A Dual-Level Model Predictive Control Scheme for Multitimescale Dynamical Systems

Xinglong Zhang, *Member, IEEE*, Wei Jiang, Shuyou Yu, Xin Xu, *Senior Member, IEEE*, and Zhizhong Li

***Abstract*—So far, many control algorithms have been developed for singularly perturbed systems. However, in many industrial processes, enforcing closed-loop fast-slow dynamics for peculiarly nonseparable ones is a prior request and a crucial issue to be resolved. Aiming at the above problem, this article presents two dual-level model predictive control (MPC) algorithms for multitimescale dynamical systems with unknown bounded disturbances and input constraints. The proposed algorithms, each one composed of two regulators working in slow and fast time scales, are designed to generate closed-loop separable dynamics at high and low levels. As a prominent feature, the proposed algorithms are not only suitable for singularly perturbed systems but also capable of imposing separable closed-loop performance for dynamics that are nonseparable and strongly coupled. The recursive feasibility and convergence properties are proven under suitable assumptions. The simulation results on controlling a boiler turbine (BT) system, including the comparisons with other classic controllers, are demonstrated, which show the effectiveness of the proposed algorithms.**

***Index Terms*—Boiler turbine (BT) control, dual-level, linear systems, model predictive control (MPC), separable dynamics.**

I. Introduction

Many industrial processes are characterized by separable fast-slow dynamics, which can be called "multitimescale dynamic systems." In a multitimescale dynamic system, for a given constant input signal, some of the output variables reach their steady-state values quickly, while others may have a longer transient period [1], [2]. A widely accepted approach for the control of such systems resorts to a hierarchical control synthesis that relies on singular perturbation theory (see [2]). A timescale separation technique is adopted therein to define regulators (controllers) at different control frequencies to guarantee the stability and performance of the dynamics associated with the adopted control channels. In addition to singularly perturbed systems, there are systems whose dynamics are not separable but must be controlled in the same way with a multirate control setting; see, for instance, the control of a boiler turbine (BT) system considered in [3]. In this case, the crucial controlled variables of the considered system must be adjusted in a faster time scale to meet the control performance requirement, while other outputs can be controlled more smoothly in a slower time scale. As another example, the control applications for mobile robots can also fit the above control problem setting. For instance, in the race car control problem [4], [5], velocity maximization is the major concern to reach the destination with minimum-time periods, while the reference tracking goal is of less importance. In urban autonomous driving applications (see [6]), lane-keeping and precise trajectory tracking are the primary concerns, while the velocity can be controlled in a smoother manner.

Model predictive control (MPC) is an advanced process control technique that is widely used in industrial processes [7], [8], [9], [10], [11], [12], [13], robotics [14], [15], urban traffics [16], [17]. In MPC, the control problem is reformulated as an optimization one solved on-line according to the receding horizon principle. Many MPC solutions have been developed based on the timescale separation technique for systems characterized by open-loop separable dynamics. Among them, a fast-slow MPC algorithm was proposed in [18] for control of nonlinear singularly perturbed systems, and the extensions to large-scale systems and to the dynamic optimization of economic cost were addressed in [19] and [20], respectively. The algorithms utilize the reduced-order models of the original system, with model couplings between fast and slow time scales disregarded, which leads to a decentralized controller design. In [21] and [22], controllers designed with a unitary slow or fast sampling period were proposed for linear singularly perturbed systems with control saturations. An MPC design with a closed-loop property guarantee was presented in [23] for continuous-time singularly perturbed systems. In [24], a decentralized controller design was proposed using input–output models. In the application aspect, notable contributions can be found in [25] for integrated wastewater treatment systems, [26] for control of a polymerization reactor, [27] for greenhouse climate management, and [28] for

Manuscript received 29 March 2022; accepted 2 September 2022. Date of publication 3 October 2022; date of current version 16 February 2023. This work was supported by the National Natural Science Foundation of China under Grant 61825305, Grant 62003361, Grant U21A20518, and Grant U1964202. This article was recommended by Associate Editor K. G. Vamvoudakis. *(Xinglong Zhang and Wei Jiang contributed equally to this work.) (Corresponding authors: Xin Xu; Shuyou Yu.)*

Xinglong Zhang was with the Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, 20133 Milan, Italy. He is now with the College of Intelligence Science and Technology, National University of Defense Technology, Changsha 410073, China (e-mail: zhangxinglong18@nudt.edu.cn).

Wei Jiang and Xin Xu are with the College of Intelligence Science and Technology, National University of Defense Technology, Changsha 410073, China (e-mail: jiangweinudt@gmail.com; xuxin_mail@263.net).

Shuyou Yu is with the State Key Laboratory of Automotive Simulation and Control and the Department of Control Science and Engineering, Jilin University at Nanling, Changchun 130025, China (e-mail: shuyou@jlu.edu.cn).

Zhizhong Li is with the State Key Laboratory of Disaster Prevention and Mitigation of Explosion and Impact, Army Engineering University, Nanjing 210007, Jiangsu, China (e-mail: lizz0607@163.com).

Color versions of one or more figures in this article are available at https://doi.org/10.1109/TSMC.2022.3207315.

Digital Object Identifier 10.1109/TSMC.2022.3207315

control of flexible joint manipulators. As a summary comment, almost all the aforementioned approaches are tailored for systems with clearly different dynamics due to their dependencies on the singular perturbation theory. Also, the closed-loop stability relies on the assumption that the open-loop dynamics is separable. As a result, the control performance of such controllers could be hampered if the real open-loop dynamical motions are nonseparable.

Motivated by the problems mentioned above, two dual-level algorithms based on MPC are proposed in this article to exhibit closed-loop separable dynamic behaviors of linear dynamical systems with unknown bounded disturbances and input constraints. Compared with the aforementioned works [18], [19], [23], [24], [27], the advantages of the proposed algorithms are as follows. First, the proposed algorithms utilize consistent models for controller designs at both time scales, making them suitable for systems with strong coupling effects, i.e., with nonseparable dynamics. Second, as opposed to the decentralized controller design in the above methods, in our case, information exchanges between the controllers at both levels are allowed, and the performance indices at both levels are optimized cooperatively.

A similar problem has been addressed in [29]. However, the control scheme described in this article shows a significant improvement for the following reasons: 1) the algorithm in [29] is proposed for systems described by finite impulse responses (FIRs), with a particular focus on the application viewpoint, while this article presents novel solutions to the theoretical developments based on a state-space formulation, with verified closed-loop recursive feasibility and stability; 2) open-loop strict stability is required in [29] due to the FIR used, while in our article, the condition is relaxed to the considered model being controllable; and 3) in [29], the input associated with the slow dynamics is only manipulated in the slow time scale. Such a design could lead to control performance degradation, especially for systems that are strongly coupled. To solve this problem, we allow the "slow" control variable to be refined in the fast time scale to improve the control performance (see the comparative results in Section V).

A two-layer control structure based on MPC has been proposed in [30], but the control problem considered is different. Indeed, it is designated to coordinate large-scale independent subsystems that must produce a constant global throughput. However, the approach proposed in this article commits to enforcing separable closed-loop dynamics for strongly coupled systems, and the robust control design under uncertainties is addressed. Hence, the control framework and technique adopted in this article are significantly different from that in [30].

The remainder of this article is organized as follows. Section II presents the problem description and the proposed control structure. The a dual-level MPC (D-MPC) algorithm is initially introduced in Section III, while an improved version of D-MPC, i.e., the *Incremental* D-MPC algorithm, is described in Section IV. A nontrivial simulation example concerning the BT control is studied in Section V, while some conclusions are drawn in Section VI. Proofs of the theoretical results are given in the Appendix.

Notation: We denote $\mathbb{C}$ as the set of the complex plane. We use $\mathbb{N}$ and $\mathbb{N}_+$ to denote the set of non-negative and positive integers, respectively. Given a matrix P, we use the symbol $P^\top$ to denote its transpose. For a generic variable z, we denote $\Delta z(k) = z(k) - z(k-1)$, where k is the discrete-time index. We use $\|x\|_Q^2$ to represent $x^\top Qx$. Given two sets A and B, we denote $A \times B$ as the Cartesian product. For a set of variables $z_i \in \mathbb{R}^{q_i}$, $i = 1, 2, \ldots, M$, we define $(z_1, z_2, \ldots, z_M) = [z_1^\top \; z_2^\top \; \cdots \; z_M^\top]^\top \in \mathbb{R}^q$, where $q = \sum_{i=1}^{M} q_i$. When given a vector v, we denote $\overrightarrow{v}(k : k+N-1)$ the sequence $v(k) \ldots v(k+N-1)$, where N is a positive integer.

II. Problem Formulation

The system to be controlled is described by a discrete-time linear system consisting of two interacting subsystems expressed as follows:

$$\begin{cases} x_s(h+1) = A_{ss}x_s(h) + A_{sf}x_f(h) + B_{ss}u_s(h) \\ \qquad\qquad +B_{sf}u_f(h) + d_s(h) \\ y_s(h) = C_{ss}x_s(h) \end{cases} \tag{1a}$$

$$\begin{cases} x_f(h+1) = A_{fs}x_s(h) + A_{ff}x_f(h) + B_{fs}u_s(h) \\ \qquad\qquad +B_{ff}u_f(h) + d_f(h) \\ y_f(h) = C_{ff}x_f(h) \end{cases} \tag{1b}$$

where $u_s \in \mathbb{R}^{m_s}$, $x_s \in \mathbb{R}^{n_s}$, $y_s \in \mathbb{R}^{p_s}$, and $d_s \in \mathscr{D}_s \subseteq \mathbb{R}^{n_s}$ are the input, state, output variables, and an unmeasured disturbance, respectively, belonged to (1a), while $u_f \in \mathbb{R}^{m_f}$, $x_f \in \mathbb{R}^{n_f}$, $y_f \in \mathbb{R}^{p_f}$, and $d_f \in \mathscr{D}_f \subseteq \mathbb{R}^{n_f}$ are the ones associated with (1b); $\mathscr{D}_s$ and $\mathscr{D}_f$ are compact sets, h is a basic discrete-time scale index, the matrices A_* and B_* (where $*$ is sf or fs in turn) represent the couplings between (1a) and (1b) through the state and input variables, respectively.

Similar to [29], in this article, models (1a) and (1b) are assumed to satisfy at least one of the following scenarios.

1) Model (1a) is characterized by a slower dynamics in contrast to (1b) in the sense that the triple (u_f, x_f, y_f) reaches their final steady-state values fast while the other one, i.e., (u_s, x_s, y_s), may have begun their main dynamic motions; see the examples in [1], [2], and [31].
2) Even if the dynamics of (1a) and (1b) might not be strictly separable, however they must be controlled in a multirate fashion, e.g., the triple (u_f, x_f, y_f) must react promptly to respond to operation (reference) variations while the triple (u_s, x_s, y_s) can be controlled in a smoother manner; see [3].

Notice that in a singularly perturbed system, couplings between different time scales are weak. Hence, the overall model is usually decomposed into decentralized reduced-order models with different time scales. However, in our case, couplings between subsystems might be strong. To cope with coupling effects, system (1) is collected as a centralized version for the controller design. Combining (1a) and (1b), the overall system is written as follows:

$$\begin{cases} x(h+1) = Ax(h) + Bu(h) + d(h) \\ y(h) = Cx(h) \end{cases} \tag{2}$$

where $u = (u_s, \; u_f) \in \mathbb{R}^m$, $m = m_s + m_f$, $x = (x_s, \; x_f) \in \mathbb{R}^n$, $n = n_s + n_f$, $y = (y_s, \; y_f) \in \mathbb{R}^p$, $p = p_s + p_f$, the unknown disturbance $d = (d_s, d_f) \in \mathscr{D}_s \times \mathscr{D}_f = \mathscr{D}$. The diagonal blocks of the collective state transition matrix A and input matrix B are A_{ss}, A_{ff} and B_{ss}, B_{ff}, respectively; whereas their nondiagonal blocks correspond to the coupling terms of the state and input variables between (1a) and (1b). The collective output matrix C is a block-diagonal matrix composed of C_{ss} and C_{ff}, i.e., $C = \text{diag}\{C_{ss}, C_{ff}\}$.

The control objectives to be achieved are introduced here.

1) *Setpoint Regulation:* For a given reference value $y_r = (y_{s,r}, y_{f,r})$, we aim to drive

$$y_s(h) \to y_{s,r} \tag{3a}$$
$$y_f(h) \to y_{f,r}. \tag{3b}$$

2) *Input Constraint:* Enforce the input constraint of the type

$$u_s(h) \in \mathscr{U}_s \tag{4a}$$
$$u_f(h) \in \mathscr{U}_f \tag{4b}$$

where $\mathscr{U}_s$ and $\mathscr{U}_f$ are convex sets, $\mathscr{U} = \mathscr{U}_s \times \mathscr{U}_f$.

The following assumption is assumed to hold.

Assumption 1:

1) The pair (A, B) is controllable;
2) $m_f = p_f$. Also, a steady-state pair (u_r, x_r) exists associated with the reference y_r, such that $x_r = Ax_r + Bu_r$, $y_r = Cx_r$, $x_r = (x_{s,r}, x_{f,r})$, and $u_r = (u_{s,r}, u_{f,r}) \in \mathscr{U}_s \times \mathscr{U}_f$.

Remark 1: Assumption 1-2) allows system (2) to be nonsquare, i.e., $m_s \neq p_s$. Specifically, given a reachable setpoint y_r, one can obtain the steady-state value x_r by calculating $(x_r, u_r) = \Phi^{\dagger}(0, y_r)$ for $m \leq p$ if Φ is full column rank where

$$\Phi = \begin{bmatrix} I - A & -B \\ C & 0 \end{bmatrix}$$

and $\Phi^{\dagger} = (\Phi^{\top}\Phi)^{-1}\Phi^{\top}$. In other cases (including the case $p < m$, where multiple steady-state solutions might exist associated with y_r), one can calculate a suitable steady-state value x_r by optimizing a user-specified performance index subject to the steady-state equality constraints $x = Ax + Bu$, $y_r = Cx$, and the control constraint $u \in \mathscr{U}$.

In principle, a centralized robust MPC problem like [32] can be designed concerning (2) to achieve the above control objectives. However, the resulting control performance (in terms of generating closed-loop separable dynamics) might be hampered due to the conflicting requirements of sampling periods and prediction horizons for (1a) and (1b), respectively.

For this reason, a D-MPC is initially proposed. As shown in Fig. 1, at the high level, a slow time scale k associated with $N \in \mathbb{N}_+$ period of the basic time scale h is adopted to define an MPC problem concerning the sampled version of (2). The computed values of the control actions, $u_f^{[N]}(k)$ and $u_s^{[N]}(k)$, are held constant within the long sampling time interval $[kN, kN + N)$, i.e., $\bar{u}_f(h) = u_f^{[N]}(k)$, $\bar{u}_s(h) = u_s^{[N]}(k)$ for all $h \in [kN, kN + N)$. At the low level, a shrinking horizon MPC is designed at the basic time scale to refine the control actions $(\bar{u}_s(h), \bar{u}_f(h))$ with additional corrections $(\delta u_f(h), \delta u_s(h))$, in order to derive satisfactory short-term transients associated with the closed-loop fast dynamics and to account for possible disturbances.

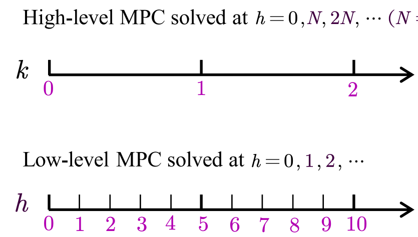


Fig. 1. Time indices adopted in different levels: $h = 1, 2, \ldots,$ denotes the basic (fast) time instant, while $h = 0, N, 2N, \ldots,$ denotes the slow time instant in the basic time scale. The above figure shows a special case with $N = 5$.

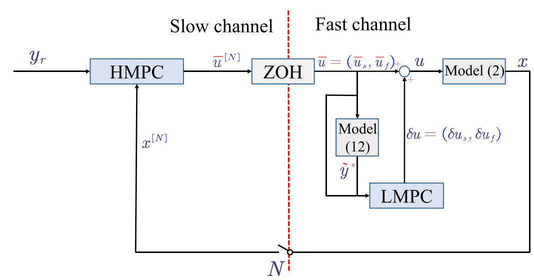


Fig. 2. Brief diagram of the proposed control scheme: HMPC (LMPC) stands for the MPC at the higher (lower) level, while ZOH is the zero-order holder.

The resulting control actions of the D-MPC regulator are summarized as follows:

$$u_s(h) = \bar{u}_s(h) + \delta u_s(h) \tag{5a}$$
$$u_f(h) = \bar{u}_f(h) + \delta u_f(h) \tag{5b}$$

where

1) the control actions $\bar{u}_s(h)$ and $\bar{u}_f(h)$ will be computed by solving an MPC problem in the slow time scale to fulfill objective (3a) and enforce constraints (4a), (4b);
2) the corrections $\delta u_s(h)$ and $\delta u_f(h)$ will be calculated by a shrinking horizon MPC regulator running in the basic time scale to fulfill objective (3b) and enforce constraints (4a), (4b).

To further improve the control performance associated with the fast controlled variables and compensate for uncertainties, an *Incremental* D-MPC algorithm is also proposed (deferred in Section IV). The *Incremental* D-MPC includes integral actions at the two control levels and a prior explicit design of the output y_f at the slow time scale to enforce y_f to the reference value or its neighbor promptly. A brief diagram of the proposed approaches is displayed in Fig. 2.

III. D-MPC Algorithm

In this section, the D-MPC algorithm, consisting of an MPC at the high level and a shrinking horizon MPC at the low level, is devised.

A. MPC at the High Level

In order to design the high-level regulator in the slow time scale k (see again Fig. 1), denote by $u_*^{[N]}$, $x_*^{[N]}$, $y_*^{[N]}$, and $d_*^{[N]}$ the samplings of u_*, x_*, y_*, and d_* (where $*$ is s or f, in turn) and by $u^{[N]} = (u_s^{[N]}, u_f^{[N]})$, $x^{[N]} = (x_s^{[N]}, x_f^{[N]})$, $y^{[N]} = (y_s^{[N]}, y_f^{[N]})$, and $d^{[N]} = (d_s^{[N]}, d_f^{[N]})$ the samplings of the overall input, state, output, and disturbance associated with the time scale k. Hence, the sampled version of (2) with N period is given as follows:

$$\begin{cases} x^{[N]}(k+1) = A^{[N]}x^{[N]}(k) + B^{[N]}u^{[N]}(k) + \tilde{d}^{[N]}(k) \\ y^{[N]}(k) = Cx^{[N]}(k) \end{cases} \tag{6}$$

where $A^{[N]} = A^N$, $B^{[N]} = \sum_{i=0}^{N-1} A^{N-i-1}B$, $\tilde{d}^{[N]}(k) = d^{[N]}(k) + f_{\delta u}^{[N]}(k)$, $d^{[N]}(k) = \sum_{i=0}^{N-1} A^{N-i-1}d(kN+i)$, $f_{\delta u}^{[N]}(k) = \sum_{i=0}^{N-1} A^{N-i-1}B\delta u(kN+i)$ is due to the control at the low level.

The following proposition can be stated for (6).

Proposition 1: The pair $(A^{[N]}, C)$ is detectable if (A, C) is detectable.

Also, the following assumption about (6) is assumed to be holding:

Assumption 2: The pair $(A^{[N]}, B^{[N]})$ is stabilizable.

Remark 2: To meet the stabilizability requirement of (6), unlike the detectability condition in Proposition 1, we require Assumption 2 to be verified *a posteriori* once the sampling period N is chosen. This is due to the fact that, starting from the controllability of (A, B), there is no guarantee the sampled pair $(A^{[N]}, B^{[N]})$ is also stabilizable. A simple example to illustrate this point is as follows: Consider a controllable single-input single-output system described by $x(h+1) = -x(h) + u(h)$. However, the $N = 2$ period sampled version $x^{[2]}(k+1) = (-1)^2x^{[2]}(k) + (-1+1)u^{[2]}(k) = x^{[2]}(k)$ is not stabilizable.

As the disturbance term $\tilde{d}^{[N]}$ is unknown, we introduce the following nominal model for prediction:

$$\begin{cases} \hat{x}^{[N]}(k+1) = A^{[N]}\hat{x}^{[N]}(k) + B^{[N]}u^{[N]}(k) \\ \hat{y}^{[N]}(k) = C\hat{x}^{[N]}(k). \end{cases} \tag{7}$$

With (7), it is now possible to state the MPC problem at the high level. At each slow time step k, we solve an optimization problem according to the receding horizon principle as follows:

$$\min_{\overrightarrow{u^{[N]}}(k:k+N_H-1)} J_H \tag{8}$$

where

$$J_H = \sum_{i=0}^{N_H-1}\left(\left\|\hat{y}^{[N]}(k+i) - y_r\right\|_{Q_H}^2 + \left\|u^{[N]}(k+i) - u_r\right\|_{R_H}^2\right) + \left\|\hat{x}^{[N]}(k+N_H) - x_r\right\|_{P_H}^2 \tag{9}$$

$N_H \in \mathbb{N}_+$ is the adopted prediction horizon. The parameters $Q_H \in \mathbb{R}^{p\times p}$ and $R_H \in \mathbb{R}^{m\times m}$ are positive definite and symmetric weighting matrices, while $P_H \in \mathbb{R}^{n\times n}$ is computed as the solution to the Lyapunov equation described by

$$F_H^\top P_H F_H - P_H = -\left(C^\top Q_H C + K_H^\top R_H K_H\right) \tag{10}$$

where the matrix $F_H = A^{[N]} + B^{[N]}K_H$ is the Schur stable and K_H is a stabilizing gain matrix. The optimization problem (8) is performed under the following constraints.

1) The dynamics (7) with $\hat{x}^{[N]}(k) = x(kN)$.

2) The input constraint

$$u^{[N]}(k+i) \in \bar{\mathscr{U}}$$

where $\bar{\mathscr{U}}$ is a tightened convex set of $\mathscr{U}$, i.e., $\bar{\mathscr{U}} \subseteq \mathscr{U}$.

3) The terminal state constraint

$$\hat{x}^{[N]}(k+N_H) \in \mathscr{X}_F^s$$

where $\mathscr{X}_F^s \subseteq \mathscr{X}_F$, the set $\mathscr{X}_F$ is chosen as a positively invariant set for system (7) controlled with the stabilizing control law $u^{[N]}(k) = K_H(\hat{x}^{[N]}(k) - x_r) + u_r$ satisfying $K_H(\mathscr{X}_F \ominus x_r) \subseteq \mathscr{U} \ominus u_r$. To guarantee the recursive feasibility under uncertainties (deferred in Theorem 1), $\mathscr{X}_F^s$ is selected such that: for any $z(k) \in \mathscr{X}_F$, the successive state under the above stabilizing control law satisfies $z(k+1) \in \mathscr{X}_F^s$.

B. Shrinking Horizon MPC at the Low Level

Assume now to be at a specific basic time instant $h = kN$ (correspond to the slow time instant k, see again Fig. 1) such that the high-level problem (8) with cost (9) has been successfully solved. Let $\overrightarrow{u^{[N]}}(k : k+N_H-1|k)$ be the optimal solution to (8). Thus, the one-step ahead state prediction $\hat{x}^{[N]}(k+1|k)$ is available. Let us focus on the output performance in the fast time scale within the interval $h \in [kN, kN+N)$. Denoting by $\tilde{y}(h) = (\tilde{y}_s(h), \tilde{y}_f(h))$ the output resulting from (2) with $d(h) = 0$ and $u(h) = u^{[N]}(\lfloor h/N \rfloor)$, the component $\tilde{y}_f(h)$ may expect undesired transient due to the long sampling period at the high level. For this reason, the overall control action associated with y_f at the low level is refined as follows:

$$u_f(h) = \bar{u}_f(h) + \delta u_f(h) \tag{11a}$$

where $\bar{u}_f(h) = u_f^{[N]}(\lfloor h/N \rfloor)$, δu_f is computed by a regulator at the low level, which is deferred in (13).

Since $\delta u_f(h)$ could influence the value of $y_s(h)$ in the fast time scale due to coupling terms A_{sf} and B_{sf} from (1a) to (1b), it is convenient to allow the correction on u_s, i.e.,

$$u_s(h) = \bar{u}_s(h) + \delta u_s(h) \tag{11b}$$

where $\bar{u}_s(h) = u_s^{[N]}(\lfloor h/N \rfloor)$, δu_s is another decision variable at the low level.

In view of (11), one can rewrite (2) with $u(h) = \bar{u}(h) + \delta u(h)$, where $\bar{u} = (\bar{u}_s, \bar{u}_f)$, $\delta u = (\delta u_s, \delta u_f)$. For prediction purpose at the basic time scale, we define a prediction model by neglecting the effect of d

$$\begin{cases} \hat{x}(h+i+1|h) = A\hat{x}(h+i|h) + Bu(h+i|h) \\ \hat{y}(h+i|h) = C\hat{x}(h+i|h) \end{cases} \tag{12}$$

where $\hat{x}(h|h) = x(h)$.

Accordingly, at any fast time instant $h = kN+t$, a shrinking horizon MPC problem can be solved at the low level

$$\min_{\overrightarrow{\delta u}(h:(k+1)N-1)} J_L \tag{13}$$

where

$$J_L = \sum_{j=0}^{N-t-1} \left\|\hat{y}(h+j|h) - \tilde{y}^*(h+j)\right\|_Q^2 + \|\delta u(h+j|h)\|_R^2 \tag{14}$$

Algorithm 1 On-Line Implementation of D-MPC

Initial condition $x^{[N]}(0) = x(0)$.
while *for any* $k \geq 0$ **do**
  h1) Compute $u^{[N]}(k|k)$ by solving (8) with (9) and update $\hat{x}^{[N]}(k+1|k)$
  h2) Generate output $\tilde{y}^*(h)$ from (2) with $d(h) = 0$ and $u(h) = u^{[N]}(\lfloor h/N \rfloor)$, for all $h \in [kN, kN+N)$
  for $h \leftarrow kN$ **to** $kN+N-1$ **do**
    l1) Compute $\delta u(h|h)$ using (13) with (14) and apply $u(h) = u^{[N]}(\lfloor h/N \rfloor) + \delta u(h|h)$ to (2)
    l2) Update $x(h+1)$, $y(h+1)$, set $\hat{x}(h+1) = x(h+1)$, $\hat{y}(h+1) = y(h+1)$
  end
  h3) $k \leftarrow k+1$
end

$\tilde{y}^*(h) = (\tilde{y}_s(h),\ \tilde{y}_f(kN+N))$, $h \in [kN, kN+N)$, $\tilde{y}_s$ and $\tilde{y}_f$ are defined above (11a).

The optimization problem (13) is performed under the following constraints.

1) Dynamics (12).
2) The input constraint
$$\begin{aligned} &u^{[N]}(\lfloor h/N \rfloor) + \delta u(h|h) \in \mathscr{U} \\ &u^{[N]}(\lfloor h/N \rfloor) + \delta u(h+j|h) \in \hat{\mathscr{U}}_t\ \forall j = 1, \ldots, N-t-1 \end{aligned} \tag{15}$$
where $\hat{\mathscr{U}}_t$ is selected such that $\bar{\mathscr{U}} \subseteq \hat{\mathscr{U}}_t \subseteq \mathscr{U}$ [see (18)].
3) The state terminal constraint
$$\hat{x}(kN+N|h) = \hat{x}^{[N]}(k+1|k). \tag{16}$$

Remark 3: The rationale of choosing signal $\tilde{y}^*(h)$ as the reference for the low level lies in the fact that $\hat{y}_f(h)$ is expected to react promptly to respond to $\tilde{y}_f(kN+N)$, while $\hat{y}_s(h)$ can be controlled to follow the smooth trajectory $\tilde{y}_s(h)$ generated from the high level.

Remark 4: It is highlighted that the structure of the proposed approach is different from that of the cascade ones; see, for instance [33]. In the cascade algorithm, the computed input from the high level is considered as the output reference to be tracked at the low level. In contrast, the proposed algorithms utilize the control action computed from the high level, i.e., $\bar{u}(h)$, to generate the possible reference profile with model (2) in an open-loop fashion.

Remark 5: Note that an identifier-critic framework with an event-triggered control mechanism was developed in [34] for decentralized control of nonlinear interconnected systems with input constraints. Our approach is different from [34] in the following two aspects: 1) our approach focuses on enforcing closed-loop separable dynamic behaviors rather than improving communication efficiency in [34] and 2) the information exchanges between the controllers at both levels are allowed in our approaches, in contrast to the decentralized control design in [34].

C. Summary of the D-MPC Algorithm

In summary, the main steps for the on-line implementation of the D-MPC are given in Algorithm 1.

Under Assumption 1, if (8) is feasible at $k = 0$
$$\left(A^{[N]}\right)^{N_H-1}\mathscr{D} \oplus \mathscr{X}_F^s \subseteq \mathscr{X}_F \tag{17}$$
and at any time $h \in [kN, kN+N-2]$
$$-A^{N-t-1}\mathscr{D} \oplus L_t\hat{\mathscr{U}}_t \subseteq \Delta L_t\mathscr{U} \oplus L_{t+1}\hat{\mathscr{U}}_{t+1} \tag{18}$$
where $L_t = \sum_{i=t+1}^{N-1} A^{N-i-1}B$, $\Delta L_t = L_t - L_{t+1}$, $t = h - kN$, then the following results can be stated.

Theorem 1 (Recursive Feasibility and Convergence of D-MPC):

1) The feasibility can be guaranteed.
   a) For the high-level problem (8) at all slow time instant $k > 0$.
   b) For the low-level problem (13) at all fast time instant $h \geq 0$.
2) Moreover, if the disturbance $d = 0$, the asymptotic convergence of the closed-loop system can be ensured.
   a) The slow-time scale system (6) enjoys the convergence property, i.e., $\lim_{k\to+\infty}(u^{[N]}(k),\ x^{[N]}(k),\ y^{[N]}(k)) = (u_r,\ x_r,\ y_r)$.
   b) Consequently, for the low-level problem (13), it holds that $\lim_{h\to+\infty}\delta u(h) = 0$. Finally, $\lim_{h\to+\infty}(u(h),\ x(h),\ y(h)) = (u_r,\ x_r,\ y_r)$.

Remark 6: A convenient but conservative choice of $\hat{\mathscr{U}}_t$ in (15) can be made, via setting $\hat{\mathscr{U}}_t = \hat{\mathscr{U}}\ \forall t = 1, \ldots, N-1$. As such, condition (18) can be replaced by
$$-M\mathscr{D} \oplus \bar{L}\hat{\mathscr{U}} \subseteq \bar{L}\mathscr{U} \tag{19}$$
where $M = \sum_{j=0}^{N-1} A^j$ and $\bar{L} = \sum_{j=0}^{N-1} A^jB$.

Terminal constraint (16) in (13) plays a crucial role for guaranteeing the closed-loop property of D-MPC. However, as the proposed control structure is an upper-bottom one, the computed value of $\hat{x}_f^{[N]}(k)$ at the high-level influences the control performance at the low level due to (16). As a consequence, the state $\hat{x}_f(h)$ associated with $\hat{y}_f(h)$ in the basic time scale might not converge to its nominal value faster than $\hat{x}_s(h)$, especially for systems that exhibit nonseparable open-loop dynamics. We solve this problem in the following section.

IV. Incremental D-MPC Algorithm

In this section, we design an *Incremental* D-MPC algorithm to improve the control performance associated with the fast output y_f and to compensate for possible time-varying piecewise constant or smooth uncertainties.

A. Design of the Incremental D-MPC

In the following, we first focus on the redesign of the MPC regulator at the high level. To this end, we partition the above-sampled system as the one with the structure similar to (1). To proceed, we rewrite matrices $A^{[N]}$, $B^{[N]}$, and vector $\tilde{d}^{[N]}$ into the following forms:
$$A^{[N]} = \begin{bmatrix} A_{ss}^{[N]} & A_{sf}^{[N]} \\ A_{fs}^{[N]} & A_{ff}^{[N]} \end{bmatrix},\quad B^{[N]} = \begin{bmatrix} B_{ss}^{[N]} & B_{sf}^{[N]} \\ B_{fs}^{[N]} & B_{ff}^{[N]} \end{bmatrix},\quad \tilde{d}^{[N]} = \begin{bmatrix} \tilde{d}_s^{[N]} \\ \tilde{d}_f^{[N]} \end{bmatrix}$$
where $A_{ss}^{[N]} \in \mathbb{R}^{n_s\times n_s}$, $B_{ss}^{[N]} \in \mathbb{R}^{n_s\times m_s}$, and $\tilde{d}_s^{[N]} \in \mathbb{R}^{n_s}$.

The sampled system (6) can be partitioned as two interacting ones as follows:

$$\begin{cases} x_s^{[N]}(k+1) = A_{ss}^{[N]}x_s^{[N]}(k) + A_{sf}^{[N]}x_f^{[N]}(k) + B_{ss}^{[N]}u_s^{[N]}(k) \\ \qquad\qquad\quad + B_{sf}^{[N]}u_f^{[N]}(k) + \tilde{d}_s^{[N]}(k) \\ y_s^{[N]}(k) = \quad C_{ss}x_s^{[N]}(k) \end{cases} \tag{20a}$$

$$\begin{cases} x_f^{[N]}(k+1) = A_{fs}^{[N]}x_s^{[N]}(k) + A_{ff}^{[N]}x_f^{[N]}(k) + B_{fs}^{[N]}u_s^{[N]}(k) \\ \qquad\qquad\quad + B_{ff}^{[N]}u_f^{[N]}(k) + \tilde{d}_f^{[N]}(k) \\ y_f^{[N]}(k) = \quad C_{ff}x_f^{[N]}(k). \end{cases} \tag{20b}$$

The following assumption about (20b) is assumed to hold.

Assumption 3: Matrix $C_{ff}B_{ff}^{[N]}$ is full rank.

With (20), with the goal of guaranteeing satisfactory control performance related to y_f in the basic time scale, it is convenient to enforce all the future predictions $y_f^{[N]}(k)\,\forall k>0$ associated with (20b) being equal to the reference value $y_{f,r}$. In this way, the real output y_f resulting from controller (13) will reach the reference $y_{f,r}$ in only one slow time step. However, this restriction might cause an infeasibility issue in case $y_{f,r}$ is far from its initial value $y_f(0)$ and constraints on the control increments are enforced. For this reason, instead of imposing $y_f^{[N]}(k) = y_{f,r}\ \forall k>0$, one can enforce the following relation:

$$y_f^{[N]}(k) = \tilde{y}_{f,r}(k)\ \forall k > 0 \tag{21}$$

where $\tilde{y}_{f,r}(k) = y_f(0) + \alpha(k)(y_{f,r} - y_f(0))$, $\alpha(k)$ is defined as an optimization variable and its value is restricted by $0 \le \alpha(k) \le 1$ and reaches 1 in finite time steps, i.e.,

$$\begin{cases} \alpha(0) = 0 \\ 0 \le \alpha(k) \le 1, \quad k \in [1,\ N_\alpha) \\ \alpha(k) = 1, \quad\quad\ \ k \ge N_\alpha \end{cases} \tag{22}$$

where N_α is a positive integer. As $\tilde{d}_f^{[N]}(k)$ is unknown at time k, (21) can be slightly relaxed, i.e., we enforce $\hat{y}_f^{[N]}(k+j) = \tilde{y}_{f,r}(k+j)$. In view of (20), it is required that

$$u_f^{[N]}(k+j) = (C_{ff}B_{ff}^{[N]})^{-1}\Big(\tilde{y}_{f,r}(k+j) - C_{ff}\Big(\begin{bmatrix} A_{fs}^{[N]} & A_{ff}^{[N]} \end{bmatrix}\hat{x}^{[N]}(k+j) + B_{fs}^{[N]}u_s^{[N]}(k+j)\Big)\Big) \tag{23}$$

where $\hat{x}^{[N]}$ is the predicted value of $x^{[N]}$. Under constraint (23), the time steps required for $\hat{y}_f^{[N]} = y_{f,r}$ can be defined via properly tuning parameter N_α.

By substituting $u_f^{[N]}$ with (23) in (20), one can write the one-step ahead state prediction at time k, i.e.,

$$\begin{cases} \hat{x}^{[N]}(k+1) = \tilde{A}^{[N]}\hat{x}^{[N]}(k) + \tilde{B}_s^{[N]}u_s^{[N]}(k) + \tilde{B}_f^{[N]}\tilde{y}_{f,r}(k) \\ \hat{y}_s^{[N]}(k) = \tilde{C}_s\hat{x}^{[N]}(k) \end{cases} \tag{24}$$

where $\tilde{A}^{[N]} = \begin{bmatrix} \tilde{A}_{ss}^{[N]} & \tilde{A}_{sf}^{[N]} \\ \tilde{A}_{fs}^{[N]} & \tilde{A}_{ff}^{[N]} \end{bmatrix}$, $\tilde{B}_s^{[N]} = \begin{bmatrix} \tilde{B}_{ss}^{[N]} \\ \tilde{B}_{fs}^{[N]} \end{bmatrix}$, $\tilde{B}_f^{[N]} = \begin{bmatrix} \tilde{B}_{sf}^{[N]} \\ \tilde{B}_{ff}^{[N]} \end{bmatrix}$, $\tilde{C}_s = \begin{bmatrix} C_{ss} \\ 0 \end{bmatrix}^\top$, and

$$\begin{aligned} \tilde{A}_{ss}^{[N]} &= A_{ss}^{[N]} - B_{sf}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}A_{fs}^{[N]} \\ \tilde{A}_{sf}^{[N]} &= A_{sf}^{[N]} - B_{sf}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}A_{ff}^{[N]} \\ \tilde{A}_{fs}^{[N]} &= A_{fs}^{[N]} - B_{ff}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}A_{fs}^{[N]} \\ \tilde{A}_{ff}^{[N]} &= A_{ff}^{[N]} - B_{ff}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}A_{ff}^{[N]} \\ \tilde{B}_{ss}^{[N]} &= B_{ss}^{[N]} - B_{sf}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}B_{fs}^{[N]} \\ \tilde{B}_{fs}^{[N]} &= B_{fs}^{[N]} - B_{ff}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}C_{ff}B_{fs}^{[N]} \\ \tilde{B}_{sf}^{[N]} &= B_{sf}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1} \\ \tilde{B}_{ff}^{[N]} &= B_{ff}^{[N]}(C_{ff}B_{ff}^{[N]})^{-1}. \end{aligned}$$

Assumption 4: The integer N is such that $(\tilde{A}^{[N]}, \tilde{B}_s^{[N]})$ is stabilizable.

To account for the model uncertainties, model (24) is reformulated in an *incremental form* and used to define an MPC including an integral action. In doing so, the closed-loop system can compensate for the influences caused by smoothly time-varying disturbances, see [35]. To this end, letting $\bar{\hat{x}}^{[N]}(k) = (\hat{y}_s^{[N]}(k),\ \Delta\hat{x}^{[N]}(k))$, from (24) we compute

$$\begin{cases} \bar{\hat{x}}^{[N]}(k+1) = \bar{A}^{[N]}\bar{\hat{x}}^{[N]}(k) + \bar{B}_s^{[N]}\Delta u_s^{[N]}(k) \\ \qquad\qquad\quad + \bar{B}_f^{[N]}\Delta\alpha(k)(y_{f,r} - y_f(0)) \\ \alpha(k) = \alpha(k-1) + \Delta\alpha(k) \\ \hat{y}_s^{[N]}(k) = \bar{C}\bar{\hat{x}}^{[N]}(k) \end{cases} \tag{25}$$

where $\bar{A}^{[N]} = \begin{bmatrix} I & \tilde{C}_s\tilde{A}^{[N]} \\ 0 & \tilde{A}^{[N]} \end{bmatrix}$, $\bar{B}_s^{[N]} = \begin{bmatrix} \tilde{C}_s\tilde{B}_s^{[N]} \\ \tilde{B}_s^{[N]} \end{bmatrix}$, $\bar{B}_f^{[N]} = \begin{bmatrix} \tilde{C}_s\tilde{B}_f^{[N]} \\ \tilde{B}_f^{[N]} \end{bmatrix}$, and $\bar{C} = \begin{bmatrix} I & 0 \end{bmatrix}$.

Proposition 2: The pair $(\bar{A}^{[N]},\ \bar{B}_s^{[N]})$ is stabilizable if and only if

- $\operatorname{rank}\left(\begin{bmatrix} \tilde{C}_s\tilde{A}^{[N]} & \tilde{C}_s\tilde{B}_s^{[N]} \\ \tilde{A}^{[N]} - I & \tilde{B}_s^{[N]} \end{bmatrix}^\top\right) = n + p_s$
- $\operatorname{rank}\left(\begin{bmatrix} 2I & 0 \\ \tilde{C}_s\tilde{A}^{[N]} & \tilde{A}^{[N]} + I \\ \tilde{C}_s\tilde{B}_s^{[N]} & \tilde{B}_s^{[N]} \end{bmatrix}^\top\right) = n + p_s.$

Under Proposition 2, it is possible to find a gain matrix $\bar{K}_{s,\mathrm{H}}$ such that $\bar{F}_{s,\mathrm{H}} = \bar{A}^{[N]} + \bar{B}^{[N]}\bar{K}_{s,\mathrm{H}}$ is a Schur stable.

Note that, it is not straightforward to write constraints on $\bar{u}_s^{[N]}$ and $\bar{u}_f^{[N]}$ using model (25). We are going to show that, in line with [35], it is possible to represent control variables by states in the *incremental form*, i.e.,

$$\begin{aligned} \bar{u}_s^{[N]}(k) &= \Gamma_{us}\Big(\bar{\hat{x}}^{[N]}(k+1) - \bar{B}_f^{[N]}\tilde{y}_{f,r}\Big) \\ \bar{u}_f^{[N]}(k) &= \Big(C_{ff}B_{ff}^{[N]}\Big)^{-1}\Big(\tilde{y}_{f,r}(k) \\ &\quad - \Gamma_{uf}\Big(\bar{\hat{x}}^{[N]}(k+1) - \bar{B}_f^{[N]}\tilde{y}_{f,r}\Big)\Big) \end{aligned} \tag{26}$$

where $\Gamma_{uf} = C_{ff}\begin{bmatrix} A_{fs}^{[N]} & A_{ff}^{[N]} & B_{fs}^{[N]} \end{bmatrix}\Gamma$, $\Gamma_{us} = \begin{bmatrix} 0_n & I_{m_s} \end{bmatrix}\Gamma$, and $\Gamma = \begin{bmatrix} \tilde{C}_s\tilde{A}^{[N]} & \tilde{C}_s\tilde{B}_s^{[N]} \\ \tilde{A}^{[N]} - I_n & \tilde{B}_s^{[N]} \end{bmatrix}^{-1}$. In view of (4a), (4b), and (26), to enforce constraints on $(\bar{u}_s^{[N]}(k), \bar{u}_f^{[N]}(k)) \in \mathscr{U}_s \times \mathscr{U}_f$, one can use

$$A_x\bar{x}^{[N]}(k+1) + b_y \in \bar{\mathscr{U}} \tag{27}$$

where $A_x = \begin{bmatrix} \Gamma_{us}^\top & -\Gamma_{uf}^\top \end{bmatrix}^\top$, $b_y = \begin{bmatrix} -(\Gamma_{us}\bar{B}_f^{[N]})^\top & (C_{ff}B_{ff}^{[N]})^{-\top} + ((C_{ff}B_{ff}^{[N]})^{-1}\Gamma_{uf}\bar{B}_f^{[N]})^\top \end{bmatrix}^\top \tilde{y}_{f,r}$.

Based on (25) and (27), now we state the *Incremental* MPC problem at the high level. At each slow time step k, we solve an optimization problem according to the receding horizon principle as follows:

$$\min_{\overrightarrow{\Delta u_s^{[N]}}(k:k+\bar{N}_H-1)} \bar{J}_H \tag{28}$$

where

$$\bar{J}_H = \sum_{i=0}^{\bar{N}_H-1} \left\| \hat{\bar{x}}^{[N]}(k+i) - \bar{C}^\top y_{s,r} \right\|^2_{\bar{Q}_{s,\mathrm{H}}} + \left\| \Delta u_s^{[N]}(k+i) \right\|^2_{\bar{R}_{s,\mathrm{H}}} + \gamma(\alpha(k+i)-1)^2 + \left\| \hat{\bar{x}}^{[N]}\left(k+\bar{N}_H\right) - \bar{C}^\top y_{s,r} \right\|^2_{\bar{P}_H} \tag{29}$$

γ is a positive scalar, $\bar{N}_H > N_\alpha$ is the adopted prediction horizon. The positive definite and symmetric weighting matrices $\bar{Q}_{s,\mathrm{H}} \in \mathbb{R}^{(n+p_s)\times(n+p_s)}$ and $\bar{R}_{s,\mathrm{H}} \in \mathbb{R}^{m_s\times m_s}$ are free design parameters, while $\bar{P}_H$ is computed as the solution to the Lyapunov equation

$$\bar{F}_{s,\mathrm{H}}^\top \bar{P}_H \bar{F}_{s,\mathrm{H}} - \bar{P}_H = -\left(Q_{s,\mathrm{H}} + \bar{K}_{s,\mathrm{H}}^\top \bar{R}_{s,\mathrm{H}} \bar{K}_{s,\mathrm{H}}\right). \tag{30}$$

The optimization problem (28) is performed under the following constraints.

1) Dynamics (25) with $\hat{x}^{[N]}(k) = x^{[N]}(k)$, constraints (22) and (27).
2) The state terminal constraint

$$\bar{x}^{[N]}(k+\bar{N}_H) \in \bar{\mathscr{X}}_F^s$$

where $\bar{\mathscr{X}}_F^s \subseteq \bar{\mathscr{X}}_F$, the set $\bar{\mathscr{X}}_F$ is a positively invariant set for the nominal system of (25), i.e.,

$$z(k+1) = \bar{A}^{[N]} z(k) + \bar{B}_s^{[N]} \hat{u}(k) \tag{31}$$

that is controlled with the stabilizing control law $\hat{u}(k) = \bar{K}_{s,\mathrm{H}}(z(k) - \bar{C}^\top y_{s,r})$ such that $\bar{F}_{s,\mathrm{H}} \bar{\mathscr{X}}_F \subseteq \bar{\mathscr{X}}_F$ under constraint (27). The set $\bar{\mathscr{X}}_F^s$ is selected such that: for any $z(k) \in \bar{\mathscr{X}}_F$, the successive state under the prescribed stabilizing control law satisfies $z(k+1) \in \bar{\mathscr{X}}_F^s$.

Let $\overrightarrow{\Delta u_s^{[N]}}(k : k+\bar{N}_H-1|k)$ be the optimal solution to optimization (28). The real input $u_s^{[N]}(k)$ at time instant k is given by $u_s^{[N]}(k) = u_s^{[N]}(k-1) + \Delta u_s^{[N]}(k|k)$. Also, from (23), we can compute the value of $u_f^{[N]}(k)$. Hence, the state $\hat{x}^{[N]}(k+1|k)$ is available by applying $u^{[N]}(k) = (u_s^{[N]}(k), u_f^{[N]}(k))$ to (6).

In principle, the fast MPC problem described in the previous section, i.e., (13) with cost (15), can be used for computing the corrections of the control input in the fast time scale. We propose an improved version in the following. Slightly different to (12) in (13), we use

$$\begin{cases} \Delta\hat{x}(h+i+1|h) = A\Delta\hat{x}(h+i|h) + B\Delta u(h+i|h) \\ \hat{y}(h+i|h) = \hat{y}(h+i-1|h) + C\Delta\hat{x}(h+i|h) \end{cases} \tag{32}$$

$h \in [kN,\ kN+N)$, to compensate for uncertainties.

Accordingly, at any fast time instant $h = kN + t$, letting $\hat{\bar{x}} = (\hat{y}, \Delta\hat{x})$, and $\bar{y} = (\tilde{y}^*, 0)$, the shrinking horizon MPC problem can be solved at the low level, i.e.,

$$\min_{\overrightarrow{\Delta u}(h:(k+1)N-1)} \bar{J}_L \tag{33}$$

$$\bar{J}_L = \sum_{j=0}^{N-t-1} \|\hat{\bar{x}}(h+j|h) - \bar{y}(h+j)\|^2_{\bar{Q}} + \|\Delta u(h+j|h)\|^2_{\bar{R}} \tag{34}$$

Algorithm 2 On-Line Implementation of Incremental D-MPC

Initial condition $x^{[N]}(0) = x(0)$, given N_α.
while *for any* $k \geq 0$ **do**
  h1) Compute $\Delta u_s^{[N]}(k|k)$ by solving (28) with (29) and update $\bar{x}^{[N]}(k+1|k)$
  if (28) *with* (29) *is infeasible* **then**
    $N_\alpha \leftarrow N_\alpha + 1$ and go back to step **h1)**; see (22)
  else
    continue
  end
  h2) Calculate $u_f^{[N]}(k)$ from (23) with $u_s^{[N]}(k) = u_s^{[N]}(k-1) + \Delta u_s^{[N]}(k|k)$, apply the control $u^{[N]}(k) = (u_s^{[N]}(k), u_f^{[N]}(k))$ to (38) and update $x^{[N]}(k+1|k)$
  h3) Generate output $\tilde{y}^*(h)$ from (2) with $d(h) = 0$ and $u(h) = u^{[N]}(\lfloor h/N \rfloor)$, for all $h \in [kN, kN+N)$
  for $h \leftarrow kN$ **to** $kN+N-1$ **do**
    l1) Compute $\Delta u(h|h)$ using (33) with (34) and apply $u(h) = u(h-1) + \Delta u(h|h)$ to (2)
    l2) Update $x(h+1)$ and $y(h+1)$, set $\hat{x}(h+1) = x(h+1)$, $\hat{y}(h+1) = y(h+1)$
  end
  h4) $k \leftarrow k+1$
end

where $\bar{Q} \in \mathbb{R}^{(n+p)\times(n+p)}$ is a positive-definite matrix. The optimization problem (33) is performed under the following constraints.

1) The dynamics (32).
2) The input constraint

$$u(h-1) + \Delta u(h|h) \in \mathscr{U}$$
$$u(h-1) + \sum_{i=0}^{j} \Delta u(h+i|h) \in \hat{\mathscr{U}}_t,\ \forall j = 1,\ldots,N-t-1. \tag{35}$$

3) The state terminal constraint

$$\hat{x}(kN+N|h) = \hat{x}^{[N]}(k+1|k). \tag{36}$$

It is noted that constraints (35) and (36) can be rewritten following the line of (27), but the design steps are neglected for the sake of simplicity.

B. Summary of the Incremental D-MPC Algorithm

To better clarify the requirements for implementing *Incremental* D-MPC and its difference with the D-MPC, the main steps for the on-line implementation are given in Algorithm 2.

Under Assumptions 1–4, if (28) is feasible at $k = 0$

$$\left(\bar{A}^{[N]}\right)^{\bar{N}_H-1} E\mathscr{D} \oplus \bar{\mathscr{X}}_F^s \subseteq \bar{\mathscr{X}}_F \tag{37}$$

where $E = \left[\tilde{C}_s^\top \ \ I^\top\right]^\top$, and (18) is satisfied, then the following results hold for the *Incremental* D-MPC.

Theorem 2 (Recursive Feasibility and Convergence of Incremental D-MPC):

1) The feasibility can be guaranteed.
   a) For the high-level problem (28) at all slow time instant $k > 0$.

b) For the low-level problem (33) at all fast time instant $h \geq 0$.

2) Moreover, if the disturbance d is constant, the asymptotic convergence of the closed-loop system can be ensured.

a) The slow-time scale system (25) enjoys the convergence property, i.e., $\lim_{k\to+\infty}(\bar{x}^{[N]}(k),\ \Delta u_s^{[N]}(k)) = (\bar{C}^\top y_{s,r},\ 0)$. Consequently, $\lim_{k\to+\infty}(x^{[N]}(k),\ u^{[N]}(k)) = (x_r,\ u_r)$.

b) For the low-level problem (33), it holds that $\lim_{h\to+\infty}\Delta u(h) = 0$. Finally, $\lim_{h\to+\infty}(u(h),\ x(h),\ y(h)) = (u_r,\ x_r,\ y_r)$.

V. Simulation Example

In this section, simulation results on a realistic BT system with extensive comparisons in different domains are reported, including comparisons in the nominal and perturbed scenarios.

A. Description of the BT Model

A 160 MW BT system in [36] is considered and its dynamic diagram is presented in Fig. 3. The input variables applied to the boiler are the fuel flow q_f (kg/s) and feedwater flow q_w (kg/s), while the controlled variables of the boiler are drum pressure P (kg/cm^2) and water level L (m). The control and controlled variables of the turbine are the steam control q_s (kg/s) and the electrical power output Q (MW). Typically, the goal of BT control is to regulate the electrical power to meet the load demand profile meanwhile minimizing the variations of internal variables, such as water level and drum pressure within their safe sets. Moreover, drum pressure must also be controlled properly in the operation range to respond to possible turbine speed changes caused by load demand variations. Many works have been addressed at this point focusing on deriving satisfactory closed-loop control performance of electrical power plants; see [37], [38], [39], [40], [41]. In this scenario, the control related to the output variables, such as electrical power and drum pressure is a major issue that must be tackled properly to respond to frequent load demand variations. At the same time, the water level can be adjusted smoothly under its constraint with the possibility to follow its desired value. Hence, it is reasonable to apply the proposed dual-level control algorithms in this scenario.

In the considered system [36], the state variables are ρ, P, and Q, where ρ is the fluid density (kg/cm^3) that establishes a static mapping to the water level. The control variables are limited by $0 \leq q_f, q_w, q_s \leq 1$ and their rate constraints are also considered, i.e., $-0.007 \leq \dot{q}_f \leq 0.007$, $-2 \leq \dot{q}_s \leq 0.2$, $-0.05 \leq \dot{q}_w \leq 0.05$. The linearized model at an operation point $(\rho_r,\ P_r,\ Q_r) = (513.6,\ 129.6,\ 105.8)$, $(q_{w,r},\ q_{f,r}$ and $q_{s,r}) = (0.663,\ 0.505,\ 0.828)$ is considered as the controlled model, i.e.,

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{38}$$

where $C = I$, the state and output variables are $y = x = (\rho - \rho_r,\ P - P_r,\ Q - Q_r)$, while the input variables are $u = (q_w - q_{w,r},\ q_f - q_{f,r},\ q_s - q_{s,r})$. The unitary step response of (38) is presented in Fig. 4, which displays that the system outputs are strongly coupled, and the dynamics are not strictly separable.

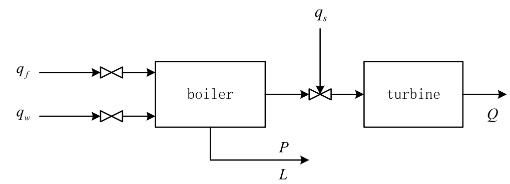


Fig. 3. Diagram of the BT dynamics.

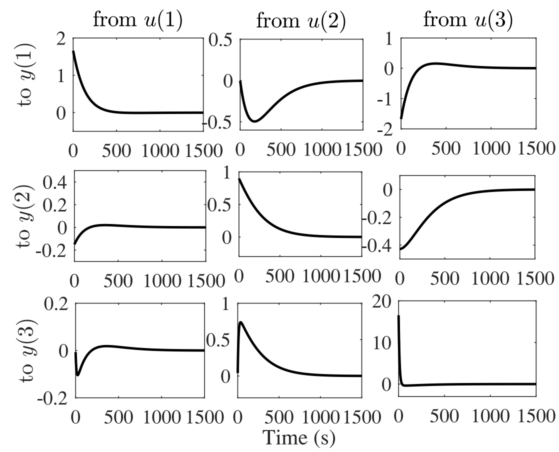


Fig. 4. Unitary impulse response of the BT dynamics.

B. Design of the D-MPC and Incremental D-MPC Regulators

In order to implement the proposed dual-level control algorithms, the system's continuous-time model (38) was sampled with $\Delta t = 1$ s to derive model (1), where the input, state, and output variables associated with (1a) to be controlled smoothly were chosen as $u_s = q_w - q_{w,r}$, $x_s = \rho - \rho_r$, and $y_s = x_s$, while the corresponding ones associated with (1b) to be controlled promptly were $u_f = (q_f - q_{f,r},\ q_s - q_{s,r})$, $x_f = (P - P_r,\ Q - Q_r)$, and $y_f = x_f$. The resulting model was resampled with $N = 20$ to obtain (6) and (20) to be used at the high level.

1) Design of the D-MPC Regulator:

1) The high-level MPC (8) with cost (9) was implemented with $Q_H = I$ and $R_H = \text{diag}\{2,\ 20,\ 20\}$, and the prediction horizon was set as $N_H = 20$. The control gain matrix K_H was selected by solving an infinite horizon LQ problem. The terminal penalty P_H was calculated according to (10). The terminal set was chosen according to the algorithm described in [42].

2) The low-level shrinking horizon MPC (13) with cost (14) was designed with $Q = I$ and $R = \text{diag}\{1,\ 1,\ 10\}$.

2) Design of the Incremental D-MPC Regulator:

1) The high-level MPC (28) with cost (29) was implemented with $N_\alpha = 2$ (see Algorithm 2), $\bar{Q}_H = I$, $\bar{R}_H = \text{diag}\{2,\ 20,\ 20\}$, and the prediction horizon was set as

TABLE I
TUNING PARAMETERS OF THE DECENTRALIZED PIDS

Control pair	Proportional (P)	Integral (I)	Derivative (D)
(u_s, y_s)	0.019	$2{\cdot}10^{-4}$	-0.07
$(u_f(1), y_f(1))$	0.24	0.006	-1
$(u_f(2), y_f(2))$	-0.035	$-4.6{\cdot}10^{-4}$	0.36

$\bar{N}_H = 20$. The control gain matrix $\bar{K}_{S,H}$ was selected by solving the corresponding infinite horizon LQ problem. The terminal penalty was calculated according to (10). Likewise, the terminal set was chosen according to the algorithm described in [42].

2) The low-level shrinking horizon MPC (33) with cost (34) was designed with $\bar{Q} = I$ and $R = \mathrm{diag}\{1,\ 1,\ 10\}$.

C. Simulation Results: Control of the Linear Nominal Model

The proposed dual-level control algorithms were applied to the linear BT system by solving an output reference tracking problem. The output set-point $y_r = (10, 2, -2)$ was initially considered; while at time $t = 400$ s, the reference value was reset according to a new load profile, i.e., $y_r = (5, 1, 4)$. The dual-level control algorithms were implemented from null initial conditions. In the following, the designs of the adopted comparative controllers are described.

1) Design of the Multirate MPC [29]: Note that, due to the usage of FIR, the model used in [29] must be strictly stable. However, the considered system has a pole on the unitary circle. Hence, a feedback compensator $u_s = ky_s + v$ was used, where v is an auxiliary control input calculated by the MPC, k is chosen as $k = -0.005$. For a fair comparison, the design parameters Q_s and R_s were selected coincident with the proposed MPC algorithms, i.e., $Q_s = \mathrm{diag}\{1, 2, \ldots, 2, 20, \ldots, 20\}$, $R_s = 2$.

2) Design of the Single-Layer MPC [43]: Two single-layer stabilizing MPC algorithms were designed to work in slow and fast time scales, respectively. The sampling periods were chosen as $\Delta t = 20$ s and $\Delta t = 1$ s, respectively; and the prediction horizon values were chosen as N_H and N, respectively. The design parameters Q and R are the same as that at the high level of the D-MPC.

3) Design of the Decentralized PID Controller: The decentralized PIDs, one for each input/output pair, were designed with all the selected tuning parameters listed in Table I.

4) Simulation Results: All the comparative simulation experiments were implemented within the MATLAB environment in a Laptop with Intel Core i9-9880H 2.30-GHz running Windows 10 operating system. Also, the MPC algorithms were implemented with an additional Yalmip toolbox [44] in the MATLAB environment. The simulation results were reported in Figs. 5–7. As shown in Fig. 5, after an initial transient, inputs, and outputs return to their nominal values, until the change of the reference occurs when the D-MPC algorithms and single-layer MPC algorithms properly react to bring the input and output variables to their new steady-state values, while the multirate MPC and decentralized PIDs react more slowly to reference variations. Note that the proposed

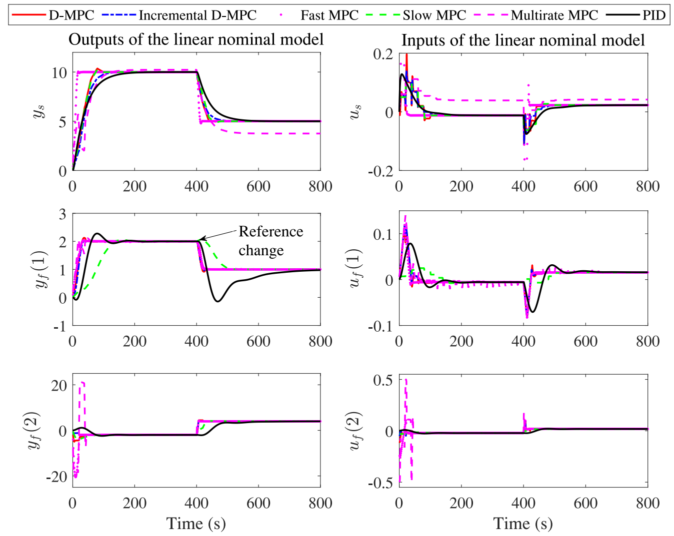


Fig. 5. Output and control variables of the controlled linear model.

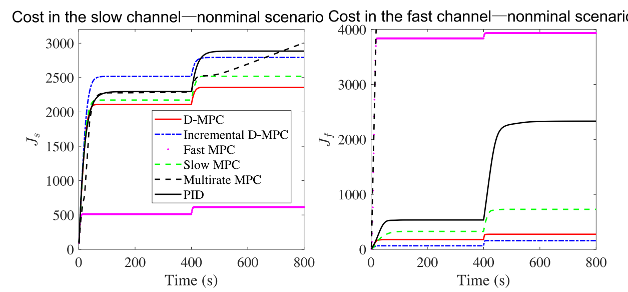


Fig. 6. Comparison of variations of cumulative square regulation errors among all the controllers in the nominal scenario: The proposed D-MPC and *Incremental* D-MPC have gained the smaller regulation costs of the fast (crucial) output y_f at the expense of larger ones of the slow (less crucial) output y_s.

dual-level algorithms exhibit better control performances than other approaches for the crucial pair (u_f, y_f), while the fast MPC performs the best for the pair (u_s, y_s) which in fact can be controlled smoothly in the considered problem. In Fig. 6, the cumulative square regulation errors $J_s = \sum_{i=1}^{N_{\mathrm{sim}}} \|y_s(i) - y_{s,r}\|^2$, and $J_f = \sum_{i=1}^{N_{\mathrm{sim}}} \|y_f(i) - y_{f,r}\|^2$ with $N_{\mathrm{sim}} = 800$ are displayed for all the approaches. The results show that, the proposed algorithms result in smaller values of cost J_f than other approaches at the expense of a larger cost on J_s. The cost J_s with the fast MPC is the lowest but at the expense of a larger cost on J_f, i.e., a degradation of the control performance on (u_f, y_f). Also, the cost J_f with the *Incremental* D-MPC is smaller than that with the D-MPC at the price of a slightly larger J_s. In other words, the proposed D-MPC and *Incremental* D-MPC show strong points in imposing separable closed-loop dynamics, i.e., controlling the pair (u_f, y_f) promptly while regulating the less crucial pair (u_s, y_s) in a smoother fashion. Also, the *Incremental* D-MPC outperforms the D-MPC in this respect. As for computational resources, the average computational time values of the proposed algorithms are slightly smaller than that of the fast MPC in the nominal scenario (see Fig. 7).

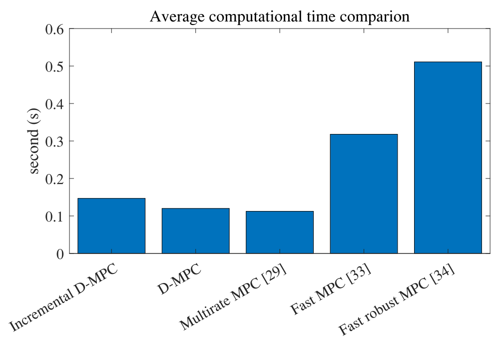


Fig. 7. Computational time comparison. The computational time values of the proposed algorithms are smaller than those of the fast single-layer MPC algorithms.

D. Simulation Results: Control of the Linear Model With Uncertainties

To further verify the capability of the proposed algorithms in dealing with disturbances. A bounded unknown step-wise disturbance, i.e., $(-0.2, 0.05, -0.1) \leq d \leq (0.2, 0.1, 0.1)$, was added to the discretized model of (38); see Fig. 8. In the proposed controller, the control constraints were properly tightened according to (18) and the terminal constraint was computed according to [45]. For comparison, the multirate MPC [29], two single-layer robust MPC regulators in [46], the decentralized PID controller, and the sliding mode controller (SMC) in [47] were used. In the robust MPC algorithms, the design parameters were chosen similar to the nominal MPC algorithms, except that the control constraints were tightened according to the robust invariant set for real constraint satisfaction under perturbations, and the optimization on the initial nominal state was considered (see [46]). In the SMC, all the parameters are fine-tuned according to the design procedures in [47]. In the simulation tests, the output set-point regulation with $y_r = (10, 2, -2)$ was considered. The corresponding simulation results are presented in Figs. 9 and 10. The results show that, the proposed *Incremental* D-MPC can realize offset-free control for all the outputs, which is not yet realized by the D-MPC, the multirate MPC, the robust MPC approaches, the PIDs, and the SMC, since most of the cumulative costs are still increasing at the terminal simulation time (see Fig. 10). Also, the multirate MPC is early terminated due to the infeasibility issue caused by disturbances. In other words, the proposed approaches, especially the *Incremental* D-MPC, outperform the multirate MPC, robust MPC algorithms, decentralized PIDs, and SMC, in enforcing satisfactory control performance for the pair (u_f, y_f) (see again Fig. 10). Also, the average computational time values with the proposed algorithms are slightly smaller than that of the fast robust MPC (see Fig. 7).

For completeness, the proposed controllers were applied to the original nonlinear systems and compared with classic nonlinear MPC algorithms and the SMC in [47]. The implementing steps and simulation results are neglected here for space limitations, interested readers may refer [48] for a detailed report. The simulation results have verified the effectiveness of the proposed approaches.

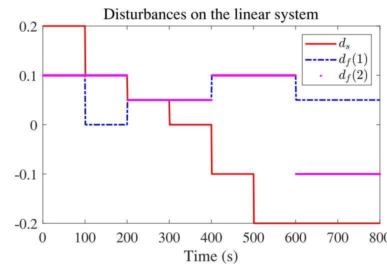


Fig. 8. Disturbances on the linear model.

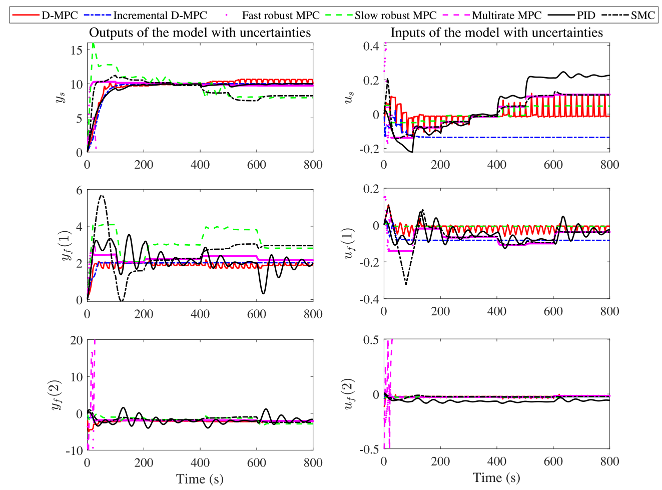


Fig. 9. Output and control variables of the controlled linear model with uncertainties.

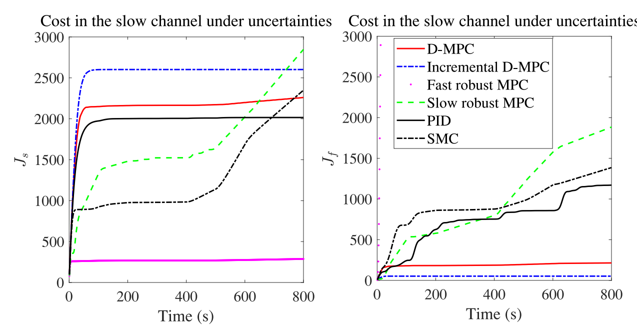


Fig. 10. Comparison of variations of cumulative square regulation errors among different controllers in the perturbed scenario: The proposed *Incremental* D-MPC is the only one that has realized offset-free control at both the fast and slow control channels. Note that the *Incremental* D-MPC has gained the smallest tracking cost of the fast (crucial) output y_f at the expense of the largest one of the slow (less crucial) output y_s.

E. Discussion

As shown in the above simulation tests, the proposed algorithms, especially *Incremental* D-MPC, have fulfilled the considered control objective, i.e., generating satisfactory fast dynamics for the fast (crucial) pair (u_f, y_f) and smooth dynamics for the slow (less crucial) pair (u_s, y_s) (see Figs. 6 and 10). The goal on the pair (u_f, y_f) is enforced at the expense of the performance degradation on the pair (u_s, y_s). However, the above control objective is not well met by the single-layer fast MPC algorithms working in the basic time scale (see again Figs. 6 and 10), where in contrast, the control performances on the less crucial (u_s, y_s) are the best ones among all the controllers. Note that the proposed algorithms might not outperform the single-layer MPC algorithms when the overall control performance is a major concern, which is not in the scope of this article. The computational time values of the proposed algorithms are slightly smaller than that of the fast MPC algorithms, due to the shrinking horizon strategy used in the basic time (see Fig. 7).

Different from output-feedback controllers, the proposed approaches do not rely on an observer for estimating disturbances. The simulation results in Figs. 9 and 10 show that the proposed approaches are robust to time-varying disturbances, and the *Incremental* D-MPC can realize offset-free control under unknown time-varying piece-wise disturbances. One possible limitation of the proposed algorithms lies in the boundedness assumption of the disturbances for the recursive feasibility guarantee of constrained control problems.

VI. Conclusion

In this article, two D-MPC control algorithms have been proposed for linear multitimescale systems with input constraints. The proposed MPC algorithms rely on clear time separation, so allow to deal with control problems in different channels. In view of their main properties, the proposed algorithms are, based on the MPC solution with a dual-level structure, suitable not only to cope with the control of singularly perturbed systems but also to impose different closed-loop dynamical performance for systems with nonseparable open-loop dynamics.

The recursive feasibility and convergence of the proposed D-MPC and *Incremental* D-MPC are proven under suitable assumptions. The effectiveness of the proposed algorithms is tested rigorously in different simulation scenarios, including numerous comparisons with different classic controllers. The simulation results show that both the proposed D-MPC and *Incremental* D-MPC are effective in imposing closed-loop separable dynamics and can deal with unknown bounded time-varying disturbances. Also, the latter can obtain offset-free control under unknown and time-varying step-wise disturbances without using a disturbance estimator.

Future work will extend the proposed framework to solving multirate control problems for large-scale dynamical systems, possibly relying on a cloud-edge computing structure; and apply the proposed approaches to multiagent control systems.

Appendix

A. Proof of Proposition 1

According to the PBH detectability rank test, the pair $(A,\ C)$ is detectable if and only if $\text{rank}(\begin{bmatrix}\lambda I - A^\top & C^\top\end{bmatrix}^\top) = n\ \forall\, \lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. An equivalent form to this condition is that $v = 0$ is the unique solution to the following linear equations:

$$\begin{cases} Av = \lambda v \\ Cv = 0 \end{cases} \tag{39}$$

$\forall\, \lambda \in \mathbb{C}$ and $|\lambda| \geq 1$. From (39), $v = 0$ is the unique solution to $\lambda^{i-1}Av = \lambda^i v,\ Cv = 0\ \forall\, i \in \mathbb{N}_+$, which is $A^i v = \lambda^i v,\ Cv = 0\ \forall\, i \in \mathbb{N}_+$. In view of this, recalling that $(A,\ C)$ is detectable, it holds that $v = 0$ is the only solution to

$$\begin{cases} A^{[N]}v = \mu v \\ Cv = 0 \end{cases}$$

where $\mu = \lambda^N$, which implies $(A^{[N]},\ C)$ is observable for all the modes that their poles $|\lambda| \geq 1$. Hence, Proposition 1 holds.

B. Proof of Theorem 1

1) Recursive Feasibility of the D-MPC [i.e., High-Level Problem (8) and Low-Level Problem (13)]: As the problem (8) is assumed to be feasible at time $k = 0$, one can prove the closed-loop recursive feasibility by verifying that: if (8) is feasible at any time k, then:

1) the low-level problem (13) is feasible at any fast time $h \in [kN, kN + N)$;
2) also, the high-level problem (8) is feasible at the subsequent slow time instant $k + 1$.

First, we show that condition 1) can be verified. To proceed, we assume that the high-level problem (8) is feasible at time k, and $\delta u(kN), \ldots, \delta u(h|h), \ldots, \delta u(kN+N-1|h)$ is a feasible solution at time $h \in [kN, kN + N - 2]$ and the terminal state constraint is verified. Letting $h = kN + t$, one has

$$\begin{aligned} \hat{x}^{[N]}(k+1|k) = &\sum_{i=0}^{N-1} A^i Bu(kN+i|h) + A^N x(kN) \\ &+ \sum_{i=0}^{t-1} A^{N-i-1} d(kN+i). \end{aligned} \tag{40}$$

One can also write at time $h + 1$

$$\begin{aligned} \hat{x}(kN+N|h+1) = &\sum_{i=0}^{N-1} A^i Bu(kN+i|h+1) + A^N x(kN) \\ &+ \sum_{i=0}^{t} A^{N-i-1} d(kN+i). \end{aligned} \tag{41}$$

To ensure the recursive feasibility, it is required to enforce $\hat{x}(kN + N|h + 1) = \hat{x}^{[N]}(k + 1|k)$. By the difference of (41) and (40), leads to

$$\begin{aligned} &\sum_{i=t+1}^{N-1} A^{N-i-1} Bu(kN+i|h+1) = -A^{N-t-1} d(h) \\ &\quad + \sum_{i=t+1}^{N-1} A^{N-i-1} Bu(kN+i|h) \in \Delta L_t \mathscr{U} \oplus L_{t+1} \hat{\mathscr{U}}_{t+1} \end{aligned} \tag{42}$$

in view of condition (18). Hence, the recursive feasibility at the low level follows.

As for 2), first note that, one can compute the gap between the real state and the predicted one, i.e.,

$$x^{[N]}(k+1) - \hat{x}^{[N]}(k+1|k) = \tilde{d}^{[N]}(k). \tag{43}$$

Note that, $\hat{x}(kN + N|kN + N - 1) = \hat{x}^{[N]}(k + 1)$ and $x(kN + N) - \hat{x}(kN + N|kN + N - 1) = d(kN + N - 1)$. One promptly has

$$\tilde{d}^{[N]}(k) = d(kN + N - 1).$$

Hence, one can also compute

$$\hat{x}^{[N]}(k+N_H|k) - \hat{x}^{[N]}(k+N_H|k+1) = \left(A^{[N]}\right)^{N_H-1} d(kN+N-1). \tag{44}$$

Assume that at any slow time instant k the optimal control sequence of (8) can be found, i.e., $\overrightarrow{u^{[N]}}(k : k+N_H-1|k) = (u^{[N]}(k|k), \ldots, u^{[N]}(k+N_H-1|k))$ such that $x^{[N]}(k+N_H|k) \in \mathscr{X}_F^s$. Let the input sequence $\overrightarrow{u^{[N],s}}(k+1 : k+N_H|k+1) = (u^{[N]}(k+1|k), \ldots, u^{[N]}(k+N_H-1|k), K_H(x^{[N]}(k+N_H|k)-x_r)+u_r)$ be a candidate choice at the next time instant $k+1$. As condition (17) is assumed, it holds that $\hat{x}(k+N_H|k+1) \in \mathscr{X}_F$. Consequently, $\hat{x}(k+N_H+1|k+1) \in \mathscr{X}_F^s$ can be verified in view of the definitions of K_H and $\mathscr{X}_F^s$. Hence, the recursive feasibility of (8) follows.

2) Convergence of the Closed-Loop System: We first prove the convergence of the high-level problem (8). As the disturbance $d = 0$, it holds that $\hat{x} = x$, $\hat{y} = y$. Denote by $J_H^o(x^{[N]}(k))$ the optimal cost associated with $\overrightarrow{u^{[N]}}(k : k+N_H-1|k)$ at time k and by $J_H^s(x^{[N]}(k+1|k))$ the suboptimal cost associated with $\overrightarrow{u^{[N],s}}(k+1 : k+N_H|k+1)$ at time $k+1$. One can compute

$$\begin{aligned} &J_H^s(x^{[N]}(k+1|k)) - J_H^o(x^{[N]}(k)) \\ &= -\left(\|y^{[N]}(k) - y_r\|_{Q_H}^2 + \|u^{[N]}(k|k) - u_r\|_{R_H}^2\right) \\ &\quad + \left\|x^{[N]}(k+N|k) - x_r\right\|^2_{F_H^\top P_H F_H - P_H + C^\top Q_H C + K_H^\top R_H K_H = 0.} \end{aligned} \tag{45}$$

In view of (10) and of $J_H^o(x^{[N]}(k+1|k)) \le J_H^s(x^{[N]}(k+1|k))$ and from (45), one has

$$\begin{aligned} &J_H^o(x^{[N]}(k+1|k)) - J_H^o(x^{[N]}(k)) \\ &\le -\left(\|y^{[N]}(k) - y_r\|_{Q_H}^2 + \|u^{[N]}(k|k) - u_r\|_{R_H}^2\right) \end{aligned} \tag{46}$$

which implies that $J_H^o(x^{[N]}(k+1|k)) - J_H^o(x^{[N]}(k))$ converges to zero. Moreover, from (46), one has $J_H^o(x^{[N]}(k)) - J_H^o(x^{[N]}(k+1|k)) \ge \|y^{[N]}(k) - y_r\|_{Q_H}^2 + \|u^{[N]}(k|k) - u_r\|_{R_H}^2$, then $\|y^{[N]}(k) - y_r\|_{Q_H}^2 + \|u^{[N]}(k|k) - u_r\|_{R_H}^2 \to 0$. Recalling the definitions of Q_H and R_H, one has $\lim_{k\to+\infty} y^{[N]}(k) = y_r$ and $\lim_{k\to+\infty} u^{[N]}(k) = u_r$. In view of Proposition 1, consequently $\lim_{k\to+\infty} x^{[N]}(k) = x_r$.

As for the convergence of the low-level problem (13), assume that the high-level system variables have reached their reference values, i.e., $u^{[N]}(k) \equiv u_r$ $x^{[N]}(k) \equiv x_r$, $y^{[N]}(k) \equiv y_r$. Define $\delta x(k) = x(kN) - x_r$ and $\delta y(k) = y(kN) - y_r$. Along the same line in [30], in view of dynamics (12) at time instant $h = kN$, the low-level dynamics at the slow time scale is defined as follows:

$$\begin{cases} \delta x(k+1) = A^N \delta x(k) + w(k) \\ \delta y(k) = C \delta x(k) \end{cases} \tag{47}$$

where $w(k) = \sum_{j=0}^{N-1} A^{N-j-1} B \delta u(kN+j)$. Since $\delta x(k) = 0\,\forall k \ge 0$ [due to (16)], it holds that $w(k) = 0$. In view of the cost function at the low level, the null sequence $\overrightarrow{\delta u}(h{:}(k+1)N-1) = 0$ solves the problem (8), which implies that $\lim_{h\to+\infty} \delta u(h) = 0$ and $\lim_{h\to+\infty} u(h) = u_r$. Finally, $\lim_{h\to+\infty} y(h) = y_r$ and $\lim_{h\to+\infty} x(h) = x_r$.

C. Proof of Proposition 2

According to the PBH stabilizability rank test, the pair $(\bar{A}^{[N]}, \bar{B}_s^{[N]})$ is stabilizable if and only if $\mathrm{rank}([\lambda I - \bar{A}^{[N]} \;\; \bar{B}_s^{[N]}]) = n + p_s$, for $\lambda \in \mathbb{C}$ and $|\lambda| \ge 1$. An equivalent form to this condition is that $v = 0$ is the unique solution to the following linear equations:

$$\begin{cases} \left(\bar{A}^{[N]}\right)^\top v = \lambda v \\ \left(\bar{B}_s^{[N]}\right)^\top v = 0 \end{cases} \tag{48}$$

where $\lambda \in \mathbb{C}$ and $|\lambda| \ge 1$.

In view of (25), it is possible to write (48) in the form

$$\begin{bmatrix} I - \lambda I & 0 \\ \tilde{C}_s \tilde{A}^{[N]} & \tilde{A}^{[N]} - \lambda I \\ \tilde{C}_s \tilde{B}_s^{[N]} & \tilde{B}_s^{[N]} \end{bmatrix}^\top v = 0. \tag{49}$$

Since $(\tilde{A}^{[N]}, \tilde{B}_s^{[N]})$ is stabilizable by Assumption 4, it is obvious to see that for $|\lambda| > 1$, $v = 0$ is the unique solution to (49). For $\lambda = 1$, $v = 0$ is the unique solution to (49) if and only if

$$\mathrm{rank}\left(\begin{bmatrix} \tilde{C}_s \tilde{A}^{[N]} & \tilde{A}^{[N]} - I \\ \tilde{C}_s \tilde{B}_s^{[N]} & \tilde{B}_s^{[N]} \end{bmatrix}^\top\right) = n + p_s.$$

As for $\lambda = -1$, $v = 0$ is the unique solution to (49) if and only if

$$\mathrm{rank}\left(\begin{bmatrix} 2I & 0 \\ \tilde{C}_s \tilde{A}^{[N]} & \tilde{A}^{[N]} + I \\ \tilde{C}_s \tilde{B}_s^{[N]} & \tilde{B}_s^{[N]} \end{bmatrix}^\top\right) = n + p_s.$$

D. Proof of Theorem 2

1) Recursive Feasibility of the Incremental D-MPC [High-Level Problem (28) and Low-Level Problem (33)]: As N_α is assumed to be reachable by Algorithm 2 such that (28) is feasible at time $k = 0$, along the same line of Appendixes-A and -B, we first prove the recursive feasibility for problem (33) in the fast time scale. To this end, note that (33) is feasible at a time instant $h \in [kN, kN+N)$ means that one can find the candidate control sequence $\Delta u(kN), \ldots, \Delta u(h|h), \ldots, \Delta u(kN+N-1|h)$ such that the terminal state constraint is verified. In line with Appendix, the above condition requires (42), which can be verified in view of (18). Hence, the recursive feasibility at the low level follows.

As for a sketch of proof for the feasibility at the high level, first note that, one can compute

$$\begin{aligned} \Delta x^{[N]}(k+1) - \Delta \hat{x}^{[N]}(k+j|k) &= x^{[N]}(k+1) - \hat{x}^{[N]}(k+1|k) \\ &= d(kN+N-1). \end{aligned} \tag{50}$$

Recalling that $\bar{x}^{[N]} = (y_s^{[N]}, \;\Delta x^{[N]})$, one also has

$$\hat{\bar{x}}^{[N]}(k+\bar{N}_H|k) - \hat{\bar{x}}^{[N]}(k+\bar{N}_H|k+1) = \left(\bar{A}^{[N]}\right)^{\bar{N}_H-1} E d(kN+N-1). \tag{51}$$

Let assume at time k that the optimal control sequence (8) is found, i.e., $\overrightarrow{\Delta u_s^{[N]}}(k : k+\bar{N}_H-1|k) = (\Delta u_s^{[N]}(k|k), \ldots, \Delta u_s^{[N]}(k+\bar{N}_H-1|k))$ such that constraint (27) is fulfilled and $\hat{\bar{x}}^{[N]}(k+\bar{N}_H|k) \in \bar{\mathscr{X}}_F^s$. Noting the fact that $\bar{N}_H \ge N_\alpha$, one has $\alpha(k+\bar{N}_H) = 1\,\forall k \ge 0$. Let $\overrightarrow{\Delta u_s^{[N],s}}(k+$

$1 : k + \bar{N}_H + 1|k + 1) = (\Delta u_s^{[N]}(k+1|k), \ldots, \Delta u_s^{[N]}(k+\bar{N}_H-1|k), \bar{K}_{s,\mathrm{H}}(\bar{x}^{[N]}(k+\bar{N}_H|k) - \bar{C}^\top y_{s,r}))$ be the candidate input sequence at the next time instant $k+1$. As condition (37) is assumed, it holds that $\tilde{\bar{x}}(k+\bar{N}_H|k+1) \in \bar{\mathscr{X}}_F$. In view of this, $\tilde{\bar{x}}(k+\bar{N}_H+1|k+1) \in \bar{\mathscr{X}}_F^s$ can be verified in view of the definitions of $\bar{K}_{s,\mathrm{H}}$ and $\bar{\mathscr{X}}_F^s$. Hence, the recursive feasibility of (28) follows.

2) Convergence of the Incremental D-MPC: In view of (22) and the feasibility result of (28) under d being constant, along the same line of Appendixes-A and -B, one can compute

$$\bar{J}_H^o(\bar{x}^{[N]}(k+1|k)) - \bar{J}_H^o(\bar{x}^{[N]}(k)) \leq -\left(\|\bar{x}_s^{[N]}(k) - \bar{C}^\top y_{s,r}\|^2_{\bar{Q}_{s,\mathrm{H}}} + \|\Delta u_s^{[N]}(k|k)\|^2_{\bar{R}_{s,\mathrm{H}}}\right) \quad (52)$$

where $\bar{J}_H^o$ is the optimal cost. Equation (52) implies that $\bar{J}_H^o(\bar{x}^{[N]}(k+1|k)) - \bar{J}_H^o(\bar{x}^{[N]}(k))$ converges to zero. Consequently, it holds that $\|\bar{x}_s^{[N]}(k) - \bar{C}^\top y_{s,r}\|^2_{\bar{Q}_H} + \|\Delta u_s^{[N]}(k|k)\|^2_{\bar{R}_H} \to 0$ as well. Recalling the definitions of $\bar{Q}_H$ and $\bar{R}_H$, it holds that $\lim_{k\to+\infty} \bar{x}_s^{[N]}(k) = \bar{C}^\top y_{s,r}$ and $\lim_{k\to+\infty} \Delta u_s^{[N]}(k) = 0$. Consequently, one has $\lim_{k\to+\infty} y^{[N]}(k) = y_r$, $\lim_{k\to+\infty} u_s^{[N]}(k) = const$. In view of Proposition 1, it promptly follows that, $\lim_{k\to+\infty} x^{[N]}(k) = x_r$, $\lim_{k\to+\infty} u_s^{[N]}(k) = u_{s,r}$. The arguments for the results $\lim_{h\to+\infty} \Delta u(h) = 0$, $\lim_{h\to+\infty} y(h) = y_r$ are similar to Appendixes-A and -B. Consequently, one has $\lim_{h\to+\infty} x(h) = x_r$, and $\lim_{h\to+\infty} u(h) = u_r$.

ACKNOWLEDGMENT

The first author wishes to thank Prof. Riccardo Scattolini and Prof. Marcello Farina from Politecnico di Milano, for their fruitful discussions.

REFERENCES

[1] D. Naidu, “Singular perturbations and time scales in control theory and applications: An overview,” *Dyn. Continuous Discr. Impulsive Syst. B*, vol. 9, no. 2, pp. 233–278, 2002.

[2] P. Kokotović, H. K. Khalil, and J. O’Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. Philadelphia, PA, USA: SIAM, 1999.

[3] J. Rodriguez-Vasquez, R. Perez, J. Moriano, and J. González, “Advanced control system of the steam pressure in a fire-tube boiler,” *IFAC Proc. Vol.*, vol. 41, no. 2, pp. 11028–11033, 2008.

[4] A. Liniger, A. Domahidi, and M. Morari, “Optimization-based autonomous racing of 1: 43 scale RC cars,” *Opt. Control Appl. Methods*, vol. 36, no. 5, pp. 628–647, 2015.

[5] R. Verschueren, S. De Bruyne, M. Zanon, J. V. Frasch, and M. Diehl, “Towards time-optimal race car driving using nonlinear MPC in real-time,” in *Proc. 53rd IEEE Conf. Decis. Control*, 2014, pp. 2505–2510.

[6] B. Paden, M. Čáp, S. Z. Yong, D. Yershov, and E. Frazzoli, “A survey of motion planning and control techniques for self-driving urban vehicles,” *IEEE Trans. Intell. Veh.*, vol. 1, no. 1, pp. 33–55, Mar. 2016.

[7] S. J. Qin and T. A. Badgwell, “A survey of industrial model predictive control technology,” *Control Eng. Pract.*, vol. 11, no. 7, pp. 733–764, 2003.

[8] M. Li, P. Zhou, H. Wang, and T. Chai, “Nonlinear multiobjective MPC-based optimal operation of a high consistency refining system in papermaking,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 3, pp. 1208–1215, Mar. 2020.

[9] Q. Jin, S. Wu, and R. Zhang, “Improved constrained model predictive tracking control for networked coke furnace systems over uncertainty and communication loss,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 5, pp. 1877–1884, May 2020.

[10] S. Roshany-Yamchi, M. Cychowski, R. R. Negenborn, B. De Schutter, K. Delaney, and J. Connell, “Kalman filter-based distributed predictive control of large-scale multi-rate systems: Application to power networks,” *IEEE Trans. Control Syst. Technol.*, vol. 21, no. 1, pp. 27–39, Jan. 2013.

[11] R. Zhang and F. Gao, “Two-dimensional iterative learning model predictive control for batch processes: A new state space model compensation approach,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 2, pp. 833–841, Feb. 2021.

[12] X. Liu, L. Ma, X. Kong, and K. Y. Lee, “Robust model predictive iterative learning control for iteration-varying-reference batch processes,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 7, pp. 4238–4250, Jul. 2021.

[13] F. Abdullah, Z. Wu, and P. D. Christofides, “Sparse-identification-based model predictive control of nonlinear two-time-scale processes,” *Comput. Chem. Eng.*, vol. 153, Oct. 2021, Art. no. 107411.

[14] J. Wang, Z. Wu, M. Tan, and J. Yu, “Model predictive control-based depth control in gliding motion of a gliding robotic dolphin,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 51, no. 9, pp. 5466–5477, Sep. 2021.

[15] F. Santoso, M. A. Garratt, S. G. Anavatti, and I. Petersen, “Robust hybrid nonlinear control systems for the dynamics of a quadcopter drone,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 50, no. 8, pp. 3059–3071, Aug. 2020.

[16] Z. Zhou, B. De Schutter, S. Lin, and Y. Xi, “Two-level hierarchical model-based predictive control for large-scale urban traffic networks,” *IEEE Trans. Control Syst. Technol.*, vol. 25, no. 2, pp. 496–508, Mar. 2017.

[17] G. S. van de Weg, M. Keyvan-Ekbatani, A. Hegyi, and S. P. Hoogendoorn, “Linear MPC-based urban traffic control using the link transmission model,” *IEEE Trans. Intell. Transp. Syst.*, vol. 21, no. 10, pp. 4133–4148, Oct. 2020.

[18] X. Chen, M. Heidarinejad, J. Liu, and P. D. Christofides, “Composite fast-slow MPC design for nonlinear singularly perturbed systems,” *AIChE J.*, vol. 58, no. 6, pp. 1802–1811, 2012.

[19] X. Chen, M. Heidarinejad, J. Liu, D. de la Peña, and P. D. Christofides, “Model predictive control of nonlinear singularly perturbed systems: Application to a large-scale process network,” *J. Process Control*, vol. 21, no. 9, pp. 1296–1305, 2011.

[20] M. Ellis, M. Heidarinejad, and P. D. Christofides, “Economic model predictive control of nonlinear singularly perturbed systems,” *J. Process Control*, vol. 23, no. 5, pp. 743–754, 2013.

[21] L. Ma, C. Cai, and X. Ma, “Slow sampling control of singularly perturbed systems subject to actuator saturation and L2 disturbance,” *Asian J. Control*, vol. 19, no. 4, pp. 1316–1328, 2017.

[22] Y. Miao, L. Ma, X. Ma, and L. Zhou, “Fast sampling control of singularly perturbed systems with actuator saturation and disturbance,” *Math. Problems Eng.*, vol. 2015, Sep. 2015, Art. no. 514583.

[23] M. Wogrin and L. Glielmo, “An MPC scheme with guaranteed stability for linear singularly perturbed systems,” in *Proc. 49th IEEE Conf. Decis. Control*, 2010, pp. 5289–5295.

[24] J. Niu, J. Zhao, Z. Xu, and J. Qian, “A two-time scale decentralized model predictive controller based on input and output model,” *J. Anal. Methods Chem.*, vol. 2009, Sep. 2009, Art. no. 164568.

[25] M. Brdys, M. Grochowski, T. Gminski, K. Konarczak, and M. Drewa, “Hierarchical predictive control of integrated wastewater treatment systems,” *Control Eng. Pract.*, vol. 16, no. 6, pp. 751–767, 2008.

[26] M. Ohshima, I. Hashimoto, H. Ohno, M. Takeda, T. Yoneyama, and F. Gotoh, “Multirate multivariable model predictive control and its application to a polymerization reactor,” *Int. J. Control*, vol. 59, no. 3, pp. 731–742, 1994.

[27] E. Van Henten and J. Bontsema, “Time-scale decomposition of an optimal control problem in greenhouse climate management,” *Control Eng. Pract.*, vol. 17, no. 1, pp. 88–96, 2009.

[28] P. Sanila and J. Jacob, “Simultaneous tracking and vibration control of flexible joint manipulator using Laguerre network based composite fast-slow MPC,” in *Proc. IEEE Int. Conf. Signal Process. Inf. Commun. Energy Syst. (SPICES)*, 2015, pp. 1–6.

[29] X. Zhang, M. Farina, S. Spinelli, and R. Scattolini, “Multi-rate model predictive control algorithm for systems with fast-slow dynamics,” *IET Control Theory Appl.*, vol. 12, no. 18, pp. 2468–2477, 2018.

[30] B. Picasso, X. Zhang, and R. Scattolini, “Hierarchical model predictive control of independent systems with joint constraints,” *Automatica*, vol. 74, pp. 99–106, Dec. 2016.

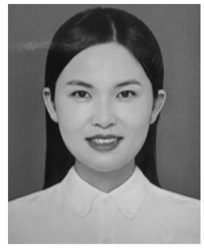
[31] E. F. Mishchenko, Y. S. Kolesov, A. Y. Kolesov, and N. K. Rozov, *Asymptotic Methods in Singularly Perturbed Systems*. New York, NY, USA: Springer, 1994.

[32] X.-H. Chang and G.-H. Yang, "Fuzzy robust constrained model predictive control for nonlinear systems," *Asian J. Control*, vol. 13, no. 6, pp. 947–955, 2011.
[33] B. Picasso, D. De Vito, R. Scattolini, and P. Colaneri, "An MPC approach to the design of two-layer hierarchical control systems," *Automatica*, vol. 46, no. 5, pp. 823–831, 2010.
[34] X. Huo, H. R. Karimi, X. Zhao, B. Wang, and G. Zong, "Adaptive-critic design for decentralized event-triggered control of constrained nonlinear interconnected systems within an identifier-critic framework," *IEEE Trans. Cybern.*, vol. 52, no. 8, pp. 7478–7491, Aug. 2022.
[35] G. Betti, M. Farina, and R. Scattolini, "A robust MPC algorithm for offset-free tracking of constant reference signals," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2394–2400, Sep. 2013.
[36] K. Åström and R. Bell, "Dynamic models for boiler-turbine alternator units: Data logs and parameter estimation for a 160 MW unit," Dept. Autom. Control, Lund Inst. Technol., Lund, Sweden, Rep. TFRT-3192, 1987.
[37] P. Chen, "Multi-objective control of nonlinear boiler-turbine dynamics with actuator magnitude and rate constraints," *ISA Trans.*, vol. 52, no. 1, pp. 115–128, 2013.
[38] X. Liu, P. Guan, and C. Chan, "Nonlinear multivariable power plant coordinate control by constrained predictive scheme," *IEEE Trans. Control Syst. Technol.*, vol. 18, no. 5, pp. 1116–1125, 2009.
[39] X. Liu and X. Kong, "Nonlinear fuzzy model predictive iterative learning control for drum-type boiler–turbine system," *J. Process Control*, vol. 23, no. 8, pp. 1023–1040, 2013.
[40] X. Liu and J. Cui, "Economic model predictive control of boiler-turbine system," *J. Process Control*, vol. 66, pp. 59–67, Jun. 2018.
[41] X. Liu and J. Cui, "Fuzzy economic model predictive control for thermal power plant," *IET Control Theory Appl.*, vol. 13, no. 8, pp. 1113–1120, 2019.
[42] X. Hu and W. Chen, "Model predictive control: Terminal region and terminal weighting matrix," *Proc. Inst. Mech. Eng. I, J. Syst. Control Eng.*, vol. 222, no. 2, pp. 69–79, 2008.
[43] J. B. Rawlings, D. Q. Mayne, and M. Diehl, *Model Predictive Control: Theory, Computation, and Design*. Madison, WI, USA: Nob Hill Publ., 2017.
[44] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proc. CACSD Conf.*, 2004, pp. 284–289.
[45] D. L. Marruedo, T. Alamo, and E. Camacho, "Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties," in *Proc. 41st IEEE Conf. Decis. Control*, vol. 4, 2002, pp. 4619–4624.
[46] I. Kiaei and S. Lotfifard, "Tube-based model predictive control of energy storage systems for enhancing transient stability of power systems," *IEEE Trans. Smart Grid*, vol. 9, no. 6, pp. 6438–6447, Nov. 2017.
[47] Z. Guo, Q. Ma, J. Guo, B. Zhao, and J. Zhou, "Performance-involved coupling effect-triggered scheme for robust attitude control of HRV," *IEEE/ASME Trans. Mechatronics*, vol. 25, no. 3, pp. 1288–1298, Jun. 2020.
[48] X. Zhang, W. Jiang, S. Yu, Z. Li, and X. Xu, "A dual-level model predictive control scheme for multi-timescale dynamical systems–extended version," 2021, *arXiv:1906.07334*.

Xinglong Zhang (Member, IEEE) received the B.E. degree in mechanical engineering from Zhejiang University, Hangzhou, China, in 2011, the M.S. degree in mechanical engineering from the PLA University of Science and Technology, Nanjing, China, in 2014, and the Ph.D. degree in system and control from the Politecnico di Milano, Milan, Italy, 2018.

He is currently an Associate Professor with the College of Intelligence Science and Technology, National University of Defense Technology, Changsha, China. His research interests include learning-based model predictive control, adaptive dynamic programming, and their applications in automotive systems.

Dr. Zhang is a member of IEEE CIS Technical Committee on Adaptive Dynamic Programming and Reinforcement Learning.



Wei Jiang received the Ph.D. degree in instrument science and Technology from the National University of Defense Technology, Changsha, China, in 2017.

Since 2019, she has been a Lecturer with the College of Intelligence Science and Technology, National University of Defense Technology. Her current research interests include the theory and technology of smart measurement and metrology, optimized control, and unmanned systems.

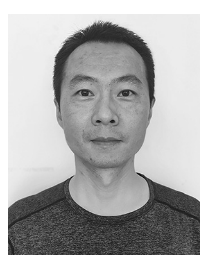
Shuyou Yu received the B.S. and M.S. degrees in control science and engineering from Jilin University, Changchun, China, in 1997 and 2005, respectively, and the Ph.D. degree in engineering cybernetics from the University of Stuttgart, Stuttgart, Germany, in 2011.

From 2010 to 2011, he was a Research and Teaching Assistant with the Institute for Systems Theory and Automatic Control, University of Stuttgart. In 2012, he joined the Faculty of the Department of Control Science and Engineering, Jilin University, where he is currently a Full Professor. His current research interests include model predictive control, robust control, and applications in mechatronic systems.

Xin Xu (Senior Member, IEEE) received the B.S. degree in electrical engineering from the Department of Automatic Control, National University of Defense Technology (NUDT), Changsha, China, in 1996, and the Ph.D. degree in control science and engineering from the College of Mechatronics and Automation, NUDT in 2002.

He has been a Visiting Professor with Hong Kong Polytechnic University, Hong Kong; the University of Alberta, Edmonton, AB, Canada; the University of Guelph, Guelph, ON, Canada; and the University of Strathclyde, Glasgow, U.K.; respectively. He is currently a Professor with the College of Intelligence Science and Technology, NUDT. He has coauthored more than 160 articles in international journals and conferences, and four books. His research interests include intelligent control, reinforcement learning, approximate dynamic programming, machine learning, robotics, and autonomous vehicles.

Dr. Xu is received the Second Class National Natural Science Award of China in 2012. He is currently an Associate Editor of the *Information Sciences*, the *Intelligent Automation and Soft Computing*, and the *Acta Automatica Sinica*, and served as a Guest Editor of the *International Journal of Adaptive Control and Signal Processing* and the IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS: SYSTEMS. He is a member of the IEEE Computational Intelligence Society Technical Committee on Approximate Dynamic Programming and Reinforcement Learning and the IEEE Robotics and Automation Society Technical Committee on Robot Learning.



Zhizhong Li received the B.S. degree in mechatronics engineering and the M.S. degree in instrument science and technology from the National University of Defense Technology, Changsha, China, in 2004 and 2010, respectively, and the Ph.D. degree in mechatronics engineering from the PLA University of Science and Technology, Nanjing, China, in 2014.

From 2015 to 2019, he was a Postdoctoral Researcher with the PLA University of Science and Technology. His current research interests include signal sampling and processing and model predictive control.