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Method for triangular fuzzy multiple attribute decision making based on two-dimensional density operator method

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Abstract: Aiming at the triangular fuzzy (TF) multi-attribute decision making (MADM) problem with a preference for the distribution density of attribute (DDA), a decision making method with TF number two-dimensional density (TFTD) operator is proposed based on the density operator theory for the decision maker (DM). Firstly, a simple TF vector clustering method is proposed, which considers the feature of TF number and the geometric distance of vectors. Secondly, the least deviation sum of squares method is used in the program model to obtain the density weight vector. Then, two TFTD operators are defined, and the MADM method based on the TFTD operator is proposed. Finally, a numerical example is given to illustrate the superiority of this method, which can not only solve the TF MADM problem with a preference for the DDA but also help the DM make an overall comparison.

Keywords: fuzzy decision making, clustering, density operator, multi-attribute decision making (MADM).

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1. Introduction

Many decision problems can be formulated as multi-criteria problems [1], and multi-attribute decision making (MADM) for incomplete information systems is an important part of modern decision science. With the development of society, research problems become complex, the decision making information is difficult to be accurate, the simple decision making theory cannot meet the practical needs, the fuzzy processing method is used to define the business function [2]. Due to more uncertainty in the decision making process, the decision making basis and decision making index system are complex and diverse [3]. Therefore, how to make scientific and effective decision for the decision maker (DM) under the uncertain and incomplete situations has become an important research topic and attracted wide attention [4].

For the fuzzy MADM problem, various methods have been concerned, i.e., ordered weighted averaging (OWA) aggregation operator [5,6], probabilistic approach [7], fuzzy multi-criteria group decision making (FMCGDM) method [8,9], technique for order preference by similarity to ideal solution (TOPSIS) method [10–12], extended compromise ratio method [13], Bonferroni mean operator [14,15], multi-objective optimization by ratio analysis plus the full multiplicative form (MULTIMOORA) method [16,17], Tomada de deciso interativa multicriterio (TODIM) method [18,19], vlsekriterijumska optimizacija I kompromisno resenje (VIKOR) method [20–22], triangular fuzzy number certitude degree (TFNCD) operator method [23], similarity programming model [24] and best-worst (BW) method [25], the superiority and inferiority ranking (SIR) method [26], and the hesitant fuzzy grey relational analysis (GRA) method [27]. However, the foregoing studies mostly focus on how to develop advanced MADM method without paying much attention to the density degree of attribute information distribution. Yi et al. presented the density intermediate operator while considering the degree of information consistency in the data group [28]. Thereafter, Liu et al. extended the density operator to the form of triangular intuitionistic fuzzy numbers [29], and Lin et al. proposed the intuitionistic fuzzy number density operator [30]. Zhang et al. developed the two-dimensional density operator and applied it to the group evaluation [31]. Clustering is a key step in the application of density operator. Li et al. pointed out most clustering algorithms cannot solve the clustering problem of uncertain information samples, and gave a clustering algorithm containing multiple triangular fuzzy (TF) normal random vari-

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ables [32].

Despite the fact that the existing researches have made important contribution to MADM studies, some insufficiencies still exist. Firstly, the two-dimensional density operator is different from the one-dimensional density operator in calculating, there is still no special research on TF number two-dimensional density (TFTD) operator. Secondly, the key problem on the application of TFTD operator is how to cluster the vector sets, but the clustering algorithm on multiple TF normal random variables are hard to deal with. Thirdly, the properties of the TFTD operator are seldom discussed. Therefore, the systematic study on the TFTD operator is valuable.

Given the above, a novel method for TF MADM based on two-dimensional density operator is developed. The main contributions are as follows.

(i) TFTD operator is proposed to solve the MADM problem in some circumstance.

(ii) A clustering algorithm on multiple TF normal random variables is presented.

(iii) The properties of the TFTD operator are given.

This paper is organized as follows. In Section 2, some definitions are formulated. In Section 3, the MADM based on TFTD operator is given. Section 4 presents an application example. Finally, the conclusions are summarized in Section 5.

2. Preliminaries

Definition 1 [8] Let X be a non-empty set.

$$A = \{< x, u_A(x) > | x \in X\} \tag{1}$$

A fuzzy set of A in X is called the TF set, in which the membership degree of A is defined as $u_A(x)$, which satisfies the condition $0 \leqslant a_l \leqslant \alpha_m \leqslant \alpha_u \leqslant 1$, a_l, α_u stand for the lower and upper value, and α_m for the modal value of the support of A respectively.

$$u_A(x) = \begin{cases} (x-\alpha_l)/(\alpha_m-\alpha_l), & \alpha_l \leqslant x < \alpha_m \\ (\alpha_u - x)/(\alpha_u-\alpha_m), & \alpha_m \leqslant x \leqslant \alpha_u \\ 0, \quad \text{others} \end{cases} \tag{2}$$

where $0 \leqslant \alpha_l \leqslant \alpha_m \leqslant \alpha_u \leqslant 1$.

$\hat{a}=(\alpha_l, \alpha_m, \alpha_u)$ is called a TF number, as shown in Fig. 1.

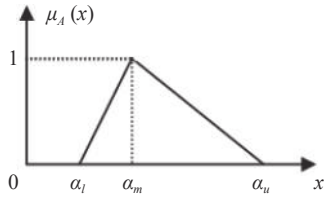

Fig. 1 Membership function of the TF number $\hat{a}$

Definition 2 [33] For the TF numbers $\hat{a}=(\alpha_l, \alpha_m, \alpha_u)$, $\hat{o}=(o_l, o_m, o_u)$ and crisp number k, the following operation rules are defined:

$$\hat{a} + \hat{o} = (\alpha_l + o_l, \alpha_m + o_m, \alpha_u + o_u), \tag{3}$$

$$\hat{a} \cdot \hat{o} = (\alpha_l \cdot o_l, \alpha_m \cdot o_m, \alpha_u \cdot o_u), \tag{4}$$

$$k \cdot \hat{a} = (k \cdot \alpha_l, k \cdot \alpha_m, k \cdot \alpha_u), \tag{5}$$

$$1/\hat{a} = (1/\alpha_u, 1/\alpha_m, 1/\alpha_l). \tag{6}$$

Definition 3 [33] $d(\hat{a}, \hat{o})$ is defined as the distance between the TF numbers $\hat{a}$ and $\hat{o}$,

$$d(\hat{a}, \hat{o}) = (|a_l - o_l| + 2|a_m - o_m| + |a_u - o_u|)/4 \tag{7}$$

where $|a_l - o_l|$ represents the absolute value of the difference between α_l and o_l.

$d_{\hat{a}}$ is used to represent the closeness of the TF number $\hat{a}$, which is relative to the TF number $\hat{o}=(0,0,0)$ and the TF number $\hat{e}=(1,1,1)$,

$$d_{\hat{a}} = d(\hat{a}, \hat{o})/d(\hat{e}, \hat{o}). \tag{8}$$

The larger $d_{\hat{a}}$, the larger $\hat{a}$.

Let $\hat{A}=\{\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n\}$ be a set consisting of n TF numbers, and $\boldsymbol{A}_a=(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$ is called the TF vector.

For $\boldsymbol{A}_a=(\hat{a}_1, \hat{a}_2, \cdots, \hat{a}_n)$, $\boldsymbol{A}_o=(\hat{o}_1, \hat{o}_2, \cdots, \hat{o}_n)$ and the crisp number k, the operation rules are defined:

$$\boldsymbol{A}_a + \boldsymbol{A}_o = (\hat{a}_1 + \hat{o}_1, \hat{a}_2 + \hat{o}_2, \cdots, \hat{a}_n + \hat{o}_n), \tag{9}$$

$$k \cdot \boldsymbol{A}_a = (k \cdot \hat{a}_1, k \cdot \hat{a}_2, \cdots, k \cdot \hat{a}_n). \tag{10}$$

Definition 4 $d(\boldsymbol{A}_a, \boldsymbol{A}_o)$ is the dissimilarity degree between two TF vectors $\boldsymbol{A}_a$ and $\boldsymbol{A}_o$,

$$d(\boldsymbol{A}_a, \boldsymbol{A}_o) = \sqrt{d(\hat{a}_1, \hat{o}_1)^2 + d(\hat{a}_2, \hat{o}_2)^2 + \cdots + d(\hat{a}_n, \hat{o}_n)^2} \tag{11}$$

where $d(\hat{a}, \hat{o})$ is the distance measure in Definition 3.

3. MADM based on TFTD operator

3.1 Clustering method

Suppose $\boldsymbol{A}=\{\boldsymbol{A}_{a1}, \boldsymbol{A}_{a2}, \cdots, \boldsymbol{A}_{al}\}$ is a set composed of l TF vectors, each TF vector is composed of n TF numbers, and the dissimilarity degree between two TF vectors can be measured. A clustering method is planned as follows:

Step 1 Calculate the dissimilarity degree of the TF vectors, i.e, the dissimilarity degree between the TF vector $\boldsymbol{A}_{ai}$ and the TF vector $\boldsymbol{A}_{aj}$ is expressed as d_{ij}.

Step 2 Construct the distance matrix $\boldsymbol{D}(d_{ij})$.

$$\boldsymbol{D}(d_{ij}) = (d_{ij})_{l\times l} = \begin{pmatrix} 0 & d_{12} & \cdots & d_{1l} \\ d_{21} & 0 & \cdots & d_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ d_{l1} & d_{l2} & \cdots & 0 \end{pmatrix}$$

Step 3 Draw the tree diagram with the minimum dissimilarity degree. Let $\boldsymbol{A}_{ai}$ (i= 1, 2,$\cdots$, l) represent the position of each point, and then paint the vertex connected graph G=($\boldsymbol{A}_{ai}$, d_{ij}), d_{ij} indicates the dissimilarity degree between $\boldsymbol{A}_{ai}$ and $\boldsymbol{A}_{aj}$. Then, in the graph G($\boldsymbol{A}_{ai}$, d_{ij}) , erase the largest edge in any loop to form a tree until the closed loop no longer exists.

Step 4 Cluster the vector sets. Select a threshold Θ and compare it with each edge in the tree graph. When d_{ij}> Θ, the denoted edges by d_{ij} are removed. Let there exist m subsets, then the TF vector set $\boldsymbol{A}$ is divided into m-group vector sets.

The clustering method has the following characteristics compared with others:

(i) It can cluster TF vectors based on the dissimilarity degree.

(ii) It is more objective because it is not necessary to determine the classification number in advance.

(iii) It is simple and easy to understand.

3.2 Density weighted vector

Suppose there is a vector set $\boldsymbol{A}$ composed of l TF vectors (each vector consists of n spatial elements in the form of TF numbers). Using clustering method in Subsection 3.1, it can be divided into m TF vector subsets, which are sorted, to form a sequential TF vector set.

Assume that k_j is the number of vector groups contained in the sequential vector set, and $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_m$. $\boldsymbol{A}=\{\boldsymbol{A}_1, \boldsymbol{A}_2, \cdots, \boldsymbol{A}_m\}$ is the set of TF vectors sorted in decrease turn, ξ_j represents the density weight of the vector set $\boldsymbol{A}_j$, there is $\xi_1, \xi_2, \cdots, \xi_m$.

Definition 5 λ is the density preference measure corresponding to the density weight $\xi_1, \xi_2, \cdots, \xi_m$.

$$\lambda = \sum_{r=1}^{m}\left(\frac{m-r}{m-1}\xi_r\right) \tag{12}$$

When $\xi_1=\xi_2=\cdots=\xi_m$, λ=0.5. When $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_m$, $\lambda \leqslant 0.5$. When $\xi_m \leqslant \xi_{m-1} \leqslant \cdots \leqslant \xi_1$, $0.5 \leqslant \lambda$. The magnitude of λ reflects the DM's preference as shown in Table 1.

Table 1 Scale table of density preference measure

λ	Feature of ξ_i	Definition
0	ξ_m=1	Complete preference for low information density attributes
(0, 0 .5)	$\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_m$	Prefer attributes with low information density
0 .5	$\xi_1 = \xi_2 = \cdots = \xi_m$	Equal preference for each attribute
(0 .5, 1)	$\xi_m \leqslant \xi_{m-1} \leqslant \cdots \leqslant \xi_1$	Prefer attributes with high information density
1	ξ_1=1	Complete preference for high information density attributes

When the density preference measure λ is greater than 0.5, the DM prefers to "subject information", and the larger the preference measure, the larger the DM prefers to "subject information". Conversely, when the density preference measure λ is less than 0.5, the DM prefers to "individual information", and the smaller the preference measure, the larger the DM prefers to "individual information".

It is not yet possible to calculate each density weight directly according to the density preference measures defined by the DM, then, the least-squares-of-deviation method is used to calculate each density weight.

Definition 6 S is the sum of deviation squares of density weights $\xi_1, \xi_2, \cdots, \xi_m$.

$$S = \sum_{r=1}^{m}\left(\xi_r - \frac{1}{m}\right)^2 \tag{13}$$

Once the density preference measure λ is known, the following model is constructed to calculate the difference between different density weights, and the density weight of each vector can be calculated. The smaller the difference among the density weights corresponding to each vector, the more consistent the information distribution, the smaller the S. Therefore, the objective function can be constructed as follows:

$$\min S = \sum_{r=1}^{m}\left(\xi_r - \frac{1}{m}\right)^2$$
$$\text{s.t.}\begin{cases} \lambda = \displaystyle\sum_{r=1}^{m}\left(\frac{m-r}{m-1}\xi_r\right) \\ \displaystyle\sum_{r=1}^{m}\xi_r = 1 \\ 0 \leqslant \xi_r \leqslant 1 \\ r = 1, 2, \cdots, m \end{cases}. \tag{14}$$

Formula (14) is a conditionally constrained nonlinear programming problem. Let λ be a known number, it is not difficult to calculate (ξ_1,ξ_2,$\cdots$,ξ_m) for the problem by computer tools. When there exists the equal number in these quential vector set, such as k_j=k_{j+1}, it is necessary to adjust each value of ξ_j and ξ_{j+1}. After adjusting, ξ'_j=ξ'_{j+1} = (ξ_j+ξ_{j+1})/2.

Example 1 Let $\boldsymbol{A}=\{\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3\}$, $\boldsymbol{A}_1 = \{\boldsymbol{A}_{a1}, \boldsymbol{A}_{a2}, \boldsymbol{A}_{a3}\}$, $\boldsymbol{A}_2 = \{\boldsymbol{A}_{a4}, \boldsymbol{A}_{a5}\}$,$\boldsymbol{A}_3=\{\boldsymbol{A}_{a6}, \boldsymbol{A}_{a7}\}$, then, k_1=3, k_2=k_3= 2, m=3. If λ is a known number as λ=0.3, we can calculate ξ_1=0.133 3, ξ_2=0.333 3, ξ_3=0.533 3 by (14), after adjusting ξ'_1=0.133 3, ξ'_2=ξ'_3=0.433 3.

3.3 TFTD operator

Definition 7 For the sequential TF vector set $\boldsymbol{A}=\{\boldsymbol{A}_1, \boldsymbol{A}_2 \cdots, \boldsymbol{A}_m\}$, suppose TFTDWA:$\mathbf{R}_n \rightarrow \mathbf{R}$, then

$$\mathrm{TFTDWA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)=\sum_{r=1}^{m}\xi_r\boldsymbol{y}(\boldsymbol{A}_r). \tag{15}$$

TFTDWA is called the TF numbers two-dimensional density arithmetic weighted average operator, in (15) $\boldsymbol{y}(\boldsymbol{A}_r)=(y_{1r},y_{2r},\cdots,y_{nr})^{\mathrm{T}}$,

$$y_{ir}=\sum_{j=1}^{r(s)}w_j^r\hat{a}_{ij}^r,$$

$$\boldsymbol{A}_r=\left\{\hat{a}_{ij}^r\|r=1,2,\cdots,m;j=1,2,\cdots,r(s)\right\},$$

$$\sum_{r=1}^{m}r(s)=l,$$

$\hat{a}_{ij}^r$ represents the ith TF number of the jth vector in the rth vector group. ξ_r represents the density weight of the r vector set, $0\leqslant\xi_r\leqslant1,\sum_{r=1}^{m}\xi_r=1,0\leqslant r\leqslant1$, w_j^r represents the importance weight of the jth vector in the rth vector set, and each weight satisfies the following conditions:

$$0\leqslant w_j^r\leqslant1,\sum_{j=1}^{r(s)}w_j^r=1.$$

Example 2 $\boldsymbol{A}_{a1}=(\hat{a}_{11},\hat{a}_{12})^{\mathrm{T}}$, $\boldsymbol{A}_{a2}=(\hat{a}_{21},\hat{a}_{22})^{\mathrm{T}}$, $\boldsymbol{A}_{a3}=(\hat{a}_{31},\hat{a}_{32})^{\mathrm{T}}$, $\hat{a}_{11}=\hat{a}_{12}=$ (0.1,0.2,0.3), $\hat{a}_{21}=\hat{a}_{22}=$(0.1,0.3,0.4), $\hat{a}_{31}=\hat{a}_{32}=$ (0.1,0.4,07), $w_1^1=0.2$, $w_1^2=0.4$, $w_1^3=0.4$, then, $0.2(0.1,0.2,0.3)+0.4(0.1,0.3,0.4)+0.4(0.1,0.4,07)=(0.1,0.32,0.5)$, Then $\boldsymbol{y}(\boldsymbol{A}_1)=((0.1,0.32,0.5),(0.1,0.32,0.5))^{\mathrm{T}}$.

Example 3 $\boldsymbol{y}(\boldsymbol{A}_1)=((0.1,0.32,0.5),(0.1,0.32,0.5))^{\mathrm{T}}$. $\boldsymbol{y}(\boldsymbol{A}_2)=((0.1,0.1,0.1),(0.1,0.3,0.5))^{\mathrm{T}}$. $\boldsymbol{y}(\boldsymbol{A}_3)=((0.1,0.2,0.3),(0.1,0.3,0.5))^{\mathrm{T}}$. $\xi_1^{'}=0.133\,3$, $\xi_2^{'}=\xi_3^{'}=0.433\,3$, then,

$$\mathrm{TFTDWA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)=$$
$$0.133\,3\begin{pmatrix}0.1&0.32&0.5\\0.1&0.32&0.5\end{pmatrix}+$$
$$0.433\,3\begin{pmatrix}0.1&0.1&0.1\\0.1&0.3&0.5\end{pmatrix}+$$
$$0.433\,3\begin{pmatrix}0.1&0.2&0.1\\0.1&0.3&0.5\end{pmatrix}=$$
$$\begin{pmatrix}0.1&0.172\,6&0.153\,3\\0.1&0.302\,6&0.5\end{pmatrix}.$$

Definition 8 For the sequential set of TF vectors $\boldsymbol{A}=\{\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m\}$, suppose TFTDWGA: $\mathbf{R}_n\rightarrow\mathbf{R}$, then

$$\mathrm{TFTDWGA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)=\prod_{r=1}^{m}\boldsymbol{y}(\boldsymbol{A}_r)^{\xi_r}. \tag{16}$$

TFTDWGA is called the TF number two-dimensional density geometric averaging operator. In (16), $\boldsymbol{y}(\boldsymbol{A}_r)=(y_{1r},y_{2r},\cdots,y_{nr})^{\mathrm{T}}$, $y_{ir}=\sum_{j=1}^{r(s)}w_j^r\hat{a}_{ij}^r$, $\boldsymbol{A}_r=\{\hat{a}_{ij}^r|r=1,2,\cdots,m;j=1,2,\cdots,r(s)\}$, $\sum_{r=1}^{m}r(s)=l$, $\hat{a}_{ij}^r$ represents the i TF number of the jth vector in the rth vector group. ξ_r represents the density weight of the rth vector group, and $0\leqslant\xi_r\leqslant1,\sum_{r=1}^{m}\xi_r=1$, $0\leqslant r\leqslant1$, w_j^r represents the importance weight of the jth vector in the rth vector group. Each weight satisfies the following conditions: $0\leqslant w_j^r\leqslant1,\sum_{j=1}^{r(s)}w_j^r=1$.

3.4 Property of TFTD operator

Property 1 Permutation invariance. Let $\boldsymbol{A}'=(\boldsymbol{A}_1',\boldsymbol{A}_2',\cdots,\boldsymbol{A}_m')$ be a permutation of $\boldsymbol{A}=(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)$, $\mathrm{TFTDWA}_{\xi}(\boldsymbol{A}_1',\boldsymbol{A}_2',\cdots,\boldsymbol{A}_m')=\mathrm{TFTDWA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)$, $\mathrm{TFTDWGA}_{\xi}(\boldsymbol{A}_1',\boldsymbol{A}_2',\cdots,\boldsymbol{A}_m')=\mathrm{TFTDWGA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_2,\cdots,\boldsymbol{A}_m)$.

Property 2 Idempotency. Let $\boldsymbol{A}=(\boldsymbol{A}_1,\boldsymbol{A}_1,\cdots,\boldsymbol{A}_1)$ be a TF vector set, then $\mathrm{TFTDWA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_1,\cdots,\boldsymbol{A}_1)=\boldsymbol{A}_1$, $\mathrm{TFTDWGA}_{\xi}(\boldsymbol{A}_1,\boldsymbol{A}_1,\cdots,\boldsymbol{A}_1)=\boldsymbol{A}_1$.

3.5 Method's steps onTFTD operator

Step 1 Collect the original data of each attribute value.
Step 2 Standardize and fuzzify the attribute values.
Step 3 Cluster partition on the attributes.
Step 4 Calculate the TF comprehensive value of each cluster.
Step 5 Calculate the density-weighted vectors $\boldsymbol{\xi}$ of each cluster.
Step 6 Use the TFTD operator to calculate the integrated value.
Step 7 Rank order.

4. Numerical example and analysis

4.1 Numerical example

Let's suppose a venture capital company wants to invest a project among three enterprises (E1, E2, E3). Experts evaluate enterprises based on the 21 attributes (F1, F2, F3, F4, F5, F6, S1, S2, S3, S4, S5, T1, T2, T3, T4, T5, P1, P2, P3, P4, P5). Among them, F4 is of cost type, and the others are of benefit type. The importance of the attribute (very low, low, medium, high, very high) is expressed as the TF number ((0,0,0.3), (0,0.3,0.5), (0.3,0.5,0.7), (0.5,0.7,1), (0.7,1,1)). After gathering experts' opinion, the final weight of each attribute is obtained below:

$W_{\mathrm{F1}}=W_{\mathrm{F2}}=W_{\mathrm{F3}}=W_{\mathrm{F4}}=W_{\mathrm{F5}}=$ (0.5, 0.725, 0.925), $W_{\mathrm{F6}}=$ (0.4, 0.6, 0.85), $W_{\mathrm{S1}}=W_{\mathrm{S2}}=$ (0.5, 0.725, 0.925), $W_{\mathrm{S3}}=W_{\mathrm{S4}}=$ (0.4, 0.6, 0.85), $W_{\mathrm{S5}}=$ (0.4, 0.6, 0.85), $W_{\mathrm{T1}}=$ (0.275, 0.5, 0.725), $W_{\mathrm{T2}}=$ (0.4, 0.6, 0.85), $W_{\mathrm{T3}}=W_{\mathrm{T4}}=$ (0.5, 0.7, 1), $W_{\mathrm{T5}}=$(0.4, 0.6, 0.85), $W_{\mathrm{P1}}=$ (0.5, 0.725, 0.925), $W_{\mathrm{P2}}=$ (0.6, 0.85, 1), $W_{\mathrm{P3}}=\mathrm{W}_{\mathrm{P4}}=W_{\mathrm{P5}}=$ (0.6, 0.85, 1).

By Step 1, the raw attribute values are collected in five periods (see Table 2).

Table 2 Company's raw attribute values

Attribute	E1					E2					E3				
	1st	2nd	3rd	4th	5th	lst	2nd	3rd	4th	5th	lst	2nd	3rd	4th	5th
F1	27.84	26.69	25.47	23.67	19.13	41.86	39.68	34.22	47.79	40.91	60.05	57.10	56.87	54.36	55.33
F2	66.52	1.05	50.26	44.70	35.57	62.10	81.79	97.61	115.87	129.04	201.70	203.33	218.06	68.04	73.31
F3	19.63	18.57	16.90	15.47	12.44	25.01	26.72	25.34	33.84	31.06	46.27	44.58	45.11	30.22	31.53
F4	42.91	48.19	49.00	48.34	45.61	51.27	45.32	40.30	47.38	43.64	39.65	39.99	41.60	57.94	52.85
F5	133.05	107.51	104.07	106.87	119.25	95.04	120.64	148.15	111.06	129.14	152.20	150.05	140.38	72.60	89.21
F6	238.92	228.73	197.32	188.85	185.91	148.34	206.15	285.22	242.46	315.47	335.88	356.11	383.43	125.16	132.49
S1	104.02	94.65	118.86	128.23	196.13	336.43	333.38	342.18	221.46	213.69	130.00	125.48	124.23	172.43	194.11
S2	96.05	0.39	114.79	121.62	189.50	314.37	315.62	317.21	207.74	202.04	121.73	119.20	117.86	166.62	185.76
S3	9.54	14.58	44.92	42.85	6.70	21.05	22.12	17.79	14.22	19.55	15.89	20.57	23.69	16.61	33.85
S4	9.93	13.85	53.39	54.95	13.15	70.82	73.75	60.88	31.50	41.78	20.66	25.81	29.43	28.65	65.70
S5	3.86	-5.65	15.87	22.01	49.01	70.28	70.00	70.78	54.85	53.20	23.08	20.31	19.50	42.00	48.48
T1	1.83	0.89	0.95	0.79	0.98	3.10	2.68	2.63	1.54	1.66	1.84	1.55	1.43	1.22	1.96
T2	0.94	1.03	1.00	1.09	1.40	2.58	2.19	1.63	1.72	2.22	3.00	3.22	3.68	3.41	3.65
T3	88.96	87.26	86.84	98.08	121.47	147.68	128.87	125.12	11.81	132.78	76.96	78.98	71.82	97.71	96.57
T4	14.14	12.07	11.28	11.99	12.64	30.13	27.96	25.56	28.12	30.61	27.89	27.06	23.85	22.34	25.19
T5	24.77	23.29	22.12	23.21	23.24	61.82	51.13	42.82	53.43	54.31	46.21	45.10	40.85	53.12	53.43
P1	3.58	8.00	11.23	14.59	25.28	12.40	9.62	5.43	12.02	13.95	10.09	6.46	5.86	14.06	14.10
P2	-3.19	-0.90	3.92	8.13	19.28	9.80	6.06	2.92	9.59	10.89	6.36	0.83	1.36	10.47	11.14
P3	17.42	22.52	23.88	28.44	45.19	18.12	13.37	12.50	14.61	18.16	17.16	9.00	13.37	13.99	20.39
P4	17.05	18.67	21.42	26.43	38.21	13.92	10.80	9.73	12.77	13.69	14.81	8.61	12.13	11.73	17.28
P5	2.41	2.25	2.42	3.17	4.83	4.19	3.02	2.49	3.59	4.19	4.13	2.33	2.89	2.62	4.35

By Step 2, the standardized attribute values are shown in Table 3.

Table 3 Standardized attribute values

Attribute	E1	E2	E3
F1	(0.268,0.330,0.358)	(0.481,0.550,0.628)	(0.714,0.764,0.800)
F2	(0.206,0.265,0.316)	(0.281,0.541,0.846)	(0.480,0.732,0.912)
F3	(0.271,0.318,0.350)	(0.445,0.555,0.706)	(0.630,0.755,0.829)
F4	(0.509,0.565,0.604)	(0.494,0.582,0.620)	(0.504,0.578,0.639)
F5	(0.454,0.554,0.627)	(0.425,0.585,0.655)	(0.426,0.571,0.681)
F6	(0.382,0.492,0.569)	(0.339,0.574,0.810)	(0.340,0.596,0.767)
S1	(0.257,0.364,0.562)	(0.612,0.805,0.905)	(0.324,0.425,0.559)
S2	(0.259,0.368,0.568)	(0.606,0.801,0.904)	(0.330,0.429,0.569)
S3	(0.169,0.534,0.891)	(0.296,0.506,0.751)	(0.345,0.564,0.854)
S4	(0.133,0.377,0.790)	(0.453,0.714,0.951)	(0.278,0.438,0.832)
S5	(−0.077,0.211,0.563)	(0.611,0.843,0.958)	(0.260,0.397,0.579)
T1	(0.276,0.352,0.453)	(0.604,0.753,0.837)	(0.455,0.536,0.713)
T2	(0.231,0.263,0.311)	(0.393,0.500,0.634)	(0.738,0.819,0.887)
T3	(0.471,0.527,0.595)	(0.628,0.708,0.782)	(0.408,0.462,0.549)
T4	(0.296,0.310,0.326)	(0.686,0.711,0.743)	(0.590,0.630,0.664)
T5	(0.292,0.313,0.350)	(0.678,0.702,0.763)	(0.570,0.637,0.674)
P1	(0.219,0.602,0.815)	(0.394,0.556,0.757)	(0.425,0.507,0.616)
P2	(−0.263,0.328,0.778)	(0.439,0.678,0.980)	(0.134,0.403,0.640)
P3	(0.572,0.770,0.856)	(0.344,0.451,0.595)	(0.325,0.424,0.564)
P4	(0.643,0.792,0.866)	(0.310,0.414,0.525)	(0.371,0.430,0.558)
P5	(0.379,0.526,0.625)	(0.542,0.618,0.682)	(0.480,0.572,0.650)

By Step 3, after a threshold value Θ=0.3 is set, 10 sorted vector sets can be formed: $\boldsymbol{A}_1$ = {F4, F5, F6, S3, P1, P5}, $\boldsymbol{A}_2$ = {F1, F2, F3}, $\boldsymbol{A}_3$={P2, P3, P4}, $\boldsymbol{A}_4$ ={S1, S2}, $\boldsymbol{A}_5$ ={T4, T5}, $\boldsymbol{A}_6$={S4}, $\boldsymbol{A}_7$={S5}, $\boldsymbol{A}_8$={T1}, $\boldsymbol{A}_9$={T2}, $\boldsymbol{A}_{10}$={T3}.

By Step 4, the closeness is calculated by normalization. For example, after normalizing the magnitude of closeness we can get d_{F1}=0.757 4 when W_{F1} = (0.5, 0.725, 0.925).

Therefore, intra-group importance weights for $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, ⋯, $\boldsymbol{A}_{10}$ of each group can be calculated as follows:

$w(\boldsymbol{A}_1)$={0.171 4, 0.171 4, 0.143 1, 0.143 1, 0.171 4, 0.199 6}, $w(\boldsymbol{A}_2)$={0.333 3, 0.333 3, 0.333 3}, $w(\boldsymbol{A}_3)$= {0.333 3, 0.333 3, 0.333 3}, $w(\boldsymbol{A}_4)$= {0.5,0.5}, $w(\boldsymbol{A}_5)$= {0.547 3,0.452 7}, $w(\boldsymbol{A}_6)$=$w(\boldsymbol{A}_7)$=$w(\boldsymbol{A}_8)$=$w(\boldsymbol{A}_9)$=$w(\boldsymbol{A}_{10})$= {1}.

The attribute values of each group is shown in Table 4.

Table 4 Attribute values after group aggregation

Group	E1	E2	E3
$\boldsymbol{A}_1$	(0.357,0.547,0.684)	(0.424,0.573,0.708)	(0.426,0.564,0.694)
$\boldsymbol{A}_2$	(0.248,0.304,0.341)	(0.402,0.549,0.727)	(0.608,0.75,0.847)
$\boldsymbol{A}_3$	(0.317,0.63,0.833)	(0. 364,0.514,0.7)	(0.277,0.419, 0.587)
$\boldsymbol{A}_4$	(0.258,0.366,0.565)	(0.609,0.803,0.905)	(0.327,0.427,0.564)
$\boldsymbol{A}_5$	(0.294,0.311,0.337)	(0.682,0.707,0.752)	(0.581,0.633,0.669)
$\boldsymbol{A}_6$	(0.133,0.377,0.79)	(0.453,0.714,0.951)	(0.278,0.438,0.832)
$\boldsymbol{A}_7$	(−0.077,0.211,0.563)	(0.611,0.843,0.958)	(0.26,0.397,0.579)
$\boldsymbol{A}_8$	(0.276,0.352,0.453)	(0.604,0.753,0.837)	(0.455,0.536,0.713)
$\boldsymbol{A}_9$	(0.231,0.263,0.311)	(0. 393,0.5,0.634)	(0.738,0.819,0.887)
$\boldsymbol{A}_{10}$	(0.471,0.527,0.595)	(0.628,0.708,0.782)	(0.408,0.462,0.549)

By Step 5, the density weight ξ_j for each classification $\boldsymbol{A}_j$ can be calculated. Let λ=0.7, which means the attributes with high information density is preferred, and the following results are obtained: ξ_1=0.198 2, ξ_2=ξ_3= 0.165 45, ξ_4=ξ_5=0.121 8, ξ_6=ξ_7=ξ_8=ξ_9=ξ_{10}=0.045 46.

By Step 6, the integrated values based on the TFTDWA operator can be got, which is shown in Table 5.

Table 5 Integrated values

Enterprise	Value
E1	(0.279,0.424,0.563)
E2	(0.49,0.633,0.767)
E3	(0.452,0.564,0.69)

By Step 7, the closeness of the comprehensive value relative to the TF number (0,0,0) and (1,1,1) is calculated, there is d_{E1}=0.422 5, d_{E2}=0.631 1, d_{E3}=0.567 7. The greater the closeness is, the better the scheme is. The final ranking is obtained as follows: E2> E3> E1.

4.2 Comparison and analysis

Compared with one-dimensional density operator, a two-dimensional density operator needs less clustering frequency and simplified operation steps, therefore, the two-dimensional density operator is superior to the one-dimensional density operators in calculating.

As shown in Table 6, when the value of the density preference measure λ changes as 0.9, 0.7, 0.5, 0.3, 0.1, the calculated density-weighted vector $\boldsymbol{\xi}$ changes too.

Table 6 Density weight under different preferences

λ	ξ_j			
	ξ_1	ξ_2=ξ_3	ξ_4=ξ_5	ξ_6=ξ_7=ξ_8=ξ_9=ξ_{10}
0.9	0.43	0.2	0.035	0
0.7	0.198 2	0.165 45	0.121 8	0.045 46
0.5	0.1	0.1	0.1	0.1
0.3	0.02	0.034 45	0.078 15	0.154 56
0.1	0	0	0	0.2

Using the TFTDWA operator to aggregate the data, we can obtain the integrated values under different density preferences, as shown in Table 7.

Table 7 Integrated values under various preference

λ	E1	E2	E3
0.9	(0.286,0.446,0.561)	(0.381,0.512,0.648)	(0.442,0.549,0.642)
0.7	(0.279,0.424,0.563)	(0.49,0.633,0.767)	(0.452,0.564,0.69)
0.5	(0.251,0.389,0.547)	(0.517,0.666,0.795)	(0.436,0.545,0.692)
0.3	(0.23,0.363,0.544)	(0.551,0.71,0.836)	(0.427,0.535,0.706)
0.1	(0.207,0.346,0.542)	(0.538,0.704,0.832)	(0.428,0.53,0.712)

When λ=0.9, the ranking of the options is E3>E2>E1, which is inconsistent with the results by the method in

[8]. When λ=0.7, 0.5, 0.3, 0.1, the ranking of the options is E2> E3> E1. In the case of complete preference for high information density attributes, the attribute with higher relevance is strengthened in the information aggregation process, the sorted result is inconsistent with others, which can help the DM to obtain richer results and make the right choice based on the different purposes.

5. Conclusions

In this paper, a TF MADM method based on TFTD operator method is proposed to handle the TF MADM problem, in which a preference for the distribution density of attribute (DDA) is necessary to take into account. The main advantages of the method include:

First, the proposed method which cares about the characteristics of DDA, can solve the TF MADM problem according to the DM's density preference.

Second, the method is useful in making an overall comparison because the result may vary with different density preferences, which can help the DM broaden his evaluation vision.

Last, the two-dimensional density operator is superior than the one-dimensional density operator in calculating, and the clustering method of TF vectors is easy to understand and apply.

In future research, we will extend the developed method by taking more fuzzy attribute values into consideration, such as intuitionistic fuzzy number, and hesitant fuzzy information. We will discuss the application of other high-dimensional density operators for the TF MADM problems too.

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