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# Time-varying sliding mode control of missile based on suboptimal method

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**Abstract:** This paper proposes a time-varying sliding mode control method to address nonlinear missile body kinematics based on the suboptimal control theory. The analytical solution of suboptimal time-varying sliding surface and the corresponding suboptimal control law are obtained by solving the state-dependent Riccati equation analytically. Then, the Lyapunov method is used to analyze the motion trend in sliding surface and the asymptotic stability of the closed-loop system is validated. The suboptimal control law is transformed to the form of pseudo-angle-of-attack feedback. The simulation results indicate that the satisfactory performance can be obtained and the control law can overcome the influence of parameter errors.

**Keywords:** suboptimal control, Riccati equation, analytic solution, sliding mode control, nonlinear control.

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## 1. Introduction

The optimal control method is a well-known method in the design of the linear time-invariant system. The optimal solution can be obtained by designing the target function and solving the corresponding Riccati equation [1−5]. The traditional design of the nonlinear control system is to select the specific operating points and linearize the original system, and then design the control law based on the approximate linear system [6,7]. The suboptimal control method which evolved from traditional optimal control has been widely used in the nonlinear control system design [8−14]. The application of suboptimal control is based on the pseudo-linear structure constructed from the original system model. The crux of system design and analysis is the stability analysis of the closed-loop system in the operational range [15].

The traditional way of getting the suboptimal control law is based on the numerical solution of the state-dependent Riccati equation, which hinders the stability analysis and prompts researchers to find the analytical solution of Riccati equation. In [16,17], the analytical solution of Riccati equation in the specific form has been used to design and analyze the nonlinear system with the particular structure. In [18], the analytical solution of the two-dimensional state-dependent Riccati equation was proposed to design the suboptimal control law and analyze the stability of the closed-loop system. The suboptimal control law proposed in [18] is composed of the feedback of acceleration and the pitch rate, ensuring good performance under the nominal model. However, when the actual value of the model parameters deviates from the nominal value, the control effect will quickly deteriorate. It is known that in actual application, it is hard to get the accurate model, indicating that the model bias always exists [19]. Hence, for the design of missile autopilot, the robustness of the control system must be considered.

Sliding mode control is a nonlinear control method widely used in engineering design [20−24]. This control method shows good robustness by designing appropriate sliding surface, ensuring satisfactory response performance in the presence of model parameter deviation [25]. To obtain better control effect, the traditional linear time-invariant sliding surface has been improved to the time-varying sliding surface [26−28]. In [29], a kind of sliding mode control method based on state transition was proposed aimed at the multivariable system, which was combined with the optimization of the indicator function.

In actual engineering, we often encounter time-varying systems in which the model parameters change with states, and the design of time-invariant sliding surfaces may not guarantee the control effect under different system states. By using the measurable system state information to adjust the parameters of the sliding surface, the design of the time-varying sliding surface can ensure that the control effect will not change significantly under different conditions. In [27] and [28], the time-varying sliding mode control law was designed for the inverted pen-

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dulum and the robotic manipulator, and achieved good control effects. In [30], the design using time-varying sliding surface overcame the influence caused by the nonlinear aerodynamic parameters.

Aimed at the nonlinear system, designing sliding surface with the suboptimal theory is an approach to improve sliding mode control. In [30], the linear approximate approach was used to acquire the suboptimal sliding mode control law based on the missile nonlinear model. However, it is difficult to analyze the stability in the sliding mode designed in [30], because only the numerical solution of the sliding surface can be obtained. Although the suboptimal sliding mode control method has good robustness, further study is needed to analyze the stability in the nonlinear sliding surface.

In this paper, aimed at the missile nonlinear model, the state transition is utilized to design the suboptimal sliding surface, and its analytical expression is solved to design the control law. On the basis of analytical solution, the stability in the nonlinear time-varying sliding surface can be acquired, and the control law can be adjusted to the form of pseudo-angle-of-attack feedback.

## 2. Missile model

In this paper, the object of study is the missile longitudinal model, and the coordinate system is shown in Fig.1 [18].

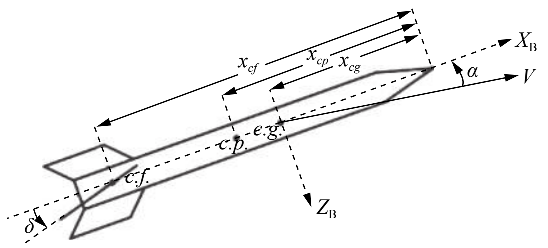


**Fig. 1 Missile longitudinal model**

Consider the following model:

$$\begin{cases} \dot{\alpha} = \omega_y + \dfrac{qS}{mV}(C_{Z\alpha}\alpha + C_{Z\delta}\delta) \\ \dot{\omega}_y = \dfrac{qSd}{I_y}(C_{M\alpha}\alpha + C_{M\delta}\delta + C_{M\omega_y}\omega_y) \end{cases} \quad (1)$$

where $\alpha, \omega_y, \delta$ represent the angle of attack, the pitch rate and the fin deflection angle, respectively. $\Omega$ is the operational range of $\alpha$, i.e., $\alpha \in \Omega$. $q, S, m, d, I_y, V$ represent the dynamic pressure, the characteristic area, the mass, the characteristic length, the moment of inertia about the pitch axis and the total velocity, respectively. $C_{Z\alpha}, C_{Z\delta}, C_{M\alpha}$, $C_{M\delta}, C_{M\omega_y}$ are aerodynamic coefficients. In this coordinate system, $C_{Z\alpha}, C_{Z\delta}, C_{M\delta}, C_{M\omega_y}$ are negative, and the sign of $C_{M\alpha}$ is dependent on the static stability of the missile, i.e., when the missile is statically stable, $C_{M\alpha}$ has a negative value, otherwise $C_{M\alpha}$ is positive.

**Assumption 1** $C_{Z\alpha}, C_{M\alpha}$ are functions of angle of attack $\alpha$, and the values of $C_{Z\delta}, C_{M\delta}, C_{M\omega_y}, m, V$ are constants.

**Remark 1** For $\alpha \in \Omega$, Assumption 1 is reasonable because of the approximation of the trigonometric function, and the application of the sliding mode can compensate the error from the approximation.

The output of the system is acceleration $a_z$, and the output equation is given as

$$a_z = \frac{qS}{m}(C_{Z\alpha}\alpha + C_{Z\delta}\delta). \quad (2)$$

Define dynamic coefficients as follows:

$$\begin{cases} a_1 = \dfrac{qS}{mV}C_{Z\alpha} \\ a_2 = \dfrac{qSd}{I_y}C_{M\alpha} \\ a_3 = \dfrac{qSd}{I_y}C_{M\omega_y} \\ b_1 = \dfrac{qS}{mV}C_{Z\delta} \\ b_2 = \dfrac{qSd}{I_y}C_{M\delta} \end{cases}. \quad (3)$$

Define $e = a_z - a_c$, $E = \int e$, where $a_c$ is acceleration command. Above-mentioned equations can be rewritten as follows:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\omega}_y \\ \dot{E} \end{bmatrix} = \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & a_3 & 0 \\ a_1 V & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \omega_y \\ E \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_1 V \end{bmatrix} \delta + \begin{bmatrix} 0 \\ 0 \\ -a_c \end{bmatrix}. \quad (4)$$

The states of the system is denoted by $\boldsymbol{x} = \begin{bmatrix} \alpha & \omega_y & E \end{bmatrix}^{\mathrm{T}}$. To facilitate analysis, redefine the state variables as follows:

$$\begin{cases} z_1 = \alpha - \dfrac{b_1}{b_2}\omega_y \\ z_2 = \omega_y - \dfrac{b_2}{b_1 V}E \\ z_3 = E \end{cases}. \quad (5)$$

Define $\boldsymbol{z} = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}^{\mathrm{T}}$, and (5) can be denoted by a matrix form $\boldsymbol{z}(t) = \boldsymbol{L}\boldsymbol{x}(t)$, where

$$\boldsymbol{L} = \begin{bmatrix} 1 & -\dfrac{b_1}{b_2} & 0 \\ 0 & 1 & -\dfrac{b_2}{b_1 V} \\ 0 & 0 & 1 \end{bmatrix}.$$

Because $b_1$, $b_2$ are constants, (4) can be represented as follows:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} a_1-\dfrac{b_1}{b_2}a_2 & \dfrac{b_1}{b_2}a_1-\dfrac{b_1^2}{b_2^2}a_2+1-\dfrac{b_1}{b_2}a_3 & \dfrac{a_1}{V}-\dfrac{b_1}{b_2V}a_2+\dfrac{b_2}{b_1V}-\dfrac{a_3}{V} \\ a_2-\dfrac{b_2}{b_1}a_1 & \dfrac{b_1}{b_2}a_2-a_1+a_3 & \dfrac{a_2}{V}-\dfrac{b_2}{b_1V}a_1+\dfrac{b_2}{b_1V}a_3 \\ a_1V & \dfrac{b_1}{b_2}a_1V & a_1 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_1V \end{bmatrix}\delta + \begin{bmatrix} 0 \\ \dfrac{b_2}{b_1V}a_c \\ -a_c \end{bmatrix}. \tag{6}$$

The system matrix of (6) is denoted as $\boldsymbol{A}(\alpha)$, and $a_{ij}$ $(i=1,2,3,\ j=1,2,3)$ represents the element of the matrix.

## 3. Suboptimal sliding mode control law design

To facilitate the design of the control law, assume that the acceleration command is 0, i.e., the last term of (6) can be neglected, and the system response will tend to be 0.

**Remark 2** The assumption that acceleration command is 0 is based on the transition from the reachability problem to the controllability problem. Although reachability is not equivalent to controllability for the nonlinear system, the approximate treatment is reasonable in the engineering design.

Noting that input control term $\delta$ only exists in the third line of the matrix of (6), the sliding surface can be defined as

$$\sigma(z,t) = c_1(t)z_1 + c_2(t)z_2 + z_3. \tag{7}$$

Under sliding mode control, the system motion state can be divided into two parts: the state variables move from outside the sliding surface to inside and then converge to origin in the sliding surface. Define $u=\delta$. The control term $u$ only directly acts on the third line of the matrix of (6) to prompt the system states to converge to the sliding surface. When states arrive, $\sigma(z,t)=0$, and the following equation holds:

$$z_3 = -c_1(t)z_1 - c_2(t)z_2. \tag{8}$$

At the moment, $z_3$ works as a control term for the first two lines of the matrix of (6).

First, design sliding mode control law $u$ to realize the state movement in the first part. Take the derivative of the sliding mode variable $\sigma$ and obtain the following equation:

$$\begin{aligned}\dot{\sigma} &= \dot{c}_1z_1 + c_1\dot{z}_1 + \dot{c}_2z_2 + c_2\dot{z}_2 + \dot{z}_3 = \\ &(c_1a_{11}+c_2a_{21}+a_{31}+\dot{c}_1)z_1 + (c_1a_{12}+c_2a_{22}+ \\ &a_{32}+\dot{c}_2)z_2 + (c_1a_{13}+c_2a_{23}+a_{33})z_3 + b_1Vu.\end{aligned} \tag{9}$$

The control law can be designed as

$$\begin{aligned}u = -\frac{1}{b_1V}\{[&(c_1a_{11}+c_2a_{21}+a_{31}+\dot{c}_1)z_1 + \\ &(c_1a_{12}+c_2a_{22}+a_{32}+\dot{c}_2)z_2 + (c_1a_{13}+ \\ &c_2a_{23}+a_{33})z_3] + k\sigma\}\end{aligned} \tag{10}$$

where $k>0$ is a constant parameter designed. Substituting (10) into (9), the following equation can be obtained:

$$\dot{\sigma} = -k\sigma. \tag{11}$$

After the states arrive at the sliding surface, the system can be denoted as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}z_3. \tag{12}$$

Define $\boldsymbol{A}'_{11} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\boldsymbol{A}'_{12} = \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$. Design the quadratic cost function as follows:

$$J = \frac{1}{2}\int_0^\infty \boldsymbol{x}^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}\mathrm{d}t \tag{13}$$

where $\boldsymbol{Q}$ is the positive definite matrix. Define $\boldsymbol{Q}' = (\boldsymbol{L}^{-1})^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{L}^{-1} = \begin{bmatrix} \boldsymbol{Q}'_{11} & \boldsymbol{Q}'_{12} \\ \boldsymbol{Q}'_{21} & \boldsymbol{Q}'_{22} \end{bmatrix}$, $z_{1,2} = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^{\mathrm{T}}$, $\boldsymbol{Q}'_{11} \in \mathbf{R}^{2\times 2}$, and the indicator function can be rewritten as

$$\begin{aligned}J &= \frac{1}{2}\int_0^\infty z^{\mathrm{T}}\boldsymbol{Q}'z\mathrm{d}t = \\ &\frac{1}{2}\int_0^\infty \left(z_{1,2}^{\mathrm{T}}\boldsymbol{Q}'_{11}z_{1,2} + 2z_{1,2}^{\mathrm{T}}\boldsymbol{Q}'_{12}z_3 + \boldsymbol{Q}'_{22}z_3^2\right)\mathrm{d}t.\end{aligned} \tag{14}$$

**Remark 3** In the indicator function of this paper, control term $u$ is not considered, because the design of the suboptimal sliding surface is based on (12) that does not contain $u$.

Construct the Hamilton function as follows:

$$\begin{aligned}H = &\frac{1}{2}\left(z_{1,2}^{\mathrm{T}}\boldsymbol{Q}'_{11}z_{1,2} + 2z_{1,2}^{\mathrm{T}}\boldsymbol{Q}'_{12}z_3 + \boldsymbol{Q}'_{22}z_3^2\right) + \\ &\boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{A}'_{11}z_{1,2} + \boldsymbol{A}'_{12}z_3)\end{aligned} \tag{15}$$

where $\boldsymbol{\lambda}$ is the Lagrange vector. According to $\dfrac{\partial H}{\partial z_3} = 0$, the following equation is obtained:

$$z_3 = -\boldsymbol{Q}'^{-1}_{22}(\boldsymbol{Q}'^{\mathrm{T}}_{12}z_{1,2} + \boldsymbol{A}'^{\mathrm{T}}_{12}\boldsymbol{\lambda}). \tag{16}$$

Assuming that $\boldsymbol{\lambda} = \boldsymbol{P}z_{1,2}$, and neglecting the derivative of $\boldsymbol{P}$, according to $\dot{\boldsymbol{\lambda}} = -\dfrac{\partial H}{\partial z_{1,2}}$, the Riccati equation can be derived as follows:

$$\boldsymbol{PA'}_{11}+\boldsymbol{A'}_{11}{}^{\mathrm T}\boldsymbol{P}+\boldsymbol{Q'}_{11}-$$
$$(\boldsymbol{PA'}_{12}+\boldsymbol{Q'}_{12})\boldsymbol{Q'}_{22}{}^{-1}(\boldsymbol{Q'}_{12}{}^{\mathrm T}+\boldsymbol{A'}_{12}{}^{\mathrm T}\boldsymbol{P})=0. \quad (17)$$

Defining $\hat{\boldsymbol{A}}'_{11}=\boldsymbol{A'}_{11}-\boldsymbol{A'}_{12}\boldsymbol{Q'}_{22}{}^{-1}\boldsymbol{Q'}_{12}{}^{\mathrm T}$, $\hat{\boldsymbol{Q}}'_{11}=\boldsymbol{Q'}_{11}-\boldsymbol{Q'}_{12}\boldsymbol{Q'}_{22}{}^{-1}\boldsymbol{Q'}_{12}{}^{\mathrm T}$, (17) can be rewritten as

$$\boldsymbol{P}\hat{\boldsymbol{A}}'_{11}+\hat{\boldsymbol{A}}'^{\mathrm T}_{11}\boldsymbol{P}+\hat{\boldsymbol{Q}}'_{11}-\boldsymbol{PA'}_{12}\boldsymbol{Q'}_{22}{}^{-1}\boldsymbol{A'}_{12}{}^{\mathrm T}\boldsymbol{P}=0. \quad (18)$$

The key to obtain the analytical expression of the suboptimal sliding surface is to solve matrix $\boldsymbol{P}$ by the analytic method.

Choose

$$\boldsymbol{Q}=\begin{bmatrix} k_1 & -\dfrac{b_1}{b_2}k_1 & 0 \\ -\dfrac{b_1}{b_2}k_1 & \dfrac{b_1^2}{b_2^2}k_1+k_2 & -\dfrac{b_2}{b_1V}k_2 \\ 0 & -\dfrac{b_2}{b_1V}k_2 & \dfrac{b_2^2}{b_1^2V^2}k_2+k_3 \end{bmatrix}$$

where $k_1, k_2, k_3$ are non-negative constants designed. Then $\boldsymbol{Q'}=\mathrm{diag}[k_1,\ k_2,\ k_3]$, i.e., $\boldsymbol{Q'}_{11}=\mathrm{diag}[k_1,\ k_2]$, $\boldsymbol{Q}'_{22}=k_3$, $\boldsymbol{Q'}_{12}=0$, $\boldsymbol{Q'}_{21}=0$. Equations (18) and (16) degrade into the following equations:

$$\boldsymbol{PA'}_{11}+\boldsymbol{A'}_{11}{}^{\mathrm T}\boldsymbol{P}+\boldsymbol{Q'}_{11}-\boldsymbol{PA'}_{12}\boldsymbol{Q}'^{-1}_{22}\boldsymbol{A'}_{12}{}^{\mathrm T}\boldsymbol{P}=0, \quad (19)$$

$$z_3=-\boldsymbol{Q}'^{-1}_{22}\boldsymbol{A'}_{12}{}^{\mathrm T}\boldsymbol{P}\boldsymbol{z}_{1,2}. \quad (20)$$

**Lemma 1** [18] The two-dimensional state dependent Riccati equation has a positive definite stabilizing solution $\boldsymbol{P}$ and the analytic solution can be obtained, if the following three conditions are satisfied on $\Omega$:

**Condition 1** All matrices in (19) are continuous matrix-valued functions.

**Condition 2** $\boldsymbol{Q'}_{11}$ and $\boldsymbol{G}$ are positive semidefinite, where $\boldsymbol{G}=\boldsymbol{A'}_{12}\boldsymbol{Q}'^{-1}_{22}\boldsymbol{A'}_{12}{}^{\mathrm T}$.

**Condition 3** The matrix pair $(\boldsymbol{A'}_{11},\ \hat{\boldsymbol{G}})$ is pointwise controllable, and the matrix pair $(\boldsymbol{A'}_{11},\ \boldsymbol{Q'}_{11})$ is pointwise observable, where $\boldsymbol{G}=\hat{\boldsymbol{G}}\hat{\boldsymbol{G}}^{\mathrm T}$.

More details about the analytic method of solving (19) can be found in [18], and there is no more tautology here.

**Assumption 2** $\dfrac{b_1}{b_2}a_2-a_1+\dfrac{3}{4}a_3$ is negative on $\Omega$.

**Remark 4** Note the following equation:

$$\frac{b_1}{b_2}a_2-a_1+\frac{3}{4}a_3=$$
$$\frac{qS}{mV}\left(\frac{C_{Z\delta}}{C_{M\delta}}C_{M\alpha}-C_{Z\alpha}+\frac{3dmV}{4I_y}C_{M\omega_y}\right). \quad (21)$$

Because $\dfrac{dmV}{I_y}$ has a much larger value, the sign of the third term in parentheses plays a dominant role in most cases. Then it is reasonable to make Assumption 2.

**Theorem 1** Choosing $k_1>0$, $k_2=0$, $k_3>0$, under the indicator function (14), the analytical expression of sliding surface (7) can be represented as follows:

$$\sigma=\frac{1}{a_{13}}\left(\sqrt{d}+a_{11}+a_{23}\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{21}a_{13}-a_{11}a_{23}}\right)z_1+$$
$$\left(\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{21}a_{13}-a_{11}a_{23}}+\frac{a_{12}}{a_{13}}\right)z_2+z_3 \quad (22)$$

where $d=(a_{11}-a_{22})^2+4a_{12}a_{21}+a_{13}^2\dfrac{k_1}{k_3}$.

**Proof** Obviously, the Riccati equation (19) satisfies Condition 1 and Condition 2 in Lemma 1. Verify Condition 3 as follows.

The controllability matrix is as follows:

$$\boldsymbol{M}_c=\begin{bmatrix} a_{13} & a_{11}a_{13}+a_{12}a_{23} \\ a_{23} & a_{21}a_{13}+a_{22}a_{23}\end{bmatrix}. \quad (23)$$

Noting that $a_{13}a_{22}-a_{12}a_{23}=0$, the determinant of $\boldsymbol{M}_c$ can be derived as follows:

$$\mathrm{Det}(\boldsymbol{M}_c)=a_{21}a_{13}^2+a_{22}a_{13}a_{23}-a_{11}a_{23}a_{13}-a_{12}a_{23}^2=$$
$$a_{13}(a_{21}a_{13}-a_{11}a_{23}). \quad (24)$$

Because $a_3<0$, under Assumption 2, $a_{13}>0$. $x_{cf}$, $x_{cp}$, $x_{cg}$ represent the distances from the nose of the missile to the aerodynamic center, the center of pressure and the center of gravity, as shown in Fig.1. Then the following equations hold [18]:

$$C_{M\alpha}=C_{Z\alpha}\frac{x_{cp}-x_{cg}}{d}, \quad (25)$$

$$C_{M\delta}=C_{Z\delta}\frac{x_{cf}-x_{cg}}{d}. \quad (26)$$

In this paper, we only study the missile with normal configuration, i.e., $x_{cp}<x_{cf}$. Using (25) and (26), the following relations hold:

$$a_2-\frac{b_2}{b_1}a_1=\frac{qSd}{I_y}\left(C_{M\alpha}-\frac{C_{M\delta}}{C_{Z\delta}}C_{Z\alpha}\right)=$$
$$\frac{qS}{I_y}C_{Z\alpha}(x_{cp}-x_{cf})>0, \quad (27)$$

$$a_{21}a_{13}-a_{11}a_{23}=\frac{b_2}{b_1V}\left(a_2-\frac{b_2}{b_1}a_1\right)>0. \quad (28)$$

Hence, the matrix-pair $(\boldsymbol{A'}_{11},\ \hat{\boldsymbol{G}})$ is pointwise controllable.

The observability matrix is as follows:

$$\boldsymbol{M}_o=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \\ k_1a_{11} & k_1a_{12} \\ k_2a_{21} & k_2a_{22}\end{bmatrix}. \quad (29)$$

Because $a_3<0$, $b_1<0$ and $b_2<0$, under Assumption 2,

$a_{12}>0$ and $\boldsymbol{M}_o$ is a column full rank matrix. Hence, the matrix-pair $(\boldsymbol{A}'_{11},\ \boldsymbol{Q}'_{11})$ is pointwise observable.

Define $\boldsymbol{P}=\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}$. Using the analytical solution of Riccati equation in Lemma 1, the matrix $\boldsymbol{P}$ in (19) can be represented as follows:

$$p_{11}=\frac{k_3}{(\mathrm{Det}(\boldsymbol{M}_c))^2}\Big\{[2a_{23}(a_{12}a_{21}-a_{11}a_{22})(a_{11}a_{23}-a_{13}a_{21})+ a_{11}(a_{11}a_{23}-a_{13}a_{21})^2+a_{23}^2a_{22}(a_{11}a_{22}-a_{12}a_{21})+ a_{12}a_{21}a_{23}(a_{11}a_{23}-a_{13}a_{21})]+\sqrt{d}[(a_{11}a_{23}-a_{13}a_{21})^2+ a_{23}^2(a_{12}a_{21}-a_{11}a_{22})]\Big\}, \tag{30}$$

$$p_{12}=\frac{k_3}{(\mathrm{Det}(\boldsymbol{M}_c))^2}\Big\{[a_{13}(a_{12}a_{21}-a_{11}a_{22})(a_{13}a_{21}- a_{23}a_{11})-a_{13}(a_{11}a_{22}-a_{12}a_{21})(a_{23}a_{22}+a_{13}a_{21})]+ \sqrt{d}[-a_{13}a_{23}(a_{12}a_{21}-a_{11}a_{22})]\Big\}, \tag{31}$$

$$p_{22}=\frac{k_3}{(\mathrm{Det}(\boldsymbol{M}_c))^2}\Big\{[a_{13}^2a_{11}(a_{11}a_{22}-a_{12}a_{21})+ a_{12}^2a_{23}(a_{11}a_{23}-a_{21}a_{13})]+\sqrt{d}[a_{13}^2(a_{12}a_{21}-a_{11}a_{22})]\Big\}. \tag{32}$$

Substituting the relation $a_{13}a_{22}-a_{12}a_{23}=0$ into (20), the following equations can be obtained:

$$c_1=\frac{1}{k_3}(a_{13}p_{11}+a_{23}p_{12})= \frac{1}{a_{13}}\left(\sqrt{d}+a_{11}+a_{23}\frac{a_{11}a_{22}-a_{12}a_{21}}{a_{21}a_{13}-a_{11}a_{23}}\right), \tag{33}$$

$$c_2=\frac{1}{k_3}(a_{13}p_{12}+a_{23}p_{22})=\frac{a_{12}a_{21}-a_{11}a_{22}}{a_{21}a_{13}-a_{11}a_{23}}+\frac{a_{12}}{a_{13}}. \tag{34}$$

The proof of Theorem 1 is completed. □

**Remark 5** In this paper, choosing $k_2=0$ is to simplify calculation. The choice of $k_1$, $k_2$, $k_3$ only needs to satisfy the conditions that at least one of $k_1$, $k_2$ is positive and $k_3>0$.

## 4. Analysis of stability in sliding surface

Substituting (33) and (34) into (8) and (12), the following relations hold:

$$\begin{bmatrix}\dot z_1\\ \dot z_2\end{bmatrix}=\begin{bmatrix}A'_{c11} & A'_{c12}\\ A'_{c21} & A'_{c22}\end{bmatrix}\begin{bmatrix} z_1\\ z_2\end{bmatrix}, \tag{35}$$

$$A'_{c11}=\frac{b_1}{b_2}a_2-a_1+a_3-\sqrt{d}, \tag{36}$$

$$A'_{c12}=-\frac{b_1}{b_2}a_1+\frac{b_1^2}{b_2^2}a_2-1+\frac{b_1}{b_2}a_3, \tag{37}$$

$$A'_{c21}=\frac{b_2}{b_1}\cdot\frac{\left(a_1-\dfrac{b_1}{b_2}a_2-a_3\right)^2+\dfrac{b_2}{b_1}\left(\dfrac{b_1}{b_2}a_2-a_1\right)}{a_1-\dfrac{b_1}{b_2}a_2-a_3+\dfrac{b_2}{b_1}}- \frac{a_2-\dfrac{b_2}{b_1}a_1+\dfrac{b_2}{b_1}a_3}{a_1-\dfrac{b_1}{b_2}a_2+\dfrac{b_2}{b_1}-a_3}\sqrt{d}, \tag{38}$$

$$A'_{c22}=-\left(\frac{b_1}{b_2}a_2-a_1+a_3\right). \tag{39}$$

Under Assumption 2, using (27), the following relations can be obtained: $A'_{c11}<0$, $A'_{c12}<0$, $A'_{c21}>0$, $A'_{c22}>0$.

**Assumption 3** $\left(\dfrac{b_1}{b_2}a_2-a_1+a_3\right)^2-\left(a_2-\dfrac{b_2}{b_1}a_1\right)$ is negative on $\Omega$.

**Remark 6** The following equation holds on $\Omega$:

$$\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2-\left(a_2-\frac{b_2}{b_1}a_1\right)=\left(\frac{qS}{mV}\right)^2\left(\frac{C_{Z\delta}}{C_{M\delta}}C_{M\alpha}- C_{Z\alpha}+\frac{dmV}{I_y}C_{M\omega_y}\right)^2-\frac{qSd}{I_y}\left(C_{M\alpha}-\frac{C_{M\delta}}{C_{Z\delta}}C_{Z\alpha}\right). \tag{40}$$

Because $V$ has a large value, $\dfrac{qSd}{I_y}\gg\left(\dfrac{qS}{mV}\right)^2$ holds in most cases. Although the value of $\left(\dfrac{C_{Z\delta}}{C_{M\delta}}C_{M\alpha}-C_{Z\alpha}+\dfrac{dmV}{I_y}C_{M\omega_y}\right)^2$ is greater than that of $\left(C_{M\alpha}-\dfrac{C_{M\delta}}{C_{Z\delta}}C_{Z\alpha}\right)$, Assumption 3 usually holds on $\Omega$ because of the amplification of $\dfrac{qSd}{I_y}$.

**Assumption 4** $-3a_3+\dfrac{k_1}{k_3}\dfrac{a_{13}^2}{a_3}-4\left(\dfrac{b_1}{b_2}a_2-a_1\right)$ is positive on $\Omega$.

**Remark 7** To keep the tracking error as small as possible, we choose that $k_3>k_1$. Considering $V$ has a large value, $\dfrac{k_1}{k_3}\dfrac{a_{13}^2}{a_3}$ has a small value. Under Assumption 2, Assumption 4 usually holds.

**Theorem 2** Under Assumptions 2 and 4, the system (35) is asymptotically stable on $\Omega$, if there exists a positive constant $\kappa$ satisfying the following relations:

$$\sup_{\alpha\in\Omega}\left\{\frac{-A'_{c11}+A'_{c22}+2\varDelta_M^{\frac{1}{4}}-\sqrt{\left(A'_{c11}-A'_{c22}-2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c21}A'_{c12}}}{4A'_{c21}}\right\}<\kappa<$$

$$\inf_{\alpha\in\Omega}\left\{\min\left(-\frac{A'_{c11}}{A'_{c21}},-\frac{A'_{c12}}{2A'_{c22}},\frac{-A'_{c11}+A'_{c22}+2\varDelta_M^{\frac{1}{4}}+\sqrt{\left(A'_{c11}-A'_{c22}-2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c21}A'_{c12}}}{4A'_{c21}}\right)\right\} \quad (41)$$

where $\varDelta_M=(a_{11}a_{22}-a_{12}a_{21})^2$.

**Proof** Consider the following Lyapunov candidate function:

$$V_L=\frac{1}{2}(z_1+\kappa z_2)^2+\frac{\kappa^2}{2}z_2^2. \quad (42)$$

Obviously, $V_L$ is positive and bounded. Take the derivative of $V_L$ and the following equation can be obtained:

$$\dot V_L=(z_1+\kappa z_2)(\dot z_1+\kappa\dot z_2)+\kappa^2z_2\dot z_2=\begin{bmatrix} z_1 & z_2\end{bmatrix}\big[(A'_{c11}+\kappa A'_{c21}+\kappa A'_{c11}+A'_{c12}+2\kappa^2A'_{c21}+$$
$$\kappa A'_{c22}+\kappa A'_{c11}+A'_{c12}+2\kappa^2A'_{c21}+\kappa A'_{c22}+\kappa A'_{c12}+2\kappa^2A'_{c22})/2\big]\begin{bmatrix} z_1\\ z_2\end{bmatrix}=z_{1,2}^{\mathrm T}A_\kappa z_{1,2}. \quad (43)$$

The characteristic polynomial of $\boldsymbol{A}_\kappa$ is as follows:

$$\lambda^2-(A'_{c11}+\kappa A'_{c21}+\kappa A'_{c12}+2\kappa^2A'_{c22})\lambda+$$
$$(A'_{c11}+\kappa A'_{c21})(\kappa A'_{c12}+2\kappa^2A'_{c22})-$$
$$\frac{(\kappa A'_{c11}+A'_{c12}+2\kappa^2A'_{c21}+\kappa A'_{c22})^2}{4}=0. \quad (44)$$

If $\dot V_L$ is negative definite, the following inequalities should be satisfied:

$$A'_{c11}+\kappa A'_{c21}<0, \quad (45)$$

$$\kappa A'_{c12}+2\kappa^2A'_{c22}<0, \quad (46)$$

$$(A'_{c11}+\kappa A'_{c21})(\kappa A'_{c12}+2\kappa^2A'_{c22})-$$
$$\frac{(\kappa A'_{c11}+A'_{c12}+2\kappa^2A'_{c21}+\kappa A'_{c22})^2}{4}>0. \quad (47)$$

When $\kappa<\min\left(-\frac{A'_{c11}}{A'_{c21}},\ -\frac{A'_{c12}}{2A'_{c22}}\right)$, (45) and (46) hold. Then, analyze the condition for the establishment of (47). Noting the relation $A'_{c11}A'_{c22}-A'_{c12}A'_{c21}=\varDelta_M^{\frac{1}{2}}$, the left-hand side of (47) can be rewritten as follows:

$$(A'_{c11}+\kappa A'_{c21})(\kappa A'_{c12}+2\kappa^2A'_{c22})-$$
$$\frac{(\kappa A'_{c11}+A'_{c12}+2\kappa^2A'_{c21}+\kappa A'_{c22})^2}{4}=$$
$$\frac{1}{4}\left[2\kappa\varDelta_M^{\frac{1}{4}}+2\kappa^2A'_{c21}-A'_{c12}+\kappa(A'_{c11}-A'_{c22})\right]\cdot$$
$$\left[2\kappa\varDelta_M^{\frac{1}{4}}-2\kappa^2A'_{c21}+A'_{c12}-\kappa(A'_{c11}-A'_{c22})\right]. \quad (48)$$

Noting that $d=a_3^2+4\left(a_2-\frac{b_2}{b_1}a_1\right)+\frac{k_1}{k_3}a_{13}^2$, the following equation can be obtained:

$$\left(A'_{c11}-A'_{c22}+2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c12}A'_{c21}=$$
$$(A'_{c11}+A'_{c22})^2+4A'_{c12}A'_{c21}+4(A'_{c11}-A'_{c22})\varDelta_M^{\frac{1}{4}}=$$
$$d+4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)\sqrt{d}-4\left(a_2-\frac{b_2}{b_1}a_1\right)-$$
$$4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2+4\left[2\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)-\sqrt{d}\right]\varDelta_M^{\frac{1}{4}}=$$
$$a_3^2+\frac{k_1}{k_3}a_{13}^2+4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)\sqrt{d}-$$
$$4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2+4\left[2\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)-\sqrt{d}\right]\varDelta_M^{\frac{1}{4}}. \quad (49)$$

Under Assumption 2, $-4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2$ and $4\left[2\left(\frac{b_1}{b_2}\cdot a_2-a_1+a_3\right)-\sqrt{d}\right]\varDelta_M^{\frac{1}{4}}$ are negative. Noting that $\sqrt{d}>-a_3$, under Assumption 4, the following equation holds:

$$a_3^2+\frac{k_1}{k_3}a_{13}^2+4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)\sqrt{d}<$$
$$a_3^2+\frac{k_1}{k_3}a_{13}^2-4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)a_3=$$
$$a_3\left[a_3+\frac{k_1}{k_3}\frac{a_{13}^2}{a_3}-4\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)\right]<0. \quad (50)$$

Then, $\left(A'_{c11}-A'_{c22}+2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c12}A'_{c21}<0$. The first term in the right-hand side of (48) is positive.

Analyze the second term in the right-hand side of (48), and the following relation holds:

$$\left(A'_{c11}-A'_{c22}-2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c12}A'_{c21}=$$

$$(A'_{c11}-A'_{c22})^2+4\varDelta_M^{\frac{1}{2}}-4(A'_{c11}-A'_{c22})\varDelta_M^{\frac{1}{4}}+8A'_{c12}A'_{c21}=$$
$$\left[(A'_{c11}-A'_{c22})^2+4\varDelta_M^{\frac{1}{2}}+4A'_{c12}A'_{c21}\right]+$$
$$\left[-4(A'_{c11}-A'_{c22})\varDelta_M^{\frac{1}{4}}+4A'_{c12}A'_{c21}\right]. \tag{51}$$

Analyze the first term in the right-hand side of (51), and the following equation can be derived:

$$(A'_{c11}-A'_{c22})^2+4\varDelta_M^{\frac{1}{2}}+4A'_{c12}A'_{c21}=$$
$$(A'_{c11}+A'_{c22})^2=d>0. \tag{52}$$

Analyze the second term in the right-hand side of (51), and the following relation can be acquired:

$$(A'_{c12}A'_{c21})^2-(A'_{c11}-A'_{c22})^2\varDelta_M^{\frac{1}{2}}=$$
$$(A'_{c11}A'_{c22}-\varDelta_M^{\frac{1}{2}})^2-(A'_{c11}-A'_{c22})^2\varDelta_M^{\frac{1}{2}}=$$
$$(A'_{c11}A'_{c22})^2+\varDelta_M-(A'^2_{11}+A'^2_{22})\varDelta_M^{\frac{1}{2}}=$$
$$\left[\left(\frac{b_1}{b_2}a_2-a_1+a_3-\sqrt{d}\right)^2-\left(a_2-\frac{b_2}{b_1}a_1\right)\right]\cdot$$
$$\left[\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2-\left(a_2-\frac{b_2}{b_1}a_1\right)\right]. \tag{53}$$

Noting the first term in the right-hand side of (53), the following inequality holds:

$$\left(\frac{b_1}{b_2}a_2-a_1+a_3-\sqrt{d}\right)^2-\left(a_2-\frac{b_2}{b_1}a_1\right)>$$
$$d-\left(a_2-\frac{b_2}{b_1}a_1\right)>0. \tag{54}$$

Under Assumption 3, $\left(\frac{b_1}{b_2}a_2-a_1+a_3\right)^2-\left(a_2-\frac{b_2}{b_1}a_1\right)$ is negative. Then, $-4(A'_{c11}-A'_{c22})\varDelta_M^{\frac{1}{4}}+4A'_{c12}A'_{c21}>0$ and $\left(A'_{c11}-A'_{c22}-2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c12}A'_{c21}>0$. The solutions of the following equation:

$$2\kappa\varDelta_M^{\frac{1}{4}}-2\kappa^2A'_{c21}+A'_{c12}-\kappa(A'_{c11}-A'_{c22})=0 \tag{55}$$

can be represented as

$$\frac{-A'_{c11}+A'_{c22}+2\varDelta_M^{\frac{1}{4}}\pm\sqrt{\left(A'_{c11}-A'_{c22}-2\varDelta_M^{\frac{1}{4}}\right)^2+8A'_{c21}A'_{c12}}}{4A'_{c21}}$$

and they are positive. Then, $\dot{V}_L$ is negative definite and bounded.

Therefore, if there exists $\kappa$ satisfying inequality (41), the system (35) is asymptotically stable on $\varOmega$. □

## 5. Control law in the form of pseudo-angle-of-attack feedback

In the third section, the suboptimal control law has been designed. In this section, the control law (10) will be transformed into the form of pseudo-angle-of-attack feedback as shown in Fig. 2.

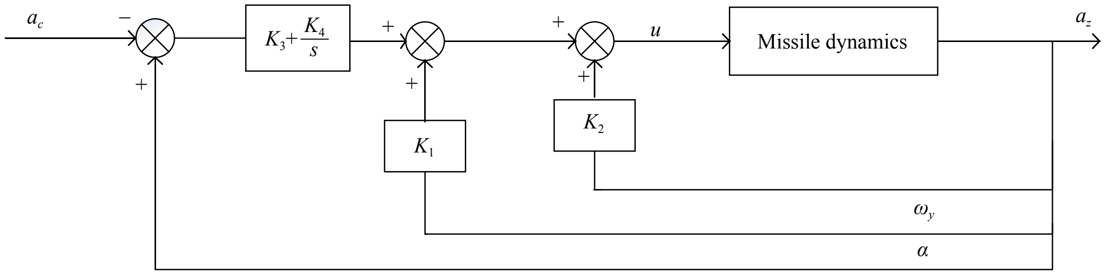


**Fig. 2 Closed-loop system with pseudo-angle-of-attack feedback control law**

**Theorem 3** Under Assumption 1, the control law (10) can be rewritten as follows:

$$u=\quad K_1\alpha+K_2\omega_y+K_3e+K_4E \tag{56}$$

where

$$K_1=-\frac{1}{3b_1V}(c_1a_{11}+a_{31}+\dot{c}_1+kc_1+c_2a_2), \tag{57}$$

$$K_2=-\frac{1}{3b_1V}\left[-\frac{b_1}{b_2}(c_1a_{11}+a_{31}+\dot{c}_1+kc_1)+\right.$$
$$\left.(c_1a_{12}+a_{32}+kc_2)+c_2a_3\right], \tag{58}$$

$$K_3=\frac{2}{3b_1V}, \tag{59}$$

$$K_4=-\frac{1}{3b_1V}\left[-\frac{b_2}{b_1V}(c_1a_{12}+a_{32}+kc_2)+\right.$$
$$\left.(c_1a_{13}+a_{33}+k)\right]. \tag{60}$$

**Proof** Using (5) and (10), the following equation can be obtained:

$$u=-\frac{1}{b_1V}\left\{[(c_1a_{11}+a_{31}+\dot{c}_1)z_1+\right.$$
$$(c_1a_{12}+a_{32}+\dot{c}_2)z_2+(c_1a_{13}+a_{33})z_3+$$

$$c_2(a_{21}z_1+a_{22}z_2+a_{23}z_3)]+k\sigma\Big\}=$$

$$-\frac{1}{b_1V}\Big\{[(c_1a_{11}+a_{31}+\dot{c}_1)z_1+(c_1a_{12}+a_{32}+\dot{c}_2)z_2+$$

$$(c_1a_{13}+a_{33})z_3+c_2\dot{z}_2]+k\sigma\Big\}=$$

$$-\frac{1}{b_1V}\Big\{\Big[(c_1a_{11}+a_{31}+\dot{c}_1)z_1+(c_1a_{12}+a_{32}+\dot{c}_2)z_2+$$

$$(c_1a_{13}+a_{33})z_3+c_2\left(a_2\alpha+a_3\omega_y+b_2u-\frac{b_2}{b_1V}e\right)\Big]+k\sigma\Big\}. \tag{61}$$

Using (34), $c_2$ is equal to $\dfrac{2b_1V}{b_2}$. Under Assumption 1, $c_2$ is a constant, i.e., $\dot{c}_2=0$. Equation (56) can be rewritten as follows:

$$u=-\frac{1}{3b_1V}\Big\{\Big[(c_1a_{11}+a_{31}+\dot{c}_1)z_1+(c_1a_{12}+a_{32})z_2+$$

$$(c_1a_{13}+a_{33})z_3+c_2\left(a_2\alpha+a_3\omega_y-\frac{b_2}{b_1V}e\right)\Big]+k\sigma\Big\}=$$

$$-\frac{1}{3b_1V}\Big\{(c_1a_{11}+a_{31}+\dot{c}_1+kc_1)z_1+(c_1a_{12}+a_{32}+kc_2)z_2+$$

$$(c_1a_{13}+a_{33}+k)z_3+c_2\left(a_2\alpha+a_3\omega_y-\frac{b_2}{b_1V}e\right)\Big\}=$$

$$-\frac{1}{3b_1V}\Big\{\alpha(c_1a_{11}+a_{31}+\dot{c}_1+kc_1+c_2a_2)+$$

$$\omega_y\left[-\frac{b_1}{b_2}(c_1a_{11}+a_{31}+\dot{c}_1+kc_1)+(c_1a_{12}+a_{32}+kc_2)+c_2a_3\right]+$$

$$\mathrm{E}\left[-\frac{b_2}{b_1V}(c_1a_{12}+a_{32}+kc_2)+(c_1a_{13}+a_{33}+k)\right]+e\left(-\frac{c_2b_2}{b_1V}\right)\Big\} \tag{62}$$

The proof of Theorem 3 is completed. □

## 6. Simulation results

The model of the missile at the altitude of 6 000 m is as follows:

$$\begin{cases} a_1=0.021\,M_a[19.373\alpha^2-31.023|\alpha|- \\ \quad 12.956(1.5-0.25M_a)] \\ a_2=1.237\,5\,M_a^2[40.440\alpha^2-64.015|\alpha|- \\ \quad 4.870(4.2-1.6M_a)] \\ a_3=1.237\,5\,M_a^2(-1.719) \\ b_1=0.021\,M_a(-1.948) \\ b_2=1.237\,5\,M_a^2(-11.803) \end{cases} \tag{63}$$

where $M_a$ is the mach number and $\Omega=\{\alpha\in\mathbf{R}|\pi/3<\alpha<\pi/3\}$. In this section, the simulation results for the missile at $2M_a$(static stable) and $3M_a$(static unstable) are given. At $2M_a$, the following relation holds:

$$0.013\,9<\kappa<0.044\,1. \tag{64}$$

At $3M_a$, the following relation holds:

$$0.009\,5<\kappa<0.024\,3. \tag{65}$$

The parameter deviation is as follows:

$$\begin{cases} a'_1=1.2a_1 \\ a'_2=0.9a_2 \\ a'_3=1.1a_1\,. \\ b'_1=0.85b_1 \\ b'_2=1.12b_2 \end{cases} \tag{66}$$

To avoid the overlarge rudder deflection angle, step inputs are processed with the transition function.

The simulation results at $2M_a$ are shown in Fig. 3–Fig. 6. The simulation results at $3M_a$ are shown in Fig. 7–Fig. 10. As shown in Fig. 3 and Fig. 7, the control laws in [18] and this paper all have good performance under nominal models. However, as shown in Fig. 5 and Fig. 8, under real models with parameter deviation, the control law in [18] will result in steady state error and the control law proposed in this paper still keeps good tracking performance. Hence, the suboptimal sliding mode control law is more robust.

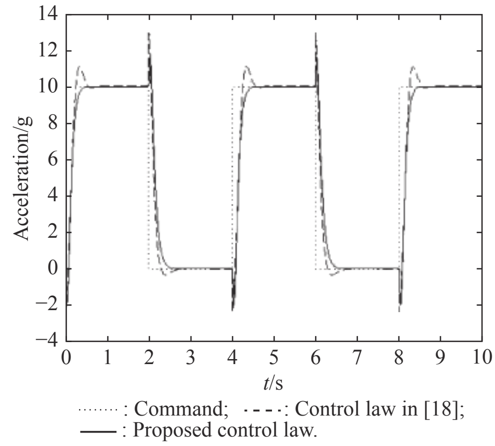


**Fig. 3 Simulation results under nominal model at $2M_a$**

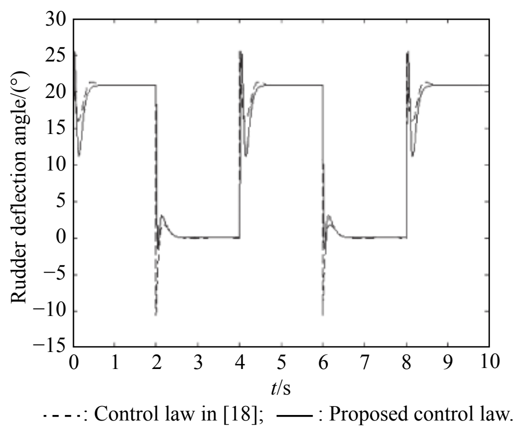


**Fig. 4 Rudder deflection angle under nominal model at $2M_a$**

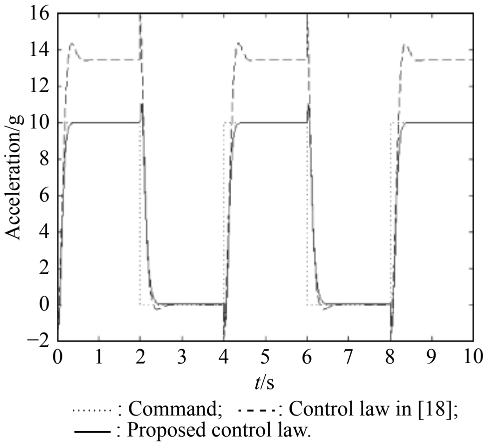


**Fig. 5 Simulation results under real model at $2M_a$**

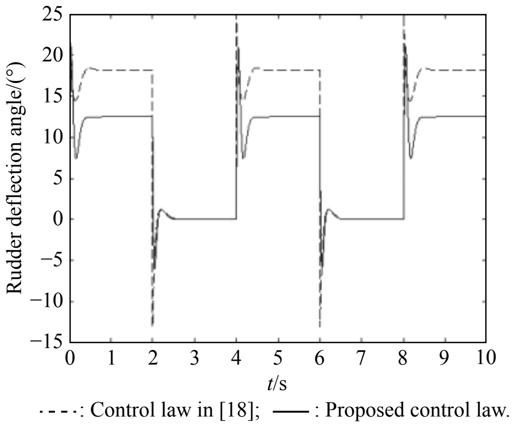


**Fig. 6 Rudder deflection angle under real model at $2M_a$**

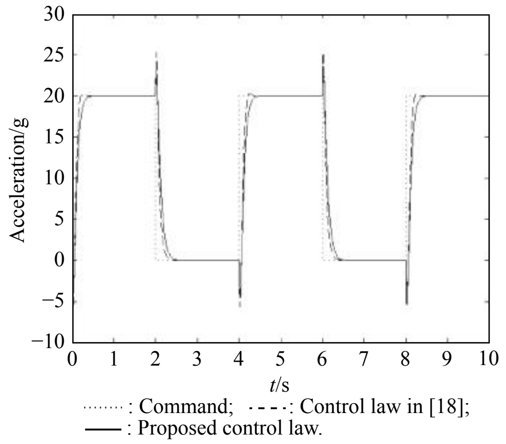


**Fig. 7 Simulation results under nominal model at $3M_a$**

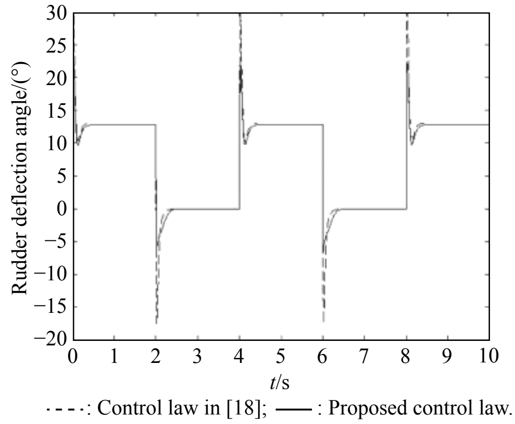


**Fig. 8 Rudder deflection angle under nominal model at $3M_a$**

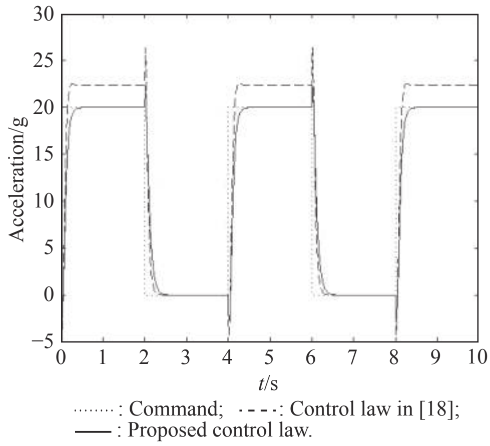


**Fig. 9 Simulation results under real model at $3M_a$**

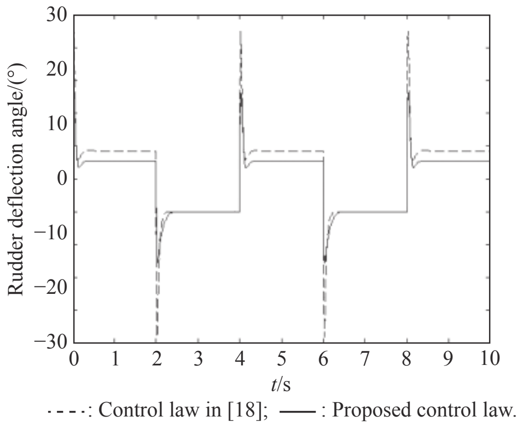


**Fig. 10 Rudder deflection angle under real model at $3M_a$**

## 7. Conclusions

In this paper, the analytical solution of the suboptimal sliding surface has been proposed and the stability in the sliding surface has been proved. The suboptimal sliding mode control law shows good performance and can overcome the effect of parameter error.

The control law designed in this paper can be written in the form of pseudo-angle-of-attack feedback, and each parameter in the control law is the function of the dynamic coefficient. Given the speed and altitude, the parameters are mainly determined by the angle of attack. The purpose of designing this control law is to overcome the influence of aerodynamic nonlinearity by using the angle of attack information, and to provide the functional relationship of the control parameters with the angle of attack (dynamic coefficient). In practical applications, the control parameters can be calculated by substituting the information of speed, altitude and angle of attack into the corresponding functions, and the control law in the form of the continuous function replaces the original control law in the form of the interpolation table.

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