 3. Illustration of $r_{\max}$ as a function of b and k, and the contour plot.

are consistent with the calculation of the contour line (with value -1) of $r_{\max}$ as a function of b and k (Fig. 3). The values of b and k for $\Gamma_1(b,k) = 0$ ensure the DCR (3) with $r_0 = 1$. Refer to Example 2 for the analytical results.

Remark 10: The digraphs $\mathcal{G}_L$ and $\mathcal{G}_H$ are not required to have a same directed spanning tree or to have a same root agent in their respective directed spanning trees to ensure the DCR (Example 1).

Proposition 5: If the eigenvalues of L are all real, then, to ensure the DCR (4), a necessary and sufficient condition on the gains is that b and k satisfy inequalities

$$\begin{cases} b \geq \max\limits_{j\in S}\left\{\frac{2r_0}{u_j}\left(1+\frac{\tilde{v}_j^2}{\tilde{u}_j^2}+\frac{|\tilde{v}_j|}{\tilde{u}_j}\sqrt{1+\frac{\tilde{v}_j^2}{\tilde{u}_j^2}}\right)\right\} \\ b \geq b_0 \\ k \in \bigcap\limits_{j\in S}\left[\zeta_j-\delta_j, \zeta_j+\delta_j\right]\bigcap(0,\infty) \\ k \geq \max\limits_{j\notin S}\left\{\frac{r_0}{\tilde{u}_j}(u_j b - r_0)\right\} \\ k > \max\limits_{j\in S}\left\{\frac{u_j^2 b^2 - 2(u_j b - 2r_0)^2}{4\tilde{u}_j}\right\}, b \in \left(b_0, \frac{4r_0 + 2^{\frac{3}{2}} r_0}{u_j}\right) \end{cases} \tag{15}$$

where S is the index set of all complex eigenvalues of H (i.e., $\tilde{v}_j \neq 0$ for $j \in S$), and

$$\zeta_j := \frac{\tilde{u}_j(u_j b - 2r_0)^2}{2\tilde{v}_j^2}$$

$$\delta_j := \frac{\tilde{u}_j(u_j b - 2r_0)}{2\tilde{v}_j^2}\sqrt{(u_j b - 2r_0)^2 - 4\frac{\tilde{v}_j^2(u_j b - r_0)r_0}{\tilde{u}_j^2}} \geq 0.$$

Remark 11: The inequalities (15) for the DCR (4) are derived by replacing the middle three inequalities of (17) with

$$\begin{cases} b \geq b_0 \\ k \in \bigcap\limits_{j\in S}\left[\zeta_j-\delta_j, \zeta_j+\delta_j\right]\bigcap(0,\infty) \\ k \geq \max\limits_{j\notin S}\left\{\frac{r_0}{\tilde{u}_j}(u_j b - r_0)\right\}. \end{cases}$$

Note that $\delta_j \geq 0$ and $\zeta_j \geq \delta_j$, due to the first inequality of (15).

Example 2: For Example 1, we determine the gains b, k to ensure the DCR (4) with $r_0 = 1$. From Proposition 5, one has

$$\begin{cases} b \geq 4 + 2\sqrt{2} \\ k \in \left[\frac{(b-2)^2}{2} - \frac{(b-2)\sqrt{b^2-8b+8}}{2}, \frac{(b-2)^2}{2} + \frac{(b-2)\sqrt{b^2-8b+8}}{2}\right] \\ k \geq \frac{b-1}{2}. \end{cases}$$

Note that $(b-2)^2 - (b-2)\sqrt{b^2-8b+8} - (b-1) > 0$, one has

$$\begin{cases} b \geq 4 + 2\sqrt{2} \\ k \in \left[\frac{(b-2)^2}{2} - \frac{(b-2)\sqrt{b^2-8b+8}}{2}, \frac{(b-2)^2}{2} + \frac{(b-2)\sqrt{b^2-8b+8}}{2}\right]. \end{cases}$$

Note that the two curves

$$k = \frac{(b-2)^2}{2} - \frac{(b-2)\sqrt{b^2-8b+8}}{2}$$
$$k = \frac{(b-2)^2}{2} + \frac{(b-2)\sqrt{b^2-8b+8}}{2}$$

constitute $\Gamma_1(b,k) = 0$ in Fig. 2.

Corollary 4: If the eigenvalues of L, H are all real. From Proposition 5, to ensure (4), a necessary and sufficient condition is

$$b \geq b_0,\ k \geq \max_{j=1,2,\ldots,n-1}\left\{\frac{r_0}{\tilde{u}_j}(u_j b - r_0)\right\}.$$

Example 3: Consider system (1) with $n = 4$ agents, and

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ H = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

The system always achieves consensus for any $b, k > 0$, since

$$\lambda(A) = \left\{\frac{-b \pm b\sqrt{1-\frac{4k}{b^2}}}{2}, \frac{-b \pm b\sqrt{1-\frac{4k}{b^2}}}{2}, \frac{-b \pm b\sqrt{1-\frac{4k}{b^2}}}{2}, 0, 0\right\}.$$

Further, to determine the gains to ensure the DCR (4) with $r_0 = 1$. Then, from Corollary 4, one has $b \geq 2,\ k \geq b - 1$.

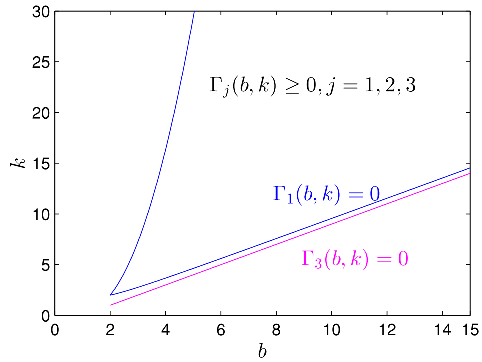

Fig. 4. Illustration of $\Gamma_j(b,k) \geq 0, j = 1, 2, 3$, from numerical calculation.

B. Results for Equal VCPC

Corollary 5: For $L = H$. To ensure the DCR (4), a necessary and sufficient condition is that: b, k satisfy inequalities (14), in which

$$\begin{aligned} \Gamma_j(b,k) &:= -v_j^2 k^2 + f_j(b)k + g_j(b) \\ C_j(b,k) &:= 4u_j k + \left(u_j^2 + v_j^2\right)b^2 - 8u_j r_0 b + 8r_0^2 \\ f_j(b) &:= u_j(u_j b - 2r_0)^2 + u_j v_j^2 b^2 \\ g_j(b) &:= -r_0(u_j b - r_0)\Big((u_j b - 2r_0)^2 + v_j^2 b^2\Big). \end{aligned}$$

Example 4: Determine b, k to ensure (4) with $r_0 = 1$

$$L = H = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Note that $\lambda_1 = 1 + \mathtt{i}$, $\lambda_2 = 1 - \mathtt{i}$, are complex eigenvalues, $\lambda_3 = 2$. Thus, $b \geq 2$, and

$$\begin{aligned} \Gamma_1(b,k) = \Gamma_2(b,k) &= -k^2 + \left(2b^2 - 4b + 4\right)k \\ &\quad - 2b^3 + 6b^2 - 8b + 4 \\ \Gamma_3(b,k) &= \left(b^2 - 4b + 4\right)k - b^3 + 5b^2 - 8b + 4 \\ C_1(b,k) = C_2(b,k) &= 4k + 2b^2 - 8b + 8 \\ C_3(b,k) &= 8k + 4b^2 - 16b + 8. \end{aligned}$$

From [1, Algorithm 1] with additional solutions of $\Gamma_j(b,k) = 0$, one gets the results of b and k as in Fig. 4, which is consistent with the calculation of the contour line (with value -1) of $r_{\max}$ (the illustration omitted here).

VII. Conclusion

In this paper, we investigate the DCR problems of coupled double-integrator agents, provide further necessary and sufficient conditions for the DCR problems, and characterize different types of the patterns of the convergence rate contours for the DCR, in terms of the damping and stiffness gains, which properties are closely related to different types of the spectra of the Laplacian matrices of the VCPC. The DCR problems have important implications for generic cooperative control problems, e.g., flocking, consensus, rendezvous, and swarming.

The nonequal VCPC case is much different from the equal VCPC case, the consensus analysis is a challenge for a general nonequal VCPC case (refer to Section III-C).

Future work includes the DCR problems of a flocking system with considerations of, e.g., coupling delays (refer to Appendix A for an example), communication constraints, and nonlinear dynamics of agents, etc. In applications, switched topologies and noise may also exist; as such, what are the results for the DCR problems? Other questions also remain to be answered: the $\mathcal{L}$-assumption reflects a relation between the topologies $\mathcal{G}_L$ and $\mathcal{G}_H$, so what is this relation (besides Theorem 1)? For a general nonequal VCPC case, what are the DCR conditions for the system if without the $\mathcal{L}$-assumption? These kinds of the problems are worth of further investigation.

Appendix A

For the DCR problems of flocking or consensus, the following is one example for the control input with coupling delay:

$$\begin{aligned} u_i = &\sum_{j \in \mathcal{M}_i} -b w_{ij}\big(\dot{x}_i(t-\tau) - \dot{x}_j(t-\tau)\big) \\ &+ \sum_{j \in \mathcal{N}_i} -k v_{ij}\big(x_i(t-\tau) - x_j(t-\tau) - a_{ij}\big) \end{aligned}$$

where $\tau > 0$ models the coupling delay, the uniform delay, and one can further consider nonuniform delays for the VCPC.

The nonequal VCPC case is much different from the equal VCPC case. For example, consider four agents with the very simple VCPC given in [4, eqs. (5) and (6)], solving the eigenvalues with respect to symbolic variables of the gains is still difficult [4], not to mention a system of a larger number of agents with general VCPC. Without the $\mathcal{L}$-assumption, how to solve the DCR is still a problem.

Appendix B

The following is some results in [1] for convenience of reference. To ensure the DCR (2), a necessary and sufficient condition is

$$\begin{cases} \Gamma_j(b,k) > 0, & j = 1, 2, \ldots, n-1 \\ C_j(b,k) > 0, & j = 1, 2, \ldots, n-1 \\ b > b_0 \\ k > 0. \end{cases} \tag{16}$$

If the eigenvalues of L are all real, to ensure the DCR (2), a necessary and sufficient condition on the gains is that

$$\begin{cases} b \geq \max\limits_{j \in S}\left\{\frac{2r_0}{u_j}\left(1 + \frac{\tilde{v}_j^2}{\tilde{u}_j^2} + \frac{|\tilde{v}_j|}{\tilde{u}_j}\sqrt{1 + \frac{\tilde{v}_j^2}{\tilde{u}_j^2}}\right)\right\} \\ b > b_0 \\ k \in \bigcap\limits_{j \in S}\big(\zeta_j - \delta_j, \zeta_j + \delta_j\big) \\ k > \max\limits_{j \notin S}\left\{\frac{r_0}{\tilde{u}_j}(u_j b - r_0)\right\} \\ k > \max\limits_{j \in S}\left\{\frac{u_j^2 b^2 - 2(u_j b - 2r_0)^2}{4\tilde{u}_j}\right\}, \ b \in \left(b_0, \frac{4r_0 + 2^{\frac{3}{2}} r_0}{u_j}\right). \end{cases} \tag{17}$$

APPENDIX C

A. Proof Preparation

Denote $x := \left[x_1^T - a_1^T, x_2^T - a_2^T, \dots, x_n^T - a_n^T\right]^T \in \mathbb{R}^{Nn}$, where $a_i \in \mathbb{R}^N$, $i = 1, 2, \dots, n$, satisfy $a_{ij} = a_i - a_j$ for all i, j [4, Proposition 1]. System (1) can be written as

$$\ddot{x} + bL \otimes I_N \dot{x} + kH \otimes I_N x = \mathbf{0}$$
$$\left(\ddot{x}^T, \dot{x}^T\right)^T = A \otimes I_N \left(\dot{x}^T, x^T\right)^T.$$

Define $y := [y_c^T, y_e^T]^T = T_1 \otimes I_N x$, where $y_c \in \mathbb{R}^N$, $y_e \in \mathbb{R}^{N(n-1)}$

$$y_e = \left[x_2^T, x_3^T, \dots, x_n^T\right]^T - \mathbf{1} \otimes x_1.$$

The shaped-subsystem is

$$\ddot{y}_e + bL_1 \otimes I_N \dot{y}_e + kH_1 \otimes I_N y_e = \mathbf{0} \tag{18}$$

which is equivalent to system (1) in terms of formation and stability. In (18), L_1 and H_1 are derived using the similarity transformations

$$T_1 L T_1^{-1} = \begin{bmatrix} 0 & * \\ \mathbf{0} & L_1 \end{bmatrix}, \; T_1 H T_1^{-1} = \begin{bmatrix} 0 & * \\ \mathbf{0} & H_1 \end{bmatrix} \tag{19}$$

where L_1 and H_1 are independent of γ [4] (one may use any other T_i, $i = 1, 2, \dots, n$, in [4] to perform the transformation).

B. Proof

Proof of Theorem 1: Without loss of generality, assume that the topology $\mathcal{G}_L$ of L is the CDS topology, and the topology $\mathcal{G}_H$ of H is arbitrary, with arbitrary edge weights. That is, L has the structure as described in Definition 2: $L = \varepsilon \mathbf{1} I_n - \mathbf{1}\varepsilon \in \mathbb{R}^{n \times n}$, and H is an *arbitrary* Laplacian matrix.

Consider the similarity transformations on L and H, we have (19), in which L_1 and H_1 are independent of γ, refer to the structure of L_1 and H_1 in [4]. Then, from Lemma 3, $L_1 = \varepsilon \mathbf{1} I_{n-1}$.

1) For matrix H_1, from Schur decomposition, there exists a unitary matrix $\tilde{U}_1 \in \mathbb{C}^{(n-1)\times(n-1)}$ (i.e., $\tilde{U}_1^* = \tilde{U}_1^{-1}$) such that $\tilde{U}_1^* H_1 \tilde{U}_1 = \tilde{U}_1^{-1} H_1 \tilde{U}_1$ is upper triangular

$$\tilde{U}_1^* H_1 \tilde{U}_1 = \tilde{U}_1^{-1} H_1 \tilde{U}_1 = \begin{bmatrix} \mu_1 & * & \dots & * \\ 0 & \mu_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{n-1} \end{bmatrix}.$$

2) While for the same unitary matrix $\tilde{U}_1$, we have

$$\tilde{U}_1^* L_1 \tilde{U}_1 = \tilde{U}_1^{-1} L_1 \tilde{U}_1 = \varepsilon \mathbf{1} I_{n-1}$$

which is the identity matrix multiplied by the coefficient $\varepsilon \mathbf{1} > 0$.

That is, L_1 and H_1 have a same unitary matrix $\tilde{U}_1$ in Schur decomposition. And from the similarity transformations on the Laplacian matrices L and H, we can conclude that the $\mathcal{L}$-assumption holds. For the Schur decompositions of L and H, refer to the following remark.

Remark 12: From the transformation (19), we have the conclusion that: the $\mathcal{L}$-assumption is equivalent to the condition that there exists a unitary matrix $U_1 \in \mathbb{C}^{(n-1)\times(n-1)}$ such that

$$U_1^* L_1 U_1 = U_1^{-1} L_1 U_1 = \begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & \lambda_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{n-1} \end{bmatrix}$$
$$U_1^* H_1 U_1 = U_1^{-1} H_1 U_1 = \begin{bmatrix} \mu_1 & * & \dots & * \\ 0 & \mu_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{n-1} \end{bmatrix}.$$

This explains why the Schur decompositions of L and H in the $\mathcal{L}$-assumption make the two zero eigenvalues λ_n and μ_n appearing as the last diagonal entries of V_1 and V_2, respectively.

Proposition A1: Denote $\lambda_j^2 - (4k/b^2)\mu_j := a_j + d_j \mathrm{i} \in \mathbb{C}$, where $a_j, d_j \in \mathbb{R}$, then

$$a_j := u_j^2 - v_j^2 - \frac{4k}{b^2}\tilde{u}_j, \; d_j := 2u_j v_j - \frac{4k}{b^2}\tilde{v}_j.$$

The real parts of the eigenvalues of A are

$$\mathrm{Re}\left\{\Lambda_{2j-1}, \Lambda_{2j}\right\} = -\frac{b}{2}u_j \pm \frac{b}{2\sqrt{2}} \cdot \sqrt{a_j + \sqrt{a_j^2 + d_j^2}}$$

where $j = 1, 2, \dots, n$, no matter $d_j = 0$ or $d_j \neq 0$.

Proof of Proposition 1: By Proposition A1, to guarantee (3), a necessary and sufficient condition is that, the maximum of the real parts of $\{\Lambda_{2j-1}, \Lambda_{2j}\}$, $j = 1, 2, \dots, n-1$, is equal to $-r_0$

$$\max_{j=1,2,\dots,n-1}\left\{-\frac{b}{2}u_j + \frac{b}{2\sqrt{2}}\sqrt{a_j + \sqrt{a_j^2 + d_j^2}}\right\} = -r_0.$$

Equivalently, there exists a nonempty index set $\mathcal{I}(b,k) \subset \{1, 2, \dots, n-1\}$, which is a function of b, k, such that

$$-\frac{b}{2}u_j + \frac{b}{2\sqrt{2}}\sqrt{a_j + \sqrt{a_j^2 + d_j^2}} = -r_0, \;\text{ for } j \in \mathcal{I}(b,k)$$

and

$$\max_{j \notin \mathcal{I}(b,k)}\left\{-\frac{b}{2}u_j + \frac{b}{2\sqrt{2}}\sqrt{a_j + \sqrt{a_j^2 + d_j^2}}\right\} < -r_0.$$

And, similarly to the procedure of [1, Proof of Th. 2], the result holds, the details are omitted here.

Proof of Proposition 2: For $\lambda(H) \in \mathbb{R}$, then $\tilde{v}_j = 0$ and $\Gamma_j(b,k) = f_j(b)k + g_j(b)$, for all j, where $g_j(b)$ remains the same in Section V, but $f_i(b)$ reduces to be $f_j(b) = \tilde{u}_j(u_j b - 2r_0)^2$. Thus

$$-\frac{g_j(b)}{f_j(b)} = \frac{r_0(u_j b - r_0)\left((u_j b - 2r_0)^2 + v_j^2 b^2\right)}{\tilde{u}_j(u_j b - 2r_0)^2}$$
$$= \frac{r_0}{\tilde{u}_j}(u_j b - r_0)\left(1 + \frac{v_j^2 b^2}{(u_j b - 2r_0)^2}\right).$$

From Proposition 1, $b \geq b_0$, and considering the definition (7), one has $u_j b \geq 2r_0$, so $u_j b > r_0$, and thus $-g_j(b)/f_j(b) > 0$ for all j.

1) From Proposition 1, condition $\Gamma_j(b,k) = 0$ for $j \in \mathcal{I}(b,k)$, one has $k = -g_j(b)/f_j(b)$; and thus $k > 0$ for all $j \in \mathcal{I}(b,k)$. Note that $C_j(b,k) \geq 0$ is equivalent to

$$k \geq -\frac{(u_j^2+v_j^2)b^2 - 8u_j r_0 b + 8r_0^2}{4\tilde{u}_j}.$$

And note that

$$\begin{aligned}&\frac{r_0}{\tilde{u}_j}(u_j b - r_0) - \left(-\frac{\left(u_j^2+v_j^2\right)b^2 - 8u_j r_0 b + 8r_0^2}{4\tilde{u}_j}\right)\\ &= \frac{1}{4\tilde{u}_j}\left(4r_0 u_j b - 4r_0^2 + \left(u_j^2+v_j^2\right)b^2 - 8u_j r_0 b + 8r_0^2\right)\\ &= \frac{1}{4\tilde{u}_j}\left(v_j^2 b^2 + (u_j b - 2r_0)^2\right) > 0.\end{aligned}$$

Thus, $k = -g_j(b)/f_j(b)$ implies $C_j(b,k) > 0$.

2) From Proposition 1, $\Gamma_j(b,k) > 0$ for $j \notin \mathcal{I}(b,k)$, then $k > -g_j(b)/f_j(b)$ and thus $k > 0$ for $j \notin \mathcal{I}(b,k)$; note that $k > -g_j(b)/f_j(b)$ implies $C_j(b,k) > 0$ [1, Proof of Th. 4].

Thus, the result holds.

Proof of Proposition 3: From (7), $b_0 = (2r_0/u_j)$ for $j \in \ell$. For $j \in \ell$, one has

$$f_j\left(\frac{2r_0}{u_j}\right) = 0,\ \ g_j\left(\frac{2r_0}{u_j}\right) = -4\frac{v_j^2}{u_j^2}r_0^4 \leq 0$$

thus, $\Gamma_j(b,k) \leq 0$ for $j \in \ell$. Thus, for $j \in \ell$, one has $\Gamma_j(b,k) = 0$ only when $v_j = 0$. Items 1) and 2) are proved.

3) For $\ell \neq \{1,2,\ldots,n-1\}$. Note that

$$f_j\left(\frac{2r_0}{u_j}\right) > 0,\ \ \text{for } j \notin \ell$$

and k needs to satisfy

$$\left\{k > 0,\ \ k \geq \max_{j\notin\ell} -\frac{g_j(b_0)}{f_j(b_0)}\right\}$$

to ensure $\Gamma_j(b_0,k) \geq 0$. For $j \in \ell$, $v_j = 0$ is required to ensure $\Gamma_j(b,k) = 0$. And, $C_j(b_0,k) \geq 0$ is required for all j. Thus

$$\begin{cases} v_j = 0, & j \in \ell\\ k \geq \max\limits_{j=1,2,\ldots,n-1, j\notin\ell}\left\{-\frac{g_j(b_0)}{f_j(b_0)}\right\} & \\ C_j(b_0,k) \geq 0, & j = 1,2,\ldots,n-1\\ k > 0. & \end{cases}$$

Note that

$$k \geq \max_{j\notin\ell}\left\{-\left(g_j(b_0)/f_j(b_0)\right)\right\}$$

implies both $k > 0$ and $C_j(b_0,k) \geq 0$, for $j \notin \ell$ (refer to [1, Proof of Th. 4]).

And, $C_j(b_0,k) \geq 0$ with $j \in \ell$ and $v_j = 0$ implies that

$$k \geq \max_{j\in\ell}\left\{-\frac{u_j^2 b_0^2 - 8u_j r_0 b_0 + 8r_0^2}{4\tilde{u}_j}\right\} = \max_{j\in\ell}\left\{\frac{r_0^2}{\tilde{u}_j}\right\}$$

where the last equal sign holds since $u_j = u_{\min}$ and $b_0 = (2r_0/u_{\min})$.

4) The result of $\ell = \{1,\ldots,n-1\}$ follows from item 3.

Proof of Proposition 4: By Proposition A1, to guarantee (4), a necessary and sufficient condition is that, the maximum of the real parts of $\{\Lambda_{2j-1}, \Lambda_{2j}\}$, $j = 1,2,\ldots,n-1$, is no large than $-r_0$

$$\max_{j=1,2,\ldots,n-1}\left\{-\frac{b}{2}u_j + \frac{b}{2\sqrt{2}}\sqrt{a_j + \sqrt{a_j^2+d_j^2}}\right\} \leq -r_0.$$

And, similarly to the procedure of [1, Proof of Th. 2], the result holds, the details are omitted here for a limited space.

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