Uncertainty entropy-based exploratory evaluation method and its applications on missile effectiveness evaluation

HU Jianwen[1], WANG Zhihui[1,2,*], GAO Yuan[3], ZHU Xiaomin[4], TIAN Yongliang[3], and LIU Hu[3]

1. Consulting Center for Strategic Assessment, Academy of Military Science, Beijing 100097, China;
2. Army Academy of Artillery and Air Defense (Nanjing Campus), Nanjing 211132, China;
3. School of Aeronautic Science and Engineering, Beihang University, Beijing 100191, China;
4. College of Systems Engineering, National University of Defense Technology, Changsha 410072, China

Abstract: Some attributes are uncertain for evaluation work because of incomplete or limited information and knowledge. It leads to uncertainty in evaluation results. To that end, an evaluation method, uncertainty entropy-based exploratory evaluation (UEEE), is proposed to guide the evaluation activities, which can iteratively and gradually reduce uncertainty in evaluation results. Uncertainty entropy (UE) is proposed to measure the extent of uncertainty. First, the belief degree distributions are assumed to characterize the uncertainty in attributes. Then the belief degree distribution of the evaluation result can be calculated by using uncertainty theory. The obtained result is then checked based on UE to see if it could meet the requirements of decision-making. If its uncertainty level is high, more information needs to be introduced to reduce uncertainty. An algorithm based on the UE is proposed to find which attribute can mostly affect the uncertainty in results. Thus, efforts can be invested in key attribute(s), and the evaluation results can be updated accordingly. This update should be repeated until the evaluation result meets the requirements. Finally, as a case study, the effectiveness of ballistic missiles with uncertain attributes is evaluated by UEEE. The evaluation results show that the target is believed to be destroyed.

Keywords: uncertainty entropy-based exploratory evaluation (UEEE), evaluation activity, belief degree distribution, missile effectiveness.

1. Introduction

The basic goal of evaluation activities is to obtain a "net" assessment [1], which means the uncertainty in the evaluation should be reduced to obtain a clear and elegant result. However, the evaluation problems are usually fraught with uncertainties (unknowns) which perplex the

whole evaluation process.

In practice, a net assessment is usually hard to be obtained. On the one hand, the evaluation object's attributes are uncertain; on the other hand, the relationship between the evaluation object's attributes and the evaluation results is also uncertain. For example, determining the attribute values of an air-surface missile in a complex operational environment can be difficult and expensive. In addition, it is hard to determine the performances of enemy air defense weapon systems. Moreover, it is difficult to generate the relationship between the missile's operational effectiveness and capability parameters. As a result, calculating the "net effectiveness" of the missile in the operational context is difficult and rarely reported in literature.

Though uncertainty leads to the difficulties in achieving net assessment, it is still common in evaluations. Cheng et al. identified the performance parameters and corresponding weights of the attack helicopter using a qualitative-to-quantitative transformation of the linguistic variable [2]. Then the analytic hierarchy process (AHP) is used to calculate the system effectiveness of the attack helicopter. Luo et al. used interval numbers to characterize uncertainty in attributes [3]. The uncertain results were described by intervals. The ranking result of alternatives was calculated based on the ranking method proposed in [4,5]. Wu et al. used interval numbers with probability distribution to characterize uncertainty [6−8]. Then the ranking results were obtained by the mean-variance method, the mid-point and half-width method, and other algorithms. Bandte et al. used the subjective probability distribution approach to characterize the uncertainty of attributes [9−11]. They then calculate the probability distribution of the evaluation result based on the experiment model (e.g., simulation model), or aggregate

operators (e.g., weighted average operator, weighted geometric operator). Finally, the mean or other statistic is used to represent the evaluation result. In addition, fuzzy sets [12,13], fuzzy variables [14], rough sets [15], and other mathematical tools are used to characterize uncertainty in evaluation. The outcome of the evaluation is still presented as a statistical value, like a mathematical expectation.

For the existing evaluation methods with uncertainty, two issues have not been well studied:

(i) Mathematical tools for characterizing human uncertainty. Uncertain attributes are traditionally characterized by subjective probability distributions [16,17] or interval numbers [18,19]. In that case, the extent of uncertainty is measured by variance [20−22] or the length of the interval [18,19]. However, this probabilistic form with associated operational rules is usually only suitable for random cases with enough samples. Liu pointed out that probability theory is no longer applicable when the estimated probability distribution is not close enough to the inherent frequency [23,24]. This is because, in the absence of many samples, estimating the inherent frequency is typically difficult. Furthermore, using fuzzy variables to describe attribute uncertainty may result in a logical paradox as discussed in [23].

(ii) Treatment of uncertain results. The evaluation results need to be used to compare which alternative is better, so uncertain results are usually replaced with certain values. Traditional methods, such as methods in [25−28], mainly rely on statistic values (e.g., expectation, maximum value). In practice, however, the statistical values often differ from the actual values, as discussed in the example of fighter jet seat design in [29]. To achieve an accurate and comprehensive evaluation result, reducing uncertainty in results should be conducted, rather than simply replacing uncertain results with a statistical value.

Evaluation methods, which can measure uncertainty and gradually reduce uncertainty in practice, are rarely reported in literature. Reducing the uncertainty should be essential to the evaluation works, otherwise the evaluation results would not be trustworthy. For example, in the development of equipment (e.g., a missile), it is necessary to evaluate the operational effectiveness of various design options with uncertainty reduced (especially in the initial design phase). Only in this manner, the net effectiveness of each design option can be determined to support equipment research and development. This is the main motivation for the works performed in this paper.

In this paper, we propose a new systematized evaluation method to reduce the uncertainty in a complex problem, which gets rid of calculating statistic values. This method is called uncertainty entropy-based exploratory evaluation (UEEE). It is based on an iterative approximation process. In this method, the belief degree distribution function [30] is introduced, which is appropriate for intuitive judgement to characterize the uncertainty in evaluation work [23,30]. Based on it, the uncertainty entropy (UE) is proposed to measure the extent of uncertainty. Furthermore, an analytical model is proposed to calculate the contribution of each attribute to overall uncertainty of the evaluation result. Based on that, reducing the uncertainty of the evaluation result can be achieved by iteratively reducing the uncertainty of the attributes with large contributions.

This paper is organized as follows: Section 2 presents the fundamentals of belief degree distribution. Section 3 proposes the evaluation method (UEEE). Section 4 applies UEEE in evaluating ballistic missile operational effectiveness under uncertainty. Section 5 concludes the paper.

2. Fundamentals of belief degree distribution

2.1 Uncertainty characterization

In general, uncertainty can be classified into two categories: aleatory uncertainty and epistemic uncertainty [31−33].

Aleatory uncertainty is referred as the uncertainty inherent in the physical behavior of the system. It is usually considered irreducible and is expressed in terms of probability (long-run frequency). For example, there is no way to reduce the uncertainty of the outcome of a dice roll or a coin toss.

Epistemic uncertainty is categorized as uncertainty caused by incomplete knowledge or limited information. This type of uncertainty can be reduced. More precise measurements or more experiments can help reduce epistemic uncertainty. However, it usually needs a great investment of manpower, materials, cost and time.

Therefore, the goal of UEEE here is to efficiently reduce epistemic uncertainty in evaluation results. Some researchers have used subjective probability to quantify epistemic uncertainty [34−36]. However, as discussed in [23,24], a fundamental premise of applying probability theory is that the estimated probability is close to the long-run frequency, otherwise, using subjective probability may result in errors.

Belief degree indicates the strength of a person's belief that an event will occur. Belief degree takes values between 0 and 1. The higher the value, the higher the probability that the person subjectively feels that the event will happen. From an axiomatic point of view, the essential difference between probability and belief degree is the difference in the product degree. Roughly speaking,

the product measure of probability theory is "multiplication", while the product measure of belief degree is "taking the smallest", i.e.,

$$P_r\{A \times B\} = P_r\{A\} \times P_r\{B\},$$
$$M\{A \times B\} = M\{A\} \wedge M\{B\},$$

where P_r is the probability measure, M is the belief degree measure. Give an example to look at how subjective probability differs from belief degree. A specific exam has four subjects (math, politics, history, and geography), and one person must pass all four to pass the exam. Assume a person takes the exam and he believes there is a possibility of 70% that he can pass math, 70% that he can pass politics, 70% that he can pass history, and 50% that he can pass geography. If the chance of passing the exam is calculated by using the probability theory, the probability of passing the exam can be calculated as $0.7 \times 0.7 \times 0.7 \times 0.5 \approx 0.17$. By applying uncertainty theory instead, the belief degree of passing the exam can be calculated as $0.7 \wedge 0.7 \wedge 0.7 \wedge 0.5 = 0.5$. Clearly, the person is just very unsure about passing the geography subject (0.5 belief degree), which cannot indicate that he believes that he will fail, nor that his chance of passing is slim. If there are more subjects, the probability of passing this exam could be close to 0. Therefore, the result obtained by subjective probability seems unreasonable.

Everyone approaches the same problem from a different angle. However, subjective probability does not correspond to the characteristics of human intuitive thinking. For this reason, the long-run frequency of an event is difficult to estimate. Long-run frequency imagination is far more difficult than natural feeling. Thus, the unknown attributes are hardly precisely described by probability distributions. However, applying probability theory requires precise distributions because the probability multiplication rule could amplify errors, resulting in incorrect decision-making. In contrast, the minimum rule of belief degree would be more reliable. As a result, we think subjective probability is useful when an expert with reasonably reliable information and a precise opinion is available. Otherwise, belief degree distribution is more reliable. That's why, in this study, only the belief degree is used to measure the attribute uncertainty.

Some attributes, such as the circular error probability (CEP) of an unknown missile, cannot be determined precisely; thus, the support of domain experts is needed to evaluate the belief degree distribution of attributes in a certain range. One possible way of modeling the belief degree is to use a fuzzy membership function [14,37]. However, it is usually inappropriate because it may lead to counter-intuitive results. In [23,38], an example of the

bridge overload is provided to explain the reasons. The characterization of the attribute "overload" should use a belief degree distribution rather than frequency due to the information reliability. Since the uncertainty stems primarily from the limitations of the measurement facility, we cannot obtain the precise/reliable value of "overload". The uncertainty distribution function of "overload" is given in Fig. 1.
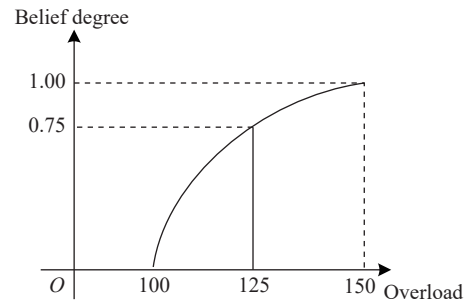

Fig. 1   Uncertainty distribution function of overload

The belief degree measure $M(\gamma)$ represents the happening belief degree of the event γ. For example, M(overload < 100) represents the belief degree of the uncertain attribute "overload" is smaller than 100. The uncertainty distribution function $\Phi(y)$ means the belief degree of the uncertain attribute $x < y$ (x means the studied attribute), i.e., $\Phi(y) = M(x < y)$. As shown in Fig. 1, it is an uncertainty distribution function curve of uncertain attribute "overload". The belief degree of overload smaller than 150 is 1, namely $\Phi(150) = 1$; the belief degree of overload smaller than 125 is 0.75, namely $\Phi(125) = 0.75$; and finally, $\Phi(100) = 0$. For the uncertain attribute vector X, if all components of X are independent with each other, then the vector belief degree can be defined as

$$M(X < Y) = \bigwedge_{i=1}^{n} M(x_i < y_i) = \bigwedge_{i=1}^{n} \Phi_i(y_i) \tag{1}$$

for any given vector Y [30].

The above is the basic principle of the belief degree distribution. It is applicable mainly when there are not enough samples (even no sample) to evaluate a probability distribution. In that case, the generation of the belief degree distribution function relies mainly on the subjective human judgment. The specific process is as follows: First, the expert uses a calibration method, which can be referred as in [39], to perform a calibration exercise. Interaction with experts is then used to collect evaluation data. Finally, the belief degree distribution function is generated by using the segmental interpolation method. The detailed steps can be found in [38,40].

2.2   Operation of belief degree distribution function

Definition 1[30]   A belief degree distribution function

$\Phi(x)$ is regular when it is a continuous and strictly increasing function with respect to x, at which $0 < \Phi(x) < 1$, and

$$\begin{cases} \lim_{x \to -\infty} \Phi(x) = 0 \\ \lim_{x \to +\infty} \Phi(x) = 1 \end{cases}. \tag{2}$$

A regular uncertainty distribution function $\Phi(x)$ has an inverse function on the range of x with $0 < \Phi(x) < 1$, and the inverse function $\Phi^{-1}(\alpha)$ exists on the open interval $(0,1)$.

Definition 2[30] Assume ξ is a variable with a regular belief degree distribution function $\Phi(x)$. The inverse function $\Phi^{-1}(\alpha)$ is then referred to as the inverse function of the belief degree distribution of ξ.

Definition 3[30] The uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are assumed to be independent if

$$M\left\{ \bigcap_{i=1}^{n} (\xi_i \in B_i) \right\} = \bigwedge_{i=1}^{n} M\{\xi_i \in B_i\} \tag{3}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Theorem 1[30] Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular belief degree distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse belief degree distribution.

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) \tag{4}$$

Similarly, after transforming, if f is a strictly decreasing function, then

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{5}$$

3. Proposed UEEE

3.1 Principles and procedures of UEEE

As discussed, the goal of UEEE is to reduce uncertainty in evaluation results and make them more certain. The extent of uncertainty in an evaluation result is primarily determined by the uncertainty in the attributes and their sensitivity to the evaluation result. Regions with high sensitivity and uncertainty are referred to as "extreme states" in [41], and they have a significant impact on the desired evaluation results.

Since the sensitivity of attributes is inherent to the evaluation model, it cannot be reduced. Therefore, the uncertainty in evaluation results can only be reduced iteratively by reducing the attribute uncertainty. However, reducing the uncertainty in all attributes is impractical in most practical problems due to the limited resources, time, and cost. As a result, the key attribute(s) should be first found which can mostly affect the uncertainty of the evaluation result.

Fig. 2 shows the general process of the proposed UEEE. The method is using UE to measure the extent of uncertainty. First, an evaluation framework is constructed, which consists of evaluation indicators, evaluation object attributes, and evaluation models. Following that, all uncertain attributes are characterized with belief degree distributions. The belief degree distribution of the evaluation result is then calculated based on uncertainty theory [38]. The belief degree distribution of the result is then checked based on UE to see if it could support decision-making.
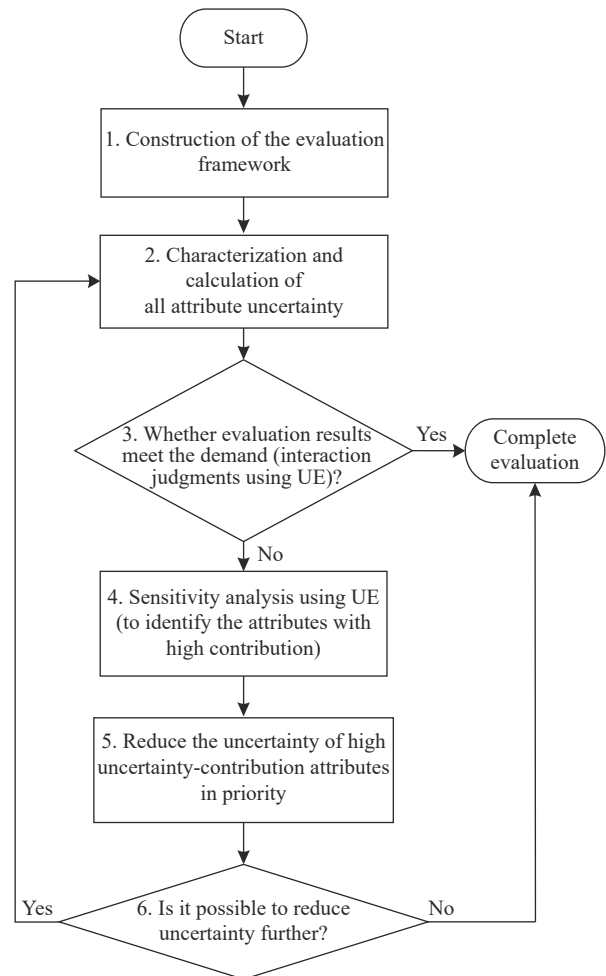

Fig. 2 Procedures of UEEE

If the level of uncertainty in the result is too high, more information and knowledge need to be introduced, for example, conducting more experiments. Then the uncertainty in attributes can be reduced. The sensitivity analysis is performed based on UE to find which attribute(s) mostly contributes to the uncertainty of the evaluation result. Thus, efforts can be made to reduce uncertainty in the obtained key attribute(s) by investing in both human and material resources. The belief degree distribution of

the evaluation result is updated with the modified attribute(s). The preceding procedures need to be carried out iteratively until the UE of the evaluation result meets the requirements.

3.2 Construction of evaluation framework

The evaluation framework mainly consists of evaluation object attributes, evaluation results and the evaluation models.

Evaluation object attributes (abbreviated as attributes in this paper) are the inherent nature of a system. Attributes represent a system's essential characteristics. Attributes may be certain or uncertain. Uncertain attributes are very common in the evaluation of practical complex problems. The evaluation result is a set of quantitative scales defined according to the evaluation objective. The evaluation model is the mapping relationship between attributes and the evaluation result. The type of an evaluation model could be an analytical model, a numerical model, a simulation model, a physical experiment model, or a subjective judgment model.

An evaluation model can be expressed as the mapping function below:

$$Y = f(\boldsymbol{X}) + \varepsilon \qquad (6)$$

where Y is the evaluation result, $\boldsymbol{X}$ is the vector of attributes, and $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)^{\mathrm{T}}$. ε is the uncertain value. $f(\cdot)$ is the evaluation model. It is worth noting that the evaluation model function $f(\cdot)$ can be assumed to be monotonic [42,43] in the most evaluation practices.

When evaluating a complex problem, the evaluation object attributes $\boldsymbol{X}$, as well as the evaluation model $f(\cdot)$, are usually uncertain for humans. Therefore, the evaluation result Y is also uncertain. The goal of UEEE is to reduce epistemic uncertainty in Y. It could be achieved by reducing the uncertainty of high uncertainty-contribution attributes iteratively.

3.3 Uncertainty metric and uncertainty contribution analysis of attributes

Information entropy is widely used in information theory to express the expectation of the amount of information about a random event. Probability theory has information entropy, and belief degree theory should also have the concept of entropy. This section gives the definition of UE on the basis of the principle of the belief degree distribution function.

Definition 4 UE is used to measure the extent of uncertainty of an uncertain variable. Let ξ be an uncertain continuous variable with a belief degree distribution function of Φ, corresponding to the UE:

$$S_\xi = \int_{L_\xi}^{U_\xi} \left(\Phi(\xi) \log_2 \frac{1}{\Phi(\xi)} + (1 - \Phi(\xi)) \log_2 \frac{1}{1 - \Phi(\xi)} \right) \mathrm{d}\xi \quad (7)$$

where ξ is an uncertain continuous variable, Φ represents the belief degree distribution function of the uncertain variable, S_ξ is the UE of the variable. $0 < S_\xi < U_\xi - L_\xi$, $U_\xi - L_\xi$ is the length of the interval.

The more the belief degree distribution of the variable ξ concentrates around $\Phi(\xi) = 0.5$, the greater S_ξ.

Let S_Y be the UE of the evaluation result Y. Then,

$$S_Y = \int_{L_Y}^{U_Y} \left(\Phi_Y(Y) \log_2 \frac{1}{\Phi_Y(Y)} + (1 - \Phi_Y(Y)) \log_2 \frac{1}{1 - \Phi_Y(Y)} \right) \mathrm{d}Y. \quad (8)$$

Let $X_i (i = 1, 2, \cdots, n)$ be an attribute and $S_{X_i} (i = 1, 2, \cdots, n)$ be the UE of X_i. Then,

$$S_{X_i} = \int_{L_i}^{U_i} \left(\Phi_i(X_i) \log_2 \frac{1}{\Phi_i(X_i)} + (1 - \Phi_i(X_i)) \log_2 \frac{1}{1 - \Phi_i(X_i)} \right) \mathrm{d}X_i. \quad (9)$$

If S_Y is large, the evaluation result is not trustworthy due to the high level of uncertainty. From (6) and (8), it can be seen that S_Y is affected by both the attributes X_i and the evaluation model $f(\cdot)$. It means S_Y can be reduced if the uncertainty of X_i or $f(\cdot)$ is reduced.

As analyzed above, after determining the evaluation model $f(\cdot)$, reducing S_Y can only be accomplished by reducing the uncertainty in attributes. It is impractical to reduce the uncertainty in all uncertain attributes especially when the number of uncertain attributes is large. This is due to the limited human, materials, and time resources. Therefore, it is necessary to identify the key attribute(s) which can mostly affect S_Y. As introduced below, the key attribute(s) can be obtained by calculating the contribution ratio.

Theorem 2 For $Y = f(X_1, X_2, \cdots, X_n)$, if $X_i (i = 1, 2, \cdots, n)$ is an uncertain variable, then Y is also an uncertain variable. S_{X_i} is the UE of X_i and S_Y is the UE of Y.

If $f(\cdot)$ is a strictly monotonically increasing function, S_Y can be decomposed as

$$S_Y = \sum_{i=1}^{n} \int_0^1 M_i(\alpha) \cdot N_i(\alpha) \mathrm{d}\alpha. \quad (10)$$

Similarly, if $f(\cdot)$ is a strictly monotonically decreasing function, S_Y can be decomposed as

$$S_Y = \sum_{i=1}^{n} \int_0^1 M_i(1 - \alpha) \cdot N_i(1 - \alpha) \mathrm{d}\alpha \quad (11)$$

where $M_i(\alpha) = \dfrac{\partial f}{\partial \Phi_i^{-1} \alpha}$ is the global sensitivity coefficient of X_i. $S_{X_i} = \int_0^1 N_i(\alpha) \mathrm{d}\alpha$ is the UE of X_i, where

$$N_i(\alpha) = \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}\right)(\Phi_i^{-1}(\alpha))'.$$

The proof of Theorem 2 is given below:

Proof   Let $Y = f(X_1, X_2, \cdots, X_n)$ be the mapping relationship for the system. Assume $f(\cdot)$ is monotonic.

Let $\Phi_Y(Y)$ be the belief degree distribution function of Y. That is, $\alpha = \Phi_Y(Y)$.

The UE of Y is

$$S_Y = \int_{-\infty}^{+\infty} \left(\Phi_Y(Y)\log_2 \frac{1}{\Phi_Y(Y)} + (1-\Phi_Y(Y))\log_2 \frac{1}{1-\Phi_Y(Y)}\right)\mathrm{d}Y,$$

$$\because (\Phi_Y^{-1}(\alpha))' = \frac{\mathrm{d}Y}{\mathrm{d}\alpha} \Rightarrow \mathrm{d}Y = \mathrm{d}\alpha(\Phi_Y^{-1}(\alpha))',$$

$$\therefore S_Y = \int_0^1 \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}\right)(\Phi_Y^{-1}(\alpha))'\mathrm{d}\alpha.$$

Let $\Phi_i(X_i)$ be the belief degree distribution function of $X_i(i = 1, 2, \cdots, n)$.

If $f(\cdot)$ is a strictly increasing function, then

$$\Phi_Y^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

If $f(\cdot)$ is a strictly decreasing function, then

$$\Phi_Y^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

Therefore,

$$(\Phi_Y^{-1}(\alpha))' = \begin{cases} \sum_{i=1}^n \frac{\partial f}{\partial \Phi_i^{-1}(\alpha)} \frac{\mathrm{d}\Phi_i^{-1}(\alpha)}{\mathrm{d}\alpha}, \\ \qquad \Phi_i \text{ is an increasing function} \\ -\sum_{i=1}^n \frac{\partial f}{\partial \Phi_i^{-1}(1-\alpha)} \frac{\mathrm{d}\Phi_i^{-1}(1-\alpha)}{\mathrm{d}(1-\alpha)}, \\ \qquad \Phi_i \text{ is a decreasing function} \end{cases}.$$

Without loss of generality, let $f(\cdot)$ be a strictly monotonically increasing function. Therefore,

$$\because (\Phi_Y^{-1}(\alpha))' = \sum_{i=1}^n \frac{\partial f}{\partial \Phi_i^{-1}(\alpha)} \frac{\mathrm{d}\Phi_i^{-1}(\alpha)}{\mathrm{d}\alpha},$$

$$\therefore S_Y = \int_0^1 \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}\right) \cdot$$

$$\sum_{i=1}^n \frac{\partial f}{\partial \Phi_i^{-1}(\alpha)} \frac{\mathrm{d}\Phi_i^{-1}(\alpha)}{\mathrm{d}\alpha}\mathrm{d}\alpha,$$

$$\therefore S_Y = \sum_{i=1}^n \int_0^1 \frac{\partial f}{\partial \Phi_i^{-1}\alpha} \cdot \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}\right) \cdot$$

$$(\Phi_i^{-1}(\alpha))'\mathrm{d}\alpha.$$

Let

$$M_i(\alpha) = \frac{\partial f}{\partial \Phi_i^{-1}(\alpha)}$$

and

$$N_i(\alpha) = \left(\alpha \log_2 \frac{1}{\alpha} + (1-\alpha)\log_2 \frac{1}{1-\alpha}\right)(\Phi_i^{-1}(\alpha))',$$

so

$$S_Y = \sum_{i=1}^n \int_0^1 M_i(\alpha) \cdot N_i(\alpha)\mathrm{d}\alpha.$$

Clearly, $S_{X_i} = \int_0^1 N_i(\alpha)\mathrm{d}\alpha$ is the UE of $X_i(i = 1, 2, \cdots, n)$. Let $C_i(i = 1, 2, \cdots, n)$ be the contribution of X_i to the UE of Y. Obviously, C_i is

$$C_i = \int_0^1 M_i(\alpha) \cdot N_i(\alpha)\mathrm{d}\alpha. \tag{12}$$

Similarly, if $f(\cdot)$ is a strictly decreasing function, then

$$\begin{cases} S_Y = -\sum_{i=1}^n \int_0^1 M_i(1-\alpha) \cdot N_i(1-\alpha)\mathrm{d}\alpha \\ C_i = -\int_0^1 M_i(1-\alpha) \cdot N_i(1-\alpha)\mathrm{d}\alpha \end{cases}. \tag{13}$$

The Sobol method is built on the basis of probability theory. The reason for its computational complexity is closely related to the fact that its first nature principle is probability theory. In probability theory, the calculation of probability distributions of multiple independent variables is very complicated. For example, the result of adding two independent random variables needs to be expressed in terms of convolution. The addition of n independent random variables will produce an $n-1$ dimensional convolution equation, that is, $n-1$ reintegration, and this calculation can only rely on Monte Carlo methods, which are computationally intensive. From a pragmatic point of view, UE is more advantageous. Using the belief degree distribution function to portray the uncertainty is more consistent with human intuitive thinking, and the calculation is simpler. □

3.4 Reducing the uncertainty of high uncertainty-contribution attributes

Based on the calculation of the contribution ratio, the attributes with the high contribution should be prioritized for the following uncertainty reduction. New knowledge and information need to be introduced. Activities such as conducting more experiments or precise measurements need to be implemented. Then the distributions of key attributes can be modified. The wisdom of crowds [44], which involves the collaboration of multiple experts, is another important method for reducing uncertainty [40]. The accuracy of subjective judgment depends not only on one's knowledge and experience, but also on one's character and state. Being too confident or too conservative can have an impact on the accuracy of judgment. In order to improve the accuracy of expert judgment and reduce judgment noise, experts need to be calibrated before they give judgment data. The calibration method can be found in [40]. In addition, the Delphi method and multi-expert belief degree integrated method [40] can help reduce judgment noise.

If the uncertainty of an attribute is hard to reduce, the attribute X_i can also be regarded as an indicator Y. Then a similar approach can be used again. It means a difficult problem can be decomposed over and over again. That is to say, UEEE can be used recursively. The evaluation results' uncertainty is reduced iteratively and recursively by applying UEEE.

4. Illustrative example: evaluation of ballistic missile operational effectiveness under uncertainty

Mission scenario: There are six ballistic missiles attacking the enemy military target (such as a command center). The operation objective is to damage the target (to make it lose operational capability). The commander needs to decide whether to launch the studied missiles. The main concern is, the location of our missiles can be discovered once the attack is launched. It means that once the missiles are launched, the target must be destroyed; otherwise, the enemy can counterattack, resulting in a passive situation. Therefore, the operational effectiveness of the six ballistic missiles needs to be evaluated in terms of the probability of destroying the enemy target.

The main factors influencing ballistic missile operational effectiveness are the number of missiles N, the penetration probability of missiles launched P_e, the circular

probability error CEP, the damage radius R of a single missile, and the probability of a single missile damaging the target P_k. They are all considered in the following evaluation works.

4.1 Evaluation framework

Evaluation result: the damage probability P of the target losing operational capability.

Evaluation object attributes: the number of ballistic missiles N, the penetration probability of missiles launched P_e, circular probability error CEP and damage radius R, damage probability of a single missile P_k.

Evaluation model: $P = f(N, \mathrm{CEP}, R, P_e, P_k)$, where f is the simulation experiment in this case study.

The simulation experiment model is built, whose principles are shown in Fig. 3. N missiles are launched. Use random simulation to choose which missiles penetrate successfully. Next, calculate the position of each missile drop point according to CEP. Determine the damage circle according to the damage radius. Then count the number of damage circles that contain the target, let t be the number. The damage probability P of the target losing operational capability can be calculated by using the following equation:
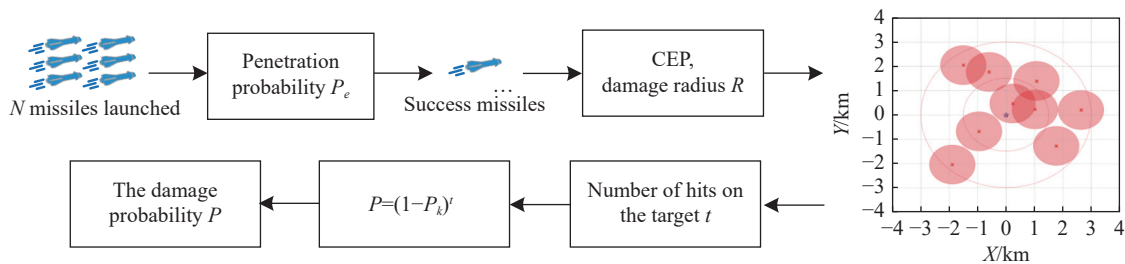
$$P = (1 - P_k)^t. \tag{14}$$

Fig. 3   Process of the simulation

Fig. 4 shows the results of a randomized simulation experiment with six ballistic missiles striking the target. The experimental parameters are set as $N = 6$, CEP $= 1$ km, $R = 1$ km, $P_e = 0.8$, $P_k = 0.6$. Fig. 4 shows that there are four missiles successfully penetrated and one missile hits the target with a damage probability of 0.6.

After 5 000 simulations, the probability (long-run frequency) of the target being destroyed is convergent at 0.57. This result can be regarded as the evaluation result of the operational effectiveness of the six ballistic missiles with determined attributes.

However, in practice, the above attributes (CEP, R, P_e, P_k) are difficult to obtain accurately. It means that they are uncertain. Domain experts need to be added to evaluate their belief degree distributions.
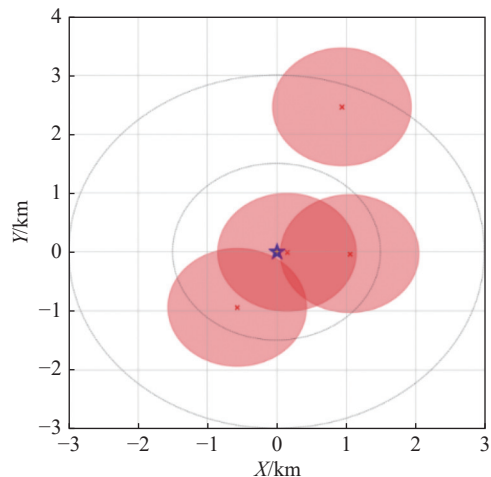

Fig. 4   Simulation of the target attacked by six ballistic missiles

4.2   Attribute belief degree distributions and evaluation results

The belief degree distributions of uncertain attributes (CEP, R, P_e, P_k) are generated as shown in Fig. 5.

From Fig. 5, P_k is highly uncertain because its belief degree is concentrated around 0.5.
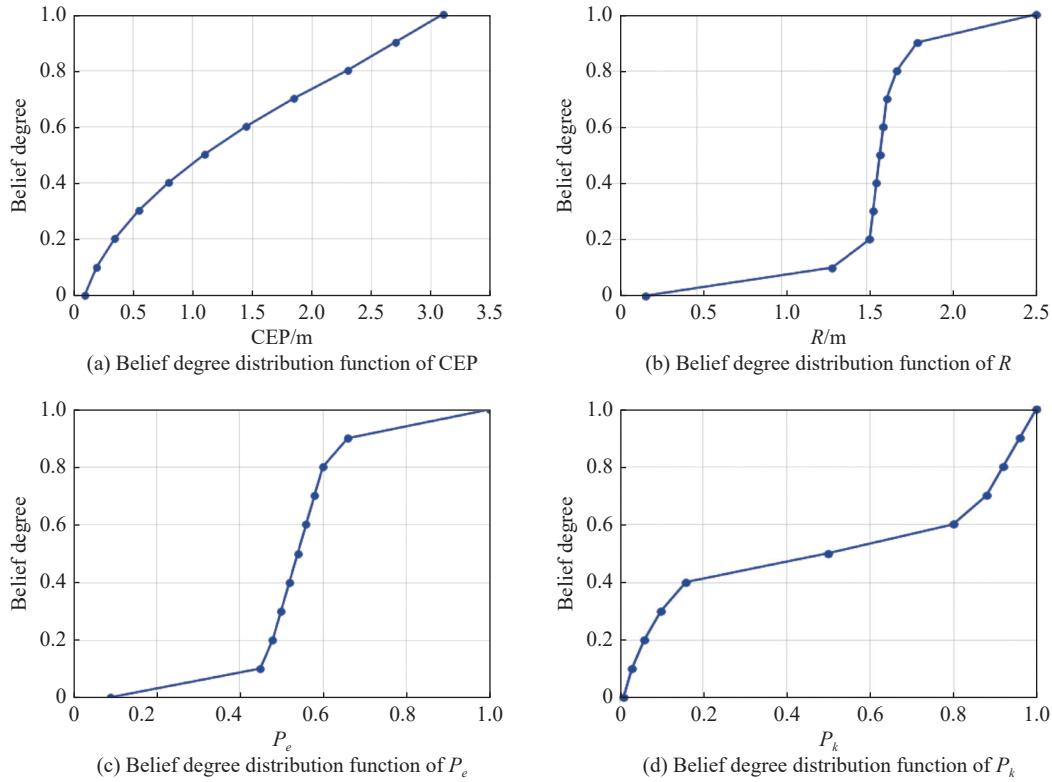

(a) Belief degree distribution function of CEP

(b) Belief degree distribution function of R

(c) Belief degree distribution function of P_e

(d) Belief degree distribution function of P_k

Fig. 5   Belief degree distributions of uncertain attributes

In the simulation experimental model, the functional relationships among R, P_e, P_k, and P can be regarded as increasing functions in their ranges, whereas the functional relationship between CEP and P can be regarded as a decreasing function in their ranges. According to Theorem 1, (4) and (5), that is,

$$\Phi_P^{-1}(\alpha) = f(N, \Phi_R^{-1}(\alpha), \Phi_{P_e}^{-1}(\alpha), \Phi_{P_k}^{-1}(\alpha), \Phi_{\text{CEP}}^{-1}(1-\alpha)) \quad (15)$$

where $P = \Phi_P^{-1}(\alpha)$ is the belief degree distribution inverse function of P, $N = 6$, $R = \Phi_R^{-1}(\alpha)$ is the belief degree distribution inverse function of R, $P_e = \Phi_{P_e}^{-1}(\alpha)$ is the belief degree distribution inverse function of P_e, $P_k = \Phi_{P_k}^{-1}(\alpha)$ is the belief degree distribution inverse function of P_k, and CEP $= \Phi_{\text{CEP}}^{-1}(\alpha)$ is the belief degree distribution inverse function of CEP.

By using (15), the belief degree distribution function of P is calculated as shown in Fig. 6.

Fig. 6 shows that the belief degree of $P < 0.99$ is 0.6. The belief degree of $P > 0.99$ is 0.4, which means the target could be damaged successfully by missiles with a 40% belief degree. It is uncertain for humans to give a decision. The epistemic uncertainty of P should be reduced.
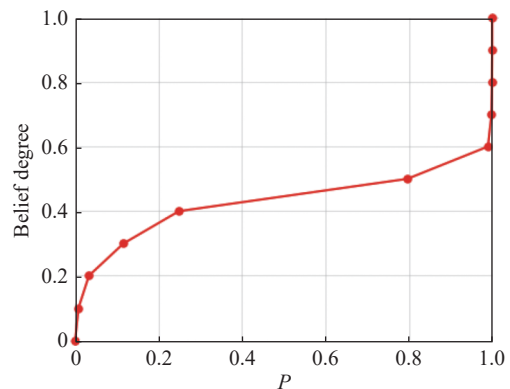

Fig. 6   Belief degree distribution of P

4.3   Reducing UE of evaluation results iteratively

As mentioned, the exploratory evaluation method reduces the UE of the evaluation result by iteratively reducing the UE of high uncertainty-contribution attribute in priority.

By using (8), the UE of evaluation result P can be given as 0.97.

Then, use (9), the obtained UE of CEP, R, P_e, P_k is 2.15, 0.83, 0.83, 0.83, respectively.

By using (12) and (13), the contribution of each input parameter to the uncertainty in P is calculated by using the numerical approach: CEP contributes 0.21, R contributes 0.03, P_e contributes 0.04, and P_k contributes 0.70. The summation of CEP, R, P_e, P_k is 0.98. It approximately equals the uncertain entropy of P ($S_P = 0.97$), indicating that the calculation is correct. The contribution ratio of each input parameter is shown in Fig. 7.
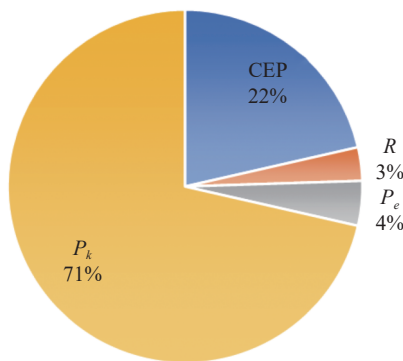
Fig. 7   Each attribute's contribution ratio

After finding the most important attribute P_k, the next step is to invest both human and material resources to reduce the uncertainty in P_k. The belief degree distribution function of P_k is modified as shown in Fig. 8. At this point, the UE of P_k decreases from 0.83 to 0.30 by using (9).
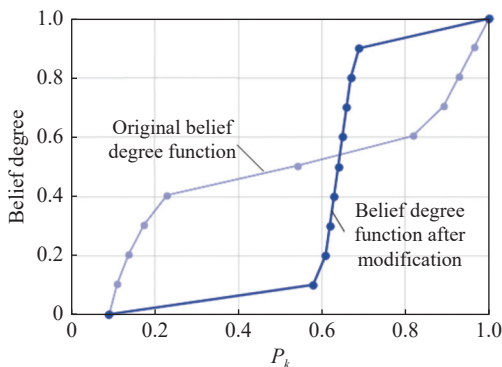

Fig. 8   Belief degree distribution of P_k after modification

Similar to the above steps, re-evaluate P after the first round of modification. The new belief degree distribution function of P updates as displayed in Fig. 9. At this point, the UE of P decreases from 0.97 to 0.65 by using (8). It is obvious that uncertainty has been greatly reduced. However, the uncertainty in P still remains so

high that we are unable to draw conclusions that they can be used to support decision-making. Instead, we need to continue to reduce the uncertainty of the key attributes.
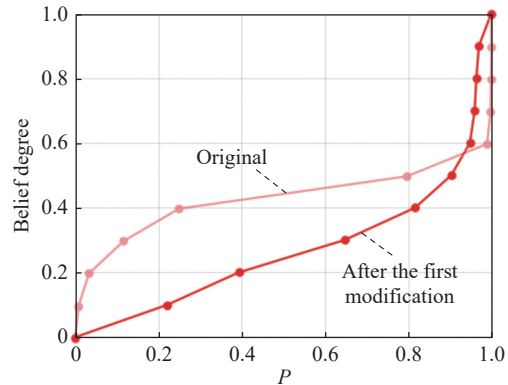

Fig. 9   Belief degree distribution of P after the first modification

The contribution of each input parameter to the uncertainty in P is updated:

CEP contributes 0.37, R contributes 0.09, P_e contributes 0.16, and P_k contributes 0.02. The summation is 0.64. It approximately equals the uncertain entropy of P ($P=0.65$), indicating that the calculation is correct. The new contribution ratio of each input attribute after the first modification is shown in Fig. 10.
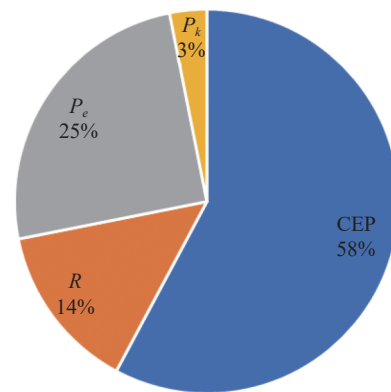

Fig. 10   Each attribute's contribution ratio after the first modification

At this point, both human and material resources are needed to reduce the uncertainty in CEP. The belief degree distribution function of CEP is modified as shown in Fig. 10. As a result, the UE of P_k decreases from 2.15 to 0.46.

Re-evaluate P after the second round of modification. The new belief degree distribution function of P is updated again (Fig. 11). At this point, the UE of P decreases from 0.65 to 0.32. It is apparent that the effect of reducing uncertainty is also beneficial.

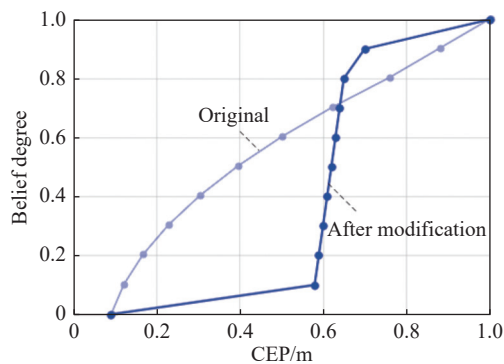**Fig. 11　Belief degree distribution of CEP after modification**

As seen in Fig. 12, the belief degree of $P < 0.8$ is 0.1. That is to say, the belief degree of $P > 0.8$ is 0.9, which means that the probability (long-run frequency) of the target destroyed by six missiles (with 90% belief degree) exceeds 0.8. Therefore, the commander has a high level of confidence to launch the attack.
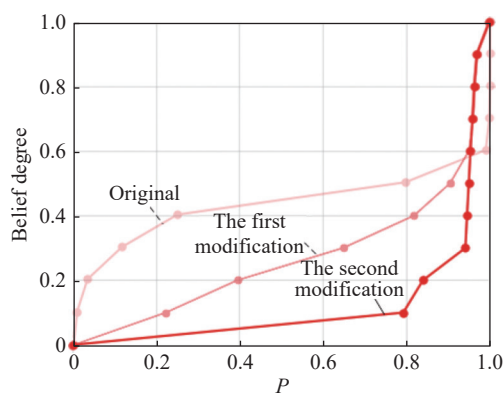

Fig. 12　Belief degree distribution of P after the second modification

From the evaluation result, it is clear that the belief degree distribution is a more reasonable mathematical tool for modeling epistemic uncertainty than subjective probability. It may be confusing to describe the unknown damage probability with a subjective probability distribution.

4.4　Discussions

From the case study of multiple missiles, when all evaluation attributes are determined, we can calculate the inherent probability (long-run frequency) of the target being destroyed through many experiments. However, these attributes are uncertain. It is difficult for us to measure them precisely, and we do not have enough samples (or even no sample) to estimate probability distributions.

In this study, domain experts are invested to evaluate their belief degree distributions. Then the belief degree distribution of the evaluation result can be calculated by using uncertainty theory. The evaluation result is then

checked based on UE to see if it could satisfy the requirement of decision-making. If the requirement is satisfied, the evaluation work can be finished. If not, a sensitivity analysis is performed to identify the key attribute that contributes the most to the UE of the evaluation result. For the key attribute, human and material resources are invested to introduce new information in order to reduce their uncertainty. Then the belief degree distribution of the key attribute can be modified. Following that, the belief distribution of the evaluation result can be updated with the modified attribute. The iterative reduction is repeated until the UE of the evaluation result meets the requirements. Finally, a reliable evaluation result that can support decision-making is obtained.

5. Conclusions

The UEEE method is proposed in this paper, which employs an iterative approximation process. The method gives specific steps to reduce the uncertainty in evaluation results in order to improve the reliability of the results. It has the following advantages:

(i) UEEE uses an exploratory iterative approach to gradually reduce uncertainty in evaluation results, rather than simply using statical values such as expectation, as the ultimate evaluation (or assessment) result.

(ii) UE can better quantify the extent of epistemic uncertainty than variance. It corresponds more to human intuitive thinking than variance.

(iii) An analytical model for UE contribution analysis is proposed, which provides an important tool for identifying the key attribute(s) that mostly affects the uncertainty of the result. UE is easy to calculate and suitable for high-dimensional space. Thus, it can be directly applied to the evaluation activities, especially those with limited resources to reduce the uncertainty in attributes. In this way, the efficiency of evaluation work can be improved.

References

[1] WANG H. Chinese maritime transformation and sea power: a net assessment. Baltimore: Johns Hopkins University, 2021.

[2] CHENG C H, YANG K L, HWANG C L. Evaluating attack helicopters by AHP based on linguistic variable weight. European Journal of Operational Research, 1999, 116(2): 423–435.

[3] LUO B Y, YE Y, YAO N, et al. Interval number ranking method based on multiple decision attitudes and its application in decision making. Soft Computing, 2021, 25(5): 4091–4101.

[4] JIANG J R, REN M, WANG J Q. Interval number multi-attribute decision-making method based on TOPSIS. Alexandria Engineering Journal, 2022, 61(7): 5059–5064.

[5] WAN S P, DONG J Y. Decision making theories and me-

thods based on interval-valued intuitionistic fuzzy sets. Berlin: Springer Nature, 2020.

[6]　WU Y N, XU H, XU C, et al. Uncertain multi-attributes decision making method based on interval number with probability distribution weighted operators and stochastic dominance degree. Knowledge-Based Systems, 2016, 113: 199–209.

[7]　HUANG F, YE Z, JIANG S H, et al. Uncertainty study of landslide susceptibility prediction considering the different attribute interval numbers of environmental factors and different data-based models. Catena, 2021, 202: 105250.

[8]　XU Z S. Dependent uncertain ordered weighted aggregation operators. Information Fusion, 2008, 9(2): 310–316.

[9]　BANDTE O. A probabilistic multi-criteria decision making technique for conceptual and preliminary aerospace systems design. Atlanta: Georgia Institute of Technology, 2000.

[10]　KIRBY M, MARIS D. A method for technology selection based on benefit, available schedule and budget resources. Proc. of the World Aviation Conference, 2000: 5563.

[11]　ALRWASHDEH S S, ALSARAIREH F M. Wind energy production assessment at different sites in Jordan using probability distribution functions. ARPN Journal of Engineering and Applied Sciences, 2018, 13(20): 8163–8172.

[12]　LIN S S, SHEN S L, ZHOU A, et al. Risk assessment and management of excavation system based on fuzzy set theory and machine learning methods. Automation in Construction, 2021, 122: 103490.

[13]　MARDANI A, HOOKER R E, OZKUL S, et al. Application of decision making and fuzzy sets theory to evaluate the healthcare and medical problems: a review of three decades of research with recent developments. Expert Systems with Applications, 2019, 137: 202–231.

[14]　HESAMIAN G, AKBARI M G. A robust multiple regression model based on fuzzy random variables. Journal of Computational and Applied Mathematics, 2021, 388: 113270.

[15]　GAO S, SUN H H, ZHAO L, et al. Dynamic assessment of island ecological environment sustainability under urbanization based on rough set, synthetic index and catastrophe progression analysis theories. Ocean & Coastal Management, 2019, 178: 104790.

[16]　KIM W, YIM C. No-reference image contrast quality assessment based on the degree of uniformity in probability distribution. IEIE Trans. on Smart Processing & Computing, 2022, 11(2): 85–91.

[17]　MEHDI K G. Assessment of distribution center locations using a multi-expert subjective−objective decision-making approach. Scientific Reports, 2021, 11(1): 1–19.

[18]　SPETZLER C, WINTER H, MEYER J. Decision quality: value creation from better business decision. New York: John Wiley & Sons, 2016.

[19]　HOWARD R A. The foundations of decision analysis. IEEE Trans. on Systems Science and Cybernetics, 1968, 4(3): 211–219.

[20]　SOBOL I M. Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates. Mathematics & Computers in Simulation, 2001, 55(1): 271–280.

[21]　KARUNANIDHI D, ARAVINTHASAMY P, SUBRAMANI T, et al. Chromium contamination in groundwater and Sobol sensitivity model based human health risk evaluation

from leather tanning industrial region of South India. Environmental Research, 2021, 199: 111238.

[22]　KARUNANIDHI D, ARAVINTHASAMY P, KUMAR D, et al. Sobol sensitivity approach for the appraisal of geomedical health risks associated with oral intake and dermal pathways of groundwater fluoride in a semi-arid region of South India. Ecotoxicology and Environmental Safety, 2020, 194: 110438.

[23]　LIU B D. Why is there a need for uncertainty theory. Journal of Uncertain Systems, 2012, 6(1): 3–10.

[24]　LIU B D. Uncertain urn problems and Ellsberg experiment. Soft Computing, 2019, 23: 6579–6584.

[25]　SIREGAR D, NURDIYANTO H, SRIADHI S, et al. Multiattribute decision making with VIKOR method for any purpose decision. Journal of Physics: Conference Series, 2018, 1019: 012034.

[26]　ZHANG K, ZHAN J M, WU W Z, et al. Fuzzy β-covering based (I, T)-fuzzy rough set models and applications to multiattribute decision-making. Computers & Industrial Engineering, 2019, 128: 605–621.

[27]　WANG J J, MA X, XU Z, et al. Three-way multi-attribute decision making under hesitant fuzzy environments. Information Sciences, 2021, 552: 328–351.

[28]　ALI Z, MAHMOOD T, SANTOS-GARCIA G. Heronian mean operators based on novel complex linear diophantine uncertain linguistic variables and their applications in multiattribute decision making. Mathematics, 2021, 9(21): 2730.

[29]　ROSE T. The end of average: how to succeed in a world that values sameness. London: Penguin UK, 2016.

[30]　LIU B D. Uncertainty theory: a branch of mathematics for modeling human uncertainty. Berlin: Springer-Verlag, 2011.

[31]　KIUREGHIAN A D, DITLEVSEN O. Aleatory or epistemic? Does it matter? Structural Safety, 2009, 31(2): 105−112.

[32]　REN B, LU Z Z, ZHOU C C, et al. A novel method for importance measure analysis in the presence of epistemic and aleatory uncertainties. Chinese Journal of Aeronautics, 2014, 27(120): 112–120.

[33]　KANG R, ZHANG Q Y, ZENG Z G, et al. Measuring reliability under epistemic uncertainty: review on non-probabilistic reliability metrics. Chinese Journal of Aeronautics, 2016, 29(3): 571–579.

[34]　DI BERNARDO R, MELLONE M, MINICHINI C, et al. Subjective probability: mathematical teachers' specialised knowledge in a betting game. Proc. of the 12th Congress of the European Society for Research in Mathematics Education, 2022. https://hal.science/hal-03748727/.

[35]　YAMAZAKI A. Recovering subjective probability distributions. Journal of Futures Markets, 2022, 42(7): 1234–1263.

[36]　YAKHYAEVA G, SKOKOVA V. Subjective expert evaluations in the model-theoretic representation of object domain knowledge. Proc. of the Russian Conference on Artificial Intelligence, 2021: 152−165.

[37]　KWAKERNAAK H. Fuzzy random variables —I. Definitions and theorems. Information Sciences, 1978, 15(1): 1–29.

[38]　LIU B D. Uncertainty theory. 4th ed. Berlin: Springer-Verlag, 2015.

[39]　HUBBARD D W. How to measure anything: finding the value of intangibles in business. New York: John Wiley & Sons, 2014.

[40] HU J W. Exploratory decision analysis. Beijing: National Defense Industry Press, 2020. (in Chinese)
[41] TALEB N N. The black swan: the impact of the highly improbable. 2nd ed. New York: Random House, 2007.
[42] HU J W, HU X, ZHANG W, et al. Monotonic indices space method and its application in the capability indices effectiveness analysis of a notional antistealth information system. IEEE Trans. on Systems, Man, and Cybernetics−Part A: Systems and Humans, 2009, 39(2): 404–413.
[43] GAO Y, TIAN Y L, LIU H, et al. Entropy based inverse design of aircraft mission success space in system-of-systems confrontation. Chinese Journal of Aeronautics, 2021, 34(12): 99–109.
[44] SURWIECKI J. The wisdom of crowds. New York: Anchor, 2005.

Biographies
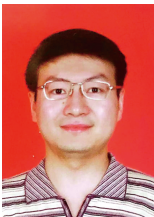
HU Jianwen was born in 1975. He received his bachelor degree from Air Force Early Warning Academy. He received his master degree and Ph.D. degree from National University of Defense Technology. He is a professor with Academy of Military Science. His research interests are exploratory decision theory and simulation experiment.
E-mail: hjwc3i@126.com

WANG Zhihui was born in 1995. He received his bachelor degree from Jiangnan University and master degree from Armored Forces Engineering Academy, PLA. He is a Ph.D. student of Academy of Military Science. He is also a lecturer at Army Academy of Artillery and Air Defense (Nanjing Campus). His research interests are exploratory decision theory, military operations, electrical engineering and autonomous systems.
E-mail: zhwang66@foxmail.com

GAO Yuan was born in 1993. He received his bachelor and master degrees from the Department of Aircraft Design, School of Aeronautic Science and Engineering, Beihang University. His research interests include machine learning, autonomous systems, and multi-agent modelling and simulation.
E-mail: gybuaa@buaa.edu.cn

ZHU Xiaomin was born in 1979. He received his Ph.D. degree in computer science from Fudan University in 2009. He is currently a professor with the College of Systems Engineering at National University of Defense Technology. His research interests include scheduling and resource management in distributed systems.
E-mail: xmzhu@nudt.edu.cn

TIAN Yongliang was born in 1985. He received his bachelor and Ph.D. degrees from Beihang University. He is now an associate professor at Beihang University. He is also a master's student advisor. His research interests include overall design of aircraft, system modeling and simulation, system engineering, and system effectiveness assessment.
E-mail: tianyl@buaa.edu.cn

LIU Hu was born in 1980. He received his bachelor and Ph.D. degrees from Beihang University. He is now a professor at Beihang University. He is also a doctoral supervisor. His main research areas are aircraft systematic design and operation support technology, including multi-mission optimization technology based on systematic effectiveness assessment, and virtual reality-based training and decision support technology.
E-mail: liuhu99@buaa.edu.cn