Hierarchical Decomposition-Based Distributed Full States Tracking Consensus for High-Order Nonlinear Multiagent Systems

Yafeng Li, Ju H. Park, *Senior Member, IEEE*, Changchun Hua, *Senior Member, IEEE*, and Xiu You

***Abstract*—This article studies the distributed adaptive leader-following control for high-order time-varying nonlinear multiagent systems (MASs) with uncertain parameters. The state feedback protocol and output feedback protocol are proposed, respectively, to render all states consensus errors to converge to zero asymptotically. First, the hierarchical decomposition algorithm is used to construct a refreshed communication graph to address the mutual dependence problem of controllers. Then, by introducing a local neighborhood consensus errors-based transformation, the leader-following consensus problem is converted into the stabilization problem for the consensus error system. Using the backstepping method and tuning function technique, the distributed adaptive state feedback controller is designed to render all followers' states to track the leader's ones. Further, by constructing the reduced-order dynamic gain k-filter to estimate unmeasured states, a distributed adaptive output feedback controller is designed. In both controller design methods, the traditional Lipschitz condition need not be satisfied any more for all time-varying nonlinear functions, and different from most of the existing results on the high-order nonlinear MASs, full states consensus can be obtained. Finally, a general numerical example is given to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Adaptive control, distributed control, leader-following, multiagent systems (MASs), time-varying systems.**

I. Introduction

IN THE past decades, the distributed cooperative control for multiagent systems (MASs) has attracted lots of researchers' attention, due to its broad range of applications, such as distributed reconfigurable sensor networks, autonomous vehicles, micro-grid, industrial systems, martial systems, and so on [1], [2], which can bring many great benefits including high adaptivity, low cost, easy maintenance, and so on. Based on the Lyapunov stability theorem and the graph theory, many kinds of MASs have been investigated, such as first-order MASs [3]–[5], second-order MASs [6]–[8], high-order linear MASs [9]–[12], and high-order nonlinear MASs [13]–[16], and the references therein. Obviously, in terms of the agent model, the high-order nonlinear MASs are more general and practical in nature, where some of the results can even cover the results in the first three types of MASs, and at the same time, the distributed control protocol design is more challenging.

Compared with the results on the first-order MASs, second-order MASs, and high-order linear MASs mainly focusing on the full states consensus problem, most of the results on the high-order nonlinear one mainly focus on the output consensus problem due to the complexity of its dynamics [14]–[20]. In [14], using the adaptive dynamic surface control, the distributed containment control problem was investigated for uncertain nonlinear strict-feedback MASs with multiple dynamic leaders under a directed graph, but the initial domain of the states was semi-global. The work [15] investigated the finite-time leader-following consensus problem for high-order nonlinear MASs under the undirected graph. In [16] and [17], the effects of unknown parameters were further considered and the adaptive distributed state feedback control protocols were proposed. In [18]–[20], the stochastic disturbance was further considered and the distributed cooperative control for stochastic MASs was addressed. There are also some results on the full states consensus for high-order nonlinear MASs, such as [21]–[23] and the reference therein.

For the high-order nonlinear MASs, the aforementioned results mainly focus on the distributed state feedback consensus and most of the results only guarantee the bounded stability of the MASs. However, the states variables in many practical engineering systems are usually hard to measure directly and even can not be measured. The distributed output feedback control problem has been always an important topic in the area of MASs control [24]–[28]. In [24] and [25], the k-filter-based distributed consensus and the dynamic-gain-based distributed consensus were studied for lower-triangular MASs, respectively, but only the output consensus can be guaranteed. The work [26] investigated the finite-time control for the chain MASs. In [27] and [28], the output feedback distributed full states consensus were further considered for more general nonlinear MASs, where the restrictive conditions were more conservative.

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Inspired by the aforementioned results, the distributed adaptive state feedback and output feedback control protocols are designed for lower-triangular time-varying nonlinear MASs with uncertain parameters, such that all states of followers can track the leader's states asymptotically. The main contributions of our paper list as follows.

1) The distributed adaptive state feedback protocol is proposed for time-varying nonlinear MASs, where the conditions on the nonlinear functions are general. Different from the results studied in [14]–[20], the full states consensus can be achieved with mild conditions.
2) The reduced dynamic gain k-filter is constructed to compensate the unmeasured states, which is more general and can be easily extended to relax some existing results [24], [29]–[33]. Based on the k-filter, the distributed adaptive output feedback protocol is designed.
3) Both of the proposed main results can guarantee that all states of the followers track the leader's ones asymptotically. One more general case is also discussed in Corollary 5.

Notations: $\mathcal{R}^n$ denotes the n dimensional Euclidean space. $\|X\| = (Tr\{X^TX\})^{(1/2)}$ is the norm of a matrix X. $\mathbf{1}_N := [1,\ldots,1]^T \in \mathcal{R}^N$. This article considers the system of $N+1$ agents (N followers and one leader). The communication graph is depicted by $G = (\upsilon, \varepsilon)$ with a finite set of $N+1$ nodes $\upsilon = \{\upsilon_i, i = 0,1,\ldots,N\}$, and a set of edges or arcs $\varepsilon \subseteq \upsilon \times \upsilon$. The set of neighbors of a node i is $N_i = \{j|(\upsilon_j, \upsilon_i) \in \varepsilon\}$, i.e., the set of nodes with arcs incoming to υ_i. The connectivity matrix is $A = [a_{ij}] \in \mathcal{R}^{n\times n}$, and the in-degree matrix is $D = \mathrm{diag}\{d_1,\ldots,d_N\}$. Define the Laplacian matrix as $L = D - A$, and pinning matrix $B = \mathrm{diag}\{b_1,\ldots,b_N\}$. The details are not presented here for saving space, referring to [25].

II. Preliminaries

A. Problem Formulation

Consider the following high-order time-varying nonlinear MASs, and the dynamic of ith agent is described by:

$$\dot{x}_{\mathrm{is}} = g_s x_{i(s+1)} + \theta f_s(t, \bar{x}_{\mathrm{is}}) \tag{1}$$

where for $i = 0,1,\ldots,N$, $s = 1,2,\ldots,n$, $x_{i(n+1)} := u_i$ is the input and $\bar{x}_{\mathrm{is}} := [x_{i1}, x_{i2},\ldots,x_{\mathrm{is}}]^T$ is the state vector. $f_s(t,\bar{x}_{\mathrm{is}}) : \mathcal{R}^{s+1} \to \mathcal{R}$ is smooth function. g_s and θ are unknown constants. The sign of g_s is known. Without loss of generality, we assume that $g_s > 0$. Following assumptions and lemma will be used in the controller design process.

Assumption 1: The graph constructed by N followers and one leader is directed. Moreover, there exists at least one directed path from the leader to each follower.

Assumption 2: The norm of the leader's states $\|\bar{x}_{0n}\|$ is bounded.

Assumption 3: For $s = 1,\ldots,n$, the function $f_s(t,\bar{x}_{\mathrm{is}})$ satisfies the properties that for $t \geq t_0$ with t_0 be the initial moment, if $\|\bar{x}_{\mathrm{is}}\| \leq \Phi_s$, where Φ_s is a positive constant, then $f_s(t,\bar{x}_{\mathrm{is}})$ and all of its first derivative, second derivative, $\ldots$, $(n-s+1)$th derivative on its variables t and $\bar{x}_{\mathrm{is}}$, are bounded, uniformly in t.

TABLE I
HIERARCHICAL DECOMPOSITION ALGORITHM

- *Input:* The directed communication graph $G = (\upsilon, \varepsilon)$.
- *Output:* Partition of the set of subsystems υ and the set of edges maintained from ε.

- *Initialization:* $j = 0$, $\upsilon_r^j \leftarrow$ {empty sequence}, $\upsilon_r^j \leftarrow \upsilon_0$, for all $\upsilon_i \in \upsilon$, setLabel(υ_i,UNVISITED).
- *while* $\upsilon_r^j \neq$ {empty sequence}
  - $\upsilon_r^{j+1} \leftarrow$ {empty sequence}
  - *for* all $\upsilon_i \in \upsilon_r^j$
    - *for* all $(\upsilon_i, \upsilon_k) \in \varepsilon$
      - *if* getLable(υ_k) =UNVISITED
        - setLable(υ_k, VISITED), $\upsilon_r^{j+1} \leftarrow \upsilon_k$
        - setLable((υ_i, υ_k) ,MAINTAINED)
      - *elseif* $\upsilon_k \in \upsilon_r^{j+1}$
        - setLable((υ_i, υ_k) ,MAINTAINED)
  - $j = j + 1$

Lemma 1 [34]: Let $\phi : \mathcal{R} \to \mathcal{R}$ be a uniformly continuous function on $(0,\infty]$. Suppose that $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ exists and is finite. Then, $\phi(t) \to 0$ as $t \to \infty$.

Remark 1: From Assumptions 2 and 3, for all $t \geq t_0$ and $\|\bar{x}_{\mathrm{is}}\| \leq \Phi_s$, the time-varying nonlinear function $f_s(t,\bar{x}_{\mathrm{is}})$ and all of its first derivative, $\ldots$, $(n-s)$th derivative on the variables t and $\bar{x}_{\mathrm{is}}$ are uniformly continuous and satisfy that $(f_s(t,\bar{x}_{\mathrm{is}}) - f_s(t,\bar{x}_{0s})) \to 0$, $([\partial f_s(t,\bar{x}_{\mathrm{is}})]/\partial t - [f_s(t,\bar{x}_{0s})]/\partial t) \to 0$, $([\partial f_s(t,\bar{x}_{\mathrm{is}})]/\partial x_{i1} - [f_s(t,\bar{x}_{0s})]/\partial x_{01}) \to 0$, $([\partial^2 f_s(t,\bar{x}_{\mathrm{is}})]/[\partial x_{i1}\partial x_{i2}] - [\partial^2 f_s(t,\bar{x}_{0s})]/[\partial x_{01}\partial x_{02}]) \to 0, \ldots$, as $(\bar{x}_{\mathrm{is}} - \bar{x}_{0s}) \to 0$, which are very general. For example, any time-invariant smooth function $f_s(\bar{x}_{\mathrm{is}})$ satisfies Assumption 3. Assumption 2 indicates that all leader's states are bounded, and similar assumptions can also be found in [14], [18]–[20], and [24]. Even though Assumptions 2 and 3 do not hold, the output consensus results can still be achieved.

B. Hierarchical Decomposition

In [4], [18], [19], and [23] and some existing results, the proposed distributed protocol of the ith agent contains the local control inputs information collected from its neighbors, which is difficult to compute without a prescribed priority. To address this issue, the original graph $G = (\upsilon, \varepsilon)$ is split into a hierarchical structure based on the hierarchical decomposition algorithm in Table I, where $B \leftarrow A$ means inserting element A into set B. Then, a refreshed graph $G_r = (\upsilon, \varepsilon_r)$ and connectivity matrix $A_r = [\bar{a}_{ij}] \in R^{n\times n}$ are formed. Refer to [24] for details. The following lemma is useful.

Lemma 2 [24]: Consider the MASs (1) under Assumption 1. Based on the hierarchical decomposition algorithm in Table I, the obtained Laplacian matrix L_r satisfies that $(L_r + B)$ has full rank and all eigenvalues of $(L_r + B)$ have positive real parts.

Remark 2: Different from the results in [24], the objective of this article is to construct the state feedback and output feedback protocols such that all the states of followers can track the leader's states asymptotically, and a reduced order dynamic gain k-filter is also constructed with mild conditions which makes the studied MASs more general.

III. Main Results

A. Distributed State Feedback Leader-Following Control

In this section, the distributed state-feedback consensus control method is proposed based on the refreshed graph G_r. First, for $s = 1, \ldots, n$, define the local neighborhood consensus error for ith agent as

$$\xi_{\text{is}} = \sum_{j=1}^{N} \bar{a}_{ij}\big(x_{\text{is}} - x_{js}\big) + b_i(x_{\text{is}} - x_{0s}). \tag{2}$$

Then, the derivative of ξ_{is} is $\dot{\xi}_{\text{is}} = g_s \xi_{i(s+1)} + \theta L_i F_s$, where L_i is the ith row of the matrix $(L + B)$, and $F_s := [f_s(t, \bar{x}_{1s}), \ldots, f_s(t, \bar{x}_{Ns})]^T - \mathbf{1}_N f_s(t, \bar{x}_{0s})$. Choose the following transformation:

$$z_{\text{is}} = \xi_{\text{is}} - \alpha_{i(s-1)} \tag{3}$$

where $\alpha_{i(s-1)}$ is designed virtual controller and $\alpha_{i0} = 0$.

In the sequel, the controller is designed using the backstepping method and the tuning function technology. Choose the Lyapunov function as

$$\begin{cases} V = \sum_{i=1}^{N} \sum_{s=1}^{n} (V_{\text{is}} + V_{gi(s-1)}) + \sum_{i=1}^{N} V_{\theta i} \\ V_{\text{is}} = \frac{1}{2} z_{\text{is}}^2 + \frac{1}{2} g_s \tilde{\delta}_{\text{is}}^2 \\ V_{gi(s-1)} = \frac{1}{2} \tilde{g}_{i(s-1)}^2, V_{\theta i} = \frac{1}{2} \tilde{\theta}_i^2, \end{cases} \tag{4}$$

where $\tilde{\delta}_{\text{is}} = g_s^{-1} - \hat{\delta}_{\text{is}}$, $\tilde{\theta}_i = \theta - \hat{\theta}_i$, $\tilde{g}_{\text{is}} = g_s - \hat{g}_{\text{is}}$, and $\hat{\delta}_{\text{is}}$, $\hat{\theta}_i$, $\hat{g}_{\text{is}}$ are the estimations of g_s^{-1}, θ, g_s in ith agent. Define $\tilde{g}_{i0} = 0$.

Step 1: The derivative of V_{i1} is

$$\dot{V}_{i1} = z_{i1}(g_1 z_{i2} + g_1 \alpha_{i1} + \theta L_i F_1) + g_1 \tilde{\delta}_{i1} \dot{\tilde{\delta}}_{i1}. \tag{5}$$

Design the virtual controller α_{i1} and adaptive law $\dot{\hat{\delta}}_{i1}$ as

$$\alpha_{i1} = -\hat{\delta}_{i1} \bar{\alpha}_{i1} \tag{6}$$

$$\dot{\hat{\delta}}_{i1} = z_{i1} \bar{\alpha}_{i1} \tag{7}$$

where $\bar{\alpha}_{i1} = \hat{\theta}_i L_i F_1 + z_{i1}$, which only includes the states information of neighbors. Using Assumptions 2 and 3, there exists smooth function $\bar{f}_{i1}(\cdot)$ such that $\bar{\alpha}_{i1} = \sum_{j=1}^{N} \bar{a}_{ij}(\bar{f}_{i1}(t, \hat{\theta}_i, z_{i1}, x_{i1}) - \bar{f}_{i1}(t, \hat{\theta}_i, z_{i1}, x_{j1})) + b_i(\bar{f}_{i1}(t, \hat{\theta}_i, z_{i1}, x_{i1}) - \bar{f}_{i1}(t, \hat{\theta}_i, z_{i1}, x_{01}))$, satisfying that if $x_{11} \to x_{01}, \ldots, x_{N1} \to x_{01}$ and $\hat{\theta}_i$ is bounded, then $\bar{\alpha}_{i1} \to 0$. $\dot{\hat{\theta}}_i$ is designed later. It follows from (4)–(7) that:

$$\begin{aligned} \dot{V}_{i1} + \dot{V}_{\theta i} &= g_1 z_{i1} z_{i2} + g_1 \tilde{\delta}_{i1} z_{i1} \bar{\alpha}_{i1} + \tilde{\theta}_i z_{i1} L_i F_1 \\ &\quad - z_{i1}^2 + g_1 \tilde{\delta}_{i1} \dot{\tilde{\delta}}_{i1} + \tilde{\theta}_i \dot{\tilde{\theta}}_i \\ &= -z_{i1}^2 + g_1 z_{i1} z_{i2} + \tilde{\theta}_i \Big(\tau_{\theta i1} - \dot{\hat{\theta}}_i\Big) \end{aligned} \tag{8}$$

where $\tau_{\theta i1} = z_{i1} L_i F_1$.

Step s: Assume that at the step $(s-1)$, there exist virtual controller $\alpha_{i(s-1)}$ and adaptive law $\dot{\hat{\delta}}_{i(s-1)}$ as follows:

$$\alpha_{i(s-1)} = -\hat{\delta}_{i(s-1)} \bar{\alpha}_{i(s-1)} \tag{9}$$

$$\dot{\hat{\delta}}_{i(s-1)} = z_{i(s-1)} \bar{\alpha}_{i(s-1)} \tag{10}$$

where there exists smooth function $\bar{f}_{i(s-1)}(\cdot)$ such that

$$\bar{\alpha}_{i(s-1)} = \sum_{j=1}^{N} \bar{a}_{ij} \left(\begin{array}{l} \bar{f}_{i(s-1)}\left(\begin{array}{c} t, \hat{\theta}_i, \bar{\hat{\delta}}_{i(s-2)}, \bar{\hat{g}}_{i(s-2)} \\ \bar{z}_{i(s-1)}, \bar{x}_{i(s-1)} \end{array} \right) \\ -\bar{f}_{i(s-1)}\left(\begin{array}{c} t, \hat{\theta}_i, \bar{\hat{\delta}}_{i(s-2)}, \bar{\hat{g}}_{i(s-2)} \\ \bar{z}_{i(s-1)}, \bar{x}_{j(s-1)} \end{array} \right) \end{array} \right)$$

$$+ b_i \left(\begin{array}{l} \bar{f}_{i(s-1)}\left(\begin{array}{c} t, \hat{\theta}_i, \bar{\hat{\delta}}_{i(s-2)}, \bar{\hat{g}}_{i(s-2)} \\ \bar{z}_{i(s-1)}, \bar{x}_{i(s-1)} \end{array} \right) \\ -\bar{f}_{i(s-1)}\left(\begin{array}{c} t, \hat{\theta}_i, \bar{\hat{\delta}}_{i(s-2)}, \bar{\hat{g}}_{i(s-2)} \\ \bar{z}_{i(s-1)}, \bar{x}_{0(s-1)} \end{array} \right) \end{array} \right) \tag{11}$$

satisfying that if $\bar{x}_{1(s-1)} \to \bar{x}_{0(s-1)}, \ldots, \bar{x}_{N(s-1)} \to \bar{x}_{0(s-1)}$, $\hat{\theta}_i$ is bounded, and the entries of vectors $\bar{\hat{\delta}}_{i(s-2)} = [\hat{\delta}_{i1}, \hat{\delta}_{i2}, \ldots, \hat{\delta}_{i(s-2)}]^T$, $\bar{\hat{g}}_{i(s-2)} = [\hat{g}_{i1}, \hat{g}_{i2}, \ldots, \hat{g}_{i(s-2)}]^T$ are bounded, then $\bar{\alpha}_{i(s-1)} \to 0$. At the same time, the derivative of $\sum_{m=1}^{s-1} (V_{im} + V_{gi(m-1)}) + V_{\theta i}$ satisfies

$$\begin{aligned} &\sum_{m=1}^{s-1} \big(\dot{V}_{im} + \dot{V}_{gi(m-1)}\big) + \dot{V}_{\theta i} \\ &= -\sum_{m=1}^{s-1} z_{im}^2 + g_{(s-1)} z_{i(s-1)} z_{\text{is}} + \tilde{g}_{i(s-2)} \Big(\tau_{gi(s-2)1} - \dot{\hat{g}}_{i(s-2)}\Big) \\ &\quad + \left(\tilde{g}_{i(s-3)} + z_{i(s-1)} \frac{\partial \alpha_{i(s-2)}}{\partial \hat{g}_{i(s-3)}}\right) \Big(\tau_{gi(s-3)2} - \dot{\hat{g}}_{i(s-3)}\Big) + \cdots \\ &\quad + \left(\tilde{g}_{i1} + \sum_{m=3}^{s-1} z_{im} \frac{\partial \alpha_{i(m-1)}}{\partial \hat{g}_{i1}}\right) \Big(\tau_{gi1(s-2)} - \dot{\hat{g}}_{i1}\Big) \\ &\quad + \left(\tilde{\theta}_i + \sum_{m=2}^{s-1} z_{im} \frac{\partial \alpha_{i(m-1)}}{\partial \hat{\theta}_i}\right) \Big(\tau_{\theta i(s-1)} - \dot{\hat{\theta}}_i\Big) \end{aligned} \tag{12}$$

where the designed functions $\tau_{gi(s-2)1}, \ldots, \tau_{gi1(s-2)}$, $\tau_{\theta i(s-1)}$ have similar properties with $\bar{\alpha}_{i(s-1)}$.

Then, calculate the derivative of V_{is},

$$\begin{aligned} \dot{V}_{\text{is}} = z_{\text{is}} \Bigg(& g_s \big(z_{i(s+1)} + \alpha_{\text{is}}\big) + \theta L_i F_s - \sum_{m=1}^{s-1} \frac{\partial \alpha_{i(s-1)}}{\partial \hat{\delta}_{im}} \dot{\hat{\delta}}_{im} \\ & - \frac{\partial \alpha_{i(s-1)}}{\partial t} - \sum_{j=1}^{N} \sum_{m=1}^{s-1} \frac{\partial \alpha_{i(s-1)}}{\partial x_{jm}} \big(g_m x_{j(m+1)} + \theta f_m(\bar{x}_{jm})\big) \\ & - \frac{\partial \alpha_{i(s-1)}}{\partial \hat{g}_{i(s-2)}} \tau_{gi(s-2)2} - \cdots - \frac{\partial \alpha_{i(s-1)}}{\partial \hat{g}_{i1}} \tau_{gi1(s-1)} \\ & - \frac{\partial \alpha_{i(s-1)}}{\partial \hat{\theta}_i} \tau_{\theta is} \Bigg) + z_{\text{is}} \frac{\partial \alpha_{i(s-1)}}{\partial \hat{g}_{i(s-2)}} \Big(\tau_{gi(s-2)2} - \dot{\hat{g}}_{i(s-2)}\Big) \\ & + \cdots + z_{\text{is}} \frac{\partial \alpha_{i(s-1)}}{\partial \hat{g}_{i1}} \Big(\tau_{gi1(s-1)} - \dot{\hat{g}}_{i1}\Big) \\ & + z_{\text{is}} \frac{\partial \alpha_{i(s-1)}}{\partial \hat{\theta}_i} \Big(\tau_{\theta is} - \dot{\hat{\theta}}_i\Big) + g_s \tilde{\delta}_{\text{is}} \dot{\tilde{\delta}}_{\text{is}} \end{aligned} \tag{13}$$

where $\tau_{gi(s-2)2} = \tau_{gi(s-2)1} - z_{\text{is}} \sum_{j=1}^{N} ([\partial \alpha_{i(s-1)}]/[\partial x_{j(s-2)}]) x_{j(s-1)}, \ldots, \tau_{gi1(s-1)} = \tau_{gi1(s-2)} - z_{\text{is}} \sum_{j=1}^{N} ([\partial \alpha_{i(s-1)}]/\partial x_{j1}) x_{j2}$, $\tau_{\theta is} = \tau_{\theta i(s-1)} + z_{\text{is}}(L_i F_s - \sum_{j=1}^{N} \sum_{m=1}^{s-1} ([\partial \alpha_{i(s-1)}]/\partial x_{jm}) f_m(\bar{x}_{jm}))$.

Design the virtual controller α_{is} and adaptive law $\dot{\hat{\delta}}_{\text{is}}$ as

$$\alpha_{\text{is}} = -\hat{\delta}_{\text{is}} \bar{\alpha}_{\text{is}} \tag{14}$$

$$\dot{\hat{\delta}}_{\text{is}} = z_{\text{is}} \bar{\alpha}_{\text{is}} \tag{15}$$

where the designed smooth function $\bar{\alpha}_{\text{is}}$ is

$$\bar{\alpha}_{\text{is}} = \hat{\theta}_i L_i F_s - \sum_{m=1}^{s-1} \frac{\partial \alpha_{i(s-1)}}{\partial \hat{\delta}_{im}} \dot{\hat{\delta}}_{im} - \sum_{j=1}^{N} \sum_{m=1}^{s-1} \frac{\partial \alpha_{i(s-1)}}{\partial x_{jm}}$$

$$\times\left(\hat{g}_{im}x_{j(m+1)}+\hat{\theta}_i f_m(\bar{x}_{jm})\right)-\frac{\partial\alpha_{i(s-1)}}{\partial t}-\frac{\partial\alpha_{i(s-1)}}{\partial\hat{g}_{i(s-2)}}\times\tau_{gi(s-2)2}-\cdots-\frac{\partial\alpha_{i(s-1)}}{\partial\hat{g}_{i1}}\tau_{gi1(s-1)}-\frac{\partial\alpha_{i(s-1)}}{\partial\hat{\theta}_i}\tau_{\theta is}$$
$$-\left(z_{i(s-1)}\frac{\partial\alpha_{i(s-2)}}{\partial\hat{g}_{i(s-3)}}\right)\left(\sum_{j=1}^{N}\frac{\partial\alpha_{i(s-1)}}{\partial x_{j(s-3)}}x_{j(s-2)}\right)-\cdots$$
$$-\left(\sum_{m=3}^{s-1}z_{im}\frac{\partial\alpha_{i(m-1)}}{\partial\hat{g}_{i1}}\right)\left(\sum_{j=1}^{N}\frac{\partial\alpha_{i(s-1)}}{\partial x_{j1}}x_{j2}\right)$$
$$+\left(\sum_{m=2}^{s-1}z_{im}\times\frac{\partial\alpha_{i(m-1)}}{\partial\hat{\theta}_i}\right)\left(L_iF_s-\sum_{j=1}^{N}\sum_{m=1}^{s-1}\frac{\partial\alpha_{i(s-1)}}{\partial x_{jm}}f_m(\bar{x}_{jm})\right)\tag{16}$$

satisfying that under Assumptions 2 and 3 if $\bar{x}_{1s}\to\bar{x}_{0s},\ldots,\bar{x}_{Ns}\to\bar{x}_{0s}$ and $\hat{\theta}_i,\bar{\hat{\delta}}_{i(s-1)},\bar{\hat{g}}_{i(s-1)}$ are bounded, then $\bar{\alpha}_{\text{is}}\to 0$. $\bar{\alpha}_{\text{is}}$ only includes the states information of neighbors.

It follows from (12)–(16) that:

$$\sum_{m=1}^{s}\left(\dot{V}_{im}+\dot{V}_{gi(m-1)}\right)+\dot{V}_{\theta i}$$
$$=-\sum_{m=1}^{s}z_{im}^2+g_sz_{\text{is}}z_{i(s+1)}+\tilde{g}_{i(s-1)}\left(\tau_{gi(s-1)1}-\dot{\hat{g}}_{i(s-1)}\right)$$
$$+\left(\tilde{g}_{i(s-2)}+z_{\text{is}}\frac{\partial\alpha_{i(s-1)}}{\partial\hat{g}_{i(s-2)}}\right)\left(\tau_{gi(s-2)2}-\dot{\hat{g}}_{i(s-2)}\right)$$
$$\vdots$$
$$+\left(\tilde{g}_{i1}+\sum_{m=3}^{s}z_{im}\frac{\partial\alpha_{i(m-1)}}{\partial\hat{g}_{i1}}\right)\left(\tau_{gi1(s-1)}-\dot{\hat{g}}_{i1}\right)$$
$$+\left(\tilde{\theta}_i+\sum_{m=2}^{s}z_{im}\frac{\partial\alpha_{i(m-1)}}{\partial\hat{\theta}_i}\right)\left(\tau_{\theta is}-\dot{\hat{\theta}}_i\right)\tag{17}$$

where $\tau_{gi(s-1)1}=z_{i(s-1)}z_{\text{is}}-z_{\text{is}}\sum_{j=1}^{N}([\partial\alpha_{i(s-1)}]/\partial x_{j(s-1)})x_{js}$.

Step n: Through the above recursive design method, if we design u_i and the other adaptive laws as follows:

$$u_i=\frac{1}{\sum_{j=1}^{N}\bar{a}_{ij}+b_i}\left(\sum_{j=1}^{N}\bar{a}_{ij}u_j+b_iu_0-\hat{\delta}_{\text{in}}\bar{\alpha}_{\text{in}}\right)\tag{18}$$
$$\dot{\hat{\delta}}_{\text{in}}=z_{\text{in}}\bar{\alpha}_{\text{in}}\tag{19}$$
$$\dot{\hat{\theta}}_i=\tau_{\theta\text{in}}\tag{20}$$
$$\dot{\hat{g}}_{i1}=\tau_{gi1(n-1)}\tag{21}$$
$$\vdots$$
$$\dot{\hat{g}}_{i(n-1)}=\tau_{gi(n-1)1}\tag{22}$$

where $\bar{\alpha}_{\text{in}},\tau_{\theta\text{in}},\tau_{gi1(n-1)},\ldots,\tau_{gi(n-1)1}$ can be easily obtained from *step s* and their details are not given any more, then, combined with (17), we have

$$\sum_{i=1}^{N}\sum_{m=1}^{n}\left(\dot{V}_{im}+\dot{V}_{gi(m-1)}\right)+\dot{V}_{\theta i}=-\sum_{i=1}^{N}\sum_{m=1}^{n}z_{im}^2.\tag{23}$$

Now, we give the first main result of our paper.

Theorem 1: For the high-order time-varying nonlinear MASs (1) satisfying Assumptions 1–3, the distributed adaptive state feedback controller (18) with the adaptive laws (7), (10), (15) and (19)–(22), can render all the tracking errors $(\bar{x}_{\text{in}}-\bar{x}_{0n})$ to converge to zero asymptotically.

Proof: Due to that (L_r+B) is the nonsingular matrix, combined with Assumption 2, we know that if for $i=1,\ldots,N$, $s=1,2,\ldots,n$, ξ_{is} are bounded, then x_{is} are bounded. Combined with (3), (23), and Assumptions and 3, we obtain that the states x_{is} and $\hat{\theta}_i$, $\bar{\hat{\delta}}_{\text{in}}$, $\bar{\hat{g}}_{i(n-1)}$ are bounded. Hence, $\dot{z}_{\text{is}}$ and z_{is} are bounded. Then, $\sum_{i=1}^{N}\sum_{m=1}^{n}z_{im}^2$ is uniformly continuous. Due to that $\int_{t_0}^{t}(\sum_{i=1}^{N}\sum_{m=1}^{n}z_{im}^2(\tau))d\tau=-\int_{t_0}^{t}\dot{V}(\tau)d\tau\le V(t_0)$ is finite, based on the Lemma 1, $z_{\text{is}}\to 0$ as $t\to\infty$. Combined with (3), Assumption 1 and the properties of the virtual controllers α_{is}, we have $(\bar{x}_{\text{in}}-\bar{x}_{0n})\to 0$ as $t\to\infty$. The proof is completed.

Based on the above controller design method and stability analysis, the following corollaries can be obtained.

Corollary 1: If $f_s(t,\bar{x}_{\text{is}})$ is time invariant, such as $f_s(t,\bar{x}_{\text{is}})=f_s(0,\bar{x}_{\text{is}})$, for the high-order nonlinear MASs (1) satisfying Assumptions 1 and 2, the distributed adaptive state feedback controller (18) can render all the tracking errors $(\bar{x}_{\text{in}}-\bar{x}_{0n})$ to converge to zero asymptotically.

Corollary 2: For the high-order time-varying nonlinear MASs (1) only satisfying Assumption 1, the distributed adaptive state feedback controller (18) can render the output tracking error $(x_{i1}-x_{01})$ to converge to zero asymptotically.

Remark 3: In this section, the distributed adaptive state feedback consensus protocol is proposed for the time-varying nonlinear MASs (1) with unknown parameters. The conditions on the nonlinear functions in the MASs are very general, which makes the proposed protocol be applicable for a wider class of MASs. The unknown parameters are compensated using an adaptive method. Different from [14] and [16]–[20], the full states consensus can be achieved. This method can also be extended to stochastic MASs.

B. Distributed Output Feedback Leader-Following Control

In this section, the distributed adaptive output feedback leader-following control method is proposed based on the refreshed graph G_r. Compared to the state feedback method in the previous section, the reduced order dynamic gain k-filter is further constructed to estimate the unmeasured states. The following transformation and assumption are given first.

For the MASs (1), letting $X_{\text{is}}=\Pi_{m=1}^{s}g_{m-1}x_{\text{is}}$ and $g_0:=1$ for $i=0,1,\ldots,N$, $s=1,2,\ldots,n$, gives

$$\begin{cases}\dot{X}_{\text{is}}=X_{i(s+1)}+\Pi_{m=1}^{s}g_{m-1}\theta f_s(t,\bar{x}_{\text{is}})\\ y_i=x_{i1}=X_{i1}\end{cases}\tag{24}$$

where $X_{i(n+1)}:=\rho u_i$, $\rho=\Pi_{m=1}^{n}g_m$ and only the output information y_i can be measured. Obviously, $(\bar{x}_{\text{in}}-\bar{x}_{0n})\to 0$ as $t\to\infty$ is equivalent to $(\bar{X}_{\text{in}}-\bar{X}_{0n})\to 0$ as $t\to\infty$. In this section, the following assumption is necessary.

Assumption 4: There exist unknown parameters σ_s, known smooth functions $\varphi_{0s}(t,y_i)$, $\varphi_{1s}(t,y_i)$ and known bounded time-varying vector function $\varphi_{2s}(t)$ such that

$$\Pi_{m=1}^{s}g_{m-1}\theta f_s(t,\bar{x}_{\text{is}})=\varphi_{0s}(t,y_i)+\sigma_s\varphi_{1s}(t,y_i)+\varphi_{2s}^{T}(t)X_i\tag{25}$$

where $\varphi_{21} = 0$, $\varphi_{2s}(t) = [\varphi_{2s2}(t), \dots, \varphi_{2ss}(t), 0, \dots, 0]^T \in \mathcal{R}^{n-1}$ and $X_i := [X_{i2}, \dots, X_{\text{in}}]^T$. $\varphi_{0s}(t, y_i)$, $\varphi_{1s}(t, y_i)$ and all of their first derivative, ..., $(n-s+1)$th derivative are bounded, uniformly in t, for all $t \geq t_0$ and $\|y_i\| \leq \bar{\Phi}$, where $\bar{\Phi}$ is a positive constant.

Remark 4: If the nonlinearities in (1) only depend on the output, then (1) can be directly transformed to (24) satisfying Assumption 4, such as the single-link robot system in [35] and [36]. The nonlinear function $\Pi_{m=1}^{s} g_{m-1}\theta f_s(t, \bar{x}_{\text{is}})$ under Assumption 4 containing more states information (i.e., $\varphi_{2s}^T(t)X_i$) is extremely different from the ones in [24] and [28]–[33], such as the parallel active suspension system, referring to [29]. Further detailed introductions can refer to the number example and Remark 7 in Section IV.

1) Reduced Order Dynamic Gain K-Filter Design: Construct the following reduced order dynamic gain k-filter to estimate the ith agent's states:

$$\begin{cases} \zeta_i = \lambda_i + lL_0 q y_i \\ \dot{\lambda}_i = (A - lL_0 q c^T)\zeta_i + \varphi_0(t, y_i) - \dot{l}L_0 q y_i \\ \quad - \dot{l}DL_0 q y_i - lL_0 q \varphi_{01}(t, y_i) + \varphi_2(t)\zeta_i \\ \dot{\Xi}_{i1} = (A - lL_0 q c^T)\Xi_{i1} - lL_0 q \varphi_{11}(t, y_i) \\ \quad + \varphi_2(t)\Xi_{i1} \\ \dot{\Xi}_{im} = (A - lL_0 q c^T)\Xi_{im} + \bar{\varphi}_{1m}(t, y_i) \\ \quad + \varphi_2(t)\Xi_{im}, 2 \leq m \leq n \\ \dot{\nu}_i = (A - lL_0 q c^T)\nu_i + Eu_i + \varphi_2(t)\nu_i \end{cases} \tag{26}$$

where ζ_i, λ_i, Ξ_{i1}, Ξ_{im}, $\nu_i \in \mathcal{R}^{n-1}$. To unify the form of the all introduced variables, define $\nu_i = [\nu_{i2}, \dots, \nu_{\text{in}}]^T$, $\zeta_i = [\zeta_{i2}, \dots, \zeta_{\text{in}}]^T$, $\lambda_i = [\lambda_{i2}, \dots, \lambda_{\text{in}}]^T$, and $\Xi_{i1} = [\Xi_{i12}, \dots, \Xi_{i1n}]^T, \dots, \Xi_{\text{in}} = [\Xi_{\text{in}2}, \dots, \Xi_{inn}]^T$. $l \geq 1$ is the dynamic gain to be designed as (31). $L_0 = \text{diag}\{1, l, \dots, l^{n-2}\}$. $\varphi_0(t, y_i) = [\varphi_{02}(t, y_i), \dots, \varphi_{0n}(t, y_i)]^T \in \mathcal{R}^{n-1}$. $\bar{\varphi}_{1m}(t, y_i) = [0, \dots, 0, \varphi_{1m}(t, y_i), 0, \dots, 0]^T \in \mathcal{R}^{n-1}$, where $\varphi_{1m}(t, y_i)$ is the $(m-1)$th entry of the vector. $E = [0, \dots, 0, 1]^T \in \mathcal{R}^{n-1}$. $D = \text{diag}\{0, 1, \dots, n-2\}$. $c = [1, 0, \dots, 0]^T \in \mathcal{R}^{n-1}$. $q = [q_2, \dots, q_n]^T$ is designed such that $(A - qc^T)$ is Hurwitz. $A \in R^{(n-1)\times(n-1)}$ and $\varphi_2 \in R^{(n-1)\times(n-1)}$ are as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\varphi_2 = \begin{bmatrix} \varphi_{222} & & \\ \vdots & \ddots & \\ \varphi_{2n2} & \cdots & \varphi_{2nn} \end{bmatrix}.$$

Then, the state estimate is formed as $\hat{X}_i = \zeta_i + \sum_{m=1}^{n} \sigma_m \Xi_{im} + \rho\nu_i$. Defining $e_i = X_i - \hat{X}_i = [e_{i2}, \dots, e_{\text{in}}]^T$, one has

$$\dot{e}_i = \left(A - lL_0 q c^T\right)e_i + \varphi_2(t)e_i. \tag{27}$$

Defining $\epsilon_i = l^{-\mu}L_0^{-1}e_i$ with μ a positive design parameter, we have

$$\begin{aligned} \dot{\epsilon}_i &= l^{-\mu}L_0^{-1}\left(A - lL_0 q c^T\right)e_i + l^{-\mu}L_0^{-1}\varphi_2(t)e_i \\ &\quad - \frac{\dot{l}}{l}(\mu I + D)l^{-\mu}L_0^{-1}e_i \\ &= l\left(A - qc^T\right)\epsilon_i + L_0^{-1}\varphi_2 L_0\epsilon_i - \frac{\dot{l}(\mu I + D)\epsilon_i}{l}. \end{aligned} \tag{28}$$

Choosing the Lyapunov function $V_{ei} = \epsilon_i^T P \epsilon_i$, where P is the positive matrix to be specified later, we have

$$\begin{aligned} \dot{V}_{ei} &\leq -l\epsilon_i^T\left(P\left(A - qc^T\right) + \left(A - qc^T\right)^T P\right)\epsilon_i \\ &\quad - \frac{\dot{l}}{l}\epsilon_i^T(2\mu P + PD + DP)\epsilon_i \\ &\quad + 2\epsilon_i^T P L_0^{-1}\varphi_2(t)L_0\epsilon_i. \end{aligned} \tag{29}$$

Design q and P such that $P(A - qc^T) + (A - qc^T)^T P \leq -I$. Due to $l \geq 1$, there exists the positive bounded time-varying function $\varphi(t)$ such that

$$2\epsilon_i^T P L_0^{-1}\varphi_2(t)L_0\epsilon_i \leq \varphi(t)\epsilon_i^T\epsilon_i. \tag{30}$$

Then, design $\dot{l}$ as

$$\dot{l} = l(-\eta l + \eta + \bar{\varphi}(t)),\ l(0) \geq 1. \tag{31}$$

where $\eta > 0$ is a constant, and $\bar{\varphi}(t) \geq 0$ is smooth bounded function to be specified later. Since P is positive, there exist positive constants μ_1, μ_2 and large enough μ such that

$$\mu_1 I \leq 2\mu P + PD + DP \leq \mu_2 I. \tag{32}$$

From (31) and (32), we have

$$\begin{aligned} &-\frac{\dot{l}}{l}\epsilon_i^T(2\mu P + PD + DP)\epsilon_i \\ &\leq \mu_2\eta l\epsilon_i^T\epsilon_i - \mu_1(\eta + \bar{\varphi}(t))\epsilon_i^T\epsilon_i. \end{aligned} \tag{33}$$

From (29)–(33), we have

$$\dot{V}_{ei} \leq -(l(1 - \mu_2\eta) + \mu_1(\eta + \bar{\varphi}(t)) - \varphi(t))\epsilon_i^T\epsilon_i. \tag{34}$$

Choose η and $\bar{\varphi}(t)$ such that $(1 - \mu_2\eta) \geq (1/2)$, $\mu_1(\eta + \bar{\varphi}(t)) - \varphi(t) \geq 0$ and $\bar{\varphi}(t) \geq 0$, then

$$\dot{V}_{ei} \leq -\frac{1}{2}l\epsilon_i^T\epsilon_i. \tag{35}$$

Since $\bar{\varphi}(t)$ is bounded, l is bounded. From (35), e_i converges to zero as $t \to \infty$. For the reduced order dynamic gain k-filter (26), we have the following lemma.

Lemma 3: The dynamic systems λ_i, ζ_i, $\Xi_{i1}, \dots, \Xi_{\text{in}}$ in (26) with input y_i and states $(\Xi_{i1}, \dots, \Xi_{\text{in}}, \lambda_i, \zeta_i)$ are input-to-state stable, and if y_i converges to zero asymptotically, then the states $(\Xi_{i1}, \dots, \Xi_{\text{in}}, \lambda_i, \zeta_i)$ converge to zero asymptotically.

Proof: From (26), we have

$$\begin{aligned} \dot{\lambda}_i &= \left(A - lL_0 q c^T\right)\lambda_i + \varphi_2(t)\lambda_i + \Psi_i\left(\dot{l}, l\right)y_i \\ &\quad + \varphi_0(t, y_i) - lL_0 q\varphi_{01}(t, y_i) \end{aligned} \tag{36}$$

where $\Psi_i(\dot{l}, l) = -\dot{l}DL_0 q - \dot{l}L_0 q + (A - lL_0 q c^T)lL_0 q + \varphi_2(t)lL_0 q$. Similar with (27)–(35), and from (36) the derivative of $V_{i\lambda} = (l^{-\mu}L_0^{-1}\lambda_i)^T P(l^{-\mu}L_0^{-1}\lambda_i)$ is

$$\begin{aligned} \dot{V}_{i\lambda} &\leq -\frac{1}{4}l\left\|l^{-\mu}L_0^{-1}\lambda_i\right\|^2 + 4\left\|\left(l^{\mu}L_0\right)P\right\|^2 \\ &\quad \times \left\|\Psi_i\left(\dot{l}, l\right)y_i + \varphi_0(t, y_i) - lL_0 q\varphi_{01}(t, y_i)\right\|^2. \end{aligned} \tag{37}$$

Due to that l and $\dot{l}$ are bounded, (36) is input-to-state stable with the input y_i and state λ_i, referring to [34, Lemma 4.7]. If

y_i converges to zero asymptotically, then λ_i converge to zero asymptotically. So is $(\Xi_{i1}, \ldots, \Xi_{\text{in}}, \zeta_i)$.

Remark 5: The constructed reduced-order k-filter can be easily extended to interconnected nonlinear systems to improve the results studied in existing works, such as [24] and [30]–[33], which will further reduce the number of introduced dynamic variables and allow the nonlinearities to be time varying. In addition, based on the analysis about Lemma 3, we can easily obtain that if $(y_i - y_0) \to 0$, then $(\Xi_{i1} - \Xi_{01}) \to 0, \ldots, (\Xi_{\text{in}} - \Xi_{01}) \to 0$, $(\lambda_i - \lambda_0) \to 0$, $(\zeta_i - \zeta_0) \to 0$.

2) Distributed Output Feedback Controller Design: In this section, based on the constructed k-filter, the distributed adaptive output feedback controller is designed by the backstepping method and tuning function technique. First, for $s = 2, \ldots, n$, define the local neighborhood consensus error of ith agent as

$$\begin{aligned} \tilde{\xi}_{i1} &= \sum_{j=1}^{N} \bar{a}_{ij}\big(x_{i1} - x_{j1}\big) + b_i(x_{i1} - x_{01}), \\ \tilde{\xi}_{\text{is}} &= \sum_{j=1}^{N} \bar{a}_{ij}\big(\nu_{\text{is}} - \nu_{js}\big) + b_i(\nu_{\text{is}} - \nu_{0s}). \end{aligned} \tag{38}$$

Choose the following transformation:

$$\tilde{z}_{i1} = \tilde{\xi}_{i1}, \;\; \tilde{z}_{\text{is}} = \tilde{\xi}_{\text{is}} - \tilde{\alpha}_{i(s-1)} \tag{39}$$

where $\tilde{\alpha}_{i(s-1)}$ is the designed virtual controller.

Choose the Lyapunov function as follows:

$$\begin{cases} \tilde{V} = \sum_{i=1}^{N}\sum_{s=1}^{n}(\tilde{V}_{\text{is}} + V_{\sigma is}) \\ \quad + \, (2\gamma+1)\sum_{i=0}^{N} V_{ei} + \sum_{i=1}^{N}(V_{\rho i} + \rho V_{\kappa i}) \\ \tilde{V}_{\text{is}} = \frac{1}{2}\tilde{z}_{\text{is}}^2, V_{\sigma is} = \frac{1}{2}\tilde{\sigma}_{\text{is}}^2, V_{\rho i} = \frac{1}{2}\tilde{\rho}_i^2, V_{\kappa i} = \frac{1}{2}\tilde{\kappa}_i^2 \end{cases} \tag{40}$$

where γ is a positive constant. $\tilde{\sigma}_{\text{is}} = \sigma_s - \hat{\sigma}_{\text{is}}$, $\tilde{\rho}_i = \rho - \hat{\rho}_i$, $\tilde{\kappa}_i = \rho^{-1} - \hat{\kappa}_i$, and $\hat{\sigma}_{\text{is}}$, $\hat{\rho}_i$, $\hat{\kappa}_i$ are the estimations of σ_s, ρ, ρ^{-1} in the ith agent.

Step 1: From (38), (39), $e_i = X_i - \hat{X}_i$ and $\hat{X}_i = \zeta_i + \sum_{m=1}^{n}\sigma_m \Xi_{im} + \rho\nu_i$, the derivative of V_{i1} is

$$\begin{aligned} \dot{\tilde{V}}_{i1} &= \tilde{z}_{i1}\dot{\tilde{z}}_{i1} = \rho\tilde{z}_{i1}(\tilde{z}_{i2} + \tilde{\alpha}_{i1}) \\ &\quad + \tilde{z}_{i1}\sum_{j=1}^{N}\bar{a}_{ij}\left(\begin{pmatrix} \zeta_{i2} + \sum_{m=1}^{n}\sigma_m\Xi_{im2} + e_{i2} \\ +\varphi_{01}(t,y_i) + \sigma_1\varphi_{11}(t,y_i) \end{pmatrix}\right. \\ &\qquad\qquad \left. - \begin{pmatrix} \zeta_{j2} + \sum_{m=1}^{n}\sigma_m\Xi_{jm2} + e_{j2} \\ +\varphi_{01}\big(t,y_j\big) + \sigma_1\varphi_{11}\big(t,y_j\big) \end{pmatrix}\right) \\ &\quad + \tilde{z}_{i1}b_i\left(\begin{pmatrix} \zeta_{i2} + \sum_{m=1}^{n}\sigma_m\Xi_{im2} + e_{i2} \\ +\varphi_{01}(t,y_i) + \sigma_1\varphi_{11}(t,y_i) \end{pmatrix}\right. \\ &\qquad\qquad \left. - \begin{pmatrix} \zeta_{02} + \sum_{m=1}^{n}\sigma_m\Xi_{0m2} + e_{02} \\ +\varphi_{01}(t,y_0) + \sigma_1\varphi_{11}(t,y_0) \end{pmatrix}\right). \end{aligned} \tag{41}$$

Using the Young's inequality, there exists smooth positive function β_{i1} such that

$$\begin{aligned} &\tilde{z}_{i1}\left(\sum_{j=1}^{N}\bar{a}_{ij}\big(e_{i2} - e_{j2}\big) + b_i(e_{i2} - e_{02})\right) \\ &\quad \le \beta_{i1}\tilde{z}_{i1}^2 + \sum_{j=0}^{N}\frac{\gamma l\epsilon_{j2}^2}{(N+1)n}. \end{aligned} \tag{42}$$

Design the virtual controller $\tilde{\alpha}_{i1}$ and adaptive law $\dot{\hat{\kappa}}_i$ as

$$\tilde{\alpha}_{i1} = -\hat{\kappa}_i\bar{\tilde{\alpha}}_{i1}, \tag{43}$$

$$\dot{\hat{\kappa}}_i = \tilde{z}_{i1}\bar{\tilde{\alpha}}_{i1} \tag{44}$$

where

$$\begin{aligned} \bar{\tilde{\alpha}}_{i1} &= \sum_{j=1}^{N}\bar{a}_{ij}\big(\zeta_{i2} - \zeta_{j2}\big) + b_i(\zeta_{i2} - \zeta_{02}) \\ &\quad + \sum_{j=1}^{N}\bar{a}_{ij}\big(\varphi_{01}(t,y_i) - \varphi_{01}\big(t,y_j\big)\big) + b_i\varphi_{01}(t,y_i) \\ &\quad - b_i\varphi_{01}(t,y_0) + \hat{\sigma}_{i1}\sum_{j=1}^{N}\bar{a}_{ij}\big(\varphi_{11}(t,y_i) - \varphi_{11}\big(t,y_j\big)\big) \\ &\quad + \hat{\sigma}_{i1}b_i(\varphi_{11}(t,y_i) - \varphi_{11}(t,y_0)) + \sum_{m=1}^{n} b_i\hat{\sigma}_{im}(\Xi_{im2} - \Xi_{0m2}) \\ &\quad + \sum_{j=1}^{N}\sum_{m=1}^{n} a_{ij}\hat{\sigma}_{im}\big(\Xi_{im2} - \Xi_{jm2}\big) + \beta_{i1}\tilde{z}_{i1} + \tilde{z}_{i1} \end{aligned}$$

which only includes the information of neighbors. Using Assumptions 2 and 4, $\bar{\tilde{\alpha}}_{i1}$ satisfies that if $x_{11} \to x_{01}, \ldots, x_{N1} \to x_{01}$, $\zeta_1 \to \zeta_0, \ldots, \zeta_N \to \zeta_0$, $\Xi_{12} \to \Xi_{02}, \ldots, \Xi_{Nn} \to \Xi_{0n}$ and $\hat{\sigma}_{i2}, \ldots, \hat{\sigma}_{\text{in}}$ are bounded, then $\bar{\tilde{\alpha}}_{i1} \to 0$. $\dot{\hat{\sigma}}_{i1}$ and $\dot{\hat{\sigma}}_{i2}$ are designed later. The structure of $\bar{\tilde{\alpha}}_{i1}$ is similar with $\bar{\alpha}_{i1}$ in the previous section.

It follows from (41), (44) that:

$$\begin{aligned} &\dot{\tilde{V}}_{i1} + \rho\dot{V}_{\kappa i} + \dot{V}_{\sigma i1} + \cdots + \dot{V}_{\sigma\text{in}} \\ &\quad \le -\tilde{z}_{i1}^2 + \rho\tilde{z}_{i1}\tilde{z}_{i2} + \sum_{j=0}^{N}\frac{\gamma l\epsilon_{j2}^2}{(N+1)n} + \tilde{\sigma}_{i1}\Big(\tau_{\sigma i11} - \dot{\hat{\sigma}}_{i1}\Big) \\ &\qquad + \cdots + \tilde{\sigma}_{\text{in}}\Big(\tau_{\sigma in1} - \dot{\hat{\sigma}}_{\text{in}}\Big) \end{aligned} \tag{45}$$

where $\tau_{\sigma i11} = \tilde{z}_{i1}b_i(\varphi_{11}(t,y_i) + \Xi_{i12} - \varphi_{11}(t,y_0) - \Xi_{j02}) + \tilde{z}_{i1}\sum_{j=1}^{N}\bar{a}_{ij}(\varphi_{11}(t,y_i) + \Xi_{i12} - \varphi_{11}(t,y_j) - \Xi_{j12})$, $\tau_{\sigma i21} = \tilde{z}_{i1}\sum_{j=1}^{N}\bar{a}_{ij}(\Xi_{i22} - \Xi_{j22}) + \tilde{z}_{i1}b_i(\Xi_{i22} - \Xi_{022}), \ldots, \tau_{\sigma in1} = \tilde{z}_{i1}\sum_{j=1}^{N}\bar{a}_{ij}(\Xi_{\text{in}2} - \Xi_{jn2}) + \tilde{z}_{i1}b_i(\Xi_{\text{in}2} - \Xi_{0n2})$, which have the similar properties with $\bar{\tilde{\alpha}}_{i1}$.

Step 2: The derivative of $\tilde{V}_{i2}$ is

$$\begin{aligned} \dot{\tilde{V}}_{i2} &= \tilde{z}_{i2}\dot{\tilde{z}}_{i2} = \tilde{z}_{i2}\Big(\dot{\tilde{\xi}}_{i2} - \dot{\tilde{\alpha}}_{i1}\Big) \\ &= \tilde{z}_{i2}\left(\tilde{\xi}_{i3} - \sum_{j=0}^{N}\frac{\partial\tilde{\alpha}_{i1}}{\partial x_{j1}}\left(\sum_{m=1}^{n}\sigma_m\Xi_{jm2} + \sigma_1\varphi_{11}\big(t,y_j\big) + \rho\nu_{j2} + e_{j2}\right)\right. \\ &\qquad - \sum_{m=1}^{n}\frac{\partial\tilde{\alpha}_{i1}}{\partial\hat{\sigma}_{im}}\dot{\hat{\sigma}}_{im} + \Psi_{i2} - \sum_{j=0}^{N}\frac{\partial\tilde{\alpha}_{i1}}{\partial\zeta_{j2}}lq_2 \\ &\qquad \left. \times\left(\sum_{m=1}^{n}\sigma_m\Xi_{jm2} + \sigma_1\varphi_{11}\big(t,y_j\big) + \rho\nu_{j2} + e_{j2}\right)\right) \end{aligned} \tag{46}$$

where Ψ_{i2} denotes all the other known terms and satisfies that if $x_{11} \to x_{01}, \ldots, x_{N1} \to x_{01}$, $\zeta_1 \to \zeta_0, \ldots, \zeta_N \to \zeta_0$, $\Xi_{12} \to \Xi_{02}, \ldots, \Xi_{Nn} \to \Xi_{0n}$ and $\hat{\kappa}_i$, $\hat{\sigma}_{i2}, \ldots, \hat{\sigma}_{\text{in}}$ are bounded, then $\Psi_{i2} \to 0$.

Using Young's inequality, there exists smooth function β_{i2} such that $-\tilde{z}_{i2}(\sum_{j=0}^{N}(\partial\tilde{\alpha}_{i1}/\partial x_{j1})e_{j2} +$

$\sum_{j=0}^{N}(\partial\tilde{\alpha}_{i1}/\partial\zeta_{j2})lq_2e_{j2}) \leq \beta_{i2}\tilde{z}_{i2}^2 + \sum_{j=0}^{N}(\gamma l\epsilon_{j2}^2/((N+1)n))$. Design the virtual controller $\tilde{\alpha}_{i2}$ as

$$\begin{aligned}\tilde{\alpha}_{i2} = -\Bigg\{&-\left(\sum_{j=0}^{N}\frac{\partial\tilde{\alpha}_{i1}}{\partial x_{j1}} + \sum_{j=0}^{N}\frac{\partial\tilde{\alpha}_{i1}}{\partial\zeta_{j2}}lq_2\right)\\ &\times\left(\sum_{m=1}^{n}\hat{\sigma}_{im}\Xi_{jm2} + \hat{\sigma}_{i1}\varphi_{11}\big(t,y_j\big) + \hat{\rho}_i\nu_{j2}\right)\\ &+\hat{\rho}_i\tilde{z}_{i1} + \beta_{i2}\tilde{z}_{i2} + \tilde{z}_{i2} + \Psi_{i2}\\ &+\frac{\partial\tilde{\alpha}_{i1}}{\partial\hat{\sigma}_{i1}}\tau_{\sigma i12} + \cdots + \frac{\partial\tilde{\alpha}_{i1}}{\partial\hat{\sigma}_{\mathrm{in}}}\tau_{\sigma\mathrm{in}2}\Bigg\} \qquad (47)\end{aligned}$$

where $\tau_{\sigma i12} = \tau_{\sigma i11} - \tilde{z}_{i2}\sum_{j=0}^{N}(\partial\tilde{\alpha}_{i1}/\partial x_{j1})(\Xi_{j12} + \varphi_{11}(t,y_j)) - \tilde{z}_{i2}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/[\partial\zeta_{j2}])lq_2(\Xi_{j12} + \varphi_{11}(t,y_j))$, $\tau_{\sigma i22} = \tau_{\sigma i21} - \tilde{z}_{i2}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/\partial x_{j1})\Xi_{j22} - \tilde{z}_{i2}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/[\partial\zeta_{j2}])lq_2\Xi_{j22}, \ldots, \tau_{\sigma\mathrm{in}2} = \tau_{\sigma in1} - \tilde{z}_{i2}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/\partial x_{j1})\Xi_{jn2} - \tilde{z}_{i2}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/\partial\zeta_{j2})lq_2\Xi_{jn2}$. $\tilde{\alpha}_{i2}$ satisfies that if $x_{11} \to x_{01}, \ldots, x_{N1} \to x_{01}$, $\zeta_1 \to \zeta_0, \ldots, \zeta_N \to \zeta_0$, $\Xi_{12} \to \Xi_{02}, \ldots, \Xi_{Nn} \to \Xi_{0n}$, $\nu_{12} \to \nu_{02}, \ldots, \nu_{N2} \to \nu_{02}$, and $\hat{\kappa}_i$, $\hat{\sigma}_{i2}, \ldots, \hat{\sigma}_{\mathrm{in}}$ are bounded, then $\tilde{\alpha}_{i2} \to 0$.

It follows from (46), (47) that:

$$\begin{aligned}&\dot{\tilde{V}}_{i1} + \dot{\tilde{V}}_{i2} + \rho\dot{V}_{\kappa i} + \dot{V}_{\rho i} + \dot{V}_{\sigma i1} + \cdots + \dot{V}_{\sigma\mathrm{in}}\\ &\leq -\tilde{z}_{i1}^2 - \tilde{z}_{i2}^2 + \tilde{z}_{i2}\tilde{z}_{i3} + \sum_{j=0}^{N}\frac{2\gamma l\epsilon_{j2}^2}{(N+1)n}\\ &\quad+\tilde{\rho}_i\Big(\tau_{\rho i1} - \dot{\hat{\rho}}_i\Big) + \left(\tilde{\sigma}_{i1} + \tilde{z}_{i2}\frac{\partial\tilde{\alpha}_{i1}}{\partial\hat{\sigma}_{i1}}\right)\Big(\tau_{\sigma i12} - \dot{\hat{\sigma}}_{i1}\Big)\\ &\quad+\cdots + \left(\tilde{\sigma}_{\mathrm{in}} + \tilde{z}_{i2}\frac{\partial\tilde{\alpha}_{i1}}{\partial\hat{\sigma}_{\mathrm{in}}}\right)\Big(\tau_{\sigma\mathrm{in}2} - \dot{\hat{\sigma}}_{\mathrm{in}}\Big) \qquad (48)\end{aligned}$$

where $\tau_{\rho i1} = \tilde{z}_{i2}(\tilde{z}_{i1} - \sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/\partial x_{j1})\nu_{i2} - \sum_{j=0}^{N}([\partial\tilde{\alpha}_{i1}]/\partial\zeta_{j2})lq_2\nu_{j2})$.

Step ($3 \leq s \leq n-1$): Similarly, design the virtual controller $\tilde{\alpha}_{\mathrm{is}}$ as follows:

$$\begin{aligned}\tilde{\alpha}_{\mathrm{is}} = -\Bigg(&\tilde{z}_{i(s-1)} + \beta_{\mathrm{is}}\tilde{z}_{\mathrm{is}} + \tilde{z}_{\mathrm{is}} + \Psi_{\mathrm{is}} + \frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\hat{\rho}_i}\tau_{\rho i(s-1)}\\ &-\left(\sum_{j=0}^{N}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial x_{j1}} + \sum_{j=0}^{N}\sum_{\varrho=2}^{s}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\zeta_{j\varrho}}l^{\varrho-1}q_\varrho\right)\\ &\times\left(\hat{\rho}_i\nu_{j2} + \sum_{m=1}^{n}\hat{\sigma}_{im}\Xi_{jm2} + \hat{\sigma}_{i1}\varphi_{11}\big(t,y_j\big)\right) + \frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\hat{\sigma}_{i1}}\tau_{\sigma i1s}\\ &+\cdots + \frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\hat{\sigma}_{\mathrm{in}}}\tau_{\sigma ins} - \left(\sum_{m=3}^{s-1}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\rho}_i}\right)\sum_{j=0}^{N}\nu_{j2}\\ &\times\left(\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial x_{j1}} + \sum_{\varrho=2}^{s}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\zeta_{j\varrho}}\frac{q_\varrho}{l^{1-\varrho}}\right) - \left(\sum_{m=2}^{s-1}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\sigma}_{i1}}\right)\\ &\times\sum_{j=0}^{N}\left(\left(\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial x_{j1}} + \sum_{\varrho=2}^{s}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\zeta_{j\varrho}}l^{\varrho-1}q_\varrho\right)\big(\Xi_{j12} + \varphi_{11}\big(t,y_j\big)\big)\right)\\ &-\left(\sum_{m=2}^{s-1}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\sigma}_{i2}}\right)\sum_{j=0}^{N}\Xi_{j22}\left(\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial x_{j1}} + \sum_{\varrho=2}^{s}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\zeta_{j\varrho}}\frac{q_\varrho}{l^{1-\varrho}}\right)\\ &-\cdots - \left(\sum_{m=2}^{s-1}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\sigma}_{\mathrm{in}}}\right)\\ &\times\sum_{j=0}^{N}\Xi_{jn2}\left(\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial x_{j1}} + \sum_{\varrho=2}^{s}\frac{\partial\tilde{\alpha}_{i(s-1)}}{\partial\zeta_{j\varrho}}\frac{q_\varrho}{l^{1-\varrho}}\right)\Bigg) \qquad (49)\end{aligned}$$

where the smooth functions $\tau_{\rho i(s-1)} = \tau_{\rho i(s-2)} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i(s-1)}]/\partial x_{j1})\nu_{j2} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}\sum_{\varrho=2}^{s}([\partial\tilde{\alpha}_{i(s-1)}]/\partial\zeta_{j\varrho})l^{\varrho-1}q_\varrho\nu_{j2}$, $\tau_{\sigma i1s} = \tau_{\sigma i1(s-1)} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i(s-1)}]/\partial x_{j1})(\Xi_{j12} + \varphi_{11}(t,y_j)) - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}\sum_{\varrho=2}^{s}([\partial\tilde{\alpha}_{i(s-1)}]/\partial\zeta_{j\varrho})l^{\varrho-1}q_\varrho(\Xi_{j12} + \varphi_{11}(t,y_j))$, and the functions $\tau_{\sigma i2s} = \tau_{\sigma i2(s-1)} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i(s-1)}]/\partial x_{j1})\Xi_{j22} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}\sum_{\varrho=2}^{s}([\partial\tilde{\alpha}_{i(s-1)}]/\partial\zeta_{j\varrho})l^{\varrho-1}q_\varrho\Xi_{j22}, \ldots, \tau_{\sigma ins} = \tau_{\sigma\mathrm{in}(s-1)} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}([\partial\tilde{\alpha}_{i(s-1)}]/\partial x_{j1})\Xi_{jn2} - \tilde{z}_{\mathrm{is}}\sum_{j=0}^{N}\sum_{\varrho=2}^{s}([\partial\tilde{\alpha}_{i(s-1)}]/\partial\zeta_{j\varrho})l^{\varrho-1}q_\varrho\Xi_{jn2}$. Ψ_{is} denotes all the other known terms. $\tilde{\alpha}_{\mathrm{is}}$ and Ψ_{is} satisfy that if $x_{11} \to x_{01}, \ldots, x_{N1} \to x_{01}$, $\zeta_1 \to \zeta_0, \ldots, \zeta_N \to \zeta_0$, $\Xi_{12} \to \Xi_{02}, \ldots, \Xi_{Nn} \to \Xi_{0n}$, $\bar{\nu}_{1s} \to \bar{\nu}_{0s}, \ldots, \bar{\nu}_{Ns} \to \bar{\nu}_{0s}$, and $\hat{\kappa}_i$, $\hat{\sigma}_{i2}, \ldots, \hat{\sigma}_{\mathrm{in}}$ are bounded, then $\tilde{\alpha}_{\mathrm{is}} \to 0$ and $\Psi_{\mathrm{is}} \to 0$.

Then, we have

$$\begin{aligned}&\dot{\tilde{V}}_{i1} + \cdots + \dot{\tilde{V}}_{\mathrm{is}} + \rho\dot{V}_{\kappa i} + \dot{V}_{\rho i} + \dot{V}_{\sigma i1} + \cdots + \dot{V}_{\sigma\mathrm{in}}\\ &\leq -\sum_{m=1}^{s}\tilde{z}_{im}^2 + \sum_{j=0}^{N}\frac{s\gamma l\epsilon_{j2}^2}{(N+1)n} + \tilde{\rho}_i\Big(\tau_{\rho i(s-1)} - \dot{\hat{\rho}}_i\Big)\\ &\quad+\tilde{z}_{\mathrm{is}}\tilde{z}_{i(s+1)} + \left(\tilde{\sigma}_{i1} + \sum_{m=2}^{s}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\sigma}_{i1}}\right)\Big(\tau_{\sigma i1s} - \dot{\hat{\sigma}}_{i1}\Big)\\ &\quad+\cdots + \left(\tilde{\sigma}_{\mathrm{in}} + \sum_{m=2}^{s}\tilde{z}_{im}\frac{\partial\tilde{\alpha}_{i(m-1)}}{\partial\hat{\sigma}_{\mathrm{in}}}\right)\Big(\tau_{\sigma ins} - \dot{\hat{\sigma}}_{\mathrm{in}}\Big). \qquad (50)\end{aligned}$$

Step n: Through the above recursive design method, if we design u_i and the other adaptive laws as follows:

$$u_i = \frac{1}{\sum_{j=1}^{N}\bar{a}_{ij} + b_i}\left(\tilde{\alpha}_{\mathrm{in}} + \sum_{j=1}^{N}\bar{a}_{ij}u_j + b_iu_0\right) \qquad (51)$$

$$\dot{\hat{\rho}}_i = \tau_{\rho i(n-1)} \qquad (52)$$

$$\dot{\hat{\sigma}}_{i1} = \tau_{\sigma i1n}, \qquad (53)$$

$$\vdots$$

$$\dot{\hat{\sigma}}_{\mathrm{in}} = \tau_{\sigma inn} \qquad (54)$$

where $\tilde{\alpha}_{\mathrm{in}}$, $\tau_{\rho i(n-1)}$, $\tau_{\sigma i1n}, \ldots, \tau_{\sigma inn}$ can be obtained as *step ($3 \leq s \leq n-1$)* and their details are not given any more. Then the derivative of $\tilde{V}$ is

$$\dot{\tilde{V}} \leq -\sum_{i=1}^{N}\sum_{m=1}^{n}\tilde{z}_{im}^2 - \sum_{i=1}^{N}\frac{1}{2}l\epsilon_i^T\epsilon_i. \qquad (55)$$

Now, we give the second main result of our paper.

Theorem 2: For the high-order time-varying nonlinear MASs (1) satisfying Assumptions 1–2 and 4, the distributed adaptive output feedback controller (51) with the adaptive laws (44), (52)–(54), can render all the tracking errors ($\bar{x}_{\mathrm{in}} - \bar{x}_{0n}$) to converge to zero asymptotically.

Proof: Due to that $\bar{\varphi}(t)$ is bounded, from (31), l is bounded. Since $(L_r + B)$ is nonsingular matrix, from Assumption 2 and inequality (55), we know that for $i = 0, 1, \ldots, N$, y_i

are bounded. Similar with the Lemma 2, all the variables Ξ_{im}, λ_i, ζ_i, and ν_i are bounded. So we can easily obtain that $\sum_{i=1}^{N}\sum_{m=1}^{n}\tilde{z}_{im}^2 + \sum_{i=1}^{N}(1/4)\epsilon_i^T\epsilon$ is uniformly continuous. Due to that $\int_{t_0}^{t}(\sum_{i=1}^{N}\sum_{m=1}^{n}\tilde{z}_{im}^2 + \sum_{i=1}^{N}(1/2)l\epsilon_i^T\epsilon)d\tau \le -\int_{t_0}^{t}\dot{\tilde{V}}(\tau)d\tau \le \tilde{V}(t_0)$ is finite, based on the Lemma 1, $\tilde{z}_{11} \to 0, \ldots, \tilde{z}_{Nn} \to 0$ and $e_0 \to 0, \ldots, e_N \to 0$ as $t \to \infty$. Since (L_r+B) is nonsingular matrix, for $i = 1, \ldots, N$, $y_i \to y_0$ as $t \to \infty$. Based on the Lemma 3, due to $(y_i - y_0) \to 0$ as $t \to \infty$, we have $(\Xi_{im} - \Xi_{0m}) \to 0$, $(\lambda_i - \lambda_0) \to 0$, $(\zeta_i - \zeta_0) \to 0$. Combined with the properties of the virtual controllers $\tilde{\alpha}_{i(s-1)}$ and Lemma 2, we can obtain that $\nu_i \to \nu_0$ as $t \to \infty$. So, $(\hat{X}_i - \hat{X}_0) \to 0$ as $t \to \infty$. Since $e_0 \to 0, \ldots, e_N \to 0$ as $t \to \infty$, $\bar{x}_{\rm in} \to \bar{x}_{0n}$ as $t \to \infty$. The proof is completed.

Based on the above controller design method and stability analysis, the following corollaries can be obtained.

Corollary 3: If $\varphi_{0s}(t, y_i)$, $\varphi_{1s}(t, y_i)$, $\varphi_{2s}^T(t)$ are time invariant, such as $\varphi_{0s}(t, y_i) = \varphi_{0s}(0, y_i)$, $\varphi_{1s}(t, y_i) = \varphi_{1s}(0, y_i)$, $\varphi_{2s}^T(t) = \varphi_{2s}^T(0)$, for the nonlinear MASs (1) satisfying Assumptions 1 and 2 and (25), the distributed adaptive output feedback controller (51) can render all the tracking errors $(\bar{x}_{\rm in} - \bar{x}_{0n})$ to converge to zero asymptotically.

Corollary 4: For the high-order time-varying nonlinear MASs (1) satisfying Assumption 1 and (25), the distributed adaptive output feedback controller (51) can render the output tracking error $(x_{i1} - x_{01})$ to converge to zero asymptotically.

In Theorem 2, the dynamic gain k-filter-based distributed output feedback protocol is proposed for the nonlinear MASs, where the control input appears in the last dynamics $\dot{X}_{\rm in}$. The results can be further extended to the following nonlinear MASs:

$$\begin{cases} \dot{X}_{i1} = X_{i2} + \theta f_1(t, \bar{x}_{i1}) \\ \quad\vdots \\ \dot{X}_{i\varpi} = \bar{\rho}u_i + X_{i(\varpi+1)} + \Pi_{m=1}^{\varpi} g_{m-1}\theta f_{\varpi}(t, \bar{x}_{i\varpi}) \\ \quad\vdots \\ \dot{X}_{\rm in} = \bar{\rho}\Pi_{\bar{m}=\varpi+1}^{n} g_{\bar{m}}u_i + \Pi_{m=1}^{n} g_{m-1}\theta f_n(t, \bar{x}_{\rm in}) \\ y_i = X_{i1} \end{cases} \tag{56}$$

where $\bar{\rho} = \Pi_{m=1}^{\varpi} g_m$. $g_1, \ldots, g_{\varpi}$ are unknown positive constants and the parameters $g_{\varpi+1}, \ldots, g_n$ are known.

Assumption 5: For $s = \varpi + 1, \ldots, n$, following system:

$$\begin{aligned} \dot{X}_{\rm is} = {} & X_{i(s+1)} + \Pi_{m=1}^{s} g_{m-1}\theta f_s(t, \bar{x}_{\rm is}) \\ & + \bar{\rho}\Pi_{\bar{m}=\varpi+1}^{s} g_{\bar{m}}\big(\tilde{y}_0 - X_{i(\varpi+1)}\big) \end{aligned} \tag{57}$$

with inputs $\tilde{y}_0, X_{i1}, \ldots, X_{i\varpi}$, and output $X_{i(\varpi+1)}$ is bounded-input to bounded-output.

Corollary 5: For the high-order time-varying nonlinear MASs (56) satisfying Assumptions 1, 2, 4, and 5, the distributed output feedback controller design method of Theorem 2 can extend to this case to solve the states $(x_{i1}, \ldots, x_{i\varpi})$ leader-following asymptotic consensus problem.

Proof: In this corollary, similar to the design process of controller (51), redesign $\dot{\nu}_i$ in (26) as $\dot{\nu}_i = (A - lL_0qc^T)\nu_i + \bar{E}u_i + \varphi_2(t)\nu_i$, where $\bar{E} = [0, \ldots, 0, 1, g_{\varpi+1}, \ldots, \Pi_{\bar{m}=\varpi+1}^{n} g_{\bar{m}}]^T \in \mathcal{R}^{n-1}$. Choose the state estimate as $\hat{X}_i = \zeta_i + \sum_{m=1}^{n}\sigma_m\Xi_{im} + \bar{\rho}\nu_i$. Through ϖ steps, the controller can be designed as $u_i = [(\tilde{\alpha}_{i\varpi} + \sum_{j=1}^{N}\bar{a}_{ij}(u_j + \nu_{j(\varpi+1)}) + b_i(u_0 + \nu_{0(\varpi+1)}))/(\sum_{j=1}^{N}\bar{a}_{ij} + b_i)] - \nu_{i(\varpi+1)}$ and the other adaptive laws is similar with (44), (52)–(54). Then, similar to the proof of the Theorem 2, the leader-following consensus of the states $(x_{i1}, \ldots, x_{i\varpi})$ can be directly addressed. Combined with (57), the boundedness of the state variables $(x_{i(\varpi+1)}, \ldots, x_{\rm in})$ can be guaranteed.

Remark 6: Assumption 5 is given to make the remaining states of the followers be bounded, which is essential to guarantee the stability of each agent.

IV. Numerical Example

To illustrate the effectiveness of the proposed methods, consider following MASs under the directed graph Fig. 1 where $b_1 = 1$, $a_{13} = a_{21} = a_{32} = 1$ and the other weight a_{ij} is zero. By the hierarchical decomposition, we have $b_1 = 1$, $\bar{a}_{21} = \bar{a}_{32} = 1$ and the other weight $\bar{a}_{ij}$ is zero. The dynamic of the ith agent is as follows:

$$\begin{cases} \dot{x}_{i1} = x_{i2} + \varphi_{01}(t, y_i) \\ \dot{x}_{i2} = \rho u_i + \sigma_2\varphi_{12}(t, y_i) + \varphi_{222}(t)x_{i2} \\ y_i = x_{i1} \end{cases} \tag{58}$$

where set $\varphi_{01}(t, y_i) = 0.8\sin(t)y_i$, $\varphi_{12}(t, y_i) = y_i^2$, $\varphi_{222}(t) = 0.8\sin(t)$. $\rho = 0.5$ and $\sigma_2 = 0.7$.

Remark 7: The dynamics of the considered agent (58) is general. If the functions $\varphi_{01}(t, y_i) = 0$, $\sigma_2\varphi_{12}(t, y_i) = 0$ and $\varphi_{222}(t)$ is chosen as a constant, then (58) is reduced to the parallel active suspension system, referring to [29]. If $\varphi_{01}(t, y_i) = \varphi_{222}(t) = 0$, and $\varphi_{12}(t, y_i) = \sin(y_i)$, then (58) is reduced to the single-link robot system [35], [36].

Design the reduced order (first order) dynamic gain k-filter to estimate the ith agent's states as (26). From (30), setting $P = 1$, we have $2\epsilon_i^T PL_0^{-1}\varphi_{222}(t)L_0\epsilon_i \le 0.8(1 + \sin(t))\epsilon_i^T\epsilon_i$. Setting $\mu_1 = \mu_2 = \mu = 0.5$ and $\eta = 2$, then design l as $\dot{l} = l(-\eta l + \eta + 0.8(1 + \sin(t)))$, $l(0) = 1$. Choosing $q_2 = 1$, we have $\dot{V}_{ei} \le -(1/2)l\epsilon_i^T\epsilon_i$. In the sequel, use the backstepping method to construct the distributed controller.

Step 1: Calculate the derivative of $\tilde{V}_{i1}$

$$\begin{aligned} \dot{\tilde{V}}_{11} &= \tilde{z}_{11}(x_{12} - x_{02} + \varphi_{01}(t, y_1) - \varphi_{01}(t, y_0)) \\ &= \rho\tilde{z}_{11}(\tilde{z}_{12} + \tilde{\alpha}_{11}) \\ &\quad + \tilde{z}_{11}((\zeta_{12} + \sigma_2\Xi_{122} + e_{12} + \varphi_{01}(t, y_1)) \\ &\quad - (\zeta_{02} + \sigma_2\Xi_{022} + e_{02} + \varphi_{01}(t, y_0))), \end{aligned} \tag{59}$$

$$\begin{aligned} \dot{\tilde{V}}_{21} &= \rho\tilde{z}_{21}(\tilde{z}_{22} + \tilde{\alpha}_{21}) \\ &\quad + \tilde{z}_{21}((\zeta_{22} + \sigma_2\Xi_{222} + e_{22} + \varphi_{01}(t, y_2)) \\ &\quad - (\zeta_{12} + \sigma_2\Xi_{122} + e_{12} + \varphi_{01}(t, y_1))), \end{aligned} \tag{60}$$

$$\begin{aligned} \dot{\tilde{V}}_{31} &= \rho\tilde{z}_{31}(\tilde{z}_{32} + \tilde{\alpha}_{31}) \\ &\quad + \tilde{z}_{31}((\zeta_{32} + \sigma_2\Xi_{322} + e_{32} + \varphi_{01}(t, y_3)) \\ &\quad - (\zeta_{22} + \sigma_2\Xi_{222} + e_{22} + \varphi_{01}(t, y_2))). \end{aligned} \tag{61}$$

Using the Young's inequality, there exist $\beta_{11} = 1$, $\beta_{21} = 1$, $\beta_{31} = 1$ such that $\tilde{z}_{11}(2e_{12} - e_{32} - e_{02}) \le \beta_{11}\tilde{z}_{11}^2 + (1/2)l\epsilon_{12}^2 + (1/2)l\epsilon_{02}^2$, $\tilde{z}_{21}(e_{22} - e_{12}) \le \beta_{21}\tilde{z}_{21}^2 + (1/2)l\epsilon_{22}^2 + (1/2)l\epsilon_{12}^2$, $\tilde{z}_{31}(e_{32} - e_{22}) \le \beta_{31}\tilde{z}_{31}^2 + (1/2)l\epsilon_{32}^2 + (1/2)l\epsilon_{22}^2$. Design the virtual controller $\tilde{\alpha}_{i1}$ and adaptive law $\dot{\hat{\kappa}}_i$ as

$$\tilde{\alpha}_{i1} = -\hat{\kappa}_i\bar{\tilde{\alpha}}_{i1},\ \dot{\hat{\kappa}}_i = \tilde{z}_{i1}\bar{\tilde{\alpha}}_{i1} \tag{62}$$

where the designed functions $\bar{\tilde{\alpha}}_{11} = (\zeta_{12} - \zeta_{02}) + (\varphi_{01}(t, y_1) - \varphi_{01}(t, y_0)) + \beta_{11}\tilde{z}_{11} + \tilde{z}_{11} + \hat{\sigma}_{12}(\Xi_{122} - \Xi_{022})$, $\bar{\tilde{\alpha}}_{21} = (\zeta_{22} - \zeta_{12}) + \varphi_{01}(t, y_2) - \varphi_{01}(t, y_1) + \hat{\sigma}_{22}(\Xi_{222} - \Xi_{122}) + \beta_{21}\tilde{z}_{21} + \tilde{z}_{21}$, and $\bar{\tilde{\alpha}}_{31} = (\zeta_{32} - \zeta_{22}) + \varphi_{01}(t, y_3) - \varphi_{01}(t, y_2) + \hat{\sigma}_{32}(\Xi_{322} - \Xi_{222}) + \beta_{31}\tilde{z}_{31} + \tilde{z}_{31}$. Obviously, $\bar{\tilde{\alpha}}_{i1}$ satisfies that if $x_{11} \to x_{01}, \ldots, x_{31} \to x_{01}$, $\zeta_1 \to \zeta_0, \ldots, \zeta_3 \to \zeta_0$, $\Xi_{12} \to \Xi_{02}, \ldots, \Xi_{32} \to \Xi_{02}$ and $\hat{\sigma}_{12}, \ldots, \hat{\sigma}_{32}$ are bounded, then $\bar{\tilde{\alpha}}_{i1} \to 0$. From (59)–(62), $\sum_{i=1}^{3} \dot{\tilde{V}}_{i1} + \dot{V}_{\kappa i} + \dot{V}_{\sigma i2} \le -\sum_{i=1}^{3} \tilde{z}_{i1}^2 + \sum_{i=1}^{3} \rho\tilde{z}_{i1}\tilde{z}_{i2} + \tilde{\sigma}_{12}(\tau_{\sigma 121} - \dot{\hat{\sigma}}_{12}) + \tilde{\sigma}_{22}(\tau_{\sigma 221} - \dot{\hat{\sigma}}_{22}) + \tilde{\sigma}_{32}(\tau_{\sigma 321} - \hat{\sigma}_{32}) + l\epsilon_{12}^2 + (1/2)l\epsilon_{22}^2 + (1/2)l\epsilon_{32}^2 + (1/2)l\epsilon_{02}^2$, where $\tau_{\sigma 121} = \tilde{z}_{11}(\Xi_{122} - \Xi_{022})$, $\tau_{\sigma 221} = \tilde{z}_{21}(\Xi_{222} - \Xi_{122})$, $\tau_{\sigma 321} = \tilde{z}_{31}(\Xi_{322} - \Xi_{222})$.

Step 2: The derivative of $\tilde{V}_{12}$ is

$$\begin{aligned}
\dot{\tilde{V}}_{12} = \tilde{z}_{12}\Bigg(& u_1 - u_0 - lq_2\tilde{\xi}_{12} + \varphi_{222}(t)\tilde{\xi}_{12} \\
& - \dot{\hat{\kappa}}_1\bar{\tilde{\alpha}}_{11} - \hat{\kappa}_1\begin{pmatrix} \varphi_{222}(t)(\zeta_{12} - \zeta_{02}) \\ +lq_2(e_{12} - e_{02}) \\ +lq_2\rho(\nu_{12} - \nu_{02}) \\ +lq_2\sigma_2(\Xi_{122} - \Xi_{022}) \end{pmatrix} \\
& - \hat{\kappa}_1\begin{pmatrix} \dot{\hat{\sigma}}_{12}(\Xi_{122} - \Xi_{022}) + \hat{\sigma}_{12} \\ \times(\varphi_{222}(t) - lq_2)(\Xi_{122} - \Xi_{022}) \\ +\hat{\sigma}_{12}(\varphi_{12}(t, y_1) - \varphi_{12}(t, y_0)) \end{pmatrix} \\
& - \hat{\kappa}_1(\beta_{11} + 1 - 0.8\sin(t))\begin{pmatrix} e_{12} - e_{02} \\ +\zeta_{12} - \zeta_{02} \\ +\rho(\nu_{12} - \nu_{02}) \\ +\sigma_3(\Xi_{132} - \Xi_{032}) \end{pmatrix} \\
& - 0.8\hat{\kappa}_1\cos(t)\tilde{z}_{11}\Bigg).
\end{aligned} \tag{63}$$

Using Young's inequality, there exists smooth function $\beta_{12} = 2(\hat{\kappa}_1(\beta_{11} + 1 - 0.8\sin(t)))^2 + 2\hat{\kappa}_1^2 q_2^2 l^2$ such that $-\tilde{z}_{12}(\hat{\kappa}_1((\beta_{11} + 1 - 0.8\sin(t)) + lq_2))(2e_{12} - e_{32} - e_{02}) \le \beta_{12}\tilde{z}_{12}^2 + 3l\epsilon_{12}^2 + (3/4)l\epsilon_{32}^2 + (3/4)l\epsilon_{02}^2$. Design

$$\begin{aligned}
\tilde{\alpha}_{12} = -\Big(& -lq_2\tilde{\xi}_{12} + \varphi_{222}(t)\tilde{\xi}_{12} - \dot{\hat{\kappa}}_1\bar{\tilde{\alpha}}_{11} \\
& - \hat{\kappa}_1(\varphi_{222}(t)(\zeta_{12} - \zeta_{02})) \\
& - \hat{\kappa}_1\big(\hat{\sigma}_{12}(\varphi_{222}(t) - lq_2)(\Xi_{122} - \Xi_{022})\big) \\
& - \hat{\kappa}_1\hat{\sigma}_{12}(\varphi_{12}(t, y_1) - \varphi_{12}(t, y_0)) \\
& - \hat{\kappa}_1(\beta_{11} + 1 - 0.8\sin(t))(\zeta_{12} - \zeta_{02}) \\
& - 0.8\hat{\kappa}_1\cos(t)\tilde{z}_{11} + \beta_{12}\tilde{z}_{12} - \hat{\kappa}_1\hat{\rho}_1 \\
& \times (lq_2 + (\beta_{11} + 1 - 0.8\sin(t)))(\nu_{12} - \nu_{02}) \\
& + \hat{\rho}_1\tilde{z}_{11} - \hat{\kappa}_1\hat{\sigma}_{12}(lq_2 + (\beta_{11} + 1 - 0.8\sin(t))) \\
& \times (\Xi_{122} - \Xi_{022}) - \tau_{\sigma 122}\hat{\kappa}_1 \\
& \times (\Xi_{122} - \Xi_{022}) + (\beta_{12} + 1)\tilde{z}_{12}\Big)
\end{aligned} \tag{64}$$

where the designed smooth functions $\tau_{\sigma 122} = \tau_{\sigma 121} - \tilde{z}_{12}\hat{\kappa}_1(lq_2 + (\beta_{11} + 1 - 0.8\sin(t)))(\Xi_{122} - \Xi_{022})$. Similar with (63), (64), design

$$\begin{aligned}
\tilde{\alpha}_{22} = -\Big(& -lq_2\tilde{\xi}_{22} + \varphi_{222}(t)\tilde{\xi}_{22} - \dot{\hat{\kappa}}_2\bar{\tilde{\alpha}}_{21} \\
& - \hat{\kappa}_2(\varphi_{222}(t)(\zeta_{22} - \zeta_{12})) \\
& - \hat{\kappa}_2\big(\hat{\sigma}_{22}(\varphi_{222}(t) - lq_2)(\Xi_{222} - \Xi_{122})\big) \\
& - \hat{\kappa}_2\hat{\sigma}_{22}(\varphi_{12}(t, y_2) - \varphi_{12}(t, y_1)) \\
& - \hat{\kappa}_2(\beta_{21} + 1 - 0.8\sin(t))(\zeta_{22} - \zeta_{12}) \\
& - 0.8\hat{\kappa}_2\cos(t)\tilde{z}_{21} + \beta_{22}\tilde{z}_{22} - \hat{\kappa}_2\hat{\rho}_2 \\
& \times (lq_2 + (\beta_{21} + 1 - 0.8\sin(t)))(\nu_{22} - \nu_{12}) \\
& + \hat{\rho}_2\tilde{z}_{21} - \hat{\kappa}_2\hat{\sigma}_{22}(lq_2 + (\beta_{21} + 1 - 0.8\sin(t))) \\
& \times (\Xi_{222} - \Xi_{122}) - \tau_{\sigma 222}\hat{\kappa}_2 \\
& \times (\Xi_{222} - \Xi_{122}) + (\beta_{22} + 1)\tilde{z}_{22}\Big)
\end{aligned} \tag{65}$$

$$\begin{aligned}
\tilde{\alpha}_{32} = -\Big(& \varphi_{222}(t)\tilde{\xi}_{32} - \dot{\hat{\kappa}}_3\bar{\tilde{\alpha}}_{31} - \hat{\kappa}_3(\varphi_{222}(t)(\zeta_{32} - \zeta_{22})) \\
& - lq_2\tilde{\xi}_{32} - \hat{\kappa}_3\big(\hat{\sigma}_{32}(\varphi_{222}(t) - lq_2)(\Xi_{322} - \Xi_{222})\big) \\
& - \hat{\kappa}_2\hat{\sigma}_{32}(\varphi_{12}(t, y_3) - \varphi_{12}(t, y_2)) - \hat{\kappa}_3(\zeta_{32} - \zeta_{22}) \\
& \times (\beta_{31} + 1 - 0.8\sin(t)) - 0.8\hat{\kappa}_3\cos(t)\tilde{z}_{31} + \beta_{32}\tilde{z}_{32} \\
& - \hat{\kappa}_3\hat{\rho}_3(lq_2 + (\beta_{31} + 1 - 0.8\sin(t)))(\nu_{32} - \nu_{22}) \\
& + \hat{\rho}_3\tilde{z}_{31} - \hat{\kappa}_3\hat{\sigma}_{32}(lq_2 + (\beta_{31} + 1 - 0.8\sin(t))) \\
& \times (\Xi_{322} - \Xi_{222}) - \tau_{\sigma 322}\hat{\kappa}_3 \\
& \times (\Xi_{322} - \Xi_{222}) + (\beta_{32} + 1)\tilde{z}_{32}\Big)
\end{aligned} \tag{66}$$

where $\beta_{22} = 2(\hat{\kappa}_2(\beta_{21} + 1 - 0.8\sin(t)))^2 + 2\hat{\kappa}_2^2 q_2^2 l^2$, $\beta_{32} = 2(\hat{\kappa}_3(\beta_{31} + 1 - 0.8\sin(t)))^2 + 2\hat{\kappa}_3^2 q_2^2 l^2$, $\tau_{\sigma 222} = \tau_{\sigma 221} - \tilde{z}_{22}\hat{\kappa}_2(lq_2 + (\beta_{21} + 1 - 0.8\sin(t)))(\Xi_{222} - \Xi_{122})$, $\tau_{\sigma 322} = \tau_{\sigma 321} - \tilde{z}_{32}\hat{\kappa}_3(lq_2 + (\beta_{31} + 1 - 0.8\sin(t)))(\Xi_{322} - \Xi_{222})$.

Design

$$u_i = \left(\sum_{j=1}^{3} \bar{a}_{ij} + b_i\right)^{-1}\left(\tilde{\alpha}_{in} + \sum_{j=1}^{3} \bar{a}_{ij}u_j + b_i u_0\right) \tag{67}$$

$$\dot{\hat{\rho}}_i = \tau_{\rho i1}, \dot{\hat{\sigma}}_{i2} = \tau_{\sigma i22} \tag{68}$$

where the positive smooth functions $\tau_{\rho 11} = -\tilde{z}_{12}\hat{\kappa}_1(lq_2 + \beta_{11} + 1 - 0.8\sin(t))(\nu_{12} - \nu_{02}) + \tilde{z}_{11}\tilde{z}_{12}$, $\tau_{\rho 21} = -\tilde{z}_{22}\hat{\kappa}_2(lq_2 + (\beta_{21} + 1 - 0.8\sin(t)))(\nu_{22} - \nu_{12}) + \tilde{z}_{21}\tilde{z}_{22}$, $\tau_{\rho 31} = -\tilde{z}_{32}\hat{\kappa}_3(lq_2 + \beta_{31} + 1 - 0.8\sin(t))(\nu_{32} - \nu_{22}) + \tilde{z}_{31}\tilde{z}_{32}$. From Steps 1 and 2, $\sum_{i=1}^{3}(\dot{\tilde{V}}_{i1} + \dot{\tilde{V}}_{i2} + \rho\dot{V}_{\kappa i} + \dot{V}_{\sigma i2} + \dot{V}_{\rho i}) + 7\sum_{i=0}^{3}\dot{V}_{ei} \le -\sum_{i=1}^{3}(\tilde{z}_{i1}^2 + \tilde{z}_{i2}^2) - \sum_{i=0}^{3}(1/2)l\epsilon_i^T\epsilon_i$.

Choose the initial values as $x_{01} = -1$, $x_{02} = 1$, $x_{11} = 0.5$, $x_{12} = -0.7$, $x_{21} = -0.7$, $x_{22} = 0.5$, $x_{31} = 0.4$, $x_{32} = 0.4$, $\lambda_{02} = 0.6$, $\Xi_{022} = 0.6$, $\nu_{02} = 0.6$, $\lambda_{12} = 0.4$, $\Xi_{122} = 0.4$, $\nu_{12} = 0.4$, $\lambda_{22} = 0.5$, $\Xi_{222} = 0.5$, $\nu_{22} = 0.5$, $\lambda_{32} = 0.4$, $\Xi_{322} = 0.4$, $\hat{\kappa}_1 = \hat{\kappa}_2 = \hat{\kappa}_3 = \hat{\rho}_1 = \hat{\rho}_2 = \hat{\rho}_3 = \hat{\sigma}_{12} = \hat{\sigma}_{22} = \hat{\sigma}_{32} = 0.3$, and set $u_0(t) = -4x_{01} - 4x_{02} + 2\sin(1.5t) - 1.4x_{01}^2$. There exists x_{02} in u_0, which is a state feedback controller. Of

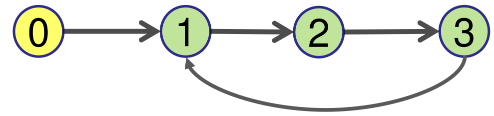

Fig. 1. Communication topology.

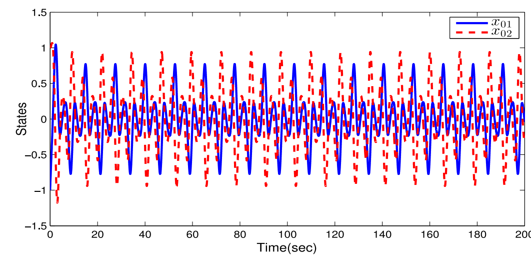

Fig. 2. Responses of the states x_{01} and x_{02}.

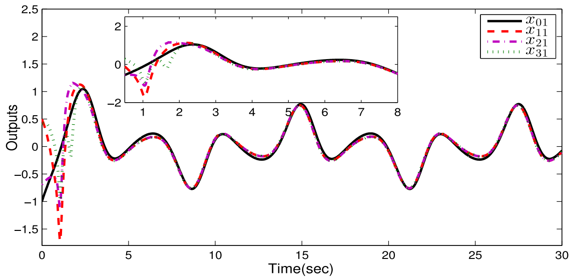

Fig. 3. Responses of the states x_{i1}.

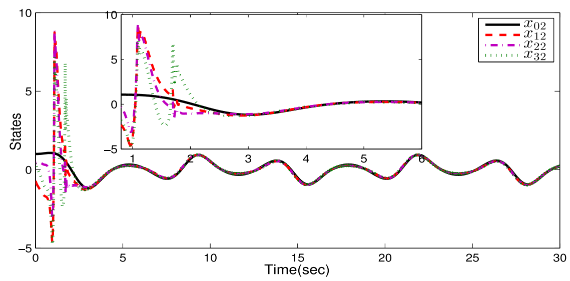

Fig. 4. Responses of the states x_{i2}.

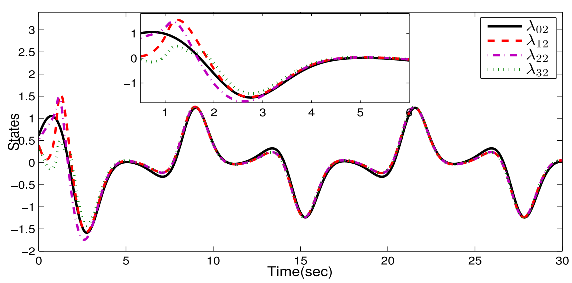

Fig. 5. Responses of the dynamics λ_{i2}.

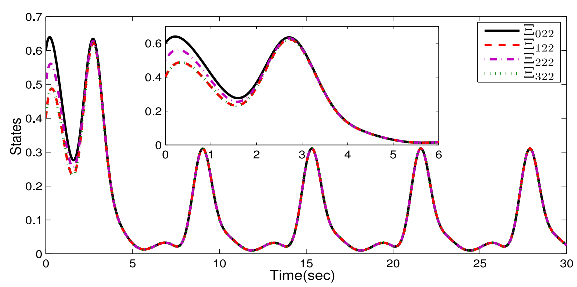

Fig. 6. Responses of the dynamics Ξ_{i22}.

course, u_0 can be designed as output feedback controller based on the constructed k-filter. This simulation mainly illustrates the effectiveness of the distributed protocol of the followers, so we just set u_0 as a simple structure. Due to that the matrix $[0.8\ 1; -2\ \ -1.2]$ is Hurwitz, it is straightforward to prove that the states of the leader are bounded under u_0. Simulation results are shown in Figs. 2–8. Fig. 2 illustrates that states of the leader are bounded. From Figs. 3 and 4, the states of followers can track states of the leader effectively based on the distributed adaptive output feedback controllers (67). Figs. 5–7 show the estimates can achieve consensus. Fig. 8 shows the responses of adaptive laws.

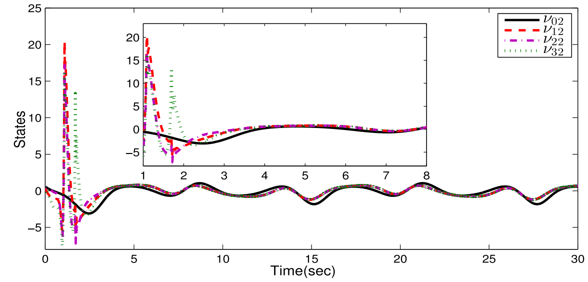

Fig. 7. Responses of the dynamics ν_{i2}.

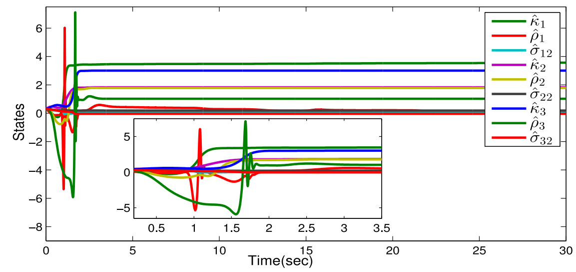

Fig. 8. Responses of the estimates.

V. Conclusion

This article investigates the distributed adaptive leader-following control for high-order time-varying nonlinear MASs with uncertain parameters under a directed communication graph. To avoid the mutually dependent controllers information produced in the design process, the hierarchical decomposition algorithm is used. Then, by introducing the consensus errors-based transformation, using the backstepping method and tuning function technique, the adaptive state feedback protocol is designed to guarantee all states of followers to track all the corresponding states of the leader asymptotically with mild conditions on nonlinearities. On the other hand, when states of the agents are unmeasured, the reduced order dynamic gain k-filter is constructed, based on which the adaptive output feedback protocol is proposed to achieve the full states consensus. All of the proposed theorems and corollaries can guarantee the global stability of the MASs. A general numerical example is given to illustrate the effectiveness of the proposed methods.

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