# Application of Generalized Inverses in the Minimum-Energy Perfect Control Theory

Tomasz Feliks, Wojciech Przemysław Hunek, and Predrag S. Stanimirović

***Abstract*—Application of generalized inverses in solving the inverse model control (IMC)-oriented minimum-energy perfect control design (PCD) problem for linear time-invariant multi-input/multi-output systems governed by the discrete-time *d*-state-space structure is presented in this article. For this reason, an appropriate class of polynomial generalized inverses is investigated. Moreover, it can be stated that the nonunique right $\sigma$-inverse, based on properly selected so-called degrees of freedom (DOFs), outperforms the well-known unique Moore–Penrose (MP) minimum-norm right *T*-inverse in terms of the energy consumption of perfect control (PC) input signals. However, the analytical confirmation of such an intriguing statement has only been established for the special class of the single-delayed plants with a zero reference value. Moreover, because of the complexity of the IMC, the objects with a time delay $d > 1$ having a nonzero setpoint have never been analytically explored in regard to the PC energy context until now. Thus, the newly introduced analytical methods defined in this article allow us to designate the proper forms of $\sigma$-inverse-related DOFs that guarantee the minimum-energy PCD for the entire set of LTI multivariable nonsquare systems with the delay $d \geq 1$. Moreover, the new original results, supported by numerical examples, strongly contest the well-established control and systems theory canons related to the optimal minimum-energy-originated peculiarity of the MP pseudoinverse.**

***Index Terms*—Feedback control, generalized inverses, inverse model control (IMC), linear multivariable systems, minimum-energy design, Moore–Penrose (MP) inverse, optimal control theory, static optimization problems, systems with time delays, time-invariant systems.**

## I. Introduction

THE INVERSE model control (IMC) strategies have been thoroughly investigated over the last decades due to their employment in a number of scientific and engineering tasks. In particular, the IMC-based explorations have discovered miscellaneous properties, such as robustness [1], [2], [3], minimum-energy maintenance [3], [4], [5], as well as speed and accuracy phenomena [6], [7], [8], [9], [10], [11]. The family of IMC-related algorithms constitutes the well-known and broadly explored stochastic minimum variance (MV) and deterministic perfect control (PC) formulas strictly dedicated to the plants defined in both the transfer function [12], [13], [14], and state-space frameworks [4], [15], [16]. The special peculiarities of such unified control law, i.e., the maximum-speed/maximum-accuracy and robust maintenance, make it desirable in many industrial real-life implementations, for example, in the quadruple tank process [9], wireless autopilot of a quadcopter [17], control of a satellite system [18], power system control [19], water distribution system [20], linear/nonlinear servo control systems [21], [22], nonlinear pendulum system [23], and refrigeration device [24]. However, the main disadvantage associated with the high energy expenditure of the control input signals frequently prevents practical applications of the discussed control strategy. It is obvious that for the square MIMO plants, including the SISO ones, we cannot affect on the behavior of closed-loop PC objects' signals. Nevertheless, in nonsquare right-invertible systems, i.e., full-rank plants with more input than output variables, we can obtain the desirable object-originated properties by involving nonunique generalized nonsquare right inverses into the PC design (PCD) processes [4], [25], [26], [27]. Nonetheless, in such a scenario, the broadly known and unique minimum-norm right *T*-inverse, which satisfies the four Moore–Penrose (MP) equations, has commonly been used in the well-established worldwide scientific literature since it has been treated as the optimal one, in general [28], [29], [30], [31], [32], [33], [34], [35].

Notwithstanding, following the heuristic studies in this matter, it should be stated that the mentioned right *T*-inverse does not guarantee the minimum-energy IMC-based MV/PCD [11], [36]. Although the heuristic approaches could be contested in some way, the novel authors' analytical investigation undeniably confirms such an intriguing statement [4]. Thus, from now on, it is clear that the recently introduced nonunique right $\sigma$-inverse, which encompasses proper degrees of freedom (DOFs), outperforms the unique MP inverse in terms of the energy of PC signals. Furthermore, because of the exceedingly complex nature of this issue, the analytical confirmation of the above statement has only been given for the single-delayed second-order LTI MISO discrete-time state-space systems having a single nonzero pole with a zero-reference value exclusively.

The minimum-energy IMC-related PCD problem structurally comes down to the selection of the appropriate $\sigma$-inverse-originated DOFs, in general. Accordingly, the

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Tomasz Feliks and Wojciech Przemysław Hunek are with the Department of Control Science and Engineering, Opole University of Technology, 45-758 Opole, Poland (e-mail: t.feliks@po.edu.pl; w.hunek@po.edu.pl).

Predrag S. Stanimirović is with the Department of Computer Science, Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia (e-mail: pecko@pmf.ni.ac.rs).

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plants possessing time delay $d > 1$ associated with a nonzero reference value/setpoint have never been examined analytically by the world control community regarding the PC energy optimization. It is due to the extremely high complexity related to the IMC methodology. However, a breakthrough in this subject is presented in this article. Through the newly defined analytical procedures it is possible to calculate correctly a suitable set of $\sigma$-inverse-related DOFs, which provide the minimum-energy PCD for the entire class of LTI MIMO $d$-step systems. Henceforth, the newly established methods allow us to consider the plants having $d \geq 1$ with a nonzero setpoint in terms of the control energy minimization, which led to a new common minimum-energy PC theory. A considerable potential of a practical use of the presented results is also corroborated by the representative numerical examples. The original observation covering the $d$-PC law breaks down the well-established IMC paradigm associated with the "pseudo-optimal" MP inverse, as a natural extension of the usual inverse [37], [38], and opens a new chapter in the control and systems theory canons.

The main contributions of this article are as follows.

1) A new concept of employing generalized inverses in the minimum-energy IMC design tasks is offered.
2) Since generalized inverses are vital from both theoretical and practical points of view, a newly introduced tool in the form of the nonunique right inverse with arbitrary selected DOFs is extensively explored.
3) A unified analytical approach to the minimum-energy design of multivariable IMC systems with different time delays is established.
4) The new solution outperforms the classical MP solution in the energy consumption of PC inputs. This phenomenon constitutes a solid background for reviewing the commonly known MP literature.
5) Henceforth, the defined methodology can successfully be employed regarding the energy optimization of physical systems.

This article is structured as follows. Fundamentals covering the PC paradigm are presented in Section II. Related generalized inverses are investigated in Section III. The notion of the PC energy problem is stated in Section IV. Section V extends the possibility of obtaining the minimum-energy solution to the single-delayed plants with zero and nonzero reference values. The discovered results undermine the optimal property of the MP inverse in Section VI. In the most important Section VII, the general minimum-energy PCD solution for LTI MIMO $d$-state-space systems is established, which is also supported by a numerical example. Following the obtained results, the progress in the minimum-energy-based IMC theory is addressed in Section VIII. Finally, in the last section of this article, the achieved outcomes are summarized, and the open problems are accented.

## II. State-Space Perfect Control Paradigm

To present the new results perspicuously, the essential abbreviations utilized in the manuscript are summarized in Table I.

TABLE I
Table of Symbols and Abbreviations

| Notation | |
|---|---|
| $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{G}$ $\mathbf{H}$, $\mathbf{K}$, $\mathbf{Q}$, $\beta$ | – parameter matrices, |
| $d$ | – time delay of a plant, |
| $E_u$ | – energy of the perfect control input variables, |
| $H$ | – time horizon, |
| $\mathbf{I_n}$ | – identity $n$-matrix, |
| $k$ | – discrete time, |
| $\frac{L(\cdot)}{M(\cdot)}$ | – rational function, |
| $q^{-1}$ | – backward shift operator, |
| $z$ | – complex operator, |
| $\mathbf{\Upsilon}(q^{-1})$ | – polynomial matrix in $q^{-1}$, |
| $\delta$ | – order of the polynomial, |
| $\lambda_1$ | – single nonzero pole, |
| $(.)^{\mathrm{R}}$ | – (non-)unique right inverse, |
| $(.)^{\mathrm{R}}_{\sigma}$ | – non-unique right $\sigma$-inverse, |
| $(.)^{\mathrm{R}}_{0}$ | – unique minimum-norm right $T$-inverse, |
| $(.)^{\dagger}$ | – Moore–Penrose inverse, |
| $(.)^{\mathrm{T}}$ | – transpose symbol, |
| $(.)^{*}$ | – conjugate transpose symbol, |
| $\Vert\cdot\Vert_2$ | – Euclidean norm symbol, |
| $\det(.)$ | – determinant, |
| $\mathrm{eig}(.)$ | – eigenvalue, |
| $\ker(.)$ | – kernel, |
| $\lim(.)$ | – limit, |
| $\mathrm{Tr}(.)$ | – trace symbol, |
| DOFs | – degrees of freedom, |
| FVT | – Final Value Theorem, |
| IMC | – Inverse Model Control, |
| JCF | – Jordan Canonical Form, |
| LTI | – linear time-invariant, |
| MIMO | – multi-input/multi-output, |
| MISO | – multi-input/single-output, |
| MVC | – minimum variance control, |
| PCD | – perfect control design, |
| (P)SVD | – (Polynomial) Singular Value Decomposition, |
| SISO | – single-input/single-output, |
| $S(\mathbf{A},\mathbf{B},\mathbf{C})$ | – state-space plant. |

The set of right inverses of $\mathbf{A}$ is marked by $\mathbf{A}^{-1}_{\{R\}} = \{\mathbf{X}|\ \mathbf{AX} = \mathbf{I}\}$, while the dual set of left inverses of $\mathbf{A}$ is defined as $\mathbf{A}^{-1}_{\{L\}} = \{\mathbf{X}|\ \mathbf{XA} = \mathbf{I}\}$, where $\mathbf{I}$ denotes an appropriate identity matrix.

Now, take into account an LTI MIMO plant $S(\mathbf{A},\mathbf{B},\mathbf{C})$ determined by the broadly known discrete-time $d$-state-space framework

$$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)q^{-d+1}, \mathbf{x}(0) = \mathbf{x_0} \qquad (1a)$$

$$\mathbf{y}(k) = \mathbf{Cx}(k). \qquad (1b)$$

The system behaves in line with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n_u}$, and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$ under the force of vectors $\mathbf{x}(k) \in \mathbb{R}^n$, $\mathbf{u}(k) \in \mathbb{R}^{n_u}$, and $\mathbf{y}(k) \in \mathbb{R}^{n_y}$. The notations $n$, $n_u$, and $n_y$ denote the number of state, input, and output variables, in that order. Naturally, the whole system proceeds in discrete time $k$ from the initial condition $\mathbf{x_0}$ in regard to the backward shift operator $q^{-1}$ and time delay $d$ of the considered object.

The IMC-oriented PC law composes the deterministic case of the well-known MV control (MVC). Thus, in the particular case, the PC strategy guarantees that the system's output $\mathbf{y}(k)$ achieves the reference value/setpoint $\mathbf{y_{ref}}(k)$ after the time delay $d$, such that

$$\mathbf{y}(k+d) = \mathbf{y_{ref}}(k+d). \qquad (2)$$

To achieve such an interesting control algorithm it is necessary to minimize the essential PC law model of the form

$$J = \min_{\mathbf{u}(k)} \left\{ \sum_{k=0}^{+\infty} \left\| \mathbf{y}(k+d) - \mathbf{y_{ref}}(k+d) \right\|_2^2 \right\} \quad (3)$$

which contains an arbitrary reference value $\mathbf{y_{ref}}(k+d) \in \mathbb{R}^{n_y}$.

After the solution of defined performance index (3), the general $d$-state-space-related PC formula is obtained [1] with any time delay $d \geq 1$

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}} \left[ \mathbf{y_{ref}}(k+d) - \mathbf{C}\left( \sum_{j=1}^{d-1} \mathbf{A}^j \mathbf{B} \mathbf{u}(k-j) + \mathbf{A}^d \mathbf{x}(k) \right) \right] \quad (4)$$

where $(\cdot)^{\mathrm{R}}$ denotes any (non)unique right inverse.

*Remark 1:* For the left-invertible plants ($n_y > n_u$), the primary requirement (3) does not hold, in general [4]. Therefore, such situations are excluded from our investigation.

*Remark 2:* It also should be noted that the PC strategy has been defined recently for the nonfull rank objects with a zero reference value $\mathbf{y_{ref}}(k) = \mathbf{0}$. In such a scenario, the unique MP $T$-pseudoinverse, supported by the so-called skeleton factorization, has to be employed in PC, or rather perfect regulation, design process. More details are available in [1] and [35].

*Remark 3:* The MVC procedure can also be provided by (4) iff the so-called expectation operator is engaged in the performance index (3).

*Remark 4:* Naturally, the general MVC algorithm operates under the structure blurred by the zero-mean Gaussian-based uncorrelated white noise sequence. However, due to the complex nature of the presented issues, we exclude this awkward instance. This is because the deterministic (PC) and stochastic (MV) scenarios can be considered interchangeably in the context of energy-oriented control results. Hence, we still proceed with intricate IMC-related PC methodology.

## III. Related Generalized Inverses

As usual, notations $\mathrm{rank}(\mathbf{A})$, $\mathcal{R}(\mathbf{A})$, $\mathcal{N}(\mathbf{A})$, and $\mathbf{A}^{\mathrm{T}}$ denote the rank, image, null space, and transpose, respectively, of $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $\mathbb{R}^{m\times n}$ stands for $m \times n$ real matrices. In addition, $\mathbb{R}^{m\times n}_r = \{\mathbf{A} \in \mathbb{R}^{m\times n} |\ \mathrm{rank}(\mathbf{A}) = r\}$. The notation $\mathbb{R}[t]$ stands for the set of polynomials with real coefficients in the unknown variable $t$, while $m \times n$ matrices with elements over $\mathbb{R}[t]$ are termed as $\mathbb{R}[t]^{m\times n}$. Moreover, $P_{U,V}$ signifies a projector onto $U$ along $V$ and $P_U$ denotes the orthogonal projector onto $U$.

The MP inverse of $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the unique matrix $\mathbf{A}^{\dagger} \in \mathbb{R}^{n\times m}$ defined by

$$\text{①}\ \mathbf{A}\mathbf{A}^{\dagger}\mathbf{A} = \mathbf{A}, \quad \text{②}\ \mathbf{A}^{\dagger}\mathbf{A}\mathbf{A}^{\dagger} = \mathbf{A}^{\dagger}$$
$$\text{③}\ \left(\mathbf{A}\mathbf{A}^{\dagger}\right)^{\mathrm{T}} = \mathbf{A}\mathbf{A}^{\dagger}, \quad \text{④}\ \left(\mathbf{A}^{\dagger}\mathbf{A}\right)^{\mathrm{T}} = \mathbf{A}^{\dagger}\mathbf{A}.$$

For given $\mathbf{A} \in \mathbb{R}^{m\times n}$, if the equation ① is satisfied with respect to unknown $\mathbf{X} \in \mathbb{R}^{n\times m}$, then $\mathbf{X}$ is an $\{1\}$-inverse of $\mathbf{A}$ and it is denoted by $\mathbf{A}^{(1)}$. The matrix $\mathbf{X}$ satisfying ② is a $\{2\}$-inverse (or outer inverse) of $\mathbf{A}$ and signified by $\mathbf{A}^{(2)}$. The outer inverse $\mathbf{X}$ is uniquely determined by the image $T$ and the null space $S$ and is marked with the standard notation $A^{(2)}_{T,S}$ if

$$\mathbf{XAX} = \mathbf{X}, \quad \mathcal{R}(\mathbf{X}) = T, \quad \mathcal{N}(\mathbf{X}) = S.$$

The following notations will be useful: $\mathbf{A}\{2\}_{T,*} = \{\mathbf{X} \in A\{2\}|\ \mathcal{R}(\mathbf{X}) = T\}$, $\mathbf{A}\{2\}_{*,S} = \{\mathbf{X} \in A\{2\}|\ \mathcal{N}(\mathbf{X}) = S\}$. For $\gamma \subseteq \{1, 2, 3, 4\}$, a $\gamma$-inverse of $\mathbf{A}$ is any matrix satisfying the equations contained in $\gamma$, and $\mathbf{A}\{\gamma\}$ represents the set of $\gamma$-inverses of $\mathbf{A}$, regarding canons ①, ②, ③, and ④. Particularly, the sets $\mathbf{A}\{1\}$ and $\mathbf{A}\{2\}$ involve inner and outer inverses of $\mathbf{A}$, respectively. Further, the set of $\gamma$-inverses of $\mathbf{A}$ with predefined rank are defined by $\mathbf{A}\{\gamma\}_s = \{\mathbf{X}|\ \mathbf{X} \in \mathbf{A}\{\gamma\},\ \mathrm{rank}(\mathbf{X}) = s\}$. A particular $\gamma$-inverse of $\mathbf{A}$ of prescribed rank $s$ is denoted by $\mathbf{A}^{(\gamma)}_s \in \mathbf{A}\{\gamma\}_s$.

In Sections III-A and III-B we investigate extensions of right and left inverses which play an important role in the minimum-energy PCD. These generalized inverses are subsets of $\{2, 3\}$ and $\{2, 4\}$ inverses and are defined by appropriate matrix polynomial terms.

### *A. Nonsquare Polynomial Right σ-Inverses*

In the available literature, it is possible to find a large number of different (non)unique generalized inverses devoted to nonsquare parameter/polynomial matrices [32], [39]. The most famous is the unique MP $T$-inverse. Still, the Smith factorization-oriented polynomial $S$-inverse and the SVD-based parameter $H$-inverse with its polynomial instance in the form of PSVD are also commonly utilized [10], [32], [40], [41]. However, the most intriguing generalized inverse is the latterly proposed polynomial matrix right $\sigma$-inverse, defined on the matrix $\mathbf{CB}$ as follows.

*Definition 1:* Consider $\mathbf{B} \in \mathbb{R}^{n\times n_u}$ and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$. The polynomial matrix right $\sigma$-inverse of the matrix $\mathbf{CB}$ is defined by

$$(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = \Upsilon\left(q^{-1}\right)\left[\mathbf{CB}\Upsilon\left(q^{-1}\right)\right]^{\dagger} \quad (5)$$

which contains the essential matrix polynomial DOFs $\Upsilon(q^{-1}) \in \mathbb{R}[q^{-1}]^{n_u\times n_y}$ with respect to the backward shift operator $q^{-1}$

$$\Upsilon\left(q^{-1}\right) = \beta_{\mathbf{0}} + \beta_{\mathbf{1}} q^{-1} + \beta_{\mathbf{2}} q^{-2} + \cdots + \beta_{\delta} q^{-\delta} \quad (6)$$

where the label $\delta$ indicates an arbitrarily chosen order of the matrix polynomial derived from the matrix coefficients $\beta_{\mathbf{i}} \in \mathbb{R}^{n_u\times n_y}$ for $i = 0, 1, \ldots, \delta$.

The dual matrix left $\sigma$-inverse is defined as follows.

*Definition 2:* Consider $\mathbf{B} \in \mathbb{R}^{n\times n_u}$ and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$. The polynomial matrix left $\sigma$-inverse of the matrix $\mathbf{CB}$ is defined by

$$(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = \left[\Upsilon\left(q^{-1}\right)\mathbf{CB}\right]^{\dagger}\Upsilon\left(q^{-1}\right) \quad (7)$$

which contains the essential matrix polynomial DOFs $\Upsilon(q^{-1})$ defined as in (6).

It should be emphasized that through the introduced DOFs $\Upsilon(q^{-1})$, it is possible to influence the behavior of the PC algorithm (4) [25]. Moreover, for any selected $\Upsilon(q^{-1})$, excluding the nonfull rank $\Upsilon(q^{-1})$, the main condition (3) always holds. Nevertheless, the mentioned DOFs only occur in the nonsquare scenarios. If $\mathbf{CB}\Upsilon$ is nonsingular, then both $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}$ defined in (5) and $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon}$ defined in (7) come down to the regular inverse of $\mathbf{CB}$.

*Remark 5:* Observe that for $\delta = 0$ we report the parameter form of the $\sigma$-inverse

$$(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_{\mathbf{0}}} = \beta_{\mathbf{0}}[\mathbf{CB}\beta_{\mathbf{0}}]^{\dagger}. \tag{8}$$

The choice $\delta = 0$ in the constant matrix case reduces (5) to a subset of $A\{2,3\}$ inverses investigated in [42].

Further, the choice $\delta = 0$ and $\beta_{\mathbf{0}} = (\mathbf{CB})^{\mathrm{T}}$ in the case $\operatorname{rank}(\mathbf{CB}) = n_y \le n_u$ leads to the unique right inverse which coincides with the MP inverse of $\mathbf{CB}$

$$(\mathbf{CB})^{\mathrm{R}}_{0} \equiv (\mathbf{CB})^{\mathrm{R}}_{\sigma|(\mathbf{CB})^{\mathrm{T}}} = (\mathbf{CB})^{\mathrm{T}}\left[\mathbf{CB}(\mathbf{CB})^{\mathrm{T}}\right]^{\dagger}. \tag{9}$$

*Remark 6:* It should be noted that the nonunique polynomial right $\sigma$-inverse (5), comprising the key DOFs (6), includes all right-inverse expressions of $\mathbf{CB}$ in the case $\operatorname{rank}(\mathbf{CB}\Upsilon) = n_y$. Furthermore, some relationships between the mentioned $\sigma$-inverse $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}$ and the broadly applied $S$-inverse can be found in [43].

In order to complete the fundamentals concerning the PC law, the stability behavior of such a control algorithm is examined below.

### *B. Properties of Nonsquare Polynomial Right $\sigma$-Inverses*

The results of this section are applicable to $\mathbf{B} \in \mathbb{R}^{n\times n_u}$, $\mathbf{C} \in \mathbb{R}^{n_y\times n}$ and $\Upsilon(q^{-1}) \in \mathbb{R}[q^{-1}]^{n_u\times n_y}$. The rank equality $\operatorname{rank}(\mathbf{A}_1) = \cdots = \operatorname{rank}(\mathbf{A}_k)$ between arbitrary matrices $\mathbf{A}_1,\ldots,\mathbf{A}_k$ will be denoted by $\Xi_{\mathbf{A}_1,\ldots,\mathbf{A}_k}$.

Representations and characterizations of the polynomial matrix right $\sigma$-inverse are investigated in Lemma 1.

*Lemma 1:* The following statements hold for the polynomial matrix right $\sigma$-inverse $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}$:

1) $\mathbf{CB}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = P_{\mathcal{R}(\mathbf{CB}\Upsilon)}$;
2) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB} = P_{\mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}),\mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB})}$;
3) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = (\mathbf{CB})^{(2,3)}_{\mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}),\mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}})} \in (\mathbf{CB})\{2,3\}$;
4) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = (\mathbf{CB})^{(2,3)}_{\mathcal{R}(\Upsilon),\mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}})} \iff \Xi_{\Upsilon,\mathbf{CB}\Upsilon}$;
5) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \in (\mathbf{CB})\{1,2,3\} \iff \Xi_{\mathbf{CB}\Upsilon,\mathbf{CB}}$;
6) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = \Upsilon((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB}\Upsilon)^{-1}(\mathbf{CB}\Upsilon)^{\mathrm{T}}$
$= \mathbf{A}^{(2,3)}_{\mathcal{R}(\Upsilon),\mathcal{N}((\mathbf{CB}\Upsilon))} \in \mathbf{A}\{2,3\}_{n_y}$
$\iff \Upsilon \in \mathbb{R}^{n_u\times n_y}_{n_y} \bigwedge \Xi_{\mathbf{CB}\Upsilon\cdot\Upsilon}$;
7) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \in (\mathbf{CB})^{-1}_{\{\mathrm{R}\}} \iff \Upsilon \in \mathbb{R}^{n_u\times n_y}_{n_y} \bigwedge \Xi_{\mathbf{CB}\Upsilon,\Upsilon}$;
8) $\Upsilon = (\mathbf{CB})^{\mathrm{T}} \bigvee \Upsilon = \mathbf{I} \Longrightarrow (\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = (\mathbf{CB})^{\dagger}$;
9) $\mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C}$ is an oblique projector;
10) $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C} =$
$$\begin{cases}\mathbf{B}^{(2)}_{\mathcal{R}(\Upsilon),*}, & \operatorname{rank}(\mathbf{CB}\Upsilon) = \operatorname{rank}(\Upsilon)\\ \mathbf{B}^{(2)}_{*,\mathcal{N}(\mathbf{C})}, & \operatorname{rank}(\mathbf{CB}\Upsilon) = \operatorname{rank}(\mathbf{C})\\ \mathbf{B}^{(2)}_{\mathcal{R}(\Upsilon^{\mathrm{T}}),\mathcal{N}(\mathbf{C})}, & \operatorname{rank}(\mathbf{CB}\Upsilon) = \operatorname{rank}(\mathbf{C}) = \operatorname{rank}(\Upsilon).\end{cases}$$

*Proof:* Consider appropriate matrices $\mathbf{B} \in \mathbb{R}^{n\times n_u}$ and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$.

1) According to the definition in (5), it follows $\mathbf{CB}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = \mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\dagger} = P_{\mathcal{R}(\mathbf{CB}\Upsilon)}$.
2) Clearly, $\mathbf{CB}^{\mathrm{R}}_{\sigma|\Upsilon} \in (\mathbf{CB})\{2\}$, which implies $(\mathbf{CB}^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB})^2 = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}$. As a consequence, $\mathbf{CB}^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB} = P_{\mathcal{R}(\mathbf{CB}^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}),\mathcal{N}(\mathbf{CB}^{\mathrm{R}}_{\Upsilon}\mathbf{CB})}$ is an oblique projector [32, Th. 8, p. 59]. On the basis of
$$\mathcal{R}\Big((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}\Big) = \mathcal{R}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big) \subseteq \mathcal{R}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\Big) = \mathcal{R}\big(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}\big)$$
in common with
$$\begin{aligned}\mathcal{R}\big(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}\big) &= \mathcal{R}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\Big)\\ &= \mathcal{R}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\Big)\\ &\subseteq \mathcal{R}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big)\\ &= \mathcal{R}\Big((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}\Big)\end{aligned}$$
it follows $\mathcal{R}((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}) = \mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}})$. Similar verification gives
$$\begin{aligned}\mathcal{N}\Big((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}\Big) &= \mathcal{N}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big)\\ &\supseteq \mathcal{N}\Big((\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big) = \mathcal{N}\big((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB}\big)\end{aligned}$$
and
$$\begin{aligned}\mathcal{N}\big((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB}\big) &= \mathcal{N}\Big((\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big)\\ &= \mathcal{N}\Big((\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big)\\ &\supseteq \mathcal{N}\Big(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB}\Big)\\ &= \mathcal{N}\Big((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}\Big)\end{aligned}$$
which gives $\mathcal{N}((\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CB}) = \mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB})$.
3) Utilizing basic properties of the MP inverse and Definition (5) it can be concluded $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \in \mathbf{CB}\{2,3\}$. Further
$$\begin{aligned}\operatorname{rank}((\mathbf{CB}\Upsilon)^{\mathrm{T}}) &\ge \operatorname{rank}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}})\\ &\ge \operatorname{rank}(\mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}) = \operatorname{rank}((\mathbf{CB}\Upsilon)^{\mathrm{T}})\end{aligned}$$
in conjunction with $\mathcal{N}(\mathbf{AB}) = \mathcal{N}(\mathbf{B}) \iff \Xi_{\mathbf{AB},\mathbf{B}}$ [32] implies $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = \mathbf{CB}^{(2,3)}_{\mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}),\mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}})} = (\mathbf{CB})^{(2,3)}_{\mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}),\mathcal{N}((\mathbf{CB}\Upsilon)^{\mathrm{T}})}$.
4) Using $\Xi_{\Upsilon,\mathbf{A}\Upsilon} \iff \Xi_{\Upsilon,(\mathbf{A}\Upsilon)^{*}} \iff \Xi_{\Upsilon,\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}}}$, this part of the proof follows from 3) and known result $\mathcal{R}(\mathbf{UV}) = \mathcal{R}(\mathbf{U}) \iff \Xi_{\mathbf{UV},\mathbf{U}}$ [32].
5) This statement is implied by 3) and the relation $\mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{CB} = \mathbf{CB} \iff \Xi_{\mathbf{CB}\Upsilon,\mathbf{CB}}$ [32].
6) This statement follows from: $(\mathbf{CB}\Upsilon)^{\dagger} = ((\mathbf{CB}\Upsilon)^{\mathrm{T}}\mathbf{CB}\Upsilon)^{-1}(\mathbf{CB}\Upsilon)^{\mathrm{T}}$ in the case $\Upsilon \in \mathbb{C}^{n_y\times n_u}_{n_u} \bigwedge \Xi_{\Upsilon\mathbf{CB},\Upsilon}$ [32, p. 57], [42].
7) The assumption $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \in (\mathbf{CB})^{-1}_{\{\mathrm{R}\}}$ initiates $\mathbf{CB}\Upsilon(\mathbf{CB}\Upsilon)^{\dagger} = \mathbf{I}_{n_y}$. According to [32, Lemma 2, p. 43], it follows $\operatorname{rank}(\mathbf{CB}\Upsilon) = n_y$, which implies $\Upsilon \in \mathbb{R}^{n_u\times n_y}_{n_u} \bigwedge \Xi_{\mathbf{CB}\Upsilon,\Upsilon}$. On the other hand, conditions $\Upsilon \in \mathbb{C}^{n_u\times n_y}_{n_u} \bigwedge \Xi_{\mathbf{CB}\Upsilon,\Upsilon}$ imply $\mathbf{CB}^{\mathrm{R}}_{\sigma|\Upsilon} = \Upsilon(\mathbf{CB}\Upsilon)^{-1} \in (\mathbf{CB})^{-1}_{\{\mathrm{R}\}}$.
8) Follows from $(\mathbf{CB})^{\dagger} = (\mathbf{CB})^{\mathrm{T}}(\mathbf{CB}(\mathbf{CB})^{\mathrm{T}})^{\dagger}$ [32].

9) Clearly, $\mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C}$ is idempotent.
10) Follows from $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C} = \Upsilon(\mathbf{CB}\Upsilon)^{\dagger}\mathbf{C}$ and the Urquhart representation of generalized inverses [32, Th. 13, p. 72]. ■

Characterizations and representations of the polynomial matrix left $\sigma$-inverse are presented in Lemma 2.

*Lemma 2:* The following statements hold for the polynomial matrix left $\sigma$-inverse $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon}$:

1) $\mathbf{CB}(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = P_{\mathcal{R}(\mathbf{A}(\Upsilon\mathbf{A})^*),\mathcal{N}((\Upsilon\mathbf{A})^*\Upsilon)}$;
2) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon}\mathbf{CB} = P_{\mathcal{R}((\Upsilon^{\mathrm{T}}(\mathbf{CB})^{\mathrm{T}})}$;
3) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = \mathbf{CB}^{(2,4)}_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}}),\mathcal{N}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon)} \in (\mathbf{CB})\{2,4\}$;
4) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = \mathbf{A}^{(2,4)}_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}}),\mathcal{N}(\Upsilon)} \iff \Xi_{\Upsilon\mathbf{CB},\Upsilon}$;
5) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} \in \mathbf{CB}\{1,2,4\} \iff \Xi_{\Upsilon\mathbf{CB},\mathbf{CB}}$;
6) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = (\Upsilon\mathbf{CB})^{\mathrm{T}}(\Upsilon\mathbf{CB}(\Upsilon\mathbf{CB})^{\mathrm{T}})^{-1}\Upsilon$
$= (\mathbf{CB})^{(2,4)}_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}}),\mathcal{N}(\Upsilon)} \in (\mathbf{CB})\{2,4\}_s$
$\iff \Upsilon \in \mathbb{C}^{n_y\times n_u}_{n_u} \bigwedge \Xi_{\Upsilon\mathbf{CB},\Upsilon}$;
7) $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} \in (\mathbf{CB})^{-1}_{\{L\}} \iff \Upsilon \in \mathbb{R}^{n_y\times n_u}_{n_y} \bigwedge \Xi_{\Upsilon\mathbf{CB},\Upsilon}$;
8) $\Upsilon = (\mathbf{CB})^{\mathrm{T}} \bigvee \Upsilon = \mathbf{I} \Longrightarrow (\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = (\mathbf{CB})^{\dagger}$;
9) $\mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C}$ is an oblique projector;
10) $\mathbf{B}(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} =$

$$\begin{cases} \mathbf{C}^{(2)}_{\mathcal{R}(\mathbf{B}),*}, & \mathrm{rank}(\mathbf{CB}\Upsilon) = \mathrm{rank}(\mathbf{B}) \\ \mathbf{C}^{(2)}_{*,\mathcal{N}(\Upsilon)}, & \mathrm{rank}(\mathbf{CB}\Upsilon) = \mathrm{rank}(\Upsilon) \\ \mathbf{C}^{(2)}_{\mathcal{R}(\mathbf{B}^{\mathrm{T}}),\mathcal{N}(\Upsilon)}, & \mathrm{rank}(\mathbf{CB}\Upsilon) = \mathrm{rank}(\mathbf{B}) = \mathrm{rank}(\Upsilon). \end{cases}$$

*Proof:* Consider appropriate matrices $\mathbf{B} \in \mathbb{R}^{n\times n_u}$ and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$.

1) Since, $(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} \in (\mathbf{CB})\{2\}$, it follows $(\mathbf{CB}(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon})^2 = \mathbf{CB}(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon}$. So, $\mathbf{CB}(\mathbf{CB})^{\mathrm{L}}_{\sigma|\Upsilon} = P_{\mathcal{R}(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}),\mathcal{N}(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon})}$ is a projector [32, Th. 8, p. 59]. On the basis of

$$\mathcal{R}\Big(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}\Big) = \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\Big) \subseteq \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Big) = \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\mathrm{T}}\Big)$$

in conjunction with

$$\begin{aligned}\mathcal{R}\big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\mathrm{T}}\big) &= \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Big) \\ &= \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Big) \\ &\subseteq \mathcal{R}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\Big) \\ &= \mathcal{R}\Big(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}\Big)\end{aligned}$$

it follows $\mathcal{R}(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}) = \mathcal{R}(\Upsilon(\mathbf{CB}\Upsilon)^{\mathrm{T}})$. On the other hand

$$\mathcal{N}\Big(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}\Big) = \mathcal{N}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\Big) \supseteq \mathcal{N}\Big((\Upsilon^{\mathrm{T}}\mathbf{CB})^{\dagger}\Upsilon\Big) = \mathcal{N}\big((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon\big)$$

and

$$\begin{aligned}\mathcal{N}\big((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon\big) &= \mathcal{N}\Big((\Upsilon\mathbf{CB})^{\dagger}\Upsilon\Big) \\ &= \mathcal{N}\Big((\Upsilon\mathbf{CB})^{\dagger}\Upsilon\mathbf{CB}(\Upsilon^{\mathrm{T}}\mathbf{CB})^{\dagger}\Upsilon^{\mathrm{T}}\Big) \\ &\supseteq \mathcal{N}\Big(\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\Big) \\ &= \mathcal{N}\Big(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}\Big)\end{aligned}$$

imply $\mathcal{N}(\mathbf{CB}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}) = \mathcal{N}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon)$.

2) According to the definition in (7), it follows $(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon}\mathbf{CB} = (\Upsilon\mathbf{CB})^{\dagger}\Upsilon\mathbf{CB} = P_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}})}$.
3) Utilizing basic properties of the MP inverse and Definition (7) it can be concluded $(\mathbf{CB})^{\mathrm{V}}_{\sigma\Upsilon} \in (\mathbf{CB})\{2,4\}$. Further

$$\begin{aligned}\mathrm{rank}((\Upsilon\mathbf{CB})^{\mathrm{T}}) &\geq \mathrm{rank}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon) \\ &\geq \mathrm{rank}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon\mathbf{CB}) = \mathrm{rank}((\Upsilon\mathbf{CB})^{\mathrm{T}})\end{aligned}$$

and $\mathcal{R}(\mathbf{UV}) = \mathcal{R}(\mathbf{U}) \iff \Xi_{\mathbf{UV},\mathbf{U}}$ [32] imply

$$\begin{aligned}(\mathbf{CB})^{\mathrm{V}}_{\sigma\Upsilon} &= \mathbf{A}^{(2,4)}_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon),\mathcal{N}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon)} \\ &= (\mathbf{CB})^{(2,4)}_{\mathcal{R}((\Upsilon\mathbf{CB})^{\mathrm{T}}),\mathcal{N}((\Upsilon\mathbf{CB})^{\mathrm{T}}\Upsilon)}.\end{aligned}$$

4) Follows from 3) and known result $\mathcal{N}(\mathbf{UV}) = \mathcal{N}(\mathbf{V}) \iff \Xi_{\mathbf{UV},\mathbf{V}}$ [32].
5) This statement is implied by 3) and the relation $\mathbf{CB}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\mathbf{CB} = \mathbf{CB} \iff \mathrm{rank}(\Upsilon\mathbf{CB}) = \mathrm{rank}(\mathbf{CB})$ [32].
6) Follows from [32, p. 57].
7) Assumption $(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon} \in (\mathbf{CB})^{-1}_{\{\mathrm{V}\}}$ initiates $(\Upsilon\mathbf{CB})^{\dagger}\Upsilon\mathbf{CB} = \mathbf{I}_{n_u}$. According to [32, Lemma 2, p. 43], it follows $\mathrm{rank}(\Upsilon\mathbf{CB}) = n_u$, which implies $\Upsilon \in \mathbb{R}^{n_u\times n_y}_{n_u} \bigwedge \Xi_{\Upsilon\mathbf{CB},\Upsilon}$.
On the other hand, conditions $\Upsilon \in \mathbb{R}^{n_u\times n_y}_{n_u} \bigwedge \Xi_{\Upsilon\mathbf{CB},\Upsilon}$ imply $(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon} = (\Upsilon\mathbf{CB})^{-1}\Upsilon \in (\mathbf{CB})^{-1}_{\{\mathrm{V}\}}$.
8) It is based on $(\mathbf{CB})^{\dagger} = ((\mathbf{CB})^{\mathrm{T}}\mathbf{CB})^{\dagger}(\mathbf{CB})^{\mathrm{T}}$ [32].
9) It follows from the fact that $\mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{C}$ is idempotent.
10) Since $\mathbf{B}(\mathbf{CB})^{\mathrm{V}}_{\sigma|\Upsilon} = \mathbf{B}(\Upsilon\mathbf{CB})^{\dagger}\Upsilon$, this statement follows from the Urquhart representation of generalized inverses [32, Th. 13, p. 72]. ■

Observe that nonsquare right inverses play fundamental roles in the PC scheme design. Indeed, generalized inverses are a useful tool for exploring desirable properties of closed-loop control structures. This phenomenon is clarified in the subsequent sections.

### *C. Perfect Control Stability Characteristic*

The stability of the IMC-related $d$-PC strategy (4), with $d \geq 1$, assuming arbitrary $\mathbf{y}_{\mathbf{ref}}(k+d)$, can be investigated in terms of the representative single-delayed plant being under $\mathbf{y}_{\mathbf{ref}}(k+d) = \mathbf{0}$ [1], [44]. Therefore, in such a scenario, our control formula in the form of

$$\mathbf{u}(k) = -\mathbf{K}\mathbf{x}(k), \quad \mathbf{K} = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA} \tag{10}$$

provides the PC stability expression in the following manner:

$$\det\Big(z\mathbf{I_n} - \mathbf{A} + \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\Big) = 0 \tag{11}$$

where $z$ denotes some complex operator. According to the inverse of the matrix product $\mathbf{CB}$, the subsequent observation should be formulated.

Proposition 1 investigates conditions for a structurally stable pole-free PCD in the case when the generalized right $\sigma$-inverse defined in the canon (5) is applied. The notation $\mathbf{A}_R^{-1}$ (resp. $\mathbf{A}_L^{-1}$) will be used to denote a particular right (resp. left) inverse of $\mathbf{A}$.

*Proposition 1:* Let $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n_u}$, and $\mathbf{C} \in \mathbb{R}^{n_y\times n}$. The structurally stable pole-free PCD can be obtained in two feasible cases.

1) $n_u = n_y = n$,
2) $n_u > n_y = n$, $\text{rank}(\mathbf{B}\Upsilon) = n$.

Configurations other than those summarized above do not satisfy the optimal PC law requirement (3) in the context of the full rank consideration.

*Proof:* Consider the right inverse of $\mathbf{B}\Upsilon$ defined in the case $n_y \geq n$, $\text{rank}(\mathbf{B}\Upsilon) = n$, by $(\mathbf{B}\Upsilon)_R^{-1} = (\mathbf{B}\Upsilon)^{\mathrm{T}}(\mathbf{B}\Upsilon(\mathbf{B}\Upsilon)^{\mathrm{T}})^{-1}$. Then, the following holds.

1) In the first examined scenario both matrices $\mathbf{B}\Upsilon$ and $\mathbf{C}$ are regular-invertible; hence

$$(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} = \Upsilon(\mathbf{B}\Upsilon)^{-1}\mathbf{C}^{-1} \tag{12}$$

which provides only the pole-free (or zero-pole) instances

$$\det\Big(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\Upsilon(\mathbf{B}\Upsilon)^{-1}\mathbf{C}^{-1}\mathbf{CA}\Big) = \det(z\mathbf{I}_n) \tag{13}$$

where $\mathbf{I}_n$ denotes $n \times n$ identity matrix.

2) In the second case, we also obtain the pole-free behavior since the element $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}$ can be rewritten as follows:

$$(\mathbf{CB})^{R}_{\sigma|\Upsilon} = \Upsilon(\mathbf{B}\Upsilon)_R^{-1}\mathbf{C}^{-1} \tag{14}$$

consequently leading to

$$\det\Big(z\mathbf{I}_n - \mathbf{A} + \mathbf{B}\Upsilon(\mathbf{B}\Upsilon)_R^{-1}\mathbf{C}^{-1}\mathbf{CA}\Big) = \det(z\mathbf{I}_n) \tag{15}$$

under $\mathbf{B}\Upsilon(\mathbf{B}\Upsilon)_R^{-1} = \mathbf{I}_n$.

In other scenarios, we do not receive straightforward results, since

$$(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \in (\mathbf{B}\Upsilon)^{-1}_{\{R\}}\mathbf{C}^{-1}_{\{L\}} \tag{16}$$

is not applicable to meet PC law (3). Please see [1]. The proof is completed in all cases. ■

Remark that the relations (16) should be considered in the context of the pole-free stable PC scenario only.

Thus, in the general case, the exemplary closed-loop PC system

$$\mathbf{x}(k+1) = \mathbf{G}\Big(\Upsilon\Big(q^{-1}\Big)\Big)\mathbf{x}(k) \tag{17}$$

such that $\mathbf{G}(\Upsilon(q^{-1})) := \mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}$, possesses the poles

$$\text{eig}(\mathbf{G}) = \Big\{\lambda_1\Big(\Upsilon\Big(q^{-1}\Big)\Big), \lambda_2\Big(\Upsilon\Big(q^{-1}\Big)\Big), \ldots, \lambda_n\Big(\Upsilon\Big(q^{-1}\Big)\Big)\Big\} \tag{18}$$

and, according to the well-known stability theory, the arbitrarily selected $\Upsilon(q^{-1})$ has to meet the crucial condition

$$\Big|\lambda_j\Big(\Upsilon\Big(q^{-1}\Big)\Big)\Big| < 1, \quad j = 1, 2, 3, \ldots, n. \tag{19}$$

Having the PC paradigm notion, we can proceed with the key PC energy issue in the next section.

## IV. Energy Problem Formulation

The IMC-oriented PC energy approach often constitutes the main problem in the discussed control scheme design process. Due to the lack of analytical methods covering the optimization of the PC energy expenditure, the heuristic solutions introduced in the literature are related to the following well-known general expression:

$$E_{\mathbf{u}}(H) = \sum_{k=0}^{H}\Big\{\mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k)\Big\} \tag{20}$$

wherein $\mathbf{u}(k)$ is determined as in (4) and $H$ denotes arbitrarily selected time horizon.

Moreover, the complementary performance index, involving the cases of $\mathbf{y}_{\mathbf{ref}}(k+d) \neq \mathbf{0}$, in the form of

$$E_{\mathbf{u}}(H) = \sum_{k=0}^{H}\Big\{[\mathbf{u}(k) - \mathbf{u}_{\mathbf{ss}}]^{\mathrm{T}}[\mathbf{u}(k) - \mathbf{u}_{\mathbf{ss}}]\Big\} \tag{21}$$

where $\mathbf{u}_{\mathbf{ss}}$ stands for the steady-state control input vector, has also been utilized.

However, because of the heuristic procedures, the time horizon $H$, in both presented energy-based indices, could at most be selected as a considerable number, yet, it naturally has to hold the condition $H \ll +\infty$. Consequently, the results obtained in the $\mathcal{L}_2$-norm-based domain are generally not representative.

Therefore, in our analytical investigation, we have to consider the general case, where $H \to +\infty$. Hence, we should proceed with the subsequent formula

$$E_{\mathbf{u}}(+\infty) = \sum_{k=0}^{+\infty}\Big\{\mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k)\Big\}. \tag{22}$$

Now, the fundamental question of the IMC theory has arisen: what kind of the matrix $\Upsilon(q^{-1})$ involved in the $\sigma$-inverse defined in (5) guarantees the IMC-oriented minimum-energy PCD (4)?

Originally, this issue could be considered as the following minimization:

$$\Upsilon_{\mathbf{opt}}\Big(q^{-1}\Big) = \arg \min_{\Upsilon(q^{-1})} \sum_{k=0}^{+\infty}\Big\{\mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k)\Big\}. \tag{23}$$

In fact, the solution to the fundamental problem (23) can be resolved analytically by

$$\frac{\mathrm{d}\Big\{\sum_{k=0}^{+\infty}\mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k)\Big\}}{\mathrm{d}\Big\{\Upsilon\Big(q^{-1}\Big)\Big\}} = \mathbf{0}. \tag{24}$$

Nevertheless, the presented operation comprising the derivatives of the matrix components has not yet been defined in an analytical manner [32].

*Remark 7:* Accordingly, due to the lack of analytical results in this field, the pseudo-optimal well-known unique MP inverse (9) has commonly been used in the IMC-oriented scheme design processes [20], [30], [32].

## V. Minimum-Energy Problem Solution

To illustrate the intricate nature of the problem concerning the PC energy optimization, we start with the broadly known single-delayed system governed by the simplified state-space framework (1) as follows:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \mathbf{x}(0) = \mathbf{x_0} \tag{25a}$$
$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k). \tag{25b}$$

In such a case, the general PC index (3) can be restated in the form

$$J = \min_{\mathbf{u}(k)} \left\{ \sum_{k=0}^{+\infty} \left\| \mathbf{y}(k+1) - \mathbf{y_{ref}}(k+1) \right\|_2^2 \right\}. \tag{26}$$

Consequently, the complex PC expression (4) boils down into the compact design

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\left[\mathbf{y_{ref}}(k+1) - \mathbf{CA}\mathbf{x}(k)\right] \tag{27}$$

which under zero reference value $\mathbf{y_{ref}}(k+1) = \mathbf{0}$ goes to

$$\mathbf{u}(k) = -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\mathbf{x}(k). \tag{28}$$

Now, for the simple representative scenario associated with (28), the control can be redefined in accordance with the structure (25) through the recursive mechanism as follows:

$$\begin{aligned}
\mathbf{u}(0) &= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\mathbf{x}(0)\\
\mathbf{u}(1) &= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\mathbf{x}(1)\\
&= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\left(\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\right)\mathbf{x}(0)\\
\mathbf{u}(2) &= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\mathbf{x}(2)\\
&= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\left(\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\right)^2\mathbf{x}(0)\\
&\vdots\\
\mathbf{u}(m) &= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\mathbf{x}(m)\\
&= -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\left(\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\right)^m\mathbf{x}(0).
\end{aligned} \tag{29}$$

The rationale for employment of the initial condition $\mathbf{x_0}$ in pieces (29) is that we can rewrite the complex formula (22) in the following compact common form:

$$\begin{aligned}
E_{\mathbf{u}}(+\infty) = \sum_{k=0}^{+\infty}\Bigg\{&\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\left(\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\right)^k\mathbf{x}(0)\right]^{\mathrm{T}}\\
&\times\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\left(\mathbf{A} - \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon}\mathbf{CA}\right)^k\mathbf{x}(0)\right]\Bigg\}
\end{aligned} \tag{30}$$

which will be useful for obtaining a $\sigma$-inverse-oriented $\Upsilon(q^{-1})$ matrix that guarantees the minimum-energy IMC-based PCD.

### A. Perfect Regulation Energy Solution

The perfect regulation term is related to the PC strategy with the zero reference value $\mathbf{y_{ref}}(k+1) = \mathbf{0}$. In such a case, the minimum-energy solution for the second-order MISO LTI discrete-time state-space systems has already been formulated in the recent paper [4].

Accordingly, the closed-loop perfect regulation-based state-space structure (17) can be transformed into the operator representation

$$z(\mathbf{X}(z) - \mathbf{X}(0)) = \mathbf{G}(\boldsymbol{\beta_0})\mathbf{X}(z) \tag{31}$$

which can be prescribed in the following way:

$$(z\mathbf{I_n} - \mathbf{G}(\boldsymbol{\beta_0}))\mathbf{X}(z) = z\mathbf{X}(0). \tag{32}$$

Now, the solution to the given relation can be combined as $\mathbf{r} + \mathbf{q}$, where the outcome $\mathbf{r}$ and an arbitrary $\mathbf{q}$ have to fulfill the consolidated expression

$$\{(z\mathbf{I_n} - \mathbf{G}(\boldsymbol{\beta_0}))\mathbf{r} = z\mathbf{X}(0) \wedge \mathbf{q} \in \ker(z\mathbf{I_n} - \mathbf{G}(\boldsymbol{\beta_0}))\}. \tag{33}$$

Therefore, the essential formula appears for the second-order state-space plants

$$\mathbf{q} \in \ker(\lambda_1(\boldsymbol{\beta_0})\mathbf{I_2} - \mathbf{G}(\boldsymbol{\beta_0})) \tag{34}$$

where $\lambda_1(\boldsymbol{\beta_0})$ denotes the single nonzero pole of the closed-loop control system (17), giving rise to the formulation of the key relation

$$\mathbf{x}(k) \in \ker(\lambda_1(\boldsymbol{\beta_0})\mathbf{I_2} - \mathbf{G}(\boldsymbol{\beta_0})), \quad k \geq 1. \tag{35}$$

Furthermore, the canon (35) can be reformulated in the following way:

$$\mathbf{G}(\boldsymbol{\beta_0})\mathbf{x}(k) = \lambda_1(\boldsymbol{\beta_0})\mathbf{x}(k), \quad k \geq 1 \tag{36}$$

and in consequence, the control formula (28) can now be designated by the sophisticated rule

$$\mathbf{u}(1+j) = -(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\left(\lambda_1(\boldsymbol{\beta_0})\right)^j\mathbf{x}(1), \quad j = 0, 1, 2, \ldots \tag{37}$$

Thus, the energy performance index (30) can now be divided into two main parts as follows:

$$E_{\mathbf{u}}(+\infty) = \mathbf{u}^{\mathrm{T}}(0)\mathbf{u}(0) + \sum_{k=1}^{+\infty}\left\{\mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k)\right\} \tag{38}$$

with the second component certainly arranging the method (37), giving rise to the following explicit form revealing the geometric sequence notion:

$$\begin{aligned}
E_{\mathbf{u}}(+\infty) = &\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(0)\right]^{\mathrm{T}}\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(0)\right]\\
&+\sum_{k=1}^{+\infty}\Bigg\{\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(1)\right]^{\mathrm{T}}\left[\left((\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(1)\right]\right.\\
&\times\left(\lambda_1(\boldsymbol{\beta_0})\right)^{2(k-1)}\Bigg\}.
\end{aligned} \tag{39}$$

After taking into account the stability condition (19) imposing $|\lambda_1(\boldsymbol{\beta_0})| < 1$ along with the geometric sequence peculiarities, we finally arrive at the following fundamental expression:

$$\begin{aligned}
E_{\mathbf{u}}(+\infty) = &\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(0)\right]^{\mathrm{T}}\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CA}\mathbf{x}(0)\right]\\
&+\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CAG}(\boldsymbol{\beta_0})\mathbf{x}(0)\right]^{\mathrm{T}}\\
&\times\left[(\mathbf{CB})^{\mathrm{R}}_{\sigma|\boldsymbol{\beta_0}}\mathbf{CAG}(\boldsymbol{\beta_0})\mathbf{x}(0)\right]\\
&\times\left[1 - [\mathrm{Tr}(\mathbf{G}(\boldsymbol{\beta_0}))]^2\right]^{-1}.
\end{aligned} \tag{40}$$

The presented formula allows us to determine the total energy of the PC input signals analytically. In addition, for the considered MISO case, the relation (24) can now be redefined in the form

$$\frac{\mathrm{d}\{E_{\mathbf{u}}(+\infty)\}}{\mathrm{d}\{\beta_{\mathbf{0}}\}} = \begin{cases} \frac{\partial\{E_{\mathbf{u}}(+\infty)\}}{\partial\{\beta_{0_1}\}} = 0 \\ \frac{\partial\{E_{\mathbf{u}}(+\infty)\}}{\partial\{\beta_{0_2}\}} = 0 \\ \vdots \\ \frac{\partial\{E_{\mathbf{u}}(+\infty)\}}{\partial\{\beta_{0_{n_u}}\}} = 0 \end{cases} \tag{41}$$

under

$$\beta_{\mathbf{0}}^{\mathrm{T}} = \begin{bmatrix} \beta_{0_1} & \beta_{0_2} & \dots & \beta_{0_{n_u}} \end{bmatrix} \tag{42}$$

finally providing the minimum-energy solution subject to the crucial index (23).

*Remark 8:* It has to be recalled that the complete analytical evidence concerning the presented complex material can be found in [4].

Still, the given results are justified only for second-order MISO plants associated with the zero reference value $\mathbf{y_{ref}}(k+1) = \mathbf{0}$. However, what about the cases with a nonzero setpoint $\mathbf{y_{ref}}(k+1) \neq \mathbf{0}$, for which the above investigation is not valid? Moreover, is it possible to consider the general scenario of the entire class of single-delayed LTI MIMO state-space systems with more than one nonzero pole? The breakthrough in these matters is presented in the subsequent key section.

### *B. Perfect Control Energy Solution*

Let us consider the PC formula (27) with the nonzero reference value $\mathbf{y_{ref}}(k+1) \neq \mathbf{0}$, being under the parameter $\sigma$-inverse canon (8), in the ensuing configuration

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_{\mathbf{0}}}\big[\mathbf{y_{ref}}(k+1) - \mathbf{CAx}(k)\big]. \tag{43}$$

Notice that in such a case, the performance indices (20) and (21) can be utilized interchangeably in the context of the minimum-energy issue. Since we are searching for the general solution, we must operate in the infinite time horizon. Therefore, the formula (21) has to be rewritten as follows:

$$E_{\mathbf{u}}(+\infty) = \sum_{k=0}^{+\infty}\big\{[\mathbf{u}(k) - \mathbf{u_{ss}}]^{\mathrm{T}}[\mathbf{u}(k) - \mathbf{u_{ss}}]\big\}. \tag{44}$$

At this point, we should formulate the following vital theorem.

*Theorem 1:* The energy-based performance expression (44) can be redefined in terms of the minimum-energy solution exploration as follows:

$$E_{\mathbf{u_{ss}}}(+\infty) = \lim_{k\to+\infty} \mathbf{u}^{\mathrm{T}}(k)\mathbf{u}(k) = \mathbf{u}_{\mathbf{ss}}^{\mathrm{T}}\mathbf{u_{ss}}. \tag{45}$$

*Proof:* The formula (44) is utterly dependent on the value of the steady-state control input vector $\mathbf{u_{ss}}$, so the transient control values (for $k < +\infty$) can be omitted under the infinite time horizon consideration, hence the proof follows. ∎

In order to exploit the crucial result of Theorem 1, we have to designate the value of the steady-state control vector $\mathbf{u_{ss}}$ as follows:

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_{\mathbf{0}}}\big[\mathbf{y_{ref}}(+\infty) - \mathbf{CAx}(+\infty)\big]. \tag{46}$$

*Remark 9:* Due to the complexity of the analytical energy-oriented investigation, we assume, w.l.o.g., that the value of the setpoint $\mathbf{y_{ref}}(k)$ is constant $\mathbf{y_{ref}}$.

So, after engaging the control algorithm (43) to the state-space framework (25), we get

$$\mathbf{x}(k+1) = \mathbf{G}(\beta_{\mathbf{0}})\mathbf{x}(k) + \Omega(\beta_{\mathbf{0}}) \tag{47}$$

with $\Omega(\beta_{\mathbf{0}}) = \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_{\mathbf{0}}}\mathbf{y_{ref}}$ and the $\mathbf{G}(\beta_{\mathbf{0}})$ derived from (17).

*Remark 10:* In order to present new results in a legible manner, we already propose the simplifying nomenclature: $\mathbf{G}_{\beta_{\mathbf{0}}} \equiv \mathbf{G}(\beta_{\mathbf{0}})$ and $\Omega_{\beta_{\mathbf{0}}} \equiv \Omega(\beta_{\mathbf{0}})$.

Observe that the structure (47) develops in the recursive way

$$\mathbf{x}(k+2) = \mathbf{G}_{\beta_{\mathbf{0}}}\mathbf{x}(k+1) + \Omega_{\beta_{\mathbf{0}}} \tag{48}$$

which comes down to

$$\mathbf{x}(k+2) = \mathbf{G}^2_{\beta_{\mathbf{0}}}\mathbf{x}(k) + \mathbf{G}_{\beta_{\mathbf{0}}}\Omega_{\beta_{\mathbf{0}}} + \Omega_{\beta_{\mathbf{0}}}. \tag{49}$$

Accordingly, in the further step we obtain

$$\mathbf{x}(k+3) = \mathbf{G}_{\beta_{\mathbf{0}}}\mathbf{x}(k+2) + \Omega_{\beta_{\mathbf{0}}} \tag{50}$$

and finally

$$\mathbf{x}(k+3) = \mathbf{G}^3_{\beta_{\mathbf{0}}}\mathbf{x}(k) + \mathbf{G}^2_{\beta_{\mathbf{0}}}\Omega_{\beta_{\mathbf{0}}} + \mathbf{G}_{\beta_{\mathbf{0}}}\Omega_{\beta_{\mathbf{0}}} + \Omega_{\beta_{\mathbf{0}}} \tag{51}$$

enabling the establishment of the yet unexplored general expression

$$\mathbf{x}(k+m) = \mathbf{G}^m_{\beta_{\mathbf{0}}}\mathbf{x}(k) + \Big[\mathbf{I_n} + \mathbf{G}_{\beta_{\mathbf{0}}} + \mathbf{G}^2_{\beta_{\mathbf{0}}} + \dots + \mathbf{G}^{m-1}_{\beta_{\mathbf{0}}}\Big]\Omega_{\beta_{\mathbf{0}}}. \tag{52}$$

Now, the stability-oriented closed-loop control matrix $\mathbf{G}_{\beta_{\mathbf{0}}}$ can be presented in the Jordan Canonical form (JCF) as

$$\mathbf{G}_{\beta_{\mathbf{0}}} = \mathbf{HQ}(\beta_{\mathbf{0}})\mathbf{H}^{-1} \tag{53}$$

with the proper matrix $\mathbf{H}$ and the $\mathbf{Q}(\beta_{\mathbf{0}})$ containing the $\mathbf{G}_{\beta_{\mathbf{0}}}$'s eigenvalues on the main diagonal.

Moreover, after taking into account the fact that all control system's poles have to be located inside the unit circle (19), we receive the following outcome:

$$\mathbf{G}^{+\infty}_{\beta_{\mathbf{0}}} = \mathbf{HQ}^{+\infty}(\beta_{\mathbf{0}})\mathbf{H}^{-1} = \mathbf{0}. \tag{54}$$

Thus, the relation (52) under the above investigation accompanied by the crucial geometric sequence property reveals

$$\Big[\mathbf{I_n} + \mathbf{G}_{\beta_{\mathbf{0}}} + \mathbf{G}^2_{\beta_{\mathbf{0}}} + \dots + \mathbf{G}^{m-1}_{\beta_{\mathbf{0}}}\Big] = \Big[\mathbf{I_n} - \mathbf{G}^m_{\beta_{\mathbf{0}}}\Big]\Big[\mathbf{I_n} - \mathbf{G}_{\beta_{\mathbf{0}}}\Big]^{-1} \tag{55}$$

which for $m \to +\infty$ goes to the following important statement:

$$\mathbf{x}(+\infty) = \big[\mathbf{I_n} - \mathbf{G}_{\beta_{\mathbf{0}}}\big]^{-1}\Omega_{\beta_{\mathbf{0}}}. \tag{56}$$

Hence, the steady-state PC (46) can now be presented in a deterministic way as follows:

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{y_{ref}} - \mathbf{CA}\big[\mathbf{I_n} - \mathbf{G}_{\beta_0}\big]^{-1}\Omega_{\beta_0}\Big] \quad (57)$$

which can be rewritten in the final complex form

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{I_{n_y}} - \mathbf{CA} \times \Big[\mathbf{I_n} - \mathbf{A} + \mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}\Big]^{-1}\mathbf{B}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big]\mathbf{y_{ref}}. \quad (58)$$

Remarkably, the introduced control methodology (58) enables to analytically designate an accurate value of the control runs in the steady state. Moreover, the above formula does not rely on the initial condition $\mathbf{x_0}$. This fact is very intriguing in the context of the general minimum-energy solution, since in the case of $\mathbf{y_{ref}} = \mathbf{0}$, see (28), the optimal $\beta$ (23) just strictly depends on the initial condition, i.e., $\beta_{\mathbf{opt}}(\mathbf{x_0})$. Such an issue additionally emphasizes the intricate nature of the PC law.

According to the previous investigation, we have to solve the following fundamental expression:

$$\frac{\mathrm{d}\{E_{\mathbf{u_{ss}}}(+\infty)\}}{\mathrm{d}\{\beta_\mathbf{0}\}} = \mathbf{0} \quad (59)$$

with the control defined as in (58).

Such an operation can now be extended to the subsequent set of relations

$$\frac{\mathrm{d}\{E_{\mathbf{u_{ss}}}(+\infty)\}}{\mathrm{d}\{\beta_\mathbf{0}\}} = \begin{cases} \frac{\partial\{E_{\mathbf{u_{ss}}}(+\infty)\}}{\partial\{\beta_{0_{11}}\}} = 0 \\ \frac{\partial\{E_{\mathbf{u_{ss}}}(+\infty)\}}{\partial\{\beta_{0_{12}}\}} = 0 \\ \vdots \\ \frac{\partial\{E_{\mathbf{u_{ss}}}(+\infty)\}}{\partial\{\beta_{0_{n_u n_y}}\}} = 0 \end{cases} \quad (60)$$

under

$$\beta_\mathbf{0} = \begin{bmatrix} \beta_{0_{11}} & \beta_{0_{12}} & \dots & \beta_{0_{1n_y}} \\ \vdots & & \ddots & \vdots \\ \beta_{0_{n_u 1}} & \dots & \beta_{0_{n_u n_y - 1}} & \beta_{0_{n_u n_y}} \end{bmatrix}. \quad (61)$$

It is striking that the presented methodology can provide the minimum-energy solution to (23). Moreover, we can define in an analytical mode the minimum-energy PCD for cases with a nonzero reference value $\mathbf{y_{ref}}(k+1) \neq \mathbf{0}$.

The given outcome significantly impacts the fundamentals covering the control and systems theory. This issue will be explained in detail in the following essential section.

## VI. Moore–Penrose Paradigm Challenge

The matrix inverse and generalized inverse formulas have been utilized in a number of scientific and engineering fields, going far beyond the control-oriented applications, to mention physics [38], medicine [45], or economics [46]. Moreover, at the start of this section, it should be recalled and strongly highlighted that the MP generalized inverses have without a doubt been treated as the optimal ones, since they minimize the Euclidean norm in every general case [31], [32], [38], [39]. However, based on the new analytical investigation conducted in the previous section, we can now challenge the optimal peculiarity of the minimum-norm MP pseudoinverse. This novelty is touched upon in the following motivation example.

### A. Motivation Example

Consider the exemplary single-delayed right-invertible LTI MIMO plant $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$ described by the discrete-time state-space framework (25) with

$$\mathbf{A} = \begin{bmatrix} 1.5 & -2.2 & 1 \\ 0.1 & 1.6 & 0.2 \\ 0.2 & 1 & -0.9 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1.2 & 1 & 1 \\ -0.3 & 1.5 & -1 \\ -1 & 0.2 & 1.4 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 2 & 0.5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

and the initial condition $\mathbf{x_0} = \mathbf{0}$ under the nonzero reference value $\mathbf{y_{ref}} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$. The crucial energy performance index (45), derived from the PC algorithm (43) in the form (58) subjected to the MP inverse (9), is equal to $E_{\mathbf{u_{ss_0}}}(+\infty) = 100.1264$, whilst $\mathbf{u_{ss_0}}(+\infty) = \begin{bmatrix} 4.1305 \\ 1.7552 \\ 8.9434 \end{bmatrix}$ with the closed-loop control system's poles $\mathrm{eig}(\mathbf{G}) = \{0.3398, 0, 0\}$, see (18).

Now, let us move on to the critical minimum-energy examination. We start with the next breakthrough theorem.

*Theorem 2:* The application of the unique MP inverse (9) to the state-space PC law (43) does not guarantee the minimum-energy behavior of the control. In other words, the minimum-norm MP pseudoinverse does not generally minimize the Euclidean-oriented norm (45), and this fact can be manifested by the equivalent explicit form

$$E_{\mathbf{u_{ss}}}(+\infty) = \left\|\mathbf{u}(+\infty)\right\|_2^2. \quad (62)$$

*Proof:* The considered state-space structure (25) can easily be transformed, under the zero initial condition $\mathbf{x_0} = \mathbf{0}$, to the equipollent input-output-related $z$-transfer-function domain in the following manner:

$$\mathbf{y} = \mathbf{C}(z\mathbf{I_n} - \mathbf{A})^{-1}\mathbf{Bu}. \quad (63)$$

Now, after taking into account the well-known final value theorem (FVT) in $z$-transform

$$\lim_{k\to+\infty} f(k) = \lim_{z\to 1}\left(1 - z^{-1}\right)F(z) \quad (64)$$

engaging the fixed reference value $\mathbf{y_{ref}}(k) = \mathbf{y_{ref}}$ for $k \geq 0$, the stable steady state reveals

$$\mathbf{Hu_{ss}} = \mathbf{y_{ref}} \quad (65)$$

where $\mathbf{H} = \mathbf{C}(\mathbf{I_n} - \mathbf{A})^{-1}\mathbf{B}$.

Notice that the given formula corresponds to the broadly known primary linear equation $\mathbf{Ax} = \mathbf{b}$ and its optimal solution for the full rank $\mathbf{A} \in \mathbb{R}^{n_y \times n_u}$ with $n_y \leq n_u$ is $\mathbf{x} = \mathbf{A}_0^{\mathrm{R}}\mathbf{b} = \mathbf{A}^{\mathrm{T}}(\mathbf{AA}^{\mathrm{T}})^{-1}\mathbf{b}$ [31], [32].

Thus, we can state with full confidence that in the nonsquare transfer-function-originated case, the unique MP inverse (9) always ensures the optimal solution in the form of

$$\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}} = \mathbf{H}^{\mathrm{T}}\left(\mathbf{H}\mathbf{H}^{\mathrm{T}}\right)^{-1}\mathbf{y}_{\mathbf{ref}} \tag{66}$$

which, for our exemplary state-space plant $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$, produces $\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}} = \begin{bmatrix} 6.9251 \\ 1.3765 \\ 3.3922 \end{bmatrix}$ with the control energy (45) equal to $E_{\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}} = 61.3584$.

It has to be mentioned here that the received result in the input-output domain provides the notion of the optimal control value in the steady state.

Notice that the given MP-based outcomes in the discussed state-space and input-output structures differ, i.e., $\mathbf{u}_{\mathbf{ss}_\mathbf{0}}(+\infty) \neq \mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}$, and consequently, the energies also vary $E_{\mathbf{u}_{\mathbf{ss}_\mathbf{0}}}(+\infty) > E_{\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}}$, so the proof follows. ■

Notwithstanding, the question remains: which inverse formula guarantees the minimum-energy PCD for the state-space plant $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$? The answer to this issue can now be established through the new methods defined in the previous section. Theorem 3 gives answer in this particular case.

*Theorem 3:* The right $\sigma$-inverse (8), containing the properly selected set of DOFs, guarantees the minimum-energy PCD for the state-space object $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$ in the case $\mathbb{R}^{2\times 3}$.

*Proof:* In the beginning, it is required to engage the parameter $\sigma$-inverse-related DOFs in the symbolic form

$$\beta_{\mathbf{0}}^{\mathrm{T}} = \begin{bmatrix} \beta_{0_{11}} & \beta_{0_{12}} & \beta_{0_{13}} \\ \beta_{0_{21}} & \beta_{0_{22}} & \beta_{0_{23}} \end{bmatrix} \tag{67}$$

to the PC formula in the steady state, see (58).

Now, after applying Theorem 1 to the key expression (60) we receive the collection of six equations. Throughout the algebraic calculations, we obtain the set of solutions

$$\beta_{\mathbf{0}_{\mathbf{opt}}} = \{\beta_{\mathbf{s1}}, \quad \beta_{\mathbf{s2}}\} \tag{68}$$

where

$$\beta_{\mathbf{s1}}^{\mathrm{T}} = \begin{bmatrix} 2.0414 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{69}$$

and

$$\beta_{\mathbf{s2}}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 2.0414 & 0.4058 & 1 \end{bmatrix}. \tag{70}$$

Amazingly, for both solutions $\beta_{\mathbf{s1}}$ and $\beta_{\mathbf{s2}}$ the state-space PC law (58) comes down to $\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0}_{\mathbf{opt}}}}(+\infty) = \begin{bmatrix} 6.9251 \\ 1.3765 \\ 3.3922 \end{bmatrix}$ and the energy index (62) goes to $E_{\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0}_{\mathbf{opt}}}}}(+\infty) = 61.3584$ under the closed-loop object's eigenvalues equal to $\mathrm{eig}(\mathbf{G}) = \{-0.7060, 0, 0\}$.

Observe that the optimal state-space-related design is established, since $\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0}_{\mathbf{opt}}}}(+\infty) = \mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}$ as well as $E_{\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0}_{\mathbf{opt}}}}}(+\infty) = E_{\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}}$, what ends the proof. ■

*Remark 11:* According to Theorem 3, the right $\sigma$-inverse, possessing the appropriate DOFs, provides the minimum-energy PCD in the case $\mathbb{R}^{2\times 3}$, and it outperforms the pseudo-optimal unique MP inverse under the state-space investigation.

Nevertheless, what about the cases with a time delay $d > 1$? Can the presented approach be employed in such instances? Moreover, is it possible to improve the control inputs behavior through the polynomial forms of the right $\sigma$-inverse (5)? The answers to these issues and the analytical extension of the presented methodology are presented in the subsequent section.

## VII. General Minimum-Energy Solution

In this vital section, we have addressed the problem of the minimum-energy IMC-based PCD for the multivariable $d$-state-space systems of the delay $d \geq 1$ and an arbitrary nonzero reference value $\mathbf{y}_{\mathbf{ref}} \in \mathbb{R}^{n_y} \setminus \{\mathbf{0}\}$.

Thus, we start with the pivotal observation covering the steady-state peculiarity, which is presented in the following theorem.

*Theorem 4:* In the steady state under the PC force (4), the key relations

$$\mathbf{x}(+\infty) = \mathbf{x}(+\infty - 1) = \mathbf{x}(+\infty - 2) = \cdots \tag{71}$$

and

$$\mathbf{u}(+\infty) = \mathbf{u}(+\infty - 1) = \mathbf{u}(+\infty - 2) = \cdots \tag{72}$$

appear in every stable general case.

*Proof:* Immediately after considering the fact that only asymptotically stable closed-loop control plants (17)–(19) are examined, the proof follows. ■

Therefore, after taking into account the above perception we can just propose the new approach to the PC energy issue.

Now, the breakthrough Theorem 1 requires the value of the PC signals in the steady state. Contrary to the authors' previous geometric-related solutions, we additionally introduce the new remarkable method to the world control society.

Thus, according to the above consideration it is clear that the state equation of the $d$-step structure (1) in the form of

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k-d+1) \tag{73}$$

comes down to

$$\mathbf{x}(+\infty) = \mathbf{A}\mathbf{x}(+\infty) + \mathbf{B}\mathbf{u}(+\infty) \tag{74}$$

under the steady-state examination subjected to the relations (71) and (72).

Moreover, the above formula can easily be rewritten as follows:

$$(\mathbf{I_n} - \mathbf{A})\mathbf{x}(+\infty) = \mathbf{B}\mathbf{u}(+\infty) \tag{75}$$

resulted in

$$\mathbf{x}(+\infty) = (\mathbf{I_n} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(+\infty) \tag{76}$$

for $(\mathbf{I_n} - \mathbf{A})$ related to the full rank $n$.

As a consequence, the PC algorithm (46) can now be presented as follows:

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_\mathbf{0}}\left[\mathbf{y}_{\mathbf{ref}} - \mathbf{CA}(\mathbf{I_n} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(+\infty)\right] \tag{77}$$

or rather

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_\mathbf{0}}\mathbf{y}_{\mathbf{ref}} - (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_\mathbf{0}}\mathbf{CA}(\mathbf{I_n} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(+\infty) \tag{78}$$

with a slight modification

$$\left[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\right]\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{79}$$

giving rise to the new remarkable formula

$$\mathbf{u}(+\infty) = \left[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\right]^{-1} \times (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}}. \tag{80}$$

Intriguingly, the presented approach also allows us to designate the accurate control value in the steady state. Moreover, the new method (80) provides the same results as in the case of the rule (58). Furthermore, the geometric-oriented methodology presented in Section V can only be applied to the single-delayed state-space plants. However, the newly introduced issue, heavily arranging the expressions (71) and (72) of Theorem 4, enables the examination of the systems with any $d \geq 1$.

Thus, to define an expanded PC paradigm, let us investigate the state-space object with $d = 2$, which can be drafted by the discrete-time state-space framework (1) as follows:

$$\mathbf{x}(k+1) = \mathbf{Ax}(k) + \mathbf{Bu}(k)q^{-1}, \mathbf{x}(0) = \mathbf{x_0} \tag{81a}$$
$$\mathbf{y}(k) = \mathbf{Cx}(k). \tag{81b}$$

In such a case, the general PC (4) presented as follows:

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\left[\mathbf{y_{ref}} - \mathbf{CABu}(k-1) - \mathbf{CA}^2\mathbf{x}(k)\right] \tag{82}$$

goes to

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\left[\mathbf{y_{ref}} - \mathbf{CABu}(+\infty-1) - \mathbf{CA}^2\mathbf{x}(+\infty)\right] \tag{83}$$

in the steady state.

Now, according to the canons (72) and (76), the above rule comes down to the item

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{y_{ref}} - \mathbf{CABu}(+\infty) - \mathbf{CA}^2(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{Bu}(+\infty)\Big] \tag{84}$$

which after simple manipulations can be rewritten to

$$\left[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\right] \times \mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{85}$$

finally providing

$$\mathbf{u}(+\infty) = \Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}}. \tag{86}$$

The same investigation can be carried out for instances with $d = 3$. In such a scenario, the steady state of the complex PC algorithm (4) is expressed as follows:

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{y_{ref}} - \mathbf{CABu}(+\infty-1) - \mathbf{CA}^2\mathbf{Bu}(+\infty-2) - \mathbf{CA}^3\mathbf{x}(+\infty)\Big] \tag{87}$$

or rather

$$\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{y_{ref}} - \mathbf{CABu}(+\infty) - \mathbf{CA}^2\mathbf{Bu}(+\infty) - \mathbf{CA}^3\mathbf{x}(+\infty)\Big]. \tag{88}$$

Thus, considering the previous study, we can write the following formula:

$$\Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2\mathbf{B} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^3(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]\mathbf{u}(+\infty) = (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{89}$$

which provides the subsequent solution for $d = 3$

$$\mathbf{u}(+\infty) = \Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2\mathbf{B} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^3(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}}. \tag{90}$$

Consequently, the following steady-state $d$-PC input values in the forms of:

$$\mathbf{u_{d=1}}(+\infty) = \Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{91}$$

and

$$\mathbf{u_{d=2}}(+\infty) = \Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{92}$$

as well as

$$\mathbf{u_{d=3}}(+\infty) = \Big[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CAB} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^2\mathbf{B} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{CA}^3(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{B}\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{93}$$

clearly identify the prevailing trend related to the crucial time delay $d$.

Therefore, the general $d$-PC methodology covering the value of the control input runs in the steady state can now be presented as follows:

$$\mathbf{u}(+\infty) = \left[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{C}\left[\left(\sum_{k=1}^{d-1}\mathbf{A}^k\right) + \mathbf{A}^d(\mathbf{I_n}-\mathbf{A})^{-1}\right]\mathbf{B}\right]^{-1} \times (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{94}$$

giving rise to the elegant complete $d$-PC formula

$$\mathbf{u}(+\infty) = \left[\mathbf{I_{n_u}} + (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Pi(d)\right]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{95}$$

where

$$\Pi(d) = \mathbf{C}\left[\left(\sum_{k=1}^{d-1}\mathbf{A}^k\right) + \mathbf{A}^d(\mathbf{I_n}-\mathbf{A})^{-1}\right]\mathbf{B}. \tag{96}$$

Remarkably, the newly introduced methods concern all LTI multivariable discrete-time $d$-state-space plants. The novel established innovation, which has never been investigated previously, significantly generalizes the knowledge corresponding to the yet undiscovered IMC theory.

To express explicitly the general $\beta_0$-oriented minimum-energy IMC-based PCD rule, we should provide a solution to the equation

$$\frac{\mathrm{d}\left\{\left\|\left[\mathbf{I}_{n_u}+\beta_0[\mathbf{CB}\beta_0]^{-1}\Pi(d)\right]^{-1}\beta_0[\mathbf{CB}\beta_0]^{-1}\mathbf{y_{ref}}\right\|_2^2\right\}}{\mathrm{d}\{\beta_0\}}=\mathbf{0}. \tag{97}$$

Unfortunately, due to the lack of analytical matrix-oriented methods, the derivative involved in (97) cannot be resolved directly [32]. Nevertheless, in such a case, we can utilize the previously defined energy-oriented analytical procedures (59)–(61).

The following question remains: can the polynomial form of the right $\sigma$-inverse (5) outperform the parameter one (8) under the nonzero setpoint consideration? This subject is touched in the subsequent theorem.

*Theorem 5:* The polynomial right $\sigma$-inverse (5), with the arbitrarily selected order $\delta$ supported by the freely chosen polynomial DOFs providing the stability conditions (17)–(19), cannot decrease the energy consumption of the PC input runs strictly derived from $\mathbf{y_{ref}} \neq \mathbf{0}$ and the parameter $\sigma$-inverse (8) employing the parameter optimal $\Upsilon_{\mathbf{0_{opt}}} = \beta_{\mathbf{0_{opt}}}$.

*Proof:* In the steady state, the crucial relations (71) and (72) hold; hence, the $q$-time-depended pieces associated with the matrix polynomial $\Upsilon(q^{-1})$ are just constant. Therefore, the polynomial DOFs of $\Upsilon(q^{-1})$ (6) can now be redefined, under the steady-state investigation, to the vital observation

$$\Upsilon\left(q^{-1}\right)=(\beta_0+\beta_1+\beta_2+\cdots+\beta_\delta)q^0 \tag{98}$$

which coincides to the parameter form of the $\sigma$-inverse with $\Upsilon_{\mathbf{0_{opt}}}$. Since $q^0 = 1$, the proof follows. ■

It is understandable that the polynomial $\sigma$-inverse does not have to be considered in the context of a minimum-energy solution, as $\Upsilon_{\mathbf{opt}}(q^{-1}) \to \beta_{\mathbf{0_{opt}}}$ and consequently $(\mathbf{CB})^{\mathrm{R}}_{\sigma|\Upsilon} \to (\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}$ under $\mathbf{y_{ref}} \in \mathbb{R}^{n_y} \setminus \{\mathbf{0}\}$.

In the end, after taking into account the fact that the parameter $\sigma$-inverse-related DOFs can comprise any value of $\beta_0 \in \mathbb{R}^{n_u \times n_y} \setminus \{\mathbf{0}\}$, including $\beta_0 = (\mathbf{CB})^{\mathrm{T}}$, we propose the following closing theorem.

*Theorem 6:* The parameter right $\sigma$-inverse (8) with appropriately selected DOFs guarantees, in every general case, the minimum-energy IMC-oriented PCD for the entire class of LTI MIMO discrete-time $d$-state-space plants.

*Proof:* Immediately, after considering the whole discussed investigation deeply relying on the original concepts of Theorems 1 and 4. ■

Now, to exhibit the decisive nature of the obtained results, we present the numerical example in the next examination.

## *A. Simulation Example*

Let us consider the exemplary LTI MIMO plant $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with $d = 3$ described by the discrete-time state-space structure (1) as follows:

$$\mathbf{x}(k+1)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)q^{-2},\ \ \mathbf{x}(0)=\mathbf{0} \tag{99a}$$

$$\mathbf{y}(k)=\mathbf{C}\mathbf{x}(k) \tag{99b}$$

being under

$$\mathbf{A}=\begin{bmatrix}1.5 & -2.2 & 1\\ 0.1 & 1.6 & 0.2\\ 0.2 & 1 & -0.9\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}1.2 & 1 & 1\\ -0.3 & 1.5 & -1\\ -1 & 0.2 & 1.4\end{bmatrix}$$

$$\mathbf{C}=\begin{bmatrix}2 & 0.5 & 0\\ 1 & 2 & -1\end{bmatrix}$$

and the nonzero reference value $\mathbf{y_{ref}} = \begin{bmatrix}-5\\ 6\end{bmatrix}$.

In conditions of such a scenario, the general PC algorithm (4) boils down to

$$\mathbf{u}(k)=(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Big[\mathbf{y_{ref}}-\mathbf{CABu}(k-1) - \mathbf{CA}^2\mathbf{Bu}(k-2)-\mathbf{CA}^3\mathbf{x}(k)\Big]. \tag{100}$$

The newly introduced energy-oriented PC methodology (95) in the form of

$$\mathbf{u}(+\infty)=\Big[\mathbf{I}_{n_u}+(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\Pi(3)\Big]^{-1}(\mathbf{CB})^{\mathrm{R}}_{\sigma|\beta_0}\mathbf{y_{ref}} \tag{101}$$

with $\Pi(3)$ specified according to (96), allows us to settle the minimum-energy issue for the considered state-space plant $S_e(\mathbf{A}, \mathbf{B}, \mathbf{C})$ with $d = 3$.

After arranging the pseudo-optimal MP inverse (9) to the product of **CB**, the essential energy performance index (45) under the PC method (101) surprisingly again goes to $E_{\mathbf{u_{ss_0}}}(+\infty) = 100.1264$, whilst $\mathbf{u_{ss_0}}(+\infty) = \begin{bmatrix}4.1305\\ 1.7552\\ 8.9434\end{bmatrix}$.

Naturally, the poles of the 3-step system in the form of $\mathrm{eig}(\mathbf{G}) = \{0.3398, 0, 0\}$ correspond to those obtained for the single-delayed instance, which is in relation to the PC stability characteristic of Section III-C.

The runs of the state, control, and output variables of the system (99) under IMC force (100) are presented in Figs. 1–3.

*Remark 12:* Notice that in the input-output scenario the general canon

$$\mathbf{y}(+\infty)=\mathbf{C}(\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{Bu}(+\infty-d+1) \tag{102}$$

changes, according to Theorem 4, to

$$\mathbf{y}(+\infty)=\mathbf{C}(z\mathbf{I_n}-\mathbf{A})^{-1}\mathbf{Bu}(+\infty) \tag{103}$$

hence the formula (66) holds for any $d$-step object.

Therefore, the transfer-function-based control expression (66) provides the issue of the optimal control value in the steady state in every $d$-step case. Thus, for any $d$ we have $\mathbf{u_{ss_{opt}}} = \begin{bmatrix}6.9251\\ 1.3765\\ 3.3922\end{bmatrix}$ with the energy (45) equal to $E_{\mathbf{u_{ss_{opt}}}} = 61.3584$.

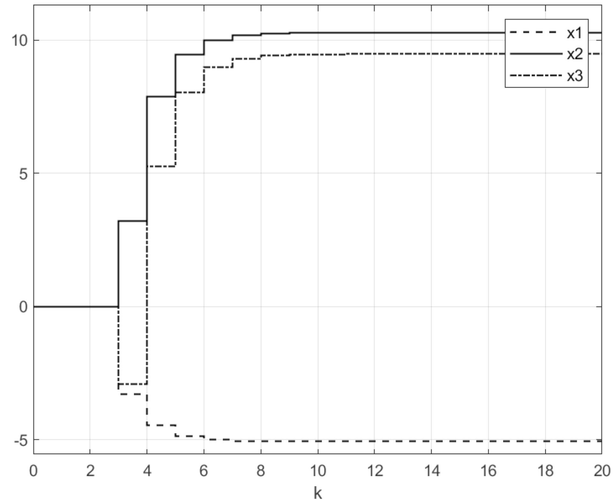

Fig. 1. PC signals of state: case $T$-inverse.

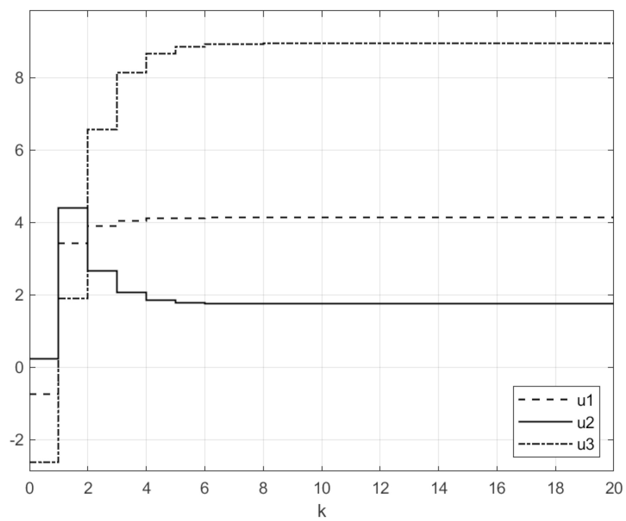

Fig. 2. PC signals of control: case $T$-inverse.

Observe that the pseudo-optimal MP minimum-norm $T$-pseudoinverse (9) again does not guarantee the minimum-energy solution to the PC strategy, resulting in $E_{\mathbf{u}_{\mathbf{ss}_0}}(+\infty) > E_{\mathbf{u}_{\mathbf{ss}_{\mathbf{opt}}}}$.

On the other hand, the authors' novel PC methods will be appreciated here to express the minimum-energy approach.

So, after employing the symbolic $\beta_{\mathbf{0}}$ in the form of

$$\beta_{\mathbf{0}}^{\mathrm{T}} = \begin{bmatrix} \beta_{0_{11}} & \beta_{0_{12}} & \beta_{0_{13}} \\ \beta_{0_{21}} & \beta_{0_{22}} & \beta_{0_{23}} \end{bmatrix} \tag{104}$$

to the PC formula (101) subjected via (62) to the partial derivatives (60), we surprisingly receive the same outcome as in the case of the single-delayed plant of Theorem 3, that is

$$\beta_{\mathbf{0_{opt}}} = \{\beta_{\mathbf{s1}}, \beta_{\mathbf{s2}}\} \tag{105}$$

with

$$\beta_{\mathbf{s1}}^{\mathrm{T}} = \begin{bmatrix} 2.0414 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{106}$$

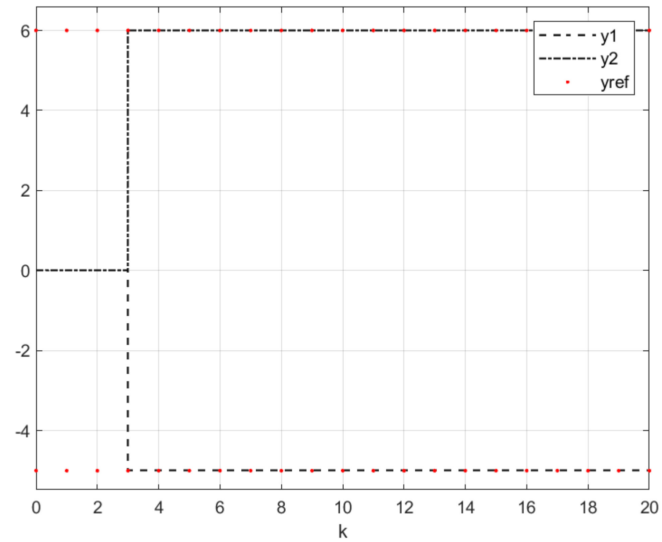

Fig. 3. PC signals of output: case $T$-inverse.

and

$$\beta_{\mathbf{s2}}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & 0 \\ 2.0414 & 0.4058 & 1 \end{bmatrix}. \tag{107}$$

It is intriguing that the same result: $\beta_{\mathbf{s1}}$ and $\beta_{\mathbf{s2}}$, guarantees the minimum-energy 3-step PCD with dual $\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0_{opt}}}}(+\infty) = \begin{bmatrix} 6.9251 \\ 1.3765 \\ 3.3922 \end{bmatrix}$ and the energy (62) equals $E_{\mathbf{u}_{\mathbf{ss}|\beta_{\mathbf{0_{opt}}}}}(+\infty) = 61.3584$ under the closed-loop system's poles $\mathrm{eig}(\mathbf{G}) = \{-0.7060, 0, 0\}$.

After comparing the results with those established in the single-delayed example of Section VI-A and many others, we propose the following conjecture.

*Conjecture 1:* It seems that the minimum-energy $d$-PC structure can be considered in terms of a single-delayed plant with corresponding triplet $S(\mathbf{A}, \mathbf{B}, \mathbf{C})$. In other words, the minimum-energy $\beta_{\mathbf{0_{opt}}}$-related PCD for the cases with $d = 1$ also provides the same energy solution for examples with $d > 1$. Indeed, the time delay $d$ can be passive in a number of scenarios, also those associated with the infinite time horizon.

Interestingly, the peculiarity in the above statement would coincide with the transfer-function-originated feature given in Remark 12 and the stability property, where the $d$-step PC systems are also discussed in terms of the single-delayed component.

Finally, the minimum-energy 3-step PC plant's behaviors are depicted in Figs. 4–6. In both cases (Figs. 3 and 6), the outputs reach the reference values just after the time delay $d \geq 3$, which achieves the fundamental PC requirement.

## VIII. Progress in the Minimum-Energy-Based IMC Theory

It should finally be emphasized that the PC methodology for the multivariable linear $d$-step systems of different domains has only been investigated through the heuristic approaches.

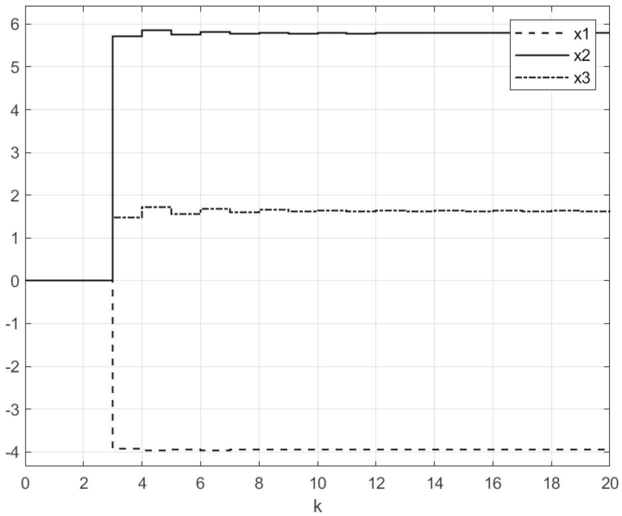


Fig. 4. PC signals of state: case $\sigma$-inverse.

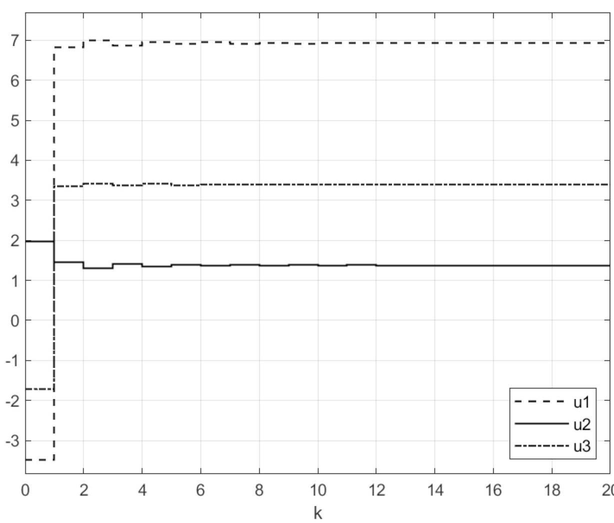


Fig. 5. PC signals of control: case $\sigma$-inverse.

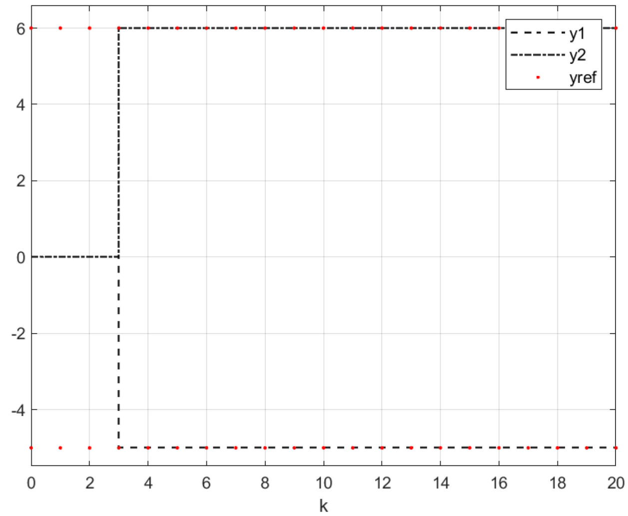


Fig. 6. PC signals of output: case $\sigma$-inverse.

Indeed, the IMC continuous-time plants have effectively been analyzed in the context of the minimum energy of control input runs subjected to an assumed finite time horizon [3], [26], [27]. On the other hand, the inverse model-based maximum-speed/maximum-accuracy control design for discrete-time systems has also been studied, and the resulted peculiarities have consequently been revealed to the control society [11], [25], [36]. Nevertheless, the received analytical outcomes in this matter have only been referred to the relatively simple objects involving a single output variable [4].

Following the notions, it has turned out that the more complex systems can constitute serious difficulties, which could be overcome by applying certain advanced analytical studies. The new methodologies presented throughout this manuscript meet these challenges. Henceforth, we can calculate the energy-originated optimal consumption analytically for every $d$-step LTI MIMO state-space plant without employing the heuristic-related computational effort. Moreover, the newly established procedures guarantee a goal of the minimum-energy function subjected to the infinite time horizon. Crucially, a completed novel set of solutions can no longer be associated with the commonly known MP inverse canon. This critical accomplishment sheds new light on the control theory and practice as a new idea never seen before.

## IX. Conclusion and Open Problems

Applications of polynomial right inverses in solving the energy-oriented IMC theory related to the LTI MIMO discrete-time $d$-step systems have been established in this article. Representations and characterizations of related generalized inverses have been investigated. It is evident that the MP paradigm can no longer be associated with the optimal design of the multivariable IMC-based $d$-state-space PC structures. This statement has been formulated analytically and proven for the first time. Moreover, it should be stated that the general IMC-oriented minimum-energy $d$-PC law is strictly related to the application of nonunique right $\sigma$-inverses. Nevertheless, the presented theory concerns the $d$-step plants with nonzero reference values. Therefore, the following crucial open problems require solving to introduce the complete unified minimum-energy PC methodology. First, the explicit analytical solution to the complex expression (97) should provide the general form of the $\beta_{0_{\text{opt}}}$. Second, the comprehensive energy-oriented analytical investigation related to the multivariable $d$-state-space plants with a zero setpoint is a crucial factor that should ultimately lead to announcing the complete IMC-based minimum-energy PCD theory. Last but not least, some generalizations of the presented results on another control algorithms, such as the generalized MVC, and their verification based on physical objects are also expected. Particularly, since the IMC procedures are sensitive in terms of the parameter descriptions, the impact of disturbances and uncertainties on the new control laws constitutes an unexplored research area worth extensive investigations in the nearest future.

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**Tomasz Feliks** was born in 1992. He received the B.Sc., M.Sc., and Ph.D. degrees (Hons.) in control and robotics from the Department of Electrical, Control and Computer Engineering, Opole University of Technology, Opole, Poland, in 2016, 2017, and 2021, respectively.

He works as an Assistant Professor with the Faculty of Electrical Engineering, Automatic Control and Informatics, Opole University of Technology. His research interests are focused on the general control and systems theory, especially related to the minimum-energy multivariable control problems.

**Wojciech Przemysław Hunek** received the Ph.D. and Habilitation degrees in electrical engineering, automatic control, and robotics from the Department of Electrical, Control, and Computer Engineering, Opole University of Technology, Opole, Poland, in 2003 and 2012, respectively.

He currently works as an Associate Professor with the Faculty of Electrical Engineering, Automatic Control and Informatics, Opole University of Technology. He has supervised six Ph.D. theses. He has also authored or coauthored about 120 papers, most of which concern the up-to-date topics in the multivariable control and systems theory.

**Predrag S. Stanimirović** received the Ph.D. degree in computer science from the Faculty of Philosophy, University of Niš, Niš, Serbia, in 1996.

He is a Full Professor with the Faculty of Sciences and Mathematics, Department of Computer Science, University of Niš. He acquired 36 years of experience in scientific research in diverse fields of mathematics and computer science, which span multiple branches of numerical linear algebra, recurrent neural networks, linear algebra, symbolic computation, and nonlinear optimization. Within recent years, he has successfully published over 320 publications in scientific journals, including five research monographs, six textbooks, five monographs, and 80 peer-reviewed research articles published in conference proceedings and book chapters. His main research topics include numerical linear algebra, operations research, recurrent neural networks, and symbolic computation.