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Fractional derivative multivariable grey model for nonstationary sequence and its application

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Abstract: Most of the existing multivariable grey models are based on the 1-order derivative and 1-order accumulation, which makes the parameters unable to be adjusted according to the data characteristics of the actual problems. The results about fractional derivative multivariable grey models are very few at present. In this paper, a multivariable Caputo fractional derivative grey model with convolution integral CFGMC(q, N) is proposed. First, the Caputo fractional difference is used to discretize the model, and the least square method is used to solve the parameters. The orders of accumulations and differential equations are determined by using particle swarm optimization (PSO). Then, the analytical solution of the model is obtained by using the Laplace transform, and the convergence and divergence of series in analytical solutions are also discussed. Finally, the CFGMC(q, N) model is used to predict the municipal solid waste (MSW). Compared with other competition models, the model has the best prediction effect. This study enriches the model form of the multivariable grey model, expands the scope of application, and provides a new idea for the development of fractional derivative grey model.

Keywords: fractional derivative of Caputo type, fractional accumulation generating operation (FAGO), Laplace transform, multivariable grey prediction model, particle swarm optimization (PSO).

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1. Introduction

Grey prediction model is the core component of the grey system theory, and its main models, such as the classical grey model GM(1,1) [1], multivariable grey model GM$(1, N)$ [2], grey Verhulst model [3,4], grey Bernoulli model [5], have made some progress. These models have been successfully applied to fields such as energy [6], finance [7] and mechanical [8], etc. The multivariate grey model has attracted the attention of scholars because it can better reflect the mutual influence and restriction among variables in the system.

The development history of GM$(1, N)$ model can be seen in Table 1. In addition, scholars have studied the background values [9], the combined model [10,11], data type [12], multicollinearity [13], time delay [14], driving factors [15], etc. These research results greatly promote the development of the GM$(1, N)$ model.

However, according to Table 1, we can see that most of the existing grey models are 1-order accumulative generation operation (AGO) and 1-order derivative. When the original data is a nonstationary sequence, the characteristics and rules of the sequence cannot be fully discovered by AGO [27], and AGO cannot embody the principle of new information priority. On the other hand,1-order derivative models are ideal memory models, which are not suitable for describing irregular phenomena. When encountering a sequence with large data fluctuation, the parameters of the model cannot be adjusted according to the data characteristics of the actual problem.

In order to solve the existing problems, fractional accumulation generating operation (FAGO) and fractional derivative should be introduced into the grey model. There are many research results on FAGO, such as the fractional discrete multivariate model [22,24], the FAGO grey Bernoulli equation [28], and the FAGO time delay model [29]. There are many research results on the univariate fractional derivative model, too. Mao et al. [30] introduced fractional derivative into the grey system based on the good degree of freedom and memory characteristics of fractional derivative, and the modeling mechanism of the fractional derivative model [31] is discussed. Yang et al. [32] used the generalized fractional grey model to forecast the per capita output of the power system. Mao et al. [33] established a nonlinear fractional grey model which combines the historical trend and the residual term. The multivariable fractional derivative grey model is only in [34].

Through the analysis of the existing model, four problems can be found.

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Table 1 Summary of the literature on GM$(1, N)$

Author	Name	Abbreviation	Application
Deng (2002) [16]	Multivariate grey model	GM$(1, N)$	Dynamic analysis of athletes' training status
Xie et al. (2009) [17]	Discrete multivariate grey model	DGM$(1, N)$	Mobile telecommunication customer
Wang (2014) [18]	Nonlinear grey multivariable model	NGM$(1, N)$	High-technology industry total output
Mao et al. (2015) [19]	Fractional accumulation time-lag model	GM$(1, N, \tau)$	Economic development of Wuhan
Ding et al. (2017) [20]	Multivariable time-delayed discrete grey model	TDDGM$(1, N)$	Output value of high-tech enterprises
Wu et al. (2018) [21]	Grey multivariable convolution model	GMCN$(1, N)$	Industrial power consumption
Wu et al. (2018) [22]	Grey multivariable model with fractional accumulation	FGMC$(1, N)$	Shandong's electricity consumption
Pei et al. (2018) [23]	Transformed model of nonlinear grey models	TNGM$(1, N)$	Pollutant emission
Ma et al. (2019) [24]	Fractional discrete multivariate model	FDGMC$(1, N)$	Industrial pollutant emission
Ma et al. (2019) [25]	Nonlinear multivariate grey Bernoulli model	NGBMC$(1, N)$	Tourist income of China
Zeng et al. (2019) [26]	Grey model of ternary interval numbers	TIGM$(1, N)$	Power generation, consumer price index
This paper	Caputo fractional derivative multivariable model	CFGMC(q, N)	Municipal solid waste (MSW) yields of Wuhan

First, the existing FAGO calculation method are extension of 1-order AGO through some deformation. It is bound to produce a certain degree of error during transformation. Second, the research of the fractional derivative grey model focuses on the single variable model, and there are few achievements in multivariable. In fact, the fractional derivative has a good global character, while the GM$(1, N)$ model reflects the influence of related factors on the system characteristics as a whole, so the multivariable prediction model is more suitable for the fractional derivative. Third, most of the existing fractional grey models only consider FAGO or only study fractional derivatives, and only [33] is used to build the grey model by combining the two. Fourth, most of the existing fractional derivative models do not discuss the type of fractional derivative, so they cannot choose the appropriate discretization method and solution method according to the type of derivative, resulting in the inability to obtain the analytical solution. Because there are many kinds of definition forms of fractional derivative, the combination of different definition forms and grey system will have different function forms and different application ranges. The types of fractional derivatives have important influence on the form and application of the model, and the iterative solutions often affect the prediction accuracy. Therefore, it is necessary to define the type of fractional derivative and find the analytical solution.

After evaluation of these problems, the solutions are respectively given. For the first aspect, FAGO and its inverse (IFAGO) are directly defined by fractional sums and difference theory, which avoids the deformation extension from 1-order AGO to FAGO. For the latter aspects, the CFGMC(q, N) model is established by combining FAGO and fractional derivative. The research object of the model is multivariable system, and its derivative type is Caputo type. The model can be discretized by Caputo fractional difference, and the analytical solution can be obtained by Laplace transform of Caputo fractional derivative.

The rest of this paper is organized as follows. Section 2 defines FAGO and IFAGO in the form of fractional sums and difference and establishes the CFGMC(q, N) model by using the Caputo fractional derivative. Then, the analytical solution of the new model is given and the existence of the solution is proved. In Section 3, the model is used to predict the MSW yields, and the validity of the new model is verified. Section 4 presents the conclusions.

2. Methodology

2.1 FAGO and IFAGO

This section defines FAGO and IFAGO by combining the sum difference theory with the grey system theory.

Definition 1 [35] Suppose r is any positive real number, n is a positive integer, and the equation

$$\nabla^{-r}x(n) = \begin{bmatrix} r \\ n \end{bmatrix} \cdot x(n) = \sum_{t=1}^{n} \begin{bmatrix} r \\ n-t \end{bmatrix} x(t) \quad (1)$$

is called the r-order sum of $x(n)$. Let $m_1 = [r]$ be the smallest integer greater than or equal to r, then $\nabla^r x(n) = \nabla^{m_1}\nabla^{-(m_1-r)}x(n)$ is the r-order difference of $x(n)$.

Definition 2 Let $\boldsymbol{\varphi}_j^{(0)} = (\varphi_j^{(0)}(1), \varphi_j^{(0)}(2), \dots, \varphi_j^{(0)}(n))$ be the vector, $k, j = \{1, 2, \dots, n\}$, if

$$\begin{cases} \varphi_j^{(0)}(k) = 0, & k \neq j \\ \varphi_j^{(0)}(k) = 1, & k = j \end{cases}. \quad (2)$$

$\boldsymbol{\varphi}_j^{(0)}$ is called the jth original generative base of AGO, and all of $\boldsymbol{\varphi}^{(0)}$ is called the original generative space of AGO. Obviously, $\boldsymbol{\varphi}^{(0)}$ is a unit matrix.

Definition 3 Let $\boldsymbol{x}^{(0)} = (x^{(0)}(1), x^{(0)}(2), \dots, x^{(0)}(n))$ be a non-negative sequence, $\boldsymbol{x}^{(r)}$ be r-order FAGO of $\boldsymbol{x}^{(0)}$, and note $\boldsymbol{x}^{(r)} = \boldsymbol{\varphi}_j^{(r)}\boldsymbol{x}^{(0)}$. Then $\boldsymbol{\varphi}_j^{(r)}$ is the generating base vector of the FAGO space of $\boldsymbol{x}^{(0)}$.

Theorem 1 Let $\boldsymbol{\varphi}_j^{(r)} = (\varphi_j^{(r)}(1), \varphi_j^{(r)}(2), \dots, \varphi_j^{(r)}(n))$

be the FAGO of $\boldsymbol{\varphi}_j^{(0)}$, then

$$\varphi_j^{(r)}(k)=\sum_{t=1}^{k}\begin{bmatrix} r \\ k-t \end{bmatrix}\varphi_j^{(0)}(t)=\begin{bmatrix} r \\ k-j \end{bmatrix}. \tag{3}$$

Proof According to (2), $\begin{bmatrix} r \\ k-t \end{bmatrix}\varphi_j^{(0)}(t)=1$ only if $t=j$, otherwise $\begin{bmatrix} r \\ k-t \end{bmatrix}\varphi_j^{(0)}(t)=0$, so $\varphi_j^{(r)}(k)=\begin{bmatrix} r \\ k-j \end{bmatrix}$.

According to the definition of the fractional sum, the vector space and the basis of FAGO can be derived. Therefore, the theory of the fractional sum can be used to calculate the FAGO. □

Definition 4 Let $x^{(r)}(k)=\nabla^{-r}x^{(0)}(k)$, then $\boldsymbol{x}^{(r)}=(x^{(r)}(1),x^{(r)}(2),\dots,x^{(r)}(n))$ is called the r-order FAGO sequence of $\boldsymbol{x}^{(0)}$.

In particular, when r is a positive integer, FAGO is consistent with the traditional AGO. After FAGO, the accumulative sequence must be processed by the corresponding inverse operation.

Theorem 2 If $\boldsymbol{x}^{(r-1)}$ is the 1-order inverse AGO sequence of $\boldsymbol{x}^{(r)}$, then $x^{(r-1)}(k)=x^{(r)}(k)-x^{(r)}(k-1)$. The process of proof is omitted.

Theorem 3 The r-order IFAGO of $\boldsymbol{x}^{(r)}$ is $\boldsymbol{x}^{(0)}=\nabla^r\boldsymbol{x}^{(r)}=\boldsymbol{A}^{-r}\boldsymbol{x}^{(r)}$, where $\boldsymbol{A}^{-r}$ is the r-order IFAGO matrix. The process of proof is omitted.

2.2 Establishment of CFGMC(q,N) model

Definition 5 [35] The equation

$${}_0^C D_t^q x(t)=\frac{1}{\Gamma(m_2-q)}\int_0^t x^{(m_2)}(\tau)(t-\tau)^{m_2-q-1}\mathrm{d}\tau \tag{4}$$

is the Caputo fractional derivative of $x(t)$, where q is the order of the derivative, $m_2=[q]+1$ is an integer, $[q]$ is the smallest integer greater than or equal to q, $x(t)$ is a differentiable function, $\Gamma(\cdot)$ is the Gamma function, and τ is a variable that is different from t.

The original GM$(1,N)$ model cannot be used for prediction because it lacks control coefficients. Tian [36] added a control variable u to the GM$(1,N)$ model and established the grey multivariate model with the convolution integral GMC$(1,N)$ which can be used for prediction. A series of new forms of GMC$(1,N)$ have been successfully used for prediction. This paper uses the same method of adding control coefficients to make the new model suitable for prediction.

Definition 6 Let $\boldsymbol{x}_1^{(0)}=(x_1^{(0)}(1),x_1^{(0)}(2),\dots,x_1^{(0)}(n))$ be a sequence of system characteristic data and $\boldsymbol{x}_i^{(0)}=(x_i^{(0)}(1),x_i^{(0)}(2),\dots,x_i^{(0)}(n))$ $(i=2,\dots,N)$ be a sequence of system factor data. $\boldsymbol{x}_1^{(r)}$ and $\boldsymbol{x}_i^{(r)}$ are the r-order FAGO sequences of $\boldsymbol{x}_1^{(0)}$ and $\boldsymbol{x}_i^{(0)}$, respectively. The differential equation

$${}_0^C D_t^q x_1^{(r)}(t)+ax_1^{(r)}(t)=\sum_{i=2}^{N}b_i x_i^{(r)}(t)+u \tag{5}$$

is called the whitening differential equation of CFGMC(q,N) model, where $0<q\leqslant 1$, a and b_i are model parameters, ${}_0^C D_t^q x_1^{(r)}(t)$ is the Caputo fractional derivative of $x_1^{(r)}(t)$ and u is the grey action coeffient.

The grey prediction model has the properties of difference, differential, and exponential compatibility. In the CFGMC(q,N) model, $x_1^{(r)}(t)$ is a continuous function. The fractional differential equation can be understood as the limit form of the fractional difference equation because the difference is an approximate calculation of the differential. Therefore, to obtain the values of the model parameters, the continuous fractional differential equation can be discretized into a difference equation.

Definition 7 [35] The equation

$${}_0^C\nabla_t^q x(t)\triangleq {}_0\nabla_t^{-(m_2-q)}[\nabla^{m_2}x(t)] \tag{6}$$

is the q-order Caputo fractional difference of $x(t)$, where ${}_0\nabla_t^{-(m_2-q)}$ is the m_2-q order sum of $\nabla^{m_2}x(t)$.

Given that ${}_0^C D_t^q x_1^{(r)}(t)$ is the Caputo fractional derivative, ${}_0^C D_t^q x_1^{(r)}(t)$ is replaced by Caputo fractional difference. According to (6), the Caputo fractional difference of $x_1^{(r)}(k)$ is ${}_1^C\nabla_k^q x_1^{(r)}(k)\triangleq {}_1\nabla_k^{-m_2+q}[\nabla^{m_2}x_1^{(r)}(k)]$. Here, because $0\leqslant q<1$, that is $m_2=1$, so it can be written as

$${}_0^C D_t^q x_1^{(r)}(t)\approx {}_1^C\nabla_k^q x_1^{(r)}(k)\triangleq {}_1\nabla_k^{-1+q}[\nabla x_1^{(r)}(k)]. \tag{7}$$

According to the definition of the integer-order difference, we can get

$$\begin{gathered}{}_1\nabla_k^{-1+q}[\nabla x_1^{(r)}(k)]= \\ {}_1\nabla_k^{-1+q}[x_1^{(r)}(k)-x_1^{(r)}(k)]= \\ {}_1\nabla_k^{-1+q}x_1^{(r-1)}(k)={}_1\nabla_k^{-1-q}x_1^{(r-1)}(k).\end{gathered} \tag{8}$$

Because $1-q>0$, ${}_1\nabla_k^{-(1-q)}x_1^{(r-1)}(k)$ represents the $1-q$ order sums of $x_1^{(r-1)}(k)$. According to (1),

$${}_1\nabla_k^{-(1-q)}x_1^{(r-1)}(k)=\sum_{t=1}^{k}\begin{bmatrix} 1-q \\ k-t \end{bmatrix}x_1^{(r-1)}(t), \tag{9}$$

because $\boldsymbol{x}_1^{(r-1)}=\boldsymbol{A}^{r-1}\boldsymbol{x}_1^{(0)}$. $\boldsymbol{A}$ is the FAGO matrix, so

$$\sum_{t=1}^{k}\begin{bmatrix} 1-q \\ k-t \end{bmatrix}x_1^{(r-1)}(t)=\sum_{t=1}^{k}\begin{bmatrix} 1-q \\ k-t \end{bmatrix}\boldsymbol{A}^{r-1}\boldsymbol{x}_1^{(0)}=$$

$$\boldsymbol{A}^{1-q}\boldsymbol{A}^{r-1}\boldsymbol{x}_1^{(0)} = \boldsymbol{A}^{r-q}\boldsymbol{x}_1^{(0)} = \boldsymbol{x}_1^{(r-q)}. \tag{10}$$

The q-order Caputo fractional difference of $x_1^{(r)}(k)$ can be obtained as ${}_1^C\nabla_k^q x_1^{(r)}(k) \triangleq x_1^{(r-q)}(k)$. Then

$${}_0^C D_t^q x_1^{(r)}(t) \approx {}_1^C\nabla_k^q x_1^{(r)}(k) \triangleq x_1^{(r-q)}(k). \tag{11}$$

Hence, the definition of the CFGMC(q,N) model is given as Definition 8.

Definition 8 The equation

$$x_1^{(r-q)}(k) + a z_1^{(r)}(k) = \sum_{i=2}^{N} b_i x_i^{(r)}(k) + u \tag{12}$$

is called the definition form of the CFGMC(q,N) model, where $\boldsymbol{z}_1^{(r)} = (z_1^{(r)}(2), z_1^{(r)}(3), \ldots, z_1^{(r)}(n))$ is the sequence of the neighboring mean of $\boldsymbol{x}_1^{(r)}$, and $\boldsymbol{x}_1^{(r-q)} = \boldsymbol{A}^{-q}\boldsymbol{x}_1^{(r)}$.

Theorem 4 The estimated parameter column $\widehat{\boldsymbol{P}} = [a, b_2, b_3, \cdots, b_N, u]^{\mathrm{T}}$ of the model of (12) obtained by the least square method is satisfied:

$$\widehat{\boldsymbol{P}} = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B})^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{Y} \tag{13}$$

where

$$\boldsymbol{Y} = \begin{bmatrix} x_1^{(r-q)}(2) & x_1^{(r-q)}(3) & \cdots & x_1^{(r-q)}(n) \end{bmatrix}_{1\times(n-1)}^{\mathrm{T}}$$

$$\boldsymbol{B} = \begin{bmatrix} -z_1^{(r)}(2) & x_2^{(r)}(2) & \cdots & x_N^{(r)}(2) & 1 \\ -z_1^{(r)}(3) & x_2^{(r)}(3) & \cdots & x_N^{(r)}(3) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -z_1^{(r)}(n) & x_2^{(r)}(n) & \cdots & x_N^{(r)}(n) & 1 \end{bmatrix}_{(n-1)\times(n+1)}$$

$$\widehat{\boldsymbol{P}} = \begin{bmatrix} a & b_2 & b_3 & \cdots & b_N & u \end{bmatrix}_{1\times(n+1)}^{\mathrm{T}}. \tag{14}$$

The process of proof is omitted.

2.3 Solving CFGMC(q,N) model by using Caputo fractional Laplace transform

In view of the superiority of Laplace transform in solving differential equations, the fractional Laplace transform method is used to solve (5).

Definition 9 [35] Laplace transform formula of Caputo fractional derivatives is

$$L({}_0^C D_t^q x(t)) = s^q X(s) - \sum_{k=0}^{m_2-1} s^{q-k-1} x^{(k)}(0) \tag{15}$$

where L is the sign of the Laplace transform, s is the variable of the Laplace transform function, and $X(s) = L(x(t))$ is the Laplace transform of $x(t)$.

Theorem 5 The solution of the CFGMC(q,N) model is presented as follows:

$$\widehat{x}_1^{(r)}(k) = \left[\frac{x_1^{(r)}(1) - M(k)Q_{k=1}}{G_{k=1}}\right] G(k) + M(k)k^q Q(k), \tag{16}$$

where

$$\begin{cases} M(k) = \displaystyle\sum_{i=2}^{N} b_i x_i^{(r)}(k) + u \\ G(k) = \displaystyle\sum_{h=0}^{\infty} \frac{(-ak^q)^h}{\Gamma(qh+1)} \\ Q(k) = \displaystyle\sum_{h=0}^{\infty} \frac{(-ak^q)^h}{\Gamma(qh+q+1)} \end{cases} \tag{17}$$

where h is the number of terms in the series.

Proof Given that $\sum_{i=2}^{N} b_i x_i^{(r)}(k) + u$ of (5) does not contain the unknown function $x_1^{(r)}(k)$, then it can be recorded as a grey variable $M(k)$, that is, $\sum_{i=2}^{N} b_i x_i^{(r)}(k) + u = M(k)$. Equation (15) is used to transform both sides of (5). Then

$$s^q X(s) - s^{q-1} x_1^{(r)}(0) + aX(s) = \frac{M(k)}{s}. \tag{18}$$

The following can be obtained by solving (18):

$$X(s) = \frac{\dfrac{M(k)}{s} + s^{q-1} x_1^{(r)}(0)}{s^q + a}.$$

The Laplace inverse transform of $X(s)$ is expressed as follows:

$$\widehat{x}_1^{(r)}(k) = L^{-1}[X(s)] = L^{-1}\left[\frac{\dfrac{M(k)}{s} + s^{q-1} x_1^{(r)}(0)}{s^q + a}\right] = M(k)L^{-1}\left[\frac{1}{s(s^q+a)}\right] + x_1^{(r)}(0)L^{-1}\left[\frac{s^{q-1}}{s^q+a}\right]. \tag{19}$$

According to the relationship of Laplace inverse transform and Mittage-leffler function, the following results can be obtained as

$$L^{-1}\left[\frac{1}{s(s^q+a)}\right] = L^{-1}\left(\frac{s^{-1}}{s^q+a}; s\right) = k^q E_{q,q+1}(-ak^q) = k^q \sum_{h=0}^{\infty} \frac{(-ak^q)^h}{\Gamma(qh+q+1)}, \tag{20}$$

$$L^{-1}\left(\frac{x_1^{(r)}(0)s^{q-1}}{s^q+a}; s\right) = x_1^{(r)}(0)E_{q,1}(-ak^q) =$$

$$x_1^{(r)}(0)\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}. \tag{21}$$

Substituting (20) and (21) into (19) yields

$$\widehat{x}_1^{(r)}(k)=x_1^{(r)}(0)\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}+$$
$$M(k)k^q\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+q+1)}. \tag{22}$$

Let $\widehat{x}_1^{(r)}(1)=x_1^{(r)}(1)$, then

$$x_1^{(r)}(0)=\frac{x_1^{(r)}(1)-M(k)\displaystyle\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+q+1)}}{\displaystyle\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+1)}}. \tag{23}$$

Substituting (23) into (22) yields

$$\widehat{x}_1^{(r)}(k)=\left[\frac{x_1^{(r)}(1)-M(k)\displaystyle\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+q+1)}}{\displaystyle\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+1)}}\right].$$
$$\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}+M(k)k^q\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+q+1)}. \tag{24}$$

Let

$$\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}=G(k),$$
$$\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+q+1)}=Q(k),$$
$$\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+1)}=G_{k=1},$$

and

$$\sum_{h=0}^{\infty}\frac{(-a)^h}{\Gamma(qh+q+1)}=Q_{k=1}.$$

Then,

$$\widehat{x}_1^{(r)}(k)=\left[\frac{x_1^{(r)}(1)-M(k)Q_{k=1}}{G_{k=1}}\right]G(k)+$$
$$M(k)k^qQ(k). \qquad \square$$

Because the solution contains four series. The convergence and divergence of these series determine whether the solution exists. Therefore, discussing the convergence and divergence of such series is necessary. Theorem 6 takes $\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}$ as an example to illustrate this problem.

Theorem 6 Series $\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}$ is convergent throughout the domain of definition.

Proof The deformation of series is

$$\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}=$$
$$\sum_{h=0}^{\infty}\frac{(-a)^h(k^q)^h}{\Gamma(qh+1)}\overset{\text{let }k^q=t}{=\!=\!=}\sum_{h=0}^{\infty}\frac{(-a)^h(t)^h}{\Gamma(qh+1)}. \tag{25}$$

$\sum_{h=0}^{\infty}\frac{(-ak^q)^h}{\Gamma(qh+1)}$ is a power series, and its coefficient is $a_n=\frac{(-a)^h}{\Gamma(qh+1)}$. According to the convergence criterion of the series, the following can be obtained:

$$\lim_{h\to\infty}\sqrt[h]{\left|\frac{(-a)^h}{\Gamma(qh+1)}\right|}=|a|\lim_{h\to\infty}\frac{1}{\sqrt[h]{\Gamma(qh+1)}}. \tag{26}$$

The denominator limit of (26) is

$$\lim_{h\to\infty}\sqrt[h]{\Gamma(qh+1)}=\lim_{h\to\infty}\mathrm{e}^{\ln(\Gamma(qh+1))^{\frac{1}{h}}}=$$
$$\mathrm{e}^{\lim\limits_{h\to\infty}\frac{\ln(\Gamma(qh+1))}{h}}. \tag{27}$$

According to the relationship between Gamma function and double Gamma function $\frac{\mathrm{d}\ln(\Gamma(qh+1))}{\mathrm{d}h}=\psi(qh+1)$, then

$$\lim_{h\to\infty}\frac{\ln(\Gamma(qh+1))}{h}=\lim_{h\to\infty}\frac{\mathrm{d}\ln(\Gamma(qh+1))}{\mathrm{d}h}=$$
$$\lim_{h\to\infty}\psi(qh+1). \tag{28}$$

According to $\psi(x+1)=\psi(x)+\frac{1}{x}$,

$$\lim_{h\to\infty}\psi(qh+1)=\lim_{h\to\infty}\psi(qh)+\lim_{h\to\infty}\frac{1}{qh}=\lim_{h\to\infty}\psi(qh). \tag{29}$$

The image of double Gamma function shows that $\lim\limits_{h\to\infty}\psi(qh)=+\infty$, $\lim\limits_{h\to\infty}\frac{\ln(\Gamma(qh+1))}{h}=+\infty$, and $\lim\limits_{h\to\infty}\sqrt[h]{\Gamma(qh+1)}=\infty$, so

$$\lim_{h\to\infty}\sqrt[h]{\left|\frac{(-a)^h}{\Gamma(qh+1)}\right|}=|a|\lim_{h\to\infty}\frac{1}{\sqrt[h]{\Gamma(qh+1)}}=0. \tag{30}$$

The convergence radius $R=(-\infty,\infty)$, that is $\sum_{k=0}^{\infty}\frac{(-at^p)^k}{\Gamma(pk+1)}$ is convergent over the entire domain. $\square$

Similarly, other series converge. It shows that the solution of the CFGMC(q,N) model exists and the model is feasible.

The predicted value of $x_1^{(r)}(k)$ is $\widehat{x}_1^{(r)}(k)$ from Theorem 5, and the predicted value $\widehat{x}_1^{(0)}(k)$ of $x_1^{(0)}(k)$ can be obtained from the IFAGO formula in Theorem 3.

2.4 Relationship of CFGMC(q, N) model and other grey models

(i) When $r=1$, $q=1$, $N=1$, and $u \neq 0$, (5) can be recorded as $\dfrac{\mathrm{d}x_1^{(1)}(t)}{\mathrm{d}t}+ax_1^{(1)}(t)=u$. This form is consistent with the GM(1,1) model.

(ii) When $r=1$, $q=1$, $N>1$, if $u=0$, (5) can be recorded as $\dfrac{\mathrm{d}x_1^{(1)}(t)}{\mathrm{d}t}+ax_1^{(1)}(t)=\sum\limits_{i=2}^{N} b_i x_i^{(1)}(t)$, and this form is consistent with the GM$(1,N)$ model; if $u \neq 0$, (5) can be recorded as $\dfrac{\mathrm{d}x_1^{(1)}(t)}{\mathrm{d}t}+ax_1^{(1)}(t)=\sum\limits_{i=2}^{N} b_i x_i^{(1)}(t)+u$, and it is the GMC$(1,N)$ model.

(iii) When $0<r<1$, $q=1$, $u \neq 0$, and $N=1$, (5) can be recorded as $\dfrac{\mathrm{d}x_1^{(r)}(t)}{\mathrm{d}t}+ax_1^{(r)}(t)=u$. This form is consistent with the fractional grey model FGM(1,1).

(iv) When $0<r<1$, $q=1$, $u \neq 0$, and $N>1$, (5) can be recorded as $\dfrac{\mathrm{d}x_1^{(r)}(t)}{\mathrm{d}t}+ax_1^{(r)}(t)=\sum\limits_{i=2}^{N} b_i x_i^{(r)}(t)+u$. This form is consistent with the fractional multivariate grey model with the convolution integral FGMC$(1,N)$.

Obviously, the CFGMC(q, N) model is the generalization of the classical GM(1,1) model, GM$(1,N)$ model, GMC$(1,N)$ model, FGM(1,1) model, and FGMC$(1,N)$ model.

2.5 PSO for determining the order r of FAGO and the order q of differential equation

Fractional derivative and FAGO are the core of the CFGMC(q, N) model. The aforementioned analysis and calculation should be conducted when r and q are known. Therefore, selecting the appropriate r and q for the model is greatly important. In this study, the PSO algorithm is used to determine the most suitable r and q to ensure the minimum mean absolute percentage error (MAPE).

3. Case analysis: predicting the yields of MSW of Wuhan, China

In this section, the validity of the CFGMC(q, N) model is verified and compared to the other grey models. The MAPE(%), mean absolute error (MAE) (10 000 t), root mean square error (RMSE) (10 000 t), and absolute percent error (APE)(%) are used to assess the modeling effect. The units of these indicators are uniform throughout the text. They are defined as follows:

$$\mathrm{MAPE}=\frac{1}{n-1}\sum_{k=2}^{n}\frac{|\widehat{x}^{(0)}(k)-x^{(0)}(k)|}{x^{(0)}(k)}\times 100\%, \quad (31)$$

$$\mathrm{MAE}=\frac{1}{n}\sum_{i=1}^{n}|\widehat{x}^{(0)}(k)-x^{(0)}(k)|, \quad (32)$$

$$\mathrm{RMSE}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}|\widehat{x}^{(0)}(k)-x^{(0)}(k)|^2}, \quad (33)$$

$$\mathrm{APE}=\frac{|\widehat{x}^{(0)}(k)-x^{(0)}(k)|}{x^{(0)}(k)}\times 100\%. \quad (34)$$

The MSW yields in China are huge and growing rapidly. The prediction of MSW yields is a small sample and uncertain system. The Ministry of Construction promulgated the industry standard "Method for calculating and forecasting the MSW yields" (CJ/T106-2016) [37], which clearly pointed out that the prediction of the MSW yields should be "based on the MSW yields of 6 – 8 consecutive years starting from the next year of the forecast year". Too much data can substitute outdated information and affect the quality of the model. The main influencing factors contain both known and unknown uncertain information and belong to an uncertain system. These two characteristics determine that the prediction of MSW yields must choose a prediction model that can deal with small samples, uncertainty and diversification. From Fig. 1, we can see that after 2012, the growth rate accelerates and the range of data fluctuation is very large. Faced with such a large range of data, traditional forecasting methods are ineffective, while the CFGMC(q, N) model has the ability to process such data. In this paper, the CFGMC(q, N) model is established based on the data of MSW yields in Wuhan from 2006 to 2012, and the MSW yields from 2013 to 2017 are predicted. Data are from Wuhan Statistical Yearbook (2006—2017).

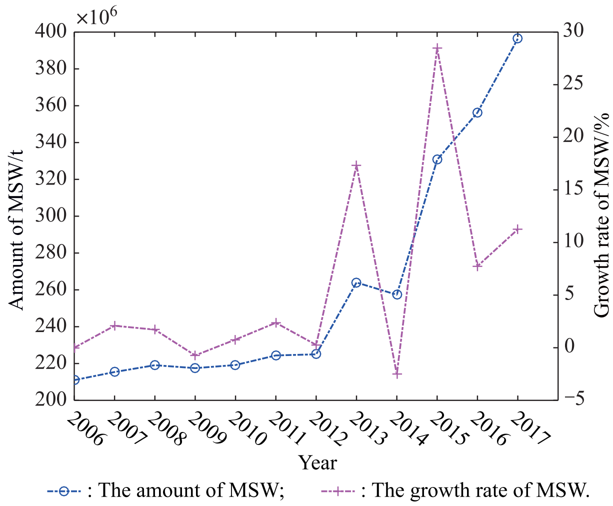

Fig. 1 Amount and growth rate of MSW produced in Wuhan in 2006—2017

3.1 Selection of influencing factors

The selection of influencing factors has an important impact on the results and rationality of modeling. In [38], the characteristics, influencing factors and components of MSW were compared and analyzed. Results showed that the main indicators of influencing factors to MSW were all of economic development levels, population and investments of government input. Based on the results of [38] and the characteristics of urban development in Wuhan, this paper takes five factors as input factors, namely, resident population at the end of the year, road sweeping area, passenger accommodation and passenger volume, per capita net income and resident consumption index. See Table 2 for the original data on the input variables. Taking the MSW yields as the output variable, the CFGMC(q, N) model is established and compared with other models.

Table 2 Original data of MSW yields and its influencing factors

Year	$y1$ MSW yields/ 10 000 tons	$y2$ Resident population/ 10 000 people	$y3$ Road sweeping area/ 100 000 m^2	$y4$ Passenger capacity/ 10 000 people	$y5$ Per capita net income/yuan	$y6$ Resident consumption index/yuan
2006	211.00	875.00	4 009.00	16 193.70	4 748.00	9 182.10
2007	215.40	891.00	5 414.00	17 338.40	5 371.00	10 600.00
2008	219.10	897.00	5 994.00	18 882.10	6 349.00	11 433.00
2009	217.50	910.00	6 479.00	21 735.60	7 161.00	12 710.00
2010	219.20	978.50	6 640.00	22 896.70	8 295.00	14 490.10
2011	224.40	1 002.00	7 130.00	25 743.20	9 814.00	17 141.00
2012	225.00	1 012.00	8 857.00	27 492.20	11 190.00	18 813.10
2013	264.00	1 022.00	10 040.00	29 621.69	12 713.00	20 157.30
2014	257.36	1 033.80	15 837.00	27 899.48	16 160.00	22 002.20
2015	330.66	1 060.77	17 730.00	27 628.71	17 722.00	23 943.05
2016	356.29	1 076.62	12 486.00	29 177.15	19 152.00	26 535.00
2017	396.38	1 089.29	13 853.00	29 950.30	20 887.00	28 546.00

3.2 Model comparison and evaluation

(i) Analysis of test results: The fitting and prediction results are shown in Table 3. In order to more intuitively distinguish the differences among the three test indicators, the histogram Fig. 2 of the three test indicators modeling stages and the histogram Fig. 3 of the prediction stage are drawn. It can be seen that although the four models have good results in simulating, and their MAPE is less than 5%, the prediction results are quite different. The test index values of the CFGMC(0.6,6) model are smaller than the other three models.

Table 3 Comparison of Wuhan's MSW by CFGMC(0.6,6) model, GM(1,6) model, FGMC(1,6) model, and MLR model

Year or Index	$y1$ MSW	CFGMC(q, N) r=0.5	q=0.6	GM$(1, N)$ r=1	q=1	FGMC$(1, N)$ r=0.8	q=1	MLR	
		Fitting	APE	Fitting	APE	Fitting	APE	Fitting	APE
2006	211.00	211.00	0.00%	211.00	0.00%	211.00	0.00%	212.24	0.59%
2007	215.40	220.32	2.28%	194.01	9.93%	206.37	4.19%	215.97	0.27%
2008	219.10	220.50	0.64%	250.54	14.35%	211.73	3.36%	216.69	1.10%
2009	217.50	216.21	0.59%	231.33	6.36%	211.63	2.70%	217.95	0.21%
2010	219.20	223.69	2.05%	223.93	2.16%	214.12	2.32%	219.42	0.10%
2011	224.40	238.06	6.09%	226.38	0.88%	219.62	2.13%	222.87	0.68%
2012	225.00	234.22	4.10%	225.42	0.19%	220.57	1.97%	226.45	0.65%
MAPE			2.25%		4.84%		2.38%		**0.51%**
MAE			4.99		10.54		5.22		**1.12**
RMSE			6.76		15.42		5.84		**1.33**
2013	264.00	237.37	10.09%	188.04	28.77%	218.87	17.10%	228.16	13.58%
2014	257.36	213.46	17.06%	131.75	48.81%	214.52	16.65%	236.10	8.26%
2015	330.66	217.06	34.35%	141.09	57.33%	208.77	36.86%	240.86	27.16%
2016	356.29	366.26	2.80%	39.13	89.02%	217.71	38.90%	237.29	33.40%
2017	396.38	393.80	0.65%	22.86	94.23%	234.59	40.82%	241.07	39.18%
MAPE			**12.99%**		63.63%		30.06%		24.32%
MAE			**39.34**		216.36		102.05		84.24
RMSE			**55.94**		243.96		113.23		98.07

(ii) Trend of prediction curve and analysis of its reasons: In order to more intuitively reflect the difference degree of the model effect and further analyze the causes, the line maps Figs. 4 – 7 of the four models in the modeling and prediction stages are drawn. Fig. 4 shows that although the multiple linear regression (MLR) model has the best fitting effect, its prediction effect is unsatisfactory. Its forecast trend of MSW is linear, which is obviously not in line with reality. The reason is that the MLR model only has better prediction results when dealing with stationary series. Fig. 5 shows that the trend predicted by the GM(1,6) model is decreasing year by year, which is opposite to the actual trend. The reason is that the order of AGO and derivative are fixed, and cannot be adjusted according to the actual situation of data types. Fig. 6 shows that the FGMC(1,6) model is closer to the original data sequence than that of the GM(1,6) model. The reason is that the order of AGO of the FGMC$(1, N)$ model is adjustable. Fig. 7 shows that the trend chart of the CFGMC(0.6,6) model is closer to the original curve than other models. The reason is that the CFGMC(q, N) model not only has better prediction effect on the stationary sequence, but also has more unique advantages for non-stationary sequences. Obviously, it is necessary and meaningful to extend AGO and integer grey derivatives to the fractional order.

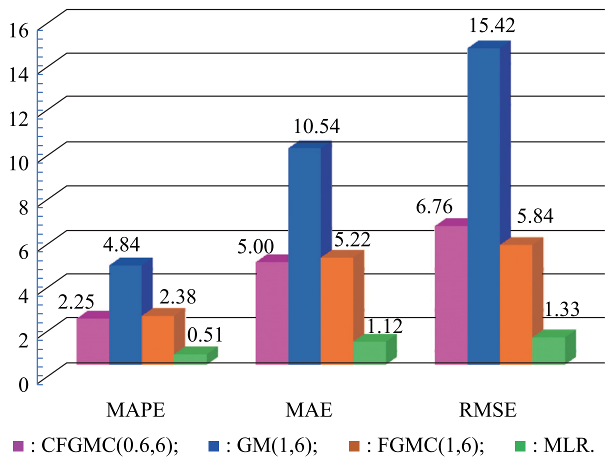

Fig. 2 Comparison of fitting results

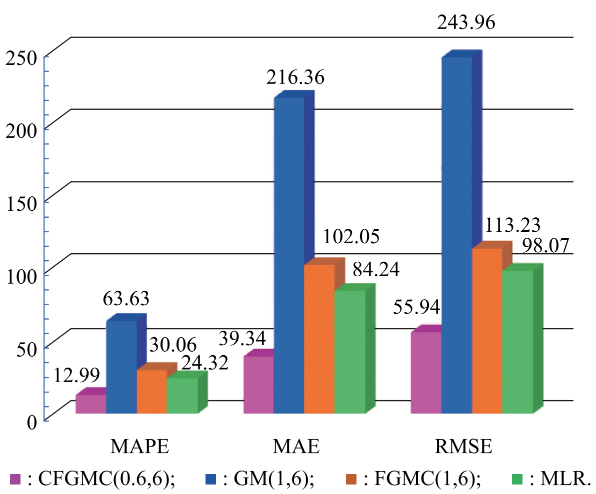

Fig. 3 Comparison of prediction results

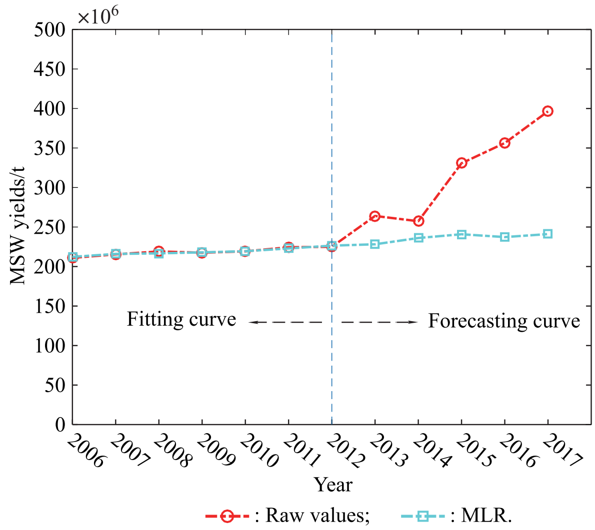

Fig. 4 Curve of the MLR model and original data

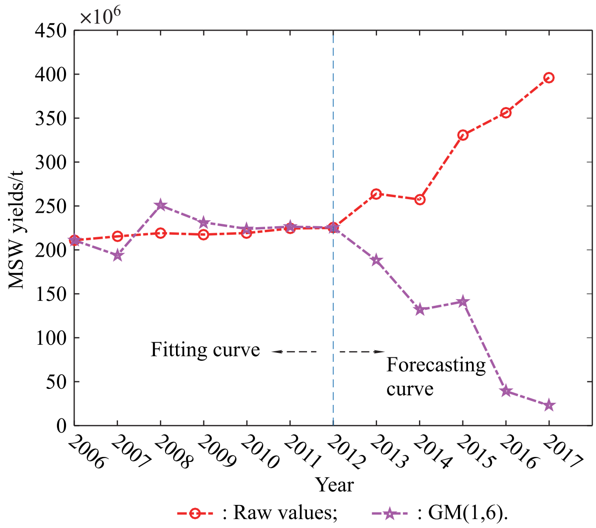

Fig. 5 Curve of the GM(1,6) model and original data

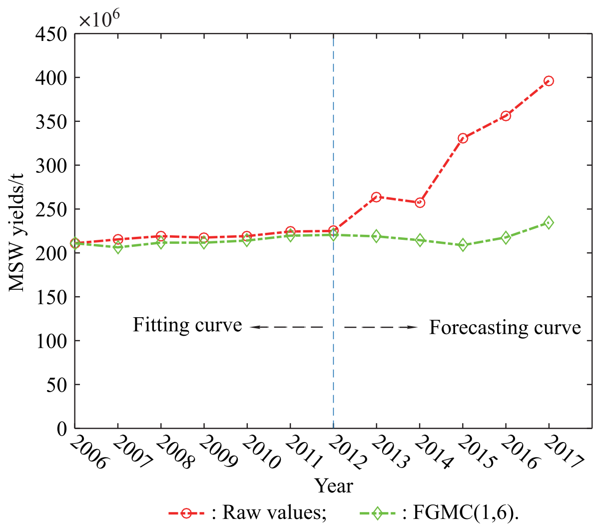

Fig. 6 Curve of the FGMC(1,6) model and original data

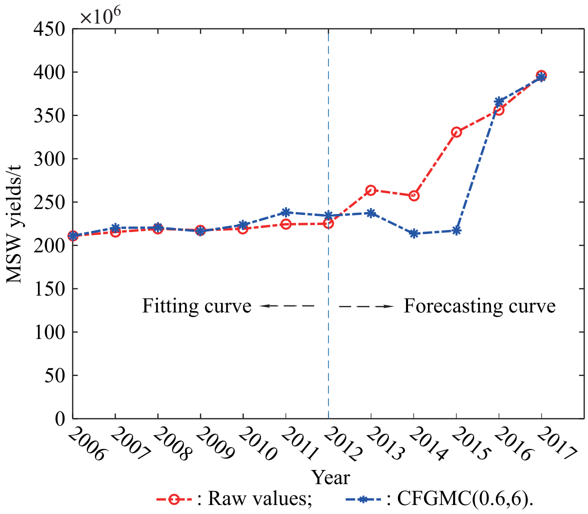

Fig. 7 Curve of the CFGMC(0.6,6) model and original data

4. Conclusions

The main contents of this paper include the following five aspects. First, FAGO and IFAGO are directly defined by fractional sums and the difference theory. Second, the type of fractional derivative used in the model is pointed out. Third, the CFGMC(q, N) model is established by combining FAGO and fractional derivative. Fourth, the analytical solution of the model is obtained. Fifth, the existence of the analytical solution is proven. The model is used to forecast the MSW yields. Through the work, we can get the following conclusions:

(i) The order of FAGO and differential equation have important influence on the accuracy of the model. The fractional grey model is a general case of the integral grey model.

(ii) The CFGMC(q, N) model can adjust the order of FAGO and the differential equation according to the data characteristics of practical problems, so that the prediction of non-stationary series can also have satisfactory results.

(iii) The establishment of the CFGMC(q, N) model extends the form and application scope of the multivariate grey prediction model, and improves the accuracy of the model to a certain extent.

(iv) FAGO and IFAGO, which are defined directly by the theory of fraction sum and difference, do not need to be transformed indirectly. They are intuitionistic, clear and accurate.

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