Journal of Systems Engineering and Electronics
Vol. 33, No. 5, October 2022, pp.1268 – 1280

Joint optimization of inspection-based and age-based preventive maintenance and spare ordering policies for single-unit systems

MA Weining[1], ZHAO Fei[1,2,3,*], LI Xin[3], HU Qiwei[1], and SHANG Bingcong[2,3]

1. Department of Equipment Command and Management Shijiazhuang Campus, Army Engineering University, Shijiazhuang 050003, China; 2. School of Business Administration, Northeastern University, Shenyang 110057, China;
3. Northeastern University at Qinhuangdao, Qinhuangdao 066099, China

Abstract: This paper presents a joint optimization policy of preventive maintenance (PM) and spare ordering for single-unit systems, which deteriorate subject to the delay-time concept with three deterioration stages. PM activities that combine a non-periodic inspection scheme with age-replacement are implemented. When the system is detected to be in the minor defective stage by an inspection for the first time, place an order and shorten the inspection interval. If the system has deteriorated to a severe defective stage, it is either repaired imperfectly or replaced by a new spare. However, an immediate replacement is required once the system fails, the maximal number of imperfect maintenance (IPM) is satisfied or its age reaches to a pre-specified threshold. In consideration of the spare's availability as needed, there are three types of decisions, i.e., an immediate or a delayed replacement by a regular ordered spare, an immediate replacement by an expedited ordered spare with a relative higher cost. Then, some mutually independent and exclusive renewal events at the end of a renewal cycle are discussed, and the optimization model of such a joint policy is further developed by minimizing the long-run expected cost rate to find the optimal inspection and age-replacement intervals, and the maximum number of IPM. A Monte-Carlo based integration method is also designed to solve the proposed model. Finally, a numerical example is given to illustrate the proposed joint optimization policy and the performance of the Monte-Carlo based integration method.

Keywords: maintenance optimization, imperfect maintenance (IPM), three-stage failure process, spare ordering.

DOI: 10.23919/JSEE.2022.000120

Manuscript received April 07, 2021.
*Corresponding author.
This work was supported by the Naitonal Natural Science Foundation of China (71701038), China Ministry of Education Humanities and Social Sciences Research Youth Fund Project (16YJC630174), the Natural Science Foundation of Hebei Province (G2019501074), the Fundamental Research Funds for the Central Universities (N2123019), and the Postgraduate Funding Project of PLA (JY2020B085).

1. Introduction

Two types of maintenance interventions, i.e., time-based maintenance (TBM) and condition-based maintenance (CBM), have been widely reported in [1]. As a classical policy of TBM, age-replacement policy (ARP) is quite well known due to ease of use and extensively treated [2,3]. Based on this policy, an operating system is replaced at a specified age regardless of the operational condition or the failure time. The inspection-based preventive maintenance policy is commonly used in CBM, in which inspection activities are performed to monitor the health state of the system and identify the hidden defects. Hence, different maintenance actions that depend on the health state are taken to prevent from a functional failure. The most common decisions to be made involve no action on the functioning system with a minor defect, imperfect maintenance (IPM) on the major defective system to restore it as a condition between "as good as new" and "as bad as old", and replacement action by a new identical one [4–7]. The result for each decision has its own costs, and has different effects on the deteriorating process. Among them, replacement leads to a renewal of the defective or failed system, that is, the deterioration process restarts after replacement; IPM can improve the system condition rather than a renewal. To model the maintenance quality of IPM, Malik [8] introduced the concept of an age reduction factor (also called as the improvement factor) in 1979. Pham and Wang [9] summarized various techniques on imperfect maintenance. In most cases, an IPM action is carried out if a failure for a binary-state deteriorating system takes place or is detected; once the number of IPM reaches to a pre-threshold, replacement is performed [10,11]. Considering the combination of CBM and TBM in the practice [12,13], this present work focuses on inspection-based and age-based preventive maintenance (PM) policy for single-unit systems that covers non-periodic inspection policy and ARP policy. Replacement is triggered upon a failure, a specified age, or the system experiencing a certain number of IPM activities. From this, it is noted that decision makers are concerned with the intervals related to inspection, and age replacement, and the maximal number of IPM preceding a replacement. In reality, the system cannot be replaced until the spare is available, and it is obvious that the demand for spare parts is driven by preventive replacement or failure. However, numerous PM studies share an assumption that spare parts are unlimitedly provided for replacement and can be delivered immediately. Therefore, the spare provisioning strategy needs to be merged with the PM policy, rather than optimizing them separately.

Herein, the joint optimization of PM and spare ordering policies for single-unit systems in the literature are only reviewed. Previously in 1976, Osaki and Yamada [14] optimized an age-replacement policy with random lead time, but it is assumed that the spare is ordered at the initial time of the unit operation. Subsequently, it is extended to the system with two types of failures by Sheu and Griffith [15]. Various modifications to the model in [14] have been proposed, such as those in [16−19]. Sheu and Chien [16] and Sheu et al. [17] delt with a generalized ARP of a system subject to shocks, and the expected cost rate function is developed to determine the optimal age. Chien et al. [18,19] incorporated a cumulative repair cost limit policy, which determines whether the system should be repaired or replaced. The above models based on an ARP with random lead time all assume that a spare unit is ordered at time 0, which is quite convenient for modeling and optimization. However, it could cause the increase of the holding cost due to keeping the spare in the inventory too early. To address it, Armstrong and Atkins [20] relaxed this assumption to make the age-replacement and ordering decisions jointly. In the aforementioned policies, the demand for spare parts is triggered randomly at failures or the times where the cumulative repair cost limit is reached, but deterministic at the certain age. However, there are few studies that have been made to look at the optimization of CBM and spare ordering policies. Wang et al. [21] proposed a condition-based order-replacement policy for a single-unit system, which aims to make condition-based replacement and spare ordering decisions jointly based on the system deteriorating level. Zhou et al. [22] considered a single-unit system that deteriorates following Gamma process, and presented the collaborative optimization model of maintenance and spare ordering using a control limit policy. In this proposed model, IPM is performed when the deterioration level reaches the PM threshold, and spare is ordered when the number of IPM exceeds a critical value and meanwhile the state reaches a spare ordering level; replacement is however required once the deterioration level is larger than a predefined failure threshold. Therefore, it aims to optimize the number of IPM and the spare ordering level. Since not all systems can be monitored continuously to obtain data on health state, discrete inspection activities for systems are regarded as one of the effective CBM policy currently [23]. It is also noted that Wang et al. [21] and Zhou et al. [22] treated the system as a continuous-state system, but the maintenance policy for a finite multi-state deteriorating system has been widely focused on according to a Markovian process [24−26], or a delay-time concept [27−30]. However, Christer et al. [31] demonstrated that the delay-time maintenance model has a strong robustness to the Markov assumption by comparing with the semi-Markov maintenance models, and confirmed that the decisions made by the delay-time models are appropriate.

Numerous delay-time-based models are developed to capture the relationship between inspection and defect identification [32]. Motivated by a three-color scheme in industry, Wang et al. [33] extended the two-stage delay-time concept with three states into a four-state failure process, which is closer to reality and provides more options for consideration by decision makers. Further, the three-stage-based models to optimize an inspection policy can be found in [34,35], in which it is more economical to shorten the inspection interval after the minor defective state is identified by inspection for the first time. It is noted, however, that spare parts can be ordered at the time window of identifying a minor defective state. Clearly, this further division for the two-stage delay-time model shows the potential of joint spare parts and maintenance policies, where the nature of the intermittent demand for spare parts can be partly explained [32]. The papers [36−38] studied a joint maintenance and spare parts policy for complex systems with a number of identical components using the two-stage delay-time concept. Based on a three-stage failure process, the joint policy of inspection-based replacement and spare ordering has been modeled in our previous work [39,40], in which a non-repairable single-unit system is considered, thereby replacement is the only maintenance action. However, we are interested in a single-unit system that is expensive and critical in companies; in general, it could require a much longer lead time for delivery. Once a severe defect is identified by inspection, IPM rather than replacement is executed immediately, as the above discussion for the chimney. Only one spare is generally ordered and stored for such a system due to high value, infrequent failure, or deterioration in storage.

This paper presents a joint policy of inspection-based and age-based PM and spare ordering for single-unit systems. The main contributions are summarized as follows: (i) the failure process is divided into three stages based on the practical application, leading to four states through the lifetime process, and different decisions are made relying on the health state; (ii) a PM policy that combines non-periodic inspection and ARP is proposed, by which the health state can be identified at inspection time to determine various maintenance actions, and replacement is preventively underdone at the preset age; (iii) at most one spare part is ordered at the first time of identifying the minor defective state to serve as a replacement and arrives after a random lead time, which is called as a regular order. On the other hand, if no spare is ordered as needed, an emergency order has to be placed; (iv) IPM actions at the time of identifying the severe defective state are described by the proportional age reduction model, and we consider multiple IPMs preceding a replacement; (v) the long-run expected cost rate function for the proposed policy is developed to optimize the optimal decision variables, i.e., the initial inspection interval, the age-replacement interval and the allowable maximum number of IPM.

The remaining of this paper proceeds as follows. Section 2 gives our problem description, and in Section 3, the optimization model is formulated. In Section 4, a Monte-Carlo based integration method is given. A numerical example is given in Section 5 and conclusions are summarized in Section 6.

2. Problem description

The PM policy of integrating inspection and age replacement in our study is motivated from the reality. As a key process in the steel-making stage, Dust-laden waste gases from converter plants operating are exhausted by cooling and washing through chimney to discharge the purified gases into the open. However, the inner lining of the chimney will be corroded with usage and time due to very strong acids from the waste gases and high temperature, which gradually decreases the thickness of inner lining and ultimately causes the cracks. Based on our investigation in a steel making company, the chimney is detected with a longer period at the early operating stage, but once there exit the cracks and slight water leakage it is necessary to detect it more frequently. If it is found that the leakage caused by cracks is serious at the detection time, then the welding for the cracks is carried out to repair it. Commonly, the chimney is replaced with new and identical one after repeated welding, or a preset age determined by maintenance personnel's experience.

2.1 Assumptions and notations

The following represents some modelling assumptions.

(i) A single-unit system deteriorates as a three-stage failure process, which involves the normal stage, the minor defective stage and the severe defective stage. These three stages are mutually independent, and there exist four conditions, i.e., normal, minor and severe defective states, failure, denoted by $S_i (i = 1, 2, 3, 4)$.

(ii) The system state is acquired at discrete inspection times, and there exist various measures for different states with different consequences. An inspection scheme with interval T is firstly carried out at the initial operation of the system, and the inspection activities are perfect. If the system is identified to be in the normal state, no maintenance activity is carried out. Once the minor defective state is detected for the first time at the inspection time T_q, then shorten the inspection interval to be halved to monitor the system more frequently for preventing failure occurring. Meanwhile, one spare is ordered at T_q and arrives in a random lead time l. Repair activity is undertaken at the time of identifying the severe defective state, but the system cannot be restored to be a condition as good as new. Then, the effect of IPM on the effective age is modeled by the proportional age reduction model [4,41].

(iii) Preventive replacement is conducted when the number of IPM reaches the pre-threshold, or the system attains a specified age T_{age}, whichever comes first. Moreover, the failure of the system is self-announced, and it has to be replaced correctively and unexpectedly with a new one at failure. The renewal cycle ends at the replacement point.

(iv) If the spare for replacement is not available and has not been ordered, an expedited replenishment (ER) should be done. It costs much more than the regular order, and it is characterized by a negligible lead time such that the delivery for the emergency replenishment is instantaneous. If the spare is not delivered as required, a delay replacement (DR) will occur till the ordered spare arrives, and then it generates the shortage cost. Should the regular ordered spare be available, an immediate replacement (IR) is carried out, and then the holding cost is generated.

As mentioned in the introduction, assumptions (i) and (ii) are initiated based on the real case of the chimney in a steel-making company, and a various inspection interval scheme can also be observed in the work [42]. Such a halved inspection interval is simple for modelling, but the inspection interval in the minor defective state can be optimized simultaneously as a decision variable. However, it is beyond the scope here in this paper. In terms of the proportional age reduction model [4], the accumulative age is $(1-\rho)\tau_k$ after IPM at τ_k, where $\rho(0 \leqslant \rho \leqslant 1)$ is an improvement factor representing the maintenance effect and

$$\rho = \begin{cases} 1, & \text{PM} \\ 0, & \text{minimal maintenance} \end{cases}.$$

For $t > \tau_k$, the effective age is $t-\rho\tau_k = (1-\rho)\tau_k + (t-\tau_k)$. The system can be restored to the normal state after IPM, but the accumulative age after IPM will influence the instantaneous rates of each stage, resulting in higher instantaneous rates than a new system. Since a new spare is used to replace the defective or failed system, it is regarded as renewal in assumption (iii). Obviously, ER is prior to a regular order when no spare is ordered, so the decision in assumption (iv) is reasonable and closer to the reality. Some notations are given for developing the optimization model. $X_n (n=1,2,3)$ are random durations of three deterioration stages. $f_{X_1,p+1}(x)$, $f_{X_2,p+1}(y)$, $f_{X_3,p+1}(z)$ are probability density functions (PDF) of X_n if the last IPM occurs at T_p. M is the maximum number of identifying the severe defective state S_3. T_{age} is age-replacement interval, and $T_{\text{age}} = N_{\text{age}} \cdot T$. T_f is the failure time. T_p is the time of the last IPM. C_I is average cost of inspection. C_p is average cost of IPM. C_F is average cost of the system failure. C_R is replacement cost by the regular ordered spare. C_{ER} is replacement cost by the expedited ordered spare, and $C_{\text{ER}} > C_R$. C_{PM} is average cost of an inspection renewal. C_{age} is average cost of an age-replacement renewal at T_{age}. C_s is penalty cost per unit time due to shortage. C_h is average cost per unit time caused by holding the spare.

2.2 Joint policy of maintenance and spare ordering considering IPM

It is noted from assumption (iii) that there are three renewal scenarios, i.e., a failure renewal caused by corrective replacement, an age-replacement renewal when the system age reaches T_{age}, and an inspection renewal when the severe defective state is detected and there have been $M-1$ IPM actions. We aim to minimize the long-run expected cost per unit time for determining an optimal joint policy $(T^*, M^*, T^*_{\text{age}})$, or $(T^*, M^*, T^*_{\text{age}})$ with $T^*_{\text{age}} = N^*_{\text{age}} \cdot T^*$. Depending on whether the minor defective state S_2 has been identified before a replacement and whether the last IPM at T_p has occurred before T_q, several mutually exclusive and exhaustive events are discussed, as outlined in Table 1.

3. Optimization model formulation

The occurrence probabilities of all possible events in Table 1 are firstly deduced, and then the expected renewal cost and length over one cycle are obtained.

Table 1 Renewal events of the proposed policy

Renewal scenario		Whether S_2 has been identified	$T_p < T_q$	Availability of the ordered spare	Decision-making
A failure renewal		No	—	—	ER
		Yes	Yes	No	DR
				Yes	IR
			No	No	DR
				Yes	IR
An age-replacement renewal at T_{age}	The system is in the state S_1	No	—	—	ER
		Yes	No	No	DR
				Yes	IR
	The system is in the state S_2	No	—	—	ER
		Yes	Yes	No	DR
				Yes	IR
			No	No	DR
				Yes	IR
	The system is in the state S_3	No	—	—	ER
		Yes	Yes	No	DR
				Yes	IR
			No	No	DR
				Yes	IR
An inspection renewal after the number of IPM reaches the pre-specified threshold		No	—	—	ER
		Yes	Yes	No	DR
				Yes	IR
			No	No	DR
				Yes	IR

3.1 Case I: failure renewal

The system fails at T_f, before which the minor defective state is not identified. Consequently, there is ordered spare for replacement. Meanwhile, the IPM may have already been performed before T_f, as shown in Fig. 1.

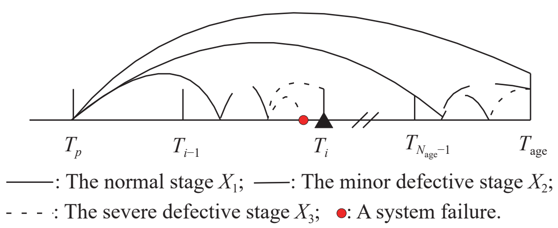

Fig. 1 System replaced by the spare through an ER

Then the failure probability of such a case can be given by

$$P(T_{i-1}-T_p<T_f<T_i-T_p)=$$
$$P(T_p)\int_{T_{i-1}-T_p}^{T_i-T_p}\int_0^{T_i-T_p-x}\int_0^{T_i-T_p-x-y} f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x \quad (1)$$

where $p=0,1,\cdots,i-1$, $i=1,2,\cdots,N_{\text{age}}$, $T_f=T_p+x+y+z$, $f_{X_n,p+1}(\cdot)=f_{X_1,p+1}(x)\cdot f_{X_2,p+1}(y)\cdot f_{X_2,p+1}(z)$, $P(T_p)$ represents the occurrence probability of the last IPM at $P(T_p)$ and no IPM arises in the case of $p=0$. Specially, the PDFs of X_n are derived as follows:

$$f_{X_1,p+1}(x)=\lambda(x+\Delta T_p)\cdot \mathrm{e}^{-\int_0^x \lambda(s+\Delta T_p)\mathrm{d}s}=\lambda(x+\Delta T_p)\cdot \mathrm{e}^{-\int_{\Delta T_p}^{x+\Delta T_p}\lambda(s)\mathrm{d}s},$$
$$f_{X_2,p+1}(y)=\varphi(y+\Delta T_p)\cdot \mathrm{e}^{-\int_0^y \varphi(s+\Delta T_p)\mathrm{d}s}=\varphi(y+\Delta T_p)\cdot \mathrm{e}^{-\int_{\Delta T_p}^{y+\Delta T_p}\varphi(s)\mathrm{d}s},$$
$$f_{X_3,p+1}(z)=\eta(z+\Delta T_p)\cdot \mathrm{e}^{-\int_0^z \eta(s+\Delta T_p)\mathrm{d}s}=\eta(z+\Delta T_p)\cdot \mathrm{e}^{-\int_{\Delta T_p}^{z+\Delta T_p}\eta(s)\mathrm{d}s},$$

where $\Delta T_p=(1-\rho)T_p$ represents the initial cumulative age after T_p, and $\lambda(\cdot)$, $\varphi(\cdot)$, $\eta(\cdot)$ are the hazard rate functions for three stages, respectively.

Considering all associated costs (including inspection, IPM, failure and replacement by an expedited ordered spare), we can then formulate the expected renewal cost $\mathrm{EC}_1(T,M,N_{\text{age}})$ and the expected renewal length $\mathrm{EL}_1(T,M,N_{\text{age}})$ are formulated as

$$\mathrm{EC}_1(T,M,N_{\text{age}})=$$
$$\sum_{i=1}^{N_{\text{age}}}\sum_{p=0}^{i-1}[(i-1)C_I+E_m(T_p)C_p+C_F+C_{\mathrm{ER}}]\cdot$$
$$P(T_{i-1}-T_p<T_f<T_i-T_p), \quad (2)$$

$$\mathrm{EL}_1(T,M,N_{\text{age}})=\sum_{i=1}^{N_{\text{age}}}\sum_{p=0}^{i-1}T_f\cdot P(T_{i-1}-T_p<T_f<T_i-T_p), \quad (3)$$

where $E_m(T_p)$ represents the average number of IPM until T_p. It is obvious that the allowable maximal number of IPM up to and including T_p is $\min(M-1,p)$. $E_m(T_p)=\sum_{m=0}^{\min(M-1,p)} m\cdot P_m(T_p)$, $P(T_p)=\sum_{m=0}^{\min(M-1,p)} P_m(T_p)$, where $P_m(T_p)$ can be calculated using the recursive calculation [4].

(i) A system failure occurs at T_f after the identification of the minor defective state S_2 at $T_q(q=1,2,\cdots,N_{\text{age}}-1)$ and the last IPM T_p. It is recognized from assumption (ii) that the spare is ordered at T_p and arrives at T_q+l; and, a halved inspection scheme is undertaken after T_q. If $T_p<T_q$, it indicates the state S_2 is found for the first time after the last IPM, then $p=0,1,\cdots,q-1$ and $i=q+1/2,q+1,\cdots,N_{\text{age}}$, as shown in Fig. 2. The probability of spare unavailability at T_f is given by

$$P(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l<T_f-T_q)=$$
$$P(T_p)\int_{T_{q-1}-T_p}^{T_q-T_p}\int_{T_{i-\frac{1}{2}}-T_p-x}^{T_i-T_p-x}\int_0^{T_i-T_p-x-y}\int_{T_f-T_q}^{\infty} f(l)f_{X_n,p+1}(\cdot)\mathrm{d}l\mathrm{d}z\mathrm{d}y\mathrm{d}x. \quad (4)$$

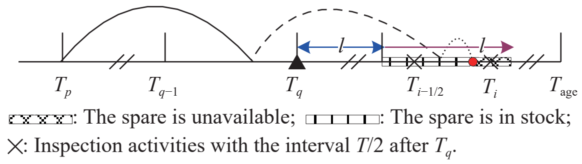

Fig. 2 A failure renewal with $T_p<T_q$

Further, we obtain the corresponding expected renewal cost and length, see (5) and (6).

$$\mathrm{EC}_2(T,M,N_{\text{age}})=\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=0}^{q-1}\sum_{i=q+\frac{1}{2}}^{N_{\text{age}}}[C_1(i,q)\cdot$$
$$P\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l<T_f-T_q\right)+$$
$$C_2(i,q)\cdot P\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l<T_f-T_q\right)\Big] \quad (5)$$

where the first term responds to the unavailability of the ordered spare, the second term shows that the ordered spare may be also available at T_f with $(T_q+l\leqslant T_f)$; $C_1(i,q)=[q+2(i-q)-1]C_I+E_m(T_p)C_p+C_F+C_R\,T_q{+}1{-}T_f\,C_s$, $C_2(i,_q)=[q{+}2i{-}q){-}1]C_I{+}E_m(T_p)C_p{+}C_F{+}C_R{+}(T_f{-}T_q{-}l)\cdot C_h$.

$$\mathrm{EL}_2(T,M,N_{\text{age}})=\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=0}^{q-1}\sum_{i=q+\frac{1}{2}}^{N_{\text{age}}}[[T_q+l]\cdot$$
$$P\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l>T_f-T_q\right)+$$
$$T_f\cdot P\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l\leqslant T_f-T_q\right)\Big]. \quad (6)$$

(ii) The system fails randomly at T_f, but $T_q<T_p$ $(q=1,\cdots,N_{\text{age}}-1;p=q+1/2,q+1,\cdots,N_{\text{age}}-1/2)$, as depicted in Fig. 3.

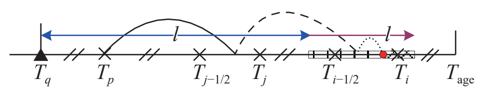

Fig. 3 A failure renewal with $T_q<T_p$

Similar to the derivation of (5) and (6), the corresponding expected cycle cost and length can be obtained as

$$\mathrm{EC}_3(T,M,N_{\text{age}})=\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\text{age}}-\frac{1}{2}}\sum_{i=p+\frac{1}{2}}^{N_{\text{age}}}[C_3(i,q)\cdot$$
$$P'\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l>T_f-T_q\right)+$$
$$C_4(i,q)\cdot P'\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l\leqslant T_f-T_q\right)\Big], \quad (7)$$

$$\mathrm{EL}_3(T,M,N_{\text{age}})=\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\text{age}}-\frac{1}{2}}\sum_{i=p+\frac{1}{2}}^{N_{\text{age}}}[[T_q+l]\cdot$$
$$P'\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l>T_f-T_q\right)+$$
$$T_f\cdot P'\left(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p,l\leqslant T_f-T_q\right)\Big], \quad (8)$$

$$P'(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p, l>T_f-T_q)=$$
$$\sum_{j=p+\frac{1}{2}}^{i}P(T_p)\int_{T_{j-\frac{1}{2}}-T_p}^{T_j-T_p}\int_{\delta(x)}^{T_i-T_p-x}\int_0^{T_i-T_p-x-y}\int_{T_f-T_q}^{\infty}f(l)f_{X_n,p+1}(\cdot)\mathrm{d}l\mathrm{d}z\mathrm{d}y\mathrm{d}x, \tag{9}$$

where

$$C_3(i,q)=(q+2(i-q)-1)C_I+E_m'(T_p)C_p+C_F+ C_R+(T_q+l-T_f)C_s,$$
$$C_4(i,q)=(q+2(i-q)-1)C_I+E_m'(T_p)C_p+C_F+ C_R+(T_f-T_q-l)C_h,$$
$$E_m'(T_p)=\sum_{m=0}^{\min(M-1,2p-q-1)}m\cdot P_m(T_p);$$

and define the function

$$\delta(x)=\begin{cases}T_{i-1/2}-T_p-x, & j=i\\ 0, & j\neq i\end{cases}.$$

$P'(T_{i-\frac{1}{2}}-T_p<T_f<T_i-T_p, l\leqslant T_f-T_q)$ is calculated similar to (9).

3.2 Case II: age-replacement renewal

In such a renewal scenario, the number of IPM over the renewal cycle is smaller than M. Three different situations are taken into account in terms of the system state at T_{age}, and for each situation, the availability of a spare is further discussed.

3.2.1 The system in the normal state S_1 at T_{age}

The system in the state S_1 can be replaced in three different ways: immediate renewal with an expedited spare, delayed renewal till the ordered spare arrives and immediate renewal with the ordered spare in stock.

(i) As shown in Fig. 1, an expedited spare is ordered to replace the system if no inspection before T_{age} identifies the minor defective state, and then the probability is given by

$$P_{S_1}^1(T_{\text{age}})=P(X_1>T_{N_{\text{age}}}-T_p)=P(T_p)\int_{T_{N_{\text{age}}}-T_p}^{\infty}f_{X_1,p+1}(x)\mathrm{d}x \tag{10}$$

where $p=0,1,\cdots,N_{\text{age}}-1$, thus the expected renewal cost is represented as

$$\mathrm{EC}_4(T,M,N_{\text{age}})=$$
$$\sum_{p=0}^{N_{\text{age}}-1}[(N_{\text{age}}-1)C_I+C_{\text{ER}}+E_m(T_p)C_p+C_{\text{age}}]\cdot P_{S_1}^1(T_{\text{age}}) \tag{11}$$

and the expected renewal length is given by

$$\mathrm{EL}_4(T,M,N_{\text{age}})=\sum_{p=0}^{N_{\text{age}}-1}T_{\text{age}}\cdot P_{S_1}^1(T_{\text{age}}). \tag{12}$$

(ii) If the spare has been ordered before T_{age}, i.e., an inspection identifies the minor defective state at $T_q(q=1,2,\cdots,N_{\text{age}}-1)$, then the last IPM must be accomplished at $T_p(p=q+1/2,\cdots,N_{\text{age}}-1/2)$ after T_q. When the ordered spare has not been delivered at $T_{\text{age}}(T_q+l>T_{\text{age}})$, the replacement has to be delayed, and then it will result in the shortage cost. Otherwise, an immediate replacement should be performed.

$$\mathrm{EC}_5(T,M,N_{\text{age}})=$$
$$\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\text{age}}-\frac{1}{2}}\int_{T_{N_{\text{age}}}-T_p}^{\infty}\Bigg\{\int_{T_{N_{\text{age}}}-T_q}^{\infty}[(q+2(N_{\text{age}}-q)-1)C_I+C_R+ E_m'(T_p)C_p+(T_q+l-T_{\text{age}})C_s+C_{\text{age}}]\cdot f(l)\mathrm{d}l+ \int_0^{T_{N_{\text{age}}}-T_q}[(q+2(N_{\text{age}}-q)-1)C_I+C_R+E_m'(T_p)C_p+ (T_{\text{age}}-T_q-l)C_h+C_{\text{age}}]\cdot f(l)\mathrm{d}l\Bigg\}\cdot P(T_p)\cdot f_{X_1,p+1}(x)\mathrm{d}x \tag{13}$$

$$EL_5(T,M,N_{\text{age}})=\sum_{q=1}^{N_{\text{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\text{age}}-\frac{1}{2}}\int_{T_{N_{\text{age}}}-T_p}^{\infty}\Bigg\{\int_{T_{N_{\text{age}}}-T_q}^{\infty}(T_q+l)\cdot f(l)dl+\int_0^{T_{N_{\text{age}}}-T_q}T_{\text{age}}\cdot f(l)\mathrm{d}l\Bigg\}\cdot P(T_p)\cdot f_{X_1,p+1}(x)\mathrm{d}x \tag{14}$$

3.2.2 The system in the minor defective state S_2 at T_{age}

(i) If no minor defective state is identified by inspection before T_{age}, no spare is ordered for replacement. This event shown in Fig. 1 occurs with the following probability.

$$P_{S_2}^1(T_{\text{age}})=$$
$$P(T_p)\int_{T_{N_{\text{age}}-1}-T_p}^{T_{N_{\text{age}}}-T_p}\int_{T_{N_{\text{age}}}-T_p-x}^{\infty}f_{X_1,p+1}(x)f_{X_2,p+1}(y)\mathrm{d}y\mathrm{d}x \tag{15}$$

where $p=0,1,\cdots,N_{\text{age}}-1$. According to assumption (iv), it results in an expedited order with the higher cost, but the lead time can be negligible compared to that of a regular order. Summing up all associated costs over the age-replacement cycle $[0,T_{\text{age}}]$, we can calculate the expected renewal cost as

$$\mathrm{EC}_6(T,M,N_{\text{age}})=$$
$$\sum_{p=0}^{N_{\text{age}}-1}[(N_{\text{age}}-1)C_I+C_{\text{ER}}+E_m(T_p)C_p+C_{\text{age}}]\cdot P_{S_2}^1(T_{\text{age}}). \tag{16}$$

Further, the expected renewal length is readily formulated as

$$\mathrm{EL}_6(T,M,N_{\mathrm{age}})=\sum_{p=0}^{N_{\mathrm{age}}-1}T_{\mathrm{age}}\cdot P_{S_2}^1(T_{\mathrm{age}}). \tag{17}$$

(ii) The minor defective state has been identified at $T_q(q=1,2,\cdots,N_{\mathrm{age}}-1)$ before T_{age}, but after the last IPM $T_p(p=0,1,\cdots,q-1)$. Then, the system is still in the minor defective state at T_{age} with the probability $P_{S_2}^2(T_{\mathrm{age}})$.

$$P_{S_2}^2(T_{\mathrm{age}})=$$
$$P(T_p)\int_{T_{q-1}-T_p}^{T_q-T_p}\int_{T_{N_{\mathrm{age}}}-T_p-x}^{\infty}f_{X_1,p+1}(x)f_{X_2,p+1}(y)\mathrm{d}y\mathrm{d}x \tag{18}$$

The replacement must be postponed to T_q+l if the spare is not delivered at T_{age}; otherwise, one cycle terminates T_{age} due to an immediate replacement by the available spare. Hence, the expected renewal cost and length are given by

$$\mathrm{EC}_7(T,M,N_{\mathrm{age}})=$$
$$\sum_{q=1}^{N_{\mathrm{age}}-1}\sum_{p=0}^{q-1}\Bigg\{\int_{T_{N_{\mathrm{age}}}-T_q}^{\infty}[(q+2(N_{\mathrm{age}}-q)-1)C_I+C_R+$$
$$E_m(T_p)C_p+(T_q+l-T_{\mathrm{age}})C_s+C_{\mathrm{age}}]\cdot f(l)\mathrm{d}l+$$
$$\int_0^{T_{N_{\mathrm{age}}}-T_q}[(q+2(N_{\mathrm{age}}-q)-1)C_I+C_R+E_m(T_p)C_p+$$
$$(T_{\mathrm{age}}-T_q-l)C_h+C_{\mathrm{age}}]\cdot f(l)\mathrm{d}l\Bigg\}\cdot P_{S_2}^2(T_{\mathrm{age}}), \tag{19}$$

$$\mathrm{EL}_7(T,M,N_{\mathrm{age}})=$$
$$\sum_{q=1}^{N_{\mathrm{age}}-1}\sum_{p=0}^{q-1}\left[\int_{T_{N_{\mathrm{age}}}-T_q}^{\infty}(T_q+l)f(l)\mathrm{d}l+\int_0^{T_{N_{\mathrm{age}}}-T_q}T_{\mathrm{age}}f(l)\mathrm{d}l\right]\cdot P_{S_2}^2(T_{\mathrm{age}}). \tag{20}$$

(iii) The minor defective stage may also be identified for the first time before the last IPM, i.e., $T_q<T_p$. It is clear that after T_p the normal stage may end within a halved inspection interval $(T_{j-1/2},T_j)$ and the length of the first two stages must be longer than $T_{\mathrm{age}}-T_p$. Therefore, we have the occurrence probability

$$P_{S_2}^3(T_{\mathrm{age}})=$$
$$\sum_{j=p+\frac{1}{2}}^{N_{\mathrm{age}}}P(T_p)\int_{T_{j-\frac{1}{2}}-T_p}^{T_j-T_p}\int_{T_{N_{\mathrm{age}}}-T_p-x}^{\infty}f_{X_1,p+1}(x)f_{X_2,p+1}(y)\mathrm{d}y\mathrm{d}x. \tag{21}$$

Using (21), we can obtain the expected cost and length over a renewal cycle $[0,T_{\mathrm{age}}]$. When the ordered spare is unavailable at T_{age}, we have

$$\mathrm{EC}_8(T,M,N_{\mathrm{age}})=$$
$$\sum_{q=1}^{N_{\mathrm{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\mathrm{age}}-\frac{1}{2}}\Bigg\{\int_{T_{N_{\mathrm{age}}}-T_q}^{\infty}[(q+2(N_{\mathrm{age}}-q)-1)C_I+C_R+$$
$$E_m'(T_p)C_p+(T_q+l-T_{\mathrm{age}})C_s+C_{\mathrm{age}}]\cdot f(l)\mathrm{d}l+$$
$$\int_0^{T_{N_{\mathrm{age}}}-T_q}[(q+2(N_{\mathrm{age}}-q)-1)C_I+C_R+E_m'(T_p)C_p+$$
$$(T_{\mathrm{age}}-T_q-l)C_h+C_{\mathrm{age}}]\cdot f(l)\mathrm{d}l\Bigg\}\cdot P_{S_2}^3(T_{\mathrm{age}}), \tag{22}$$

$$\mathrm{EL}_8(T,M,N_{\mathrm{age}})=$$
$$\sum_{q=1}^{N_{\mathrm{age}}-1}\sum_{p=q+\frac{1}{2}}^{N_{\mathrm{age}}-\frac{1}{2}}\left[\int_{T_{N_{\mathrm{age}}}-T_q}^{\infty}(T_q+l)f(l)\mathrm{d}l+\int_0^{T_{N_{\mathrm{age}}}-T_q}T_{\mathrm{age}}f(l)\mathrm{d}l\right]\cdot P_{S_2}^3(T_{\mathrm{age}}). \tag{23}$$

3.2.3 The system in the severe defective state S_3 at T_{age}

Similar to the situation that the state S_2 is identified at T_{age}, three subcases are considered.

(i) It is noted from Fig. 1 that if no inspection before T_{age} identifies the minor defective state and the last IPM is performed at T_p, the occurrence probability of the system in the severe defective stage at T_{age} is given by

$$P_{S_3}^1(T_{\mathrm{age}})=$$
$$P(T_p)\int_{T_{N_{\mathrm{age}}-1}-T_p}^{T_{N_{\mathrm{age}}}-T_p}\int_0^{T_{N_{\mathrm{age}}}-T_p-x}\int_{T_{N_{\mathrm{age}}}-T_p-x-y}^{\infty}f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x. \tag{24}$$

An expedited order is placed to replace the severe defective system at T_{age}, and consequently, the expected renewal cost and length can be obtained as

$$\mathrm{EC}_9(T,M,N_{\mathrm{age}})=$$
$$\sum_{p=0}^{N_{\mathrm{age}}-1}[(N_{\mathrm{age}}-1)C_I+C_{\mathrm{ER}}+E_m(T_p)C_p+C_{\mathrm{age}}]\cdot P_{S_3}^1(T_{\mathrm{age}}), \tag{25}$$

$$\mathrm{EL}_9(T,M,N_{\mathrm{age}})=\sum_{p=0}^{N_{\mathrm{age}}-1}T_{\mathrm{age}}\cdot P_{S_3}^1(T_{\mathrm{age}}). \tag{26}$$

(ii) The minor defective state is identified at an inspection T_q for the first time after the last IPM, ends within the last halved inspection interval $(T_{N_{\mathrm{age}}-\frac{1}{2}},T_{N_{\mathrm{age}}})$. Then, it worsens to the severe defective state S_3. The occurrence probability of such an event is given by

$$P_{S_3}^2(T_{\mathrm{age}})=$$
$$P(T_p)\int_{T_{q-1}-T_p}^{T_q-T_p}\int_{T_{N_{\mathrm{age}}-\frac{1}{2}}-T_p-x}^{T_{N_{\mathrm{age}}}-T_p-x}\int_{T_{N_{\mathrm{age}}}-T_p-x-y}^{\infty}f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x. \tag{27}$$

It is noted that replacement needs to be delayed till

T_q+l once the spare ordered at T_q has not arrived yet at $T_{\rm age}$, and the severe defective system is immediately replaced by the ordered spare on hand. Then, the expected renewal cost and length are formulated as

$$\mathrm{EC}_{10}(T,M,N_{\rm age})=\sum_{q=1}^{N_{\rm age}-1}\sum_{p=0}^{q-1}\left\{\int_{T_{N_{\rm age}}-T_q}^{\infty}[(q+2(N_{\rm age}-q)-1)C_I+C_R+E_m(T_p)C_p+(T_q+l-T_{\rm age})C_s+C_{\rm age}]\cdot f(l)\mathrm{d}l+\int_0^{T_{N_{\rm age}}-T_q}[(q+2(N_{\rm age}-q)-1)C_I+C_R+E_m(T_p)C_p+(T_{\rm age}-T_q-l)C_h+C_{\rm age}]\cdot f(l)\mathrm{d}l\right\}\cdot P_{S_3}^2(T_{\rm age}), \tag{28}$$

$$\mathrm{EL}_{10}(T,M,N_{\rm age})=\sum_{q=1}^{N_{\rm age}-1}\sum_{p=0}^{q-1}\left[\int_{T_{N_{\rm age}}-T_q}^{\infty}(T_q+l)\cdot f(l)\mathrm{d}l+\int_0^{T_{N_{\rm age}}-T_q}T_{\rm age}f(l)\mathrm{d}l\right]\cdot P_{S_3}^2(T_{\rm age}). \tag{29}$$

(iii) The last IPM occurs after identifying the state S_2 for the first time, namely, $T_q<T_p$. Moreover, after T_p, the minor defective may start within the interval $(T_{j-1/2},T_j)$, and must end within $(T_{N_{\rm age}-\frac{1}{2}},T_{N_{\rm age}})$. The corresponding probability is given by

$$P_{S_3}^3(T_{\rm age})=\sum_{j=p+\frac{1}{2}}^{N_{\rm age}}P(T_p)\int_{T_{j-\frac{1}{2}}-T_p}^{T_j-T_p}\int_{\delta'(x)}^{T_{N_{\rm age}}-T_p-x}\int_{T_{N_{\rm age}}-T_p-x-y}^{\infty}f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x \tag{30}$$

where $\delta'(x)=\begin{cases}T_{N_{\rm age}-\frac{1}{2}}-T_p-x\ ,\ j\neq N_{\rm age}\\ 0,\ j=N_{\rm age}\end{cases}$.

Hence, using (30), we can get the total expected renewal cost and length of the delayed replacement in this situation, see (31) and (32).

$$\mathrm{EC}_{11}(T,M,N_{\rm age})=\sum_{q=1}^{N_{\rm age}-1}\sum_{p=q+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}}\left\{\int_{T_{N_{\rm age}}-T_q}^{\infty}[(q+2(N_{\rm age}-q)-1)C_I+C_R+E_m'(T_p)C_p+(T_q+l-T_{\rm age})C_s+C_{\rm age}]\cdot f(l)\mathrm{d}l+\int_0^{T_{N_{\rm age}}-T_q}[(q+2(N_{\rm age}-q)-1)C_I+C_R+E_m'(T_p)C_p+(T_{\rm age}-T_q-l)C_h+C_{\rm age}]\cdot f(l)\mathrm{d}l\right\}\cdot P_{S_3}^3(T_{\rm age}) \tag{31}$$

$$\mathrm{EL}_{11}(T,M,N_{\rm age})=\sum_{q=1}^{N_{\rm age}-1}\sum_{p=q+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}}\left[\int_{T_{N_{\rm age}}-T_q}^{\infty}(T_q+l)f(l)\mathrm{d}l+\int_0^{T_{N_{\rm age}}-T_q}T_{\rm age}f(l)\mathrm{d}l\right]\cdot P_{S_3}^3(T_{\rm age}) \tag{32}$$

3.3 Case III: inspection renewal

From assumption (iii), there is an inspection renewal once the severe defective stage is identified for M times by inspections.

(i) As shown in Fig. 1, if the system is found to be in the severe defective state at an inspection T_i ($i=M,M+1,\cdots,N_{\rm age}-1$), before which the IPM has been performed $M-1$ times and no inspection identifies the minor defective state. On the basis of assumption (iv), spare supply should be conducted from an expedited order at T_i, and the probability is given by

$$P_{S_3}^1(T_i)=P'(T_p^{M-1})\int_{T_{i-1}-T_p^{M-1}}^{T_i-T_p^{M-1}}\int_0^{T_i-T_p^{M-1}-x}\int_0^{T_i-T_p^{M-1}-x-y}f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x \tag{33}$$

where T_p^{M-1} means the time of the $(M-1)$th IPM and $P'(T_p^{M-1})=P_{M-1}(T_p^{M-1})$. Thus, we get the expected renewal cost and length as shown below:

$$\mathrm{EC}_{12}(T,M,N_{\rm age})=\sum_{i=M}^{N_{\rm age}-1}\sum_{p=M-1}^{i-1}[iC_I+C_{\rm ER}+(M-1)C_p+C_{\rm PM}]\cdot P_{S_3}^1(T_i), \tag{34}$$

$$\mathrm{EL}_{12}(T,M,N_{\rm age})=\sum_{i=M}^{N_{\rm age}-1}\sum_{p=M-1}^{i-1}T_i\cdot P_{S_3}^1(T_i). \tag{35}$$

(ii) The minor defective state may be also identified for the first time after the $(M-1)$th IPM, i.e., $T_p^{M-1}<T_q$. Subsequently, the severe defective state is identified at T_i, which leads to an inspection renewal. The probability under such an event is expressed as

$$P_{S_3}^2(T_i)=P'(T_p^{M-1})\int_{T_{q-1}-T_p^{M-1}}^{T_q-T_p^{M-1}}\int_{T_{i-\frac{1}{2}}-T_p^{M-1}-x}^{T_i-T_p^{M-1}-x}\int_{T_i-T_p^{M-1}-x-y}^{\infty}f_{X_n,p+1}(\cdot)\mathrm{d}z\mathrm{d}y\mathrm{d}x. \tag{36}$$

Then, the total expected renewal cost and length of a replacement cycle in this situation can be expressed as

$$\mathrm{EC}_{13}(T,M,N_{\rm age})=\sum_{q=M}^{N_{\rm age}-1}\sum_{i=q+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}}\sum_{p=M-1}^{q-1}\left[\int_{T_i-T_q}^{\infty}C_5(i,q)\cdot f(l)\mathrm{d}l+\int_0^{T_i-T_q}C_6(i,q)\cdot f(l)\mathrm{d}l\right]\cdot P_{S_3}^2(T_i), \tag{37}$$

$$\mathrm{EL}_{13}(T,M,N_{\rm age})=\sum_{q=M}^{N_{\rm age}-1}\sum_{i=q+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}}\sum_{p=M-1}^{q-1}\left[\int_{T_i-T_q}^{\infty}(T_q+l)\cdot f(l)\mathrm{d}l+\int_0^{T_i-T_q}T_i\cdot f(l)\mathrm{d}l\right]\cdot P_{S_3}^2(T_i), \tag{38}$$

where

$$C_5(i,q) = (q+2(i-q)-1)C_I + C_R + (M-1)C_p + (T_q + l - T_i)C_s + C_{\rm PM},$$

$$C_6(i,q) = (q+2(i-q)-1)C_I + C_R + (M-1)C_p + (T_i - T_q - l)C_h + C_{\rm PM}.$$

(iii) The $(M-1)$th IPM can be carried out after the first time of identifying the minor defective state, i.e., $T_q < T_p^{M-1}$ $(q = 1,2,\cdots,N_{\rm age}-2\,;\, p = q+\frac{r(q)}{2}\frac{r(q)+1}{2},\cdots,N_{\rm age}-1;\ r(q) = \begin{cases} 1,\ q \geqslant M-1 \\ M-q,\ \text{otherwise} \end{cases})$. Then, the system is recognized in the state S_3 at an inspection T_i. The probability in such a case is

$$P_{S_3}^3(T_i) = \sum_{j=p+\frac{1}{2}}^{i} P'(T_p^{M-1}) \int_{T_{j-\frac{1}{2}}-T_p^{M-1}}^{T_j-T_p^{M-1}} \int_{\delta''(x)}^{T_i-T_p^{M-1}-x} \int_0^{T_i-T_p^{M-1}-x-y} f_{X_n,p+1}(\cdot){\rm d}z{\rm d}y{\rm d}x \tag{39}$$

where

$$\delta''(x) = \begin{cases} T_{i-\frac{1}{2}} - T_p^{M-1} - x,\ j \neq i \\ 0,\ j = i \end{cases}.$$

Further, the expected cost and length summations of the delayed renewal and the immediate renewal can be calculated by

$$\begin{aligned} {\rm EC}_{14}(T,M,N_{\rm age}) = &\sum_{q=1}^{N_{\rm age}-2} \sum_{i=p+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}} \sum_{p=q+\frac{r(q)}{2}}^{N_{\rm age}-1} \left\{ \int_{T_i-T_q}^{\infty} C_5(i,q)\cdot f(l){\rm d}l + \int_0^{T_i-T_q} [(q+2(i-q))C_I + C_R + \right. \\ &\left. (M-1)C_p + (T_i - T_q - l)C_h + C_{\rm PM}]\cdot f(l){\rm d}l \right\} \cdot P_{S_3}^3(T_i), \end{aligned} \tag{40}$$

$${\rm EL}_{14}(T,M,N_{\rm age}) = \sum_{q=1}^{N_{\rm age}-2} \sum_{i=p+\frac{1}{2}}^{N_{\rm age}-\frac{1}{2}} \sum_{p=q+\frac{r(q)}{2}}^{N_{\rm age}-1} \left[\int_{T_i-T_q}^{\infty} (T_q+l)\cdot f(l){\rm d}l + \int_0^{T_i-T_q} T_i\cdot f(l){\rm d}l \right] \cdot P_{S_3}^3(T_i). \tag{41}$$

Based on the renewal reward theory [43], the long-run expected cost rate, (42), is calculated as the measure criteria to find the optimal decision.

$$\text{Min}\ \ C(T,M,N_{\rm age}) = \sum_{h=1}^{14} {\rm EC}_h(T,M,N_{\rm age}) \Big/ \sum_{h=1}^{14} {\rm EL}_h(T,M,N_{\rm age}) \tag{42}$$

4. Monte-Carlo based integration method for model solution

It is noted from (42) that the calculation for the multiple integrals is time-consuming. Here, we propose the Monte-Carlo integration method based on the generation of random numbers, which is repeated many times to count the mean value, but all terms in (42) should be firstly transformed into the forms of definite integral.

Step 1 Without loss of generality, the quadslope indefinite integral is given by

$$I = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} \int_{r(x,y,z)}^{\infty} f(x,y,z) f(l){\rm d}z{\rm d}y{\rm d}x \tag{43}$$

where $f(x,y,z)$ is a continuous distribution in the 3-D domain of integration $\Omega = \{(x,y,z)\,|\,a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), h_1(x,y) \leqslant z \leqslant h_2(x,y)\}$. Consider the pair of one-to-one relationship between (x,y,z,l) and (u,v,w,ϑ) $(0 \leqslant u \leqslant 1, 0 \leqslant v \leqslant 1, 0 \leqslant w \leqslant 1, 0 \leqslant \vartheta \leqslant 1)$ with $u = \dfrac{x-a}{b-a}$, $v = \dfrac{y-g_1(x)}{g_2(x)-g_1(x)}$, $w = \dfrac{z-h_1(x,y)}{h_2(x,y)-h_1(x,y)}$, $\vartheta = \dfrac{1}{r(x,y,z)}$. Then, the unique inverse is derived as $x = x(u)$, $y = y(u,v)$, $z = z(u,v,w)$, $l = l(u,v,w,\vartheta)$.

Step 2 Transformation of triple indefinite integral. The Jacobian operator of the transformation is formulated as $J = \dfrac{\partial(x,y,z,l)}{\partial(u,v,w,\vartheta)}$ and we have ${\rm d}x{\rm d}y{\rm d}z{\rm d}l = \dfrac{\partial(x,y,z,l)}{\partial(u,v,w,\vartheta)}{\rm d}u{\rm d}v{\rm d}w{\rm d}\vartheta$. Then, (43) can be expressed as

$$I = \int_0^1\int_0^1\int_0^1\int_0^1 f(x(u),y(u,v),z(u,v,w),l(u,v,w,\vartheta))\cdot \frac{\partial(x,y,z,l)}{\partial(u,v,w,\vartheta)}{\rm d}u{\rm d}v{\rm d}w{\rm d}\vartheta. \tag{44}$$

Step 3 Generate a uniformly distributed random samples (u,v,w,ϑ) with the size H between (0,1), then the value of $g(x,y,z,l) = f(x(u),y(u,v),z(u,v,w),l(u,v,w,\vartheta))\dfrac{\partial(x,y,z,l)}{\partial(u,v,w,\vartheta)}$ can be determined. Followed by it, we sum all of these values and divide by H to get the mean value as the approximate value of I, i.e., $\bar{I} = \dfrac{1}{H}\sum_{h=1}^{H} g(x,y,z,l)$.

5. A numerical example

To illustrate the proposed joint policy of PM and spare ordering, and ensure that the Monte-Carlo based integration method works to find the optimal solution, a numerical example is investigated and the results are discussed and analyzed.

5.1 Modeling parameters

The two-parameter Weibull distribution has been widely applied to represent the deterioration process, so it is chosen to characterize the duration of three stages with the failure rate form $h(t;\alpha,\beta) = \alpha\beta(\alpha t)^{\beta-1} (\alpha > 0, \beta > 0, t > 0)$. α and β are the scale parameter and the shape parameter, respectively. Table 2 gives the initial failure rate parameters, which are determined from experience, and the shape parameters are larger than 1, indicating that the deterioration process for three stages is worn out. These parameters can also be estimated using the field data in a case study. The cost incurred by inspection is $C_I = 20$, the average cost per IPM is a linear function of the improvement factor ($C_P = 100\rho$), the replacement cost by a regular ordered spare is 150 ($C_R = 150$), the replacement cost by a regular ordered spare is 600 ($C_{ER} > C_R$), $C_{PM} = 400$, $C_{age} = 1\,000$, $C_F = 6\,000$, $C_h = 4$, $C_s = 10$; and the delivery time of the regular ordered spare l follows the normal distribution with the form $f(l) = (1/\sigma\sqrt{2\pi})e^{-\frac{(l-u)^2}{2\sigma}}$, in which u and σ are the mean and standard deviations and we have $u = 30$, $\sigma = 7$.

Table 2 Initial failure rate parameters

Stage	$\varphi_0(y)$	$\eta_0(z)$
$\lambda_0(x)$	0.008	1.3
$\varphi_0(y)$	0.013	1.7
$\eta_0(z)$	0.021	2.3

5.2 Optimal solutions

The Monte-Carlo based integration method is used to obtain the optimal solutions of the proposed model. The initial decision variables are set and enumerated by an increment of 1, and simultaneously the objective function is calculated to check the minimal expected cost rate. Fig. 4 shows a 3-D diagram showing the long-term expected cost rate as a variety of T and M at $\rho = 0.5$ and $N_{age} = 6$.

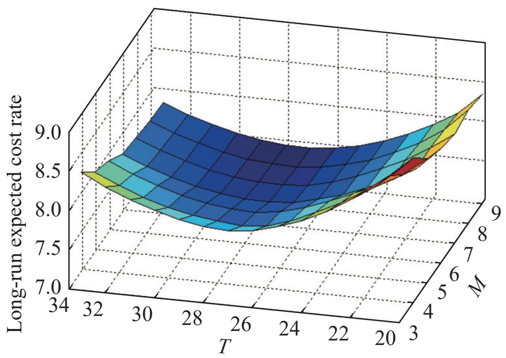

Fig. 4 Results from the enumeration algorithm with $\rho = 0.5$ and $N_{age} = 6$

It is noted that when the inspection interval T is fixed, the long-run expected cost per unit time decreases firstly and then increases with the increase of M. This can be explained that the system's availability is improved when M increases in the initial range, resulting in the decrease of the objective value. If M is much larger, the system that have been repaired many times is prone to fail, which leads to a relatively higher expected cost rate due to the higher failure cost. On the other hand, as T increases, the output changes with the same trend as mentioned above. It is because that much more frequent inspections with a smaller T increases the cost caused by inspection activities, and less frequent inspection with a larger T could cause the system failure and costs much more. The minimal expected cost rate is searched under $T^* = 27$ and $M^* = 9$. In order to further analyze the optimal solutions in the case of the improvement factor $\rho = 0.5$, we calculate situations under various N_{age} values and the results are shown in Fig. 5. It draws the long-run expected cost rate in terms of the inspection interval T when N_{age} equals 5, 6 and 7, respectively. It shoud be pointed out specially, that the curve for the given N_{age} in Fig. 5 is solved when the maximum number of identifying the state S_3 is the optimal solution M^*. It can be readily seen that the minimal expected cost rate is $C(T^*, M^*, N^*_{age}) = 7.489$ with the optimal decision variables $(T^*, M^*, N^*_{age}) = (27, 9, 6)$. This indicates that the optimal interval of the initial inspection activities T^* is 27, and the system is preventively replaced at either the 9th time of detecting the state S_3 or the age threshold 162. Moreover, when N_{age} varies from 5 to 7, the optimal inspection interval T^* is 32, 27 and 23, respectively. Interestingly, the optimal inspection interval T^* decreases with the increase of N_{age}; also, the corresponding age-replacement intervals T^*_{age} ($T^*_{age} = N^*_{age} \cdot T^*$) are 160, 162 and 161. Obviously, the difference among T^*_{age} is slight, which implies that the optimal age-replacement interval T^*_{age} is approximately the same.

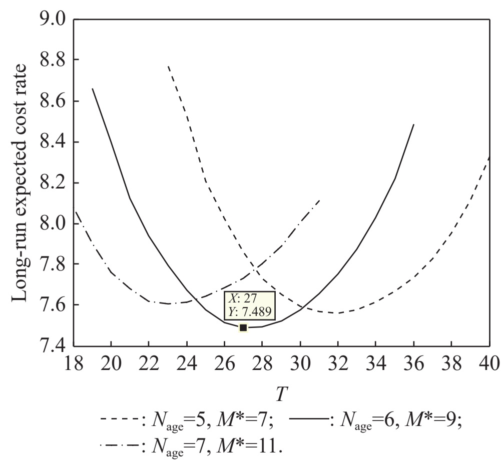

Fig. 5 Long-run expected cost rate in terms of T ($\rho = 0.5$)

5.3 Discussions of the results from various ρ

We present various improvement factors ρ contributing to the long-run expected cost rate and the corresponding optimal decisions.

Fig. 6 depicts a 3-D diagram of the long-run expected cost rate as a function of T as well as M when $\rho = 0$ and $N_{age} = 5$. It can be readily seen that the long-run expected cost rate in Fig. 6 changes with the same trend in Fig. 4. Moreover, the minimal expected cost rate 9.316 is found at $(T^*, M^*) = (27, 7)$.

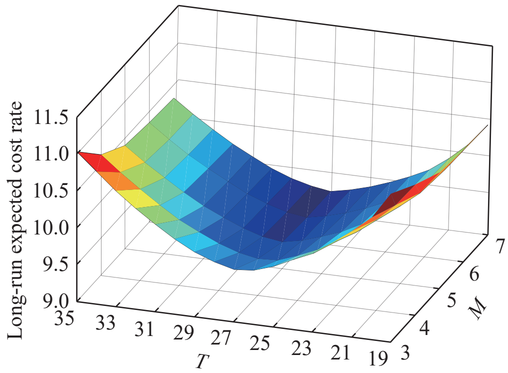

Fig. 6 Expected cost rate in terms of T and M with $\rho = 0$ and N_{age}=5

Further, we run the proposed algorithm under different values N_{age} to search the optimal solutions and draw Fig. 7. Note that, if a minimal repair ($\rho = 0$) is performed once the system is detected to be in the severe defective state by an inspection, it gives rise to the minimal expected cost rate 9.316. Evidently, it is much larger than 7.489 in case of $\rho = 0.5$, thereby the decision of a minimal repair is not economical compared to IPM. In addition, we can observe that the optimal age-replacement interval $T^*_{age} = 135$ at $\rho = 0$ is shorter than 162 at $\rho = 0.5$, which is as we expect. This is reasonable since the system after repair would be restored to a better state with the increase of ρ, resulting in prolonging the age-replacement interval.

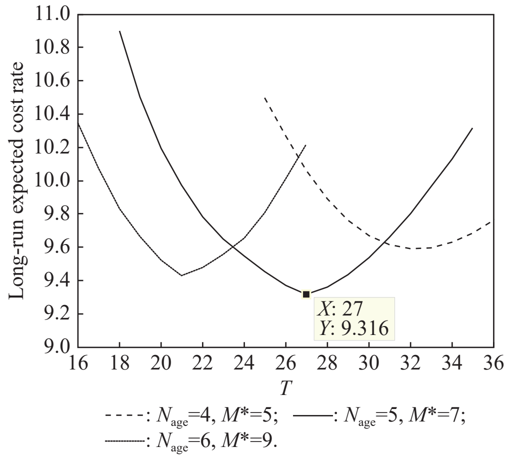

Fig. 7 Long-run expected cost rate in terms of T ($\rho = 0$)

When no repair is performed at the time of identifying the severe defective state, i.e., replacement is the only option, renewal is required at an inspection identifying the state S_3 preventively, the pre-determined age preventively, or the failure time correctively. It also means that perfect maintenance is only taken into consideration, neither IPM or minimal repair. For such a situation, we calculate the long-run expected cost rate with regard to T and N_{age}, and then obtain the results shown in Fig. 8. Clearly, the optimal decision variables are $T^* = 46$, $N^*_{age} = 2$, respectively. With them, the minimum of the objective function is 18.678, which is much larger than those when $\rho = 0$ and $\rho = 0.5$. This is because of the following reasons. The first is that the replacement cost, regardless of a regular or expedited order, is relatively higher than the IPM cost C_p; and the second is that the expected length over a renewal is reduced due to an immediate replacement.

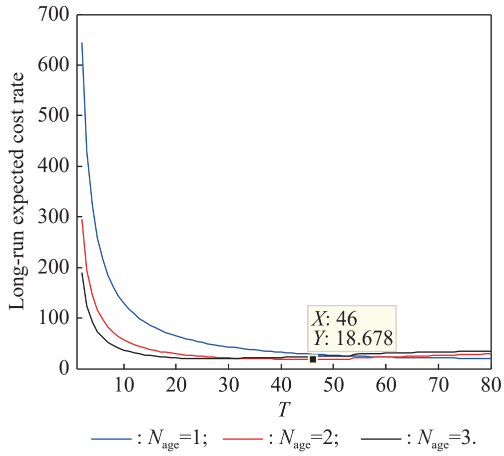

Fig. 8 Long-run expected cost rate with regard to T

In order to demonstrate the effectiveness of IPM to reduce the long-run expected cost rate, Fig. 9 illustrates the minimal expected cost rate when the improvement factor ρ ranges from 0 to 1. It can be observed that the long-run expected cost rate firstly decreases and then increases, with the increase of the improvement factor. Therefore, it is reasonable for critical or expensive systems that decision makers give priority to imperfect maintenance when an inspection detects it being in the severe defective state.

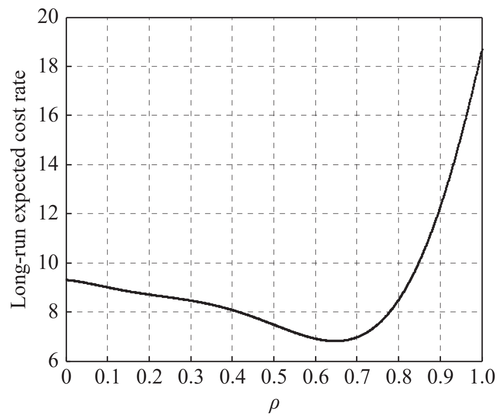

Fig. 9 The minimal expected cost rate under different improvement factors

6. Conclusions

A joint policy of maintenance and spare ordering along with imperfect maintenance is proposed herein for single-unit systems under non-periodic inspection and age-based replacement. The system deteriorates subject to the three-stage failure process, and different activities are carried out depending on the system's health state. Such a policy is mainly to derive how to schedule the inspection and age replacement intervals optimally, and how to choose the appropriate number of IPM before a renewal to reduce the long-run expected cost rate. In view of the assumption that a spare is ordered at the first time of detecting the minor defect by inspection, as a result, the optimization for the inspection interval implies selecting the proper spare ordering policy. The analytical model of the proposed joint policy is constructed to minimize the long-run expected cost rate. It is time-consuming to solve the exact solutions due to multiple integral; consequently, the Monte-Carlo based integration method is designed. The analyses for the results of different improvement factors also show the decision of IPM at the time of identifying the severe defective state is much more economical to an immediate replacement. A case study will be explored in the future, and the proposed joint optimization should be extended to investigate multi-component systems when components have different importance and deterioration processes.

References

[1] AHMAD R, KAMARUDDIN S. An overview of time-based and condition-based maintenance in industrial application. Computers & Industrial Engineering, 2012, 63(1): 135–149.

[2] CAVALCANTE C A, LOPES R S, SCARF P A. A general inspection and opportunistic replacement policy for one-component systems of variable quality. European Journal of Operational Research, 2018, 266(3): 911–919.

[3] JIANG R. Performance evaluation of seven optimization models of age replacement policy. Reliability Engineering & System Safety, 2018, 180: 302–311.

[4] WANG L, HU H J, WANG Y Q, et al. The availability model and parameters estimation method for the delay time model with imperfect maintenance at inspection. Applied Mathematical Modelling, 2011, 35(6): 2855–2863.

[5] AHMADI R. Optimal maintenance scheduling for a complex manufacturing system subject to deterioration. Annals of Operations Research, 2014, 217(1): 1–29.

[6] DO P, VOISIN A, LEVRAT E, et al. A proactive condition-based maintenance strategy with both perfect and imperfect maintenance actions. Reliability Engineering & System Safety, 2015, 133: 22–32.

[7] YANG L, YE Z S, LEE C G, et al. A two-phase preventive maintenance policy considering imperfect repair and postponed replacement. European Journal of Operational Research, 2019, 274(3): 966–977.

[8] MALIK M A K. Reliable preventive maintenance scheduling. AIIE Transactions, 1979, 11(3): 221–228.

[9] PHAM H, WANG H. Imperfect maintenance. European Journal of Operational Research, 1996, 94(3): 425–438.

[10] BISWAS A, SARKAR J. Availability of a system maintained through several imperfect repairs before a replacement or a perfect repair. Statistics & Probability Letters, 2000, 50(2): 105–114.

[11] BISWAS A, SARKAR J, SARKAR S. Availability of a periodically inspected system, maintained under an imperfect-repair policy. IEEE Trans. on Reliability, 2003, 52(3): 311–318.

[12] WEIDE J, PANDEY M D. Stochastic analysis of shock process and modeling of condition-based maintenance. Reliability Engineering & System Safety, 2011, 96(6): 619–626.

[13] WEIDE J, PANDEY M D, NOORTWIJK J. Discounted cost model for condition-based maintenance optimization. Reliability Engineering & System Safety, 2010, 95(3): 236–246.

[14] OSAKI S, YAMADA S. Age replacement with lead time. IEEE Trans. on Reliability, 1976, R-25(5): 344–345.

[15] SHEU S H, GRIFFITH W S. Optimal age-replacement policy with age-dependent minimal-repair and random-lead-time. IEEE Trans. on Reliability, 2001, 50(3): 302–309.

[16] SHEU S H, CHIEN Y H. Optimal age-replacement policy of a system subject to shocks with random lead-time. European Journal of Operational Research, 2004, 159(1): 132–144.

[17] SHEU S H, ZHANG Z G, CHIEN Y H, et al. Age replacement policy with lead-time for a system subject to non-homogeneous pure birth shocks. Applied Mathematical Modelling, 2013, 37(14/15): 7717–7725.

[18] CHIEN Y H, CHANG C C, SHEU S H. Optimal age-replacement model with age-dependent type of failure and random lead time based on a cumulative repair-cost limit policy. Annals of Operations Research, 2010, 181(1): 723–744.

[19] CHIEN Y H, SHEU S H, CHANG C C. Optimal age-replacement time with minimal repair based on cumulative repair cost limit and random lead time. International Journal of Systems Science, 2009, 40(7): 703–715.

[20] ARMSTRONG M J, ATKINS D R. Joint optimization of maintenance and inventory policies for a simple system. IIE Transactions, 1996, 28(5): 415–424.

[21] WANG L, CHU J, MAO W J. A condition-based order-replacement policy for a single-unit system. Applied Mathematical Modelling, 2008, 32(11): 2274–2289.

[22] ZHOU W, WANG D F, SHENG J Y, et al. Collaborative optimization of maintenance and spare ordering of continuously degrading systems. Journal of Systems Engineering and Electronics, 2012, 23(1): 63–70.

[23] ALASWAD S, XIANG Y. A review on condition-based maintenance optimization models for stochastically deteriorating system. Reliability Engineering & System Safety, 2017, 157: 54–63.

[24] BLOCH-MERCIER S. A preventive maintenance policy with sequential checking procedure for a Markov deteriorating system. European Journal of Operational Research, 2002, 142(3): 548–576.

[25] KANG K C, SUBRAMANIAM V. Integrated control policy of production and preventive maintenance for a deteriorating manufacturing system. Computers & Industrial Engineering, 2018, 118: 266–277.

[26] MOGHADDASS R, ERTEKIN E. Joint optimization of ordering and maintenance with condition monitoring data. Annals of Operations Research, 2018, 263(1/2): 271–310.

[27] WANG W B. An inspection model based on a three-stage failure process. Reliability Engineering & System Safety, 2011, 96(7): 838–848.

[28] BERRADE M D, SCARF P A, CAVALCANTE C A V. A study of postponed replacement in a delay time model. Reliability Engineering & System Safety, 2017, 168: 70–79.
[29] YANG L, MA X B, ZHAO Y. A condition-based maintenance model for a three-state system subject to degradation and environmental shocks. Computers & Industrial Engineering, 2017, 105: 210–222.
[30] TAGHIPOUR S, AZIMPOOR S. Joint optimization of jobs sequence and inspection policy for a single system with two-stage failure process. IEEE Trans. on Reliability, 2018, 67(1): 156–169.
[31] CHRISTER A H, WANG W, CHOI K, et al. The robustness of the semi-Markov and delay time single-component inspection models to the markov assumption. IMA Journal of Management Mathematics, 2001, 12(1): 75–88.
[32] WANG W B. An overview of the recent advances in delay-time-based maintenance modelling. Reliability Engineering & System Safety, 2012, 106(5): 165–178.
[33] WANG W, SCARF P A, SMITH M A J. On the application of a model of condition-based maintenance. Journal of the Operational Research Society, 2000, 51(11): 1218–1227.
[34] WANG W B, ZHAO F, PENG R. A preventive maintenance model with a two-level inspection policy based on a three-stage failure process. Reliability Engineering & System Safety, 2014, 121(1): 207–220.
[35] WANG H Y, WANG W B, PENG R. A two-phase inspection model for a single component system with three-stage degradation. Reliability Engineering & System Safety, 2017, 158: 31–40.
[36] WANG W B. A joint spare part and maintenance inspection optimization model using the delay-time concept. Reliability Engineering & System Safety, 2011, 96(11): 1535–1541.
[37] WANG W B, SYNTETOS A A. Spare parts demand: linking forecasting to equipment maintenance. Transportation Research Part E: Logistics & Transportation Review, 2011, 47(6): 1194–1209.
[38] WANG W B. A stochastic model for joint spare parts inventory and planned maintenance optimization. European Journal of Operational Research, 2012, 216(1): 127–139.
[39] SHI C H, XIE F F, ZHAO F, et al. A joint inspection-based preventive maintenance and spare ordering optimization policy using a three-stage failure process. Complexity, 2017, 2017: 1–19.
[40] ZHAO F, LIU X J, PENG R, et al. Joint optimization of inspection and spare ordering policy with multi-level defect information. Computers & Industrial Engineering, 2020, 139: 106205.
[41] YEH R H, CHANG W L, LO H C. Optimal threshold values of age and two-phase maintenance policy for leased equipments using age reduction method. Annals of Operations Research, 2010, 181(1): 171–183.
[42] CAI J, LI X, CHEN X. Joint optimization of maintenance inspection and spare provisioning for aircraft deteriorating parts. Journal of Systems Engineering and Electronics, 2017, 28(6): 1133–1140.
[43] ROSS S M. Introduction to probability models. London: Academic Press, 2015.

Biographies

MA Weining was born in 1982. He received his B.S. and M.S. degrees in management engineering from the Army Engineering University of PLA, Shijiazhuang, in 2006 and 2009, respectively. Now he is a Ph.D. candidate in maintenance engineering. He is currently a lecturer at the Shijiazhuang Campus of Army Engineering University of PLA. His research interests include equipment maintenance engineering, maintenance management of complex system, and optimization algorithm.
E-mail: maweiningxq@126.com

ZHAO Fei was born in 1984. She received her B.S. and M.S. degrees in School of Economics and Management from Hebei University of Engineering, Handan, China, in 2008 and 2011, respectively. She received her Ph.D. degree from University of Science and Technology Beijing, China, in 2015. She is currently an associate professor with the School of Business Administration, Northeastern University. Her research interests include system reliability, maintenance, and operations research and optimization.
E-mail: zhaofei@neuq.edu.cn

LI Xin was born in 1996. He received his B.S. degree in School of Electronic Information Engineering from Taiyuan University of Science and Technology, in 2018. He is currently a postgraduate from Northeastern University at Qinhuangdao. His research interests include system reliability, and maintenance and spare parts optimization.
E-mail: 15735169458@163.com

HU Qiwei was born in 1979. He received his M.S. and Ph. D. degrees in management engineering from the Army Engineering University of PLA, Shijiazhuang, in 2005 and 2016, respectively. Now he is currently an associate professor at the Shijiazhuang Campus of Army Engineering University of PLA. His research interests include maintenance optimization, and spare parts management.
E-mail: huqiweiaeu@163.com

SHANG Bingcong was born in 1989. She received her B.S. and M.S. degrees from School of Information Science and Engineering, Northeastern University, Shenyang, China, in 2011 and 2015. She is currently a lecturer with the School of Business Administration, Northeastern University. Her research interests include system reliability, and operations research and optimization.
E-mail: shangbingcong@163.com