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Robust output regulation problem with prescribed performance for nonlinear strict feedback systems

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Abstract: This paper investigates the problem of robust output regulation control with prospected transient property for strict feedback systems. By employing the internal model principle, the robust output regulation problem with a prospected property can be transformed to a robust stabilization problem with a new output constraint. Then, by constructing the speed function and adopting barrier Lyapunov function technique, the dynamic feedback controller can be designed not only to drive error output of the closed-loop system entering into a prescribed performance bound within a given finite time, but also to achieve that the error output converges to zero asymptotically. The effectiveness of the results is illustrated by a simulation example.

Keywords: robust output regulation, nonlinear system, prescribed performance bound, speed function, finite time.

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1. Introduction

Robust output regulation (ROR) control aims at designing a feedback controller for a class of system with uncertain parameters to achieve asymptotic tracking for desired control inputs and/or asymptotic rejecting undesired signal. As a fundamental problem in control theory, the solvability conditions of the ROR problem are extensively studied in the overviews and monographs on this topic for details [1−5]. The key to tackle the ROR problem is the internal model principle. More recently, the solvability conditions for cooperative ROR problems of some multi-agent systems are also investigated by the internal model method [6−10]. However, the solvability conditions for ROR problem only address the steady state performance of regulation. To satisfy the constraints on system states and error outputs, the transient performance of the closed-loop system is also an important issue in the design of controllers for the ROR problem. Thus, the methods and techniques to guarantee prescribed transient performance are much more interesting and also more challenging in a lot of practical system.

In recent years, some control approaches have been constructed to restrict error output keeping in a prescribed performance bound [11−14]. Prescribed performance bound is adopted to classify the transient and steady state performance. The error output is confined to a given zone to satisfy the required limits. The barrier Lyapunov function (BLF) method is a powerful and widely used method to solve these prscribed constraints. For example, in [15,16], BLF was used to constructed adaptive controllers for tracking control problems of strict-feedback systems with time-unvarying and time-varying output constraints respectively. Also, BLF based adaptive control techniques have been applied to strict-feedback systems with partial state constraints [17] and full state constraints [18]. These BLF methods require that the initial value of the system should be inside the prescribed performance bound. However, many physical systems, e.g., robot manipulators, may require that error output converges to a desired bicompact zone in a prospected time interval to perform complicated tasks [19,20]. In [21], uncertain Euler-Lagrange systems with state constraints were designed to drive the error output to converge to a designed bicompact zone in a prospected time interval by constructing a speed function and adopting error transformation technique.

In this paper, the traditional ROR problem is extended to an ROR problem with finite-time prescribed performance. Not only the error of the system is regulated to zero asymptotically as for the traditional ROR problem, but also the error output is driven into a preset constraint bound after a given time interval to achieve the desired transient performance. A framework to tackle the ROR with finite-time prescribed performance is established by combining the speed function technique and the BLF

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technique. By employing the internal model principle and adopting a state transformation, the proposed ROR problem is transformed to a robust stabilization problem with output constraint. Based on a speed function and a BLF, a finite time prescribed performance control scheme is devised to deal with the robust stabilization problem, which also solves the ROR problem with finite-time preseted constraint. Under the framework of the ROR problem, not only tracking control problem but also disturbance rejection problem can be handled conveniently by finite time prescribed performance control. Compared to the existing results, the features of this paper can be highlighted as follows.

(i) Prescribed performance control technique is introduced into the output regulation problem which can tackle the tracking control problem and the disturbance rejection problem simultaneously. Comparing with most of the results on tracking control with prescribed performance which keep the tracking error in a prescribed performance bound but not converge to zero, the tracking error in output regulation framework tends to zero asymptotically.

(ii) The finite time prescribed performance control problem considered in this paper is more general than that in [22,23] by noting that some practical systems are required to follow the prospected paths with a desired convergence rate in a finite time.

(iii) The existing BLF methods require that the initial value of error output needs to be restricted in prescribed performance bound [11−13]. By designing finite time feedback controller based on the speed function and BLF, the initial value of error output can be given arbitrarily, which enlarges the degree of freedom in the design.

This paper is organized as follows. The ROR problem with finite time prescribed performance is formulated for a class of strict feedback nonlinear systems in Section 2. Section 3 establishes a framework to solve the ROR problem with finite time prescribed performance. In Section 4, a numerical example is presented to demonstrate the validity of our results. Section 5 gives concluding remarks.

2. Problem formulation and preliminaries

Consider a nonlinear uncertain strict feedback system described by

$$\begin{cases}\dot{z} = f(z,\bar{x}_1,v,\omega)\\ \dot{\bar{x}}_1 = f_1(z,\bar{x}_1,v,\omega)+b_1(v,\omega)\bar{x}_2\\ \dot{\bar{x}}_i = f_i(z,\bar{x}_1,\cdots,\bar{x}_i,v,\omega)+b_i(v,\omega)\bar{x}_{i+1},\ i=2,3,\cdots,n-1\\ \dot{\bar{x}}_n = f_n(z,\bar{x}_1,\cdots,\bar{x}_n,v,\omega)+b_n(v,\omega)u\end{cases} \tag{1}$$

where $z\in\mathbf{R}^m$, $\bar{x}_i\in\mathbf{R}^i(i=1,2,\cdots,n)$, is the system state, $\omega\in\mathbf{R}^\omega$ is the uncertain parameter, $u\in\mathbf{R}$ is the control input, $v\in\mathbf{R}^q$ is the exogenous signal representing the disturbance and reference input which is given by

$$\dot{v} = \boldsymbol{S}v. \tag{2}$$

Without loss of generality in the literature of nonlinear output regulation problem, the exosystem (2) is assumed to be neutrally stable, i.e., the eigenvalues of $\boldsymbol{S}$ are simple and have zero real parts. Also, it is assumed that, for $\forall i=1,2,\cdots,n$, the functions $f(z,\bar{x}_1,v,\omega)$ and $f_i(z,\bar{x}_1,\cdots,\bar{x}_i,v,\omega)$ are smooth with $f_i(0,\cdots,0,0,\omega)=0$ and $b_i(v,\omega)>0$. Define the error output of the system as

$$e = y-y_d = \bar{x}_1-y_d(v). \tag{3}$$

The control objective is to design a feedback control law

$$\begin{cases}\dot{\eta}_i = \boldsymbol{M}_i\eta_i+\boldsymbol{N}_i\bar{x}_{i+1},\ \ i=1,2,\cdots,n-1\\ \dot{\eta}_n = \boldsymbol{M}_n\eta_n+\boldsymbol{N}_n u\\ u = k(\eta_1,\cdots,\eta_n,\bar{x}_1,\cdots,\bar{x}_n,e)\end{cases} \tag{4}$$

such that the output of the closed-loop system $y=\bar{x}_1$ will asymptotically track a reference trajectory $y_d(v)$ with desired transient performance, where $k(\cdot)$ is a nonlinear function and $(\boldsymbol{M}_i,\boldsymbol{N}_i)$ is a pair of matrices to be designed later. The control problem can be formulated as an output regulation problem with finite time prescribed performance as follows.

Robust output regulation with finite time prescribed performance control: consider the uncertain nonlinear strict feedback system (1), the exosystem (2), and the error output (3), design a feedback control law in the form of (4) such that the closed-loop system consisting of (1), (2), and (4) has the following three properties:

Property 1 The trajectories of the closed-loop system are bounded for all $t\geqslant 0$.

Property 2 The system output $y=x_1$ tracks a reference trajectory $y_d(v)$ asymptotically, i.e.,

$$\lim_{t\to\infty}e(t) = \lim_{t\to\infty}(y(t)-y_d(v)) = 0.$$

Property 3 Within a given finite time T, the transient performance of the tracking error $e(t)$ satisfies a prescribed performance bound as follows:

$$\begin{cases}-c\rho(t)<e(t)<c\rho(t), \quad \forall t\geqslant T\\ \rho(t) = (\rho_0-\rho_\infty)\exp(-lt)+\rho_\infty\end{cases}$$

where ρ_0, ρ_∞ and l are strictly positive real numbers, and $0<c\ll 1$ is a design parameter.

Remark 1 The ROR problem with finite time prescribed performance is an extension of the traditional output regulation problem by addressing the transient performance of the closed-loop output error $e(t)$. In fact, without Property 3, it is exactly a traditional prospected time interval problem which is extensively investigated in the literature. Property 3 defines a performance boundary $c\rho(t)$ for the transient performance of the closed-loop output error $e(t)$ and a prescribed finite time interval T. The error output $e(t)$ is required to be constrained in a given compact which is predefined by the performance bound function $c\rho(t)$ in a desired time interval T at the pre-set rate of convergence.

The following assumptions are necessary to derive the main results in this paper.

Assumption 1 There exists a sufficiently smooth function $z(v,\omega)$ with $z(0,0)=0$ such that, for all $v\in\mathbf{R}^q$, $\omega\in\mathbf{R}^\omega$,

$$\frac{\partial z(v,\omega)}{\partial v}\boldsymbol{S}v = f(z(v,\omega),y_d(v),v,\omega),\quad \forall v\in\mathbf{R}^q. \tag{5}$$

Under Assumption 1, let $\bar{x}_1(v,\omega)=y_d(v)$, and we can define

$$\bar{x}_{i+1}(v,\omega)=\frac{1}{b_i(v,\omega)}\{\frac{\partial\bar{x}_i(v,\omega)}{\partial v}\boldsymbol{S}v- f_i(z(v,\omega),\bar{x}_1(v,\omega),\cdots,\bar{x}_i(v,\omega),v,\omega)\},\ \ 1\leqslant i\leqslant n.$$

Assumption 2 For $i=1,2,\cdots,n$, there exist an integer m_i and a sufficiently smooth function $\pi_i:\mathbf{R}^{q+\omega}\mapsto\mathbf{R}^{m_i}$ vanishing at the origin, and matrices $\boldsymbol{F}_i\in\mathbf{R}^{m_i\times m_i}$, column vector $\boldsymbol{\chi}_i\in\mathbf{R}^{m_i\times 1}$ such that for all trajectories $v(t)$ of the exosystem (2) and all $\omega\in\mathbf{R}^\omega$, there is

$$\frac{\mathrm{d}\pi_i(v,\omega)}{\mathrm{d}t}=\boldsymbol{F}_i\pi_i(v,\omega),\ \ \bar{x}_{i+1}(v,\omega)=\boldsymbol{\chi}_i\pi_i(v,\omega).$$

Furthermore, the pair $(\boldsymbol{F}_i,\boldsymbol{\chi}_i)$ is observable and all the eigenvalues of $\boldsymbol{F}_i$ are simple with zero real part.

Remark 2 Assumption 1 and Assumption 2 are quite standard in the literature of robust nonlinear output regulation. Assumption 1 is to guarantee the existence of the solution of the regulator equation which is necessary for solvability of the nonlinear output regulation problem. Assumption 2 is to construct an internal model to handle the uncertainties of the robust nonlinear output regulation problem.

For $i=1,2,\cdots,n$, choose a Hurwitz matrix $\boldsymbol{M}_i\in\mathbf{R}^{m_i\times m_i}$ and a column vector $\boldsymbol{N}_i\in\mathbf{R}^{m_i\times 1}$ satisfying $(\boldsymbol{M}_i,\boldsymbol{N}_i)$ is controllable. $\boldsymbol{T}_i$ is the solution of the Sylvester equation $\boldsymbol{T}_i\boldsymbol{F}_i-\boldsymbol{M}_i\boldsymbol{T}_i=\boldsymbol{N}_i\boldsymbol{\chi}_i$, which is a nonsingular matrix of dimension m_i. Let $\boldsymbol{\theta}_i=\boldsymbol{T}_i\pi_i$, and $\boldsymbol{\gamma}_i(\boldsymbol{\theta}_i)=\boldsymbol{\chi}_i\boldsymbol{T}_i^{-1}\boldsymbol{\theta}_i$. Then an internal model of system in (1) with output $\bar{x}_{i+1}$ are given as follows:

$$\dot{\boldsymbol{\eta}}_i=\boldsymbol{M}_i\eta_i+\boldsymbol{N}_i\bar{x}_{i+1},\ \ i=1,2,\cdots,n. \tag{6}$$

Next, we perform on the system (1) and (6) the following input and coordinate transformation:

$$\begin{cases} z_0=z-z(v,\omega)\\ \bar{\boldsymbol{\eta}}_i=\boldsymbol{\eta}_i-\boldsymbol{\theta}_i(v,\omega),\ \ i=1,2,\cdots,r,\\ x_1=\bar{x}_1-\bar{x}_1(v,\omega)=e\\ x_{i+1}=\bar{x}_{i+1}-\gamma_i(\eta_i),\ \ i=1,2,\cdots,n-1\\ z_i=\boldsymbol{\eta}_i-\boldsymbol{\theta}_i(v,\omega)-\boldsymbol{N}_ib_i^{-1}(v,\omega)x_i,\ \ i=1,2,\cdots,n\\ \bar{\boldsymbol{u}}=\boldsymbol{u}-\boldsymbol{\gamma}_n(\boldsymbol{\eta}_n)\end{cases}, \tag{7}$$

the following system can be obtained

$$\begin{cases}\dot{z}_0=f_0(z_0,x_1,d(t))\\ \dot{z}_i=H_i(z_0,z_1,\cdots,z_i,x_1,\cdots,x_i,d(t))\\ \dot{x}_i=f_i(z_0,z_1,\cdots,z_i,x_1,\cdots,x_i,d(t))+b_i(v,\omega)x_{i+1}\end{cases} \tag{8}$$

where $i=1,2,\cdots,n$, $d(t)=(v(t),\omega)$, $x_{n+1}=\bar{u}$, and

$$\begin{cases} f_0(z_0,x_1,d(t))=f(z_0+z(v,\omega),x_1+\bar{x}_1(v,\omega),v,\omega)-\\ \quad f(z(v,\omega),\bar{x}_1(v,\omega),v,\omega)\\ H_i(z_0,z_1,\cdots,z_i,x_1,\cdots,x_i,d(t))=\boldsymbol{M}_iz_i+\boldsymbol{M}_i\boldsymbol{N}_ib_i^{-1}x_i-\\ \quad \boldsymbol{N}_i\frac{\partial b_i^{-1}(v,\omega)}{\partial v}\boldsymbol{S}vx_i-\boldsymbol{N}_ib_i^{-1}(v,\omega)\dot{x}_i\\ f_1(z_0,z_1,x_1,d(t))=\bar{f}_1(z_0+z(v,\omega),x_1+\bar{x}_1(v,\omega),v,\omega)-\\ \quad \bar{f}_1(z(v,\omega),\bar{x}_1(v,\omega),v,\omega)+b_1(v,\omega)\gamma_1(\eta_1)-\\ \quad b_1(v,\omega)\boldsymbol{\gamma}_1(\boldsymbol{\theta}_1)\\ f_i(z_0,z_1,\cdots,z_i,x_1,\cdots,x_i,d(t))=\bar{f}_i(z_0+z(v,\omega),x_1+\\ \quad \bar{x}_1(v,\omega),x_2+\boldsymbol{\gamma}_1(z_1+\theta_2+\boldsymbol{N}_2b_2^{-1}),\cdots,x_i+\boldsymbol{\gamma}_{i-1}(z_{i-1}+\\ \quad \boldsymbol{\theta}_i+\boldsymbol{N}_ib_i^{-1}))-\frac{\partial\boldsymbol{\gamma}_{i-1}(\boldsymbol{\theta}_{i-1})}{\partial\boldsymbol{\theta}_{i-1}}\dot{\boldsymbol{\theta}}_{i-1}\end{cases} \tag{9}$$

where the functions f_0,H_i,f_i are sufficiently smooth in their arguments and we can get that $H_i(0,0,\cdots,0,x_1,d(t))=0, f_i(0,0,\cdots,0,d(t))=0$. Thus, the ROR problem with finite time prescribed performance is converted to the constraint stabilization problem in (8).

Remark 3 After the aforementioned coordinate and input transformation, if we can make the equilibrium of the system (8) at $x=0$ globally asymptotically stable for any $d(t)\in V_0\times\mathbf{R}^\omega$ and $-c\rho(t)<x_1(t)<c\rho(t),\ t\geqslant T$, the output regulation problem with finite time prospected property can be achieved.

Assumption 3 There exists a C^1 function $\bar{V}_0(z_0,t)$ satisfying $\underline{\mu}(\|z_0\|)\leqslant\bar{V}_0(z_0,t)\leqslant\bar{\mu}(\|z_0\|)$ for some class $\mathcal{K}_\infty$ function $\underline{\mu}(\cdot)$ and $\bar{\mu}(\cdot)$, such that, along the trajectory of $\dot{z}_0=f_0(z_0,x_1,d(t))$,

$$\frac{\mathrm{d}\bar{V}_0(z_0,t)}{\mathrm{d}t}\leqslant-\|z_0\|^2+\delta_0(x_1) \tag{10}$$

for some known smooth positive definite function $\delta_0(x_1)$.

Assumption 4 For $i=1,2,\cdots,n$, there exists a C^1 function $\bar{V}_i(z_i,t)$ satisfying $\underline{\mu}(\|z_i\|)\leqslant\bar{V}_i(z_i,t)\leqslant\bar{\mu}(\|z_i\|)$ for some class $\mathcal{K}_\infty$ function $\underline{\mu}_i(\cdot)$ and $\bar{\mu}_i(\cdot)$, such that, along the trajectory of

$$\begin{cases}\dot{z}_i = H_i(z_0, z_1, \cdots, z_{i-1}, x_1, \cdots, x_i, d(t)) \\ \dfrac{\mathrm{d}\bar{V}_i(z_i, t)}{\mathrm{d}t} \leqslant -\|z_i\|^2 + \delta_i(z_0, z_1, \cdots, z_{i-1}, x_1, \cdots, x_i)\end{cases} \tag{11}$$

for some known smooth positive definite function $\delta_i(z_0, z_1, \cdots, z_i, x_1, \cdots, x_i)$.

Remark 4 According to Assumption 3 and Assumption 4, it can be obtained that the system $\dot{z}_0 = f_0(z_0, x_1, d(t))$ and $\dot{z}_i = H_i(z_0, z_1, \cdots, z_i, x_1, \cdots, x_i, d(t))$ are input-to-state stable. From changing supply rate technique, given any smooth function $\triangle_0(z_0) \geqslant 0$, there exists a C^1 function $V_0(z_0, t)$, satisfying $\underline{\xi}_0(\|z_0\|) \leqslant \bar{V}_0(z_0, t) \leqslant \bar{\xi}_0(\|z_0\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\xi}_0$ and $\bar{\xi}_0$, such that, along trajectory of system

$$\begin{cases}\dot{z}_0 = f_0(z_0, x_1, d(t)) \\ \dfrac{\mathrm{d}V_0(z_0, t)}{\mathrm{d}t} \leqslant -\|z_0\|^2 \triangle_0(z_0) + x_1^2 s_{01}(x_1)\end{cases} \tag{12}$$

for some known smooth function $s_{01}(x_1) \geqslant 1$. Similarly, for $i = 1, 2, \cdots, n$, given any smooth function $\triangle_i(z_i) \geqslant 0$, there exists a C^1 function $V_i(z_i, t)$, satisfying $\underline{\xi}_i(\|z_i\|) \leqslant \bar{V}_i(z_i, t) \leqslant \bar{\xi}_i(\|z_i\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\xi}_i$ and $\bar{\xi}_i$, such that, along trajectory of system

$$\begin{cases}\dot{z}_i = H_i(z_0, z_1, \cdots, z_i, x_1, \cdots, x_i, d(t)) \\ \dfrac{\mathrm{d}V_i(z_i, t)}{\mathrm{d}t} \leqslant -\|z_i\|^2 \triangle_i(z_i) + \displaystyle\sum_{j=0}^{i-1} \|z_j\|^2 \bar{h}_{ij}(z_j) + \sum_{j=1}^{i} x_j^2 \bar{s}_{ij}(x_j)\end{cases} \tag{13}$$

for some known smooth functions $\bar{h}_{ij}(z_j) \geqslant 1$, $s_{ij}(x_j) \geqslant 1$.

3. Main results

In this part, we construct a state feedback controller to solve the ROR with prescribed performance in a given finite time for the system (1), (2), and (3).

To proceed, we give some notations and an assumption which are used later.

A rate function is given as follows:

$$\bar{\chi}(t) = \begin{cases}\left(\dfrac{T}{T-t}\right)^4 \chi(t), & 0 \leqslant t < T \\ \infty, & t \geqslant T\end{cases} \tag{14}$$

where $0 < T < \infty$ is the preset time interal, and $\chi(t)$ is a smooth and non-decreasing function with $\chi(0) = 1$ and $\dot{\chi} \geqslant 0$. It needs to be noticed that, when $t \geqslant T$, $\bar{\chi}(t) = \infty$.

Assumption 5 The finite time T is designed to satisfy $T > T_c$. T_c is a small time interval which is necessary for signal computing and transmission.

With (14) and Assumption 5, the speed function $\beta(t)$ is given as follows:

$$\beta(t) = \frac{1}{(1-c)\bar{\chi}(t)^{-1} + c} \tag{15}$$

where c is a given parameter meeting $0 < c \ll 1$. Based on the expression of $\bar{\chi}(t)$ in (14), we obtain

$$\beta(t) = \begin{cases}\dfrac{T^4 \chi(t)}{(1-c)(T-t)^4 + cT^4 \chi(t)}, & 0 \leqslant t < T \\ \dfrac{1}{c}, & t \geqslant T\end{cases}. \tag{16}$$

The properties of the speed function $\beta(t)$ are listed in [21].

To achieve the proposed control objectives, performing on the system (8) the following coordinate transformation

$$\begin{cases}\widehat{x}_i = x_i - \alpha_{i-1} \\ \widetilde{x}_1 = \beta(t)\widehat{x}_1 \\ \widetilde{x}_i = \widehat{x}_i,\ i = 2, 3, \cdots, n\end{cases}, \tag{17}$$

we obtain

$$\begin{cases}\dot{\widetilde{x}}_1 = \dot{\beta}(t)(x_1 - \alpha_0) + \beta(t)(\dot{x}_1 - \dot{\alpha}_0) \\ \dot{\widetilde{x}}_i = \dot{x}_i - \dot{\alpha}_{i-1},\ i = 2, 3, \cdots, n\end{cases} \tag{18}$$

where $\alpha_0 = 0$ and α_{i-1} are the virtual control input which will be given later. $\beta(t)$ is the speed function which is introduced to guarantee the error output converges to a designed set in a finite time.

Let $\boldsymbol{X}_i = \mathrm{col}(z_0, z_1, \cdots, z_i, \widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_i)$ for $1 \leqslant i \leqslant n$, the control scheme is designed by the following steps.

Step 1 Define $U_1(\boldsymbol{X}_1, t) = V_0(z_0, t) + V_1(z_1, t) + \widetilde{V}_1(\widetilde{x}_1)$.

Consider the following Lyapunov function candidate:

$$\widetilde{V}_1 = \log \frac{\rho^2(t)}{\rho^2(t) - \widetilde{x}_1^2(t)} \tag{19}$$

for $\widetilde{x}_1 \in V_1$, where $V_1 = \{\widetilde{x}_1 \in \mathbf{R} \mid |\widetilde{x}_1| < \rho\}$. $V_1(\widetilde{x}_1)$ is positive definite and C^1 continuous for all $\widetilde{x}_1 \in V_1$, and $V_1(\widetilde{x}_1) \to \infty$ as $|\varepsilon_1| \to \rho$. Then, the derivative of $\widetilde{V}_1$ is given by

$$\dot{\widetilde{V}}_1 = \frac{2\widetilde{x}_1}{\rho^2(t) - \widetilde{x}_1^2}\left(\dot{\widetilde{x}}_1 - \widetilde{x}_1 \frac{\dot{\rho}(t)}{\rho(t)}\right) = \frac{2\beta(t)\widetilde{x}_1}{\rho^2(t) - \widetilde{x}_1^2}\left(\frac{\dot{\beta}}{\beta^2}\widetilde{x}_1 + f_1\left(z_0, z_1, \frac{\widetilde{x}_1}{\beta}, d(t)\right) + b_1(v, \omega) \cdot (\widetilde{x}_2 + \alpha_1) - \dot{\alpha}_0 - \widetilde{x}_1 \frac{\dot{\rho}(t)}{\rho(t)}\right). \tag{20}$$

Let $g_1(z_0, z_1, \widetilde{x}_1, d(t)) = f_1\left(z_0, z_1, \dfrac{\widetilde{x}_1}{\beta}, d(t)\right) - \dot{\alpha}_0$ It is obvious that $g_1(0, 0, 0, d(t)) = 0$. According to Lemma 7.8 in [2], we can obtain

$$|g_1(z_0, z_1, \widetilde{x}_1, d(t))| \leqslant \|z_0\|\psi_{10}(z_0) + \|z_1\|\psi_{11}(z_1) + \|\widetilde{x}_1\|\phi_{11}(\widetilde{x}_1) \tag{21}$$

where $\psi_{10}(z_0)$, $\psi_{11}(z_1)$, and $\phi_{11}(z_1)$ are smooth positive functions.

Thus, according to (12), (13), and (20) with $i=1$, we can obtain

$$\dot{U}_1 = \dot{V}_0 + \dot{V}_1 + \dot{\widetilde{V}}_1 \leqslant -\|z_0\|^2 \triangle_0(z_0) + x_1^2 s_{01}(x_1) - \|z_1\|^2 \triangle_1(z_1) + \|z_0\|^2 \bar{h}_{10}(z_0) + x_1^2 \bar{s}_{11}(x_1) + \frac{2\beta\widetilde{x}_1}{\rho^2(t)-\widetilde{x}_1^2}(\|z_0\|\psi_{10}(z_0) + \|z_1\|\psi_{11}(z_1) + \|\widetilde{x}_1\|\phi_{11}(\widetilde{x}_1) + b_1(v,\omega)(\widetilde{x}_2+\alpha_1) - x_1\frac{\dot{\rho}(t)}{\rho(t)} + \frac{\dot{\beta}}{\beta^2}\widetilde{x}_1). \tag{22}$$

Design the stabilizing function $\alpha_1(\widetilde{x}_1)$ as

$$\alpha_1(\widetilde{x}_1) = -\frac{\rho^2(t)-\widetilde{x}_1^2}{2}k_1\zeta_1(\widetilde{x}_1)\widetilde{x}_1 \tag{23}$$

where k_1 is a positive constant and $\zeta_1(\widetilde{x}_1)$ is some smooth nonnegative function to be given later. Substituting (23) into (22), we have

$$\dot{U}_1 \leqslant -(\triangle_0(z_0) - \bar{h}_{10}(z_0) - \psi_{10}^2(z_0))\|z_0\|^2 - (\triangle_1(z_1) - \psi_{11}^2(z_1))\|z_1\|^2 + \{s_{01}(x_1) + \bar{s}_{11}(x_1) + \frac{2\dot{\beta}}{\beta(\rho^2(t)-\widetilde{x}_1^2)} + \frac{2\beta^2+\beta^2\bar{b}_1^2}{(\rho^2(t)-\widetilde{x}_1^2)^2} + \frac{2\beta\rho(t)\phi_{11}(\widetilde{x}_1) - 2\dot{\rho}(t)}{\rho(t)(\rho^2(t)-\widetilde{x}_1^2)} - k_1\beta^2\underline{b}_1\zeta_1(\widetilde{x}_1)\}\widetilde{x}_1^2 + \widetilde{x}_2^2. \tag{24}$$

Then, according to Remark 4, we can choose $\triangle_0(z_0) \geqslant \bar{h}_{10}(z_0) + \psi_{10}^2(z_0) + 1$, $\triangle_1(z_1) \geqslant \psi_{11}^2(z_1) + 1$, and let

$$k_1\zeta_1(\widetilde{x}_1) \geqslant \frac{1}{\underline{b}_1}(s_{01}(x_1) + \bar{s}_{11}(x_1) + \frac{2\dot{\beta}}{\beta(\rho^2(t)-\widetilde{x}_1^2)} + \frac{2\beta^2+\beta^2 b_1^2}{(\rho^2(t)-\widetilde{x}_1^2)^2} + \frac{2\beta\rho(t)\phi_{11}(\widetilde{x}_1) - 2\dot{\rho}(t)}{\rho(t)(\rho^2(t)-\widetilde{x}_1^2)} + 1) \tag{25}$$

where $\underline{b}_1 \leqslant b_1$. It yields

$$\dot{U}_1 \leqslant -\|\boldsymbol{X}_1\|^2 + \widetilde{x}_2^2. \tag{26}$$

Based on changing supply rate technique and given any smooth function $\widetilde{\triangle}_1(X_1) \geqslant 0$, there exists a C^1 function $\widetilde{U}_1(\boldsymbol{X}_1,t)$, satisfying $\underline{\varrho}_{10}(\|\boldsymbol{X}_1\|) \leqslant \widetilde{U}_1(\boldsymbol{X}_1,t) \leqslant \bar{\varrho}_{10}(\|\boldsymbol{X}_1\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\varrho}_{10}$ and $\bar{\varrho}_{10}$, such that

$$\dot{\widetilde{U}}_1 \leqslant -\widetilde{\triangle}_1\|\boldsymbol{X}_1\|^2 + l_1(\widetilde{x}_2)\widetilde{x}_2^2, \tag{27}$$

for some known smooth function $l_1(\widetilde{x}_2) \geqslant 1$.

Step 2 Let $\widetilde{V}_2(\widetilde{x}_2) = \widetilde{x}_2^2$, the derivative of $\widetilde{V}_2$ is given by

$$\dot{\widetilde{V}}_2 = 2\widetilde{x}_2(f_2(z_0,z_1,z_2,\frac{\widetilde{x}_1}{\beta},\widetilde{x}_2+\alpha_1,d(t)) + b_2(v,\omega)(\widetilde{x}_3+\alpha_2) - \dot{\alpha}_1) \leqslant 2|\widetilde{x}_2|\|f_2(z_0,z_1,z_2,\frac{\widetilde{x}_1}{\beta},\widetilde{x}_2+\alpha_1,d(t)) - \dot{\alpha}_1| + 2\bar{b}_2|\widetilde{x}_2||\widetilde{x}_3| + 2b_2\alpha_2\widetilde{x}_2 \tag{28}$$

where $b_2 \leqslant \bar{b}_2$. Let $g_2(z_0,z_1,\widetilde{x}_1/\beta,\widetilde{x}_2+\alpha_1,d(t)) = f_2(z_0,z_1,\widetilde{x}_1/\beta,\widetilde{x}_2+\alpha_1,d(t)) - \dot{\alpha}_1$. Similar to Step 1, we can obtain

$$|g_2(z_0,z_1,\frac{\widetilde{x}_1}{\beta},\widetilde{x}_2+\alpha_1,d(t))| \leqslant \sum_{j=0}^{2}\psi_{2j}(z_j)\|z_j\| + \sum_{j=0}^{2}\phi_{2j}\widetilde{x}_j\|\widetilde{x}_j\| \tag{29}$$

where $\psi_{2j}(z_j)$ and $\phi_{2j}(z_j)$ are smooth positive functions. Thus

$$\dot{\widetilde{V}}_2 \leqslant \sum_{j=0}^{2}\psi_{2j}(z_j)\|z_j\|^2 + \sum_{j=0}^{2}\phi_{2j}\widetilde{x}_j\|\widetilde{x}_j\|^2 + \widetilde{x}_3^2 + 2b_2(t)\alpha_2\widetilde{x}_2. \tag{30}$$

Define $U_2(\boldsymbol{X}_2,t) = \widetilde{U}_1(\boldsymbol{X}_1,t) + V_2(z_2,t) + \widetilde{V}_2(\widetilde{x}_2)$ and design the stabilizing function

$$\alpha_2(\widetilde{x}_2) = -k_2\zeta_2(\widetilde{x}_2)\widetilde{x}_2 \tag{31}$$

where k_2 is a positive constant and $\zeta_2(\widetilde{x}_2)$ is some smooth nonnegative function to be given later. By inequality (13), (27), and (30) with $i=2$, the derivative of $U_2(\boldsymbol{X}_2,t)$ is given by

$$\dot{U}_2 = \dot{\widetilde{U}}_1(\boldsymbol{X}_1,t) + \dot{V}_2 + \dot{\widetilde{V}}_2 \leqslant -\widetilde{\triangle}_1\|\boldsymbol{X}_1\|^2 + l_1(\widetilde{x}_2)\widetilde{x}_2^2 - \|z_2\|^2\triangle_2(z_2) + \sum_{j=0}^{1}\|z_j\|^2\bar{h}_{2j}(z_j) + \sum_{j=1}^{2}x_j^2\bar{s}_{2j}(x_j) + \sum_{j=0}^{2}\psi_{2j}(z_j)\cdot\|z_j\|^2 + \sum_{j=1}^{2}\phi_{2j}\widetilde{x}_j\|\widetilde{x}_j\|^2 - 2\underline{b}_2(t)k_2\zeta_2(\widetilde{x}_2)\widetilde{x}_2^2 + \widetilde{x}_3^2 \leqslant -(\widetilde{\triangle}_1 - \bar{h}_{20}(z_0) - \psi_{20}(z_0) - \bar{h}_{21}(z_1) - \psi_{21}(z_1) - \bar{s}_{21}(x_1) - \phi_{21}(\widetilde{x}_1))\|\boldsymbol{X}_1\|^2 - (\triangle_2(z_2) - \psi_{22}(z_2))\cdot\|z_2\|^2 + (l_1(\widetilde{x}_2) + \bar{s}_{22}(\widetilde{x}_2) + \phi_{22}(\widetilde{x}_2) - 2\underline{b}_2(t)k_2\zeta_2(\widetilde{x}_2))\widetilde{x}_2^2 + \widetilde{x}_3^2. \tag{32}$$

Then, we choose $\widetilde{\triangle}_1 \geqslant \bar{h}_{20}(z_0) - \psi_{20}(z_0) - \bar{h}_{21}(z_1) - \psi_{21}(z_1) - \bar{s}_{21}(\widetilde{x}_1) - \phi_{21}(\widetilde{x}_1) + 1$, $\triangle_2(z_2) \geqslant \psi_{22}(z_2) + 1$, and let

$$k_2\zeta_2(\widetilde{x}_2) \geqslant \frac{1}{2\underline{b}_2(t)}(l_1(\widetilde{x}_2) + \bar{s}_{22}(\widetilde{x}_2) + \phi_{22}(\widetilde{x}_2) + 1)$$

where $\underline{b}_2 \leqslant b_2$. It obtains

$$\dot{U}_2 \leqslant -\|\boldsymbol{X}_2\|^2 + \widetilde{x}_3^2. \tag{33}$$

Similarly, based on changing supply rate technique and given any smooth function $\widetilde{\triangle}_2(\boldsymbol{X}_2) \geqslant 0$, there exists a C^1 function $\widetilde{U}_2(\boldsymbol{X}_2,t)$, satisfying $\underline{\varrho}_{20}(\|\boldsymbol{X}_2\|) \leqslant \widetilde{U}_2(\boldsymbol{X}_2,t) \leqslant \bar{\varrho}_{20}(\|\boldsymbol{X}_2\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\varrho}_{20}$ and $\bar{\varrho}_{20}$, such that

$$\dot{\widetilde{U}}_2 \leqslant -\widetilde{\triangle}_2\|\boldsymbol{X}_2\|^2 + l_2(\widetilde{x}_3)\widetilde{x}_3^2 \tag{34}$$

for some known smooth function $l_2(\widetilde{x}_3) \geqslant 1$.

Step *i* (3⩽*i*⩽*n* − 1) Let $\widetilde{V}_i(\widetilde{x}_i) = \widetilde{x}_i^2$, similarly to Step 2, we can obtain $\dot{\widetilde{x}}_i$ satisfying inequality as follows:

$$\dot{\widetilde{V}}_i \leqslant \sum_{j=0}^{i}\psi_{ij}(z_j)\|z_j\|^2 + \sum_{j=0}^{i}\phi_{ij}(\widetilde{x}_j)\|\widetilde{x}_j\|^2 + 2b_2(t)\alpha_i\widetilde{x}_i + \widetilde{x}_{i+1}^2 \tag{35}$$

where $\psi_{ij}(z_j)$ and $\phi_{ij}(z_j)$ are smooth positive functions.

Define $U_i(\boldsymbol{X}_i,t) = \widetilde{U}_{i-1}(\boldsymbol{X}_{i-1},t) + V_i(z_i,t) + \widetilde{V}_i(\widetilde{x}_i)$ and design the stabilizing function

$$\alpha_i(\widetilde{x}_i) = -k_i\zeta_i(\widetilde{x}_i)\widetilde{x}_i \tag{36}$$

where k_i is a positive constant and $\zeta_i(\widetilde{x}_i)$ is some smooth nonnegative function to be given later. According to inequality (13), (34), and (35), we can obtain the derivative of $U_i(X_i,t)$

$$\begin{aligned}\dot{U}_i &= \dot{\widetilde{U}}_{i-1}(\boldsymbol{X}_{i-1},t) + \dot{V}_i + \dot{\widetilde{V}}_i \leqslant \\ &-\widetilde{\triangle}_{i-1}\|\boldsymbol{X}_{i-1}\|^2 + l_{i-1}(\widetilde{x}_i)\widetilde{x}_i^2 - \|z_i\|^2\triangle_i(z_i) + \\ &\sum_{j=0}^{i-1}\|z_j\|^2\bar{h}_{ij}(z_j) + \sum_{j=1}^{i}x_j^2\bar{s}_{ij}(x_j) + \sum_{j=0}^{i}\psi_{ij}(z_j)\|z_j\|^2 + \\ &\sum_{j=1}^{i}\phi_{ij}(\widetilde{x}_j)\|\widetilde{x}_j\|^2 - 2\underline{b}_i(t)k_i\zeta_i(\widetilde{x}_i)\widetilde{x}_i^2 + \widetilde{x}_{i+1}^2 \leqslant \\ &-(\widetilde{\triangle}_{i-1} - \sum_{j=0}^{i-1}(\bar{h}_{ij}(z_j) + \psi_{ij}(z_j)) - \sum_{j=1}^{i}(x_j^2\bar{s}_{ij}(x_j) + \\ &\phi_{ij}(\widetilde{x}_j)))\|\boldsymbol{X}_{i-1}\|^2 - (\triangle_i(z_i) - \psi_{ii}(z_i)))\|z_i\|^2 + \\ &(l_{i-1}(\widetilde{x}_i) + \bar{s}_{ii}(\widetilde{x}_i) + \phi_{ii}(\widetilde{x}_i) - 2\underline{b}_i(t)k_i\zeta_i(\widetilde{x}_i))\widetilde{x}_i^2 + \widetilde{x}_{i+1}^2.\end{aligned} \tag{37}$$

Then, we choose $\widetilde{\triangle}_{i-1} \geqslant \sum_{j=0}^{i-1}(\bar{h}_{ij}(z_j) + \psi_{ij}(z_j)) + \sum_{j=1}^{i}(x_j^2\bar{s}_{ij}(x_j) + \phi_{ij}(\widetilde{x}_j)) + 1$, $\triangle_i(z_i) \geqslant \psi_{ii}(z_i) + 1$, and let

$$k_i\zeta_i(\widetilde{x}_i) \geqslant \frac{1}{2\underline{b}_i(t)}(l_{i-1}(\widetilde{x}_i) + \bar{s}_{ii}(\widetilde{x}_i) + \phi_{ii}(\widetilde{x}_i) + 1)$$

where $\underline{b}_i \leqslant b_i$. It obtains

$$\dot{U}_i \leqslant -\|\boldsymbol{X}_i\|^2 + \widetilde{x}_{i+1}^2. \tag{38}$$

Similarly, based on changing supply rate technique and given any smooth function $\widetilde{\triangle}_i(\boldsymbol{X}_i) \geqslant 0$, there exists a C^1 function $\widetilde{U}_i(\boldsymbol{X}_i,t)$, satisfying $\underline{\varrho}_{i0}(\|\boldsymbol{X}_i\|) \leqslant \widetilde{U}_i(\boldsymbol{X}_i,t) \leqslant \bar{\varrho}_{i0}(\|\boldsymbol{X}_i\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\varrho}_{i0}$ and $\bar{\varrho}_{i0}$, such that

$$\dot{\widetilde{U}}_i \leqslant -\widetilde{\triangle}_i\|\boldsymbol{X}_i\|^2 + l_i(\widetilde{x}_{i+1})\widetilde{x}_{i+1}^2 \tag{39}$$

for some known smooth function $l_i(\widetilde{x}_{i+1}) \geqslant 1$.

Step *n* Let $\widetilde{V}_n(\widetilde{x}_n) = \widetilde{x}_n^2$, similarly, we can obtain $\dot{\widetilde{x}}_n$ satisfying inequality as follows:

$$\dot{\widetilde{V}}_n \leqslant \sum_{j=0}^{n}\psi_{nj}(z_j)\|z_j\|^2 + \sum_{j=0}^{n}\phi_{nj}(\widetilde{x}_j)\|\widetilde{x}_j\|^2 + 2b_2(t)\alpha_n\widetilde{x}_n \tag{40}$$

where $\psi_{nj}(z_j)$ and $\phi_{nj}(z_j)$ are smooth positive functions.

Define $U_n(\boldsymbol{X}_n,t) = \widetilde{U}_{n-1}(\boldsymbol{X}_{n-1},t) + V_n(z_n,t) + \widetilde{V}_n(\widetilde{x}_n)$ and design the control law

$$\alpha_n(\widetilde{x}_n) = -k_n\zeta_n(\widetilde{x}_n)\widetilde{x}_n \tag{41}$$

where k_n is a positive constant and $\zeta_n(\widetilde{x}_n)$ is some smooth nonnegative function to be given later. According to inequality (13), (38), and (40), we can obtain the derivative of $U_n(\boldsymbol{X}_n,t)$ is

$$\begin{aligned}\dot{U}_n &= \dot{\widetilde{U}}_{n-1}(\boldsymbol{X}_{n-1},t) + \dot{V}_n + \dot{\widetilde{V}}_n \leqslant \\ &-\widetilde{\triangle}_{n-1}\|\boldsymbol{X}_{n-1}\|^2 + l_{n-1}(\widetilde{x}_n)\widetilde{x}_n^2 - \|z_n\|^2\triangle_n(z_n) + \\ &\sum_{j=0}^{n-1}\|z_j\|^2\bar{h}_{nj}(z_j) + \sum_{j=1}^{n}x_j^2\bar{s}_{nj}(x_j) + \sum_{j=0}^{n}\psi_{nj}(z_j)\|z_j\|^2 + \\ &\sum_{j=1}^{i}\phi_{ij}(\widetilde{x}_j)\|\widetilde{x}_j\|^2 - 2\underline{b}_n(t)k_n\zeta_m(\widetilde{x}_n)\widetilde{x}_n^2 \leqslant \\ &-(\widetilde{\triangle}_{n-1} - \sum_{j=0}^{n-1}(\bar{h}_{nj}(z_j) + \psi_{nj}(z_j)) - \sum_{j=1}^{n}(x_j^2\bar{s}_{nj}(x_j) + \\ &\phi_{nj}(\widetilde{x}_j)))\|\boldsymbol{X}_{n-1}\|^2 - (\triangle_n(z_n) - \psi_{nn}(z_n)))\|z_n\|^2 + \\ &(l_{n-1}(\widetilde{x}_n) + \bar{s}_{nn}(\widetilde{x}_n) + \phi_{nn}(\widetilde{x}_n) - \\ &2\underline{b}_n(t)k_n\zeta_n(\widetilde{x}_n))\widetilde{x}_n^2.\end{aligned} \tag{42}$$

Then, we choose $\widetilde{\triangle}_{n-1} \geqslant \sum_{j=0}^{n-1}(\bar{h}_{nj}(z_j) + \psi_{nj}(z_j)) + \sum_{j=1}^{n}(x_j^2\bar{s}_{nj}(x_j) + \phi_{nj}(\widetilde{x}_j)) + 1$, $\triangle_n(z_n) \geqslant \psi_{nn}(z_n) + 1$, and let

$$k_n\zeta_n(\widetilde{x}_n) \geqslant \frac{1}{2\underline{b}_n(t)}(l_{n-1}(\widetilde{x}_n) + \bar{s}_{nn}(\widetilde{x}_n) + \phi_{nn}(\widetilde{x}_n) + 1)$$

where $\underline{b}_n \leqslant b_n$. It obtains

$$\dot{U}_n \leqslant -\|\boldsymbol{X}_n\|^2. \tag{43}$$

Remark 5 Though the constraints considered in this paper are symmetric, we would like to point out that it is not difficult to extend our method to the asymmetric state constraints by using the modified BLFs. Actually, to consider the asymmetric state constraints $(-\rho_b,\rho_a)$ for the states $\widetilde{x}_1$, the control rule can be designed by using the following BLF:

$$\widetilde{V}_1 = \frac{1}{2}p(\widetilde{x}_1)\log\frac{\rho_a^2(t)}{\rho_a^2(t)-\widetilde{x}_1^2(t)} + \frac{1}{2}(1-p(\widetilde{x}_1))\log\frac{\rho_b^2(t)}{\rho_b^2(t)-\widetilde{x}_1^2(t)}$$

with

$$p(\widetilde{x}_1) = \begin{cases}1, & \widetilde{x}_1 > 0\\ 0, & \widetilde{x}_1 < 0\end{cases}.$$

Under the above discussion, we can formulate the main result as follows:

Theorem 1 Under Assumptions 1−5, given any initial system conditions $x(0)\in\mathbf{R}^n$ and $v(0)\in\mathbf{R}^q$, the ROR problem with finite time prescribed performance for the system consisting of (1), (2), and (3) is solved by a dynamic state feedback controller

$$\begin{cases}u = -k_n\zeta_n(\widetilde{x}_n)\widetilde{x}_n + \gamma_n(\eta_n)\\ \dot{\boldsymbol{\eta}}_i = \boldsymbol{M}_i\eta_i + \boldsymbol{N}_i\bar{x}_{i+1},\ i=1,2,\cdots,n\\ \widetilde{x}_1 = \beta(t)e\\ \widetilde{x}_i = \bar{x}_i - \gamma_i(\eta_i) - \alpha_{i-1},\ i=2,3,\cdots,n\end{cases} \tag{44}$$

where α_i $(i=1,2,\cdots,n-1)$ are defined by (23), (31) and (36).

Proof With input and state transformation (7), the closed-loop system is given by (8). Thus, it is sufficient to prove that the closed system (8) satisfies Properties 1−3.

According to (43), we can obtain that $U_n(X_n,t)$ converges to zero, when t tends to infinity. Thus, the closed system in (8) is globally asymptotically stable. Obviously, Propertiy 1 and Properity 2 are verified. Next, we will show Property 3 is also satisfied.

According to (43), we can know that

$$\dot{U}_n < 0 \Rightarrow U_n(t) \leqslant U_n(0),\ t \geqslant 0. \tag{45}$$

Based on the expression of $U_n(X_n,t) = \widetilde{U}_{n-1}(X_{n-1},t) + V_n(z_n,t) + \widetilde{V}_n(\widetilde{x}_n)$, we obtain $\widetilde{U}_{n-1}(X_{n-1},t) + V_n(z_n,t) + \widetilde{V}_n(\widetilde{x}_n) \leqslant \widetilde{U}_{n-1}(X_{n-1},0) + V_n(z_n,0) + \widetilde{V}_n(\widetilde{x}_n,0)$, thus $\widetilde{U}_{n-1}(X_{n-1},t) \leqslant \widetilde{U}_{n-1}(X_{n-1},0)$. According to (38) and (39), we know that $\widetilde{U}_{n-1}(X_{n-1},t)$ and $U_{n-1}(X_{n-1},t)$ have same function property. Thus, we can obtain $U_{n-1}(X_{n-1},t) \leqslant U_{n-1}(X_{n-1},0)$. By iteratively computing, we can get

$$U_1(t) \leqslant U_1(0),\ t \geqslant 0. \tag{46}$$

Based on the expression of $U_1(\boldsymbol{X}_1,t)$, we obtain

$$\log\frac{\rho^2(t)}{\rho^2(t)-\widetilde{x}_1^2(t)} \leqslant \widetilde{V}_1(0) \tag{47}$$

which implies that $\rho^2(t) \leqslant \widetilde{x}_1^{\widetilde{V}_1(0)}(\rho^2(t)-\widetilde{x}_1^2(t))$. Therefore,

$$\|\widetilde{x}_1\| \leqslant \rho(t)\sqrt{1-\widetilde{x}_1^{-\widetilde{V}_1(0)}} \leqslant \rho(t),\quad \forall t>0. \tag{48}$$

Note that $x_1 = \beta^{-1}\widetilde{x}_1$, thus, we can obtain

$$\|x_1\| \leqslant \beta^{-1}\rho(t). \tag{49}$$

Then, according to (16), we obtain

$$\|x_1\| \leqslant (1-c)\frac{T-t^4}{T}\chi^{-1}\rho + c\rho,\ 0\leqslant t<T, \tag{50}$$

$$\|x_1\| \leqslant c\rho,\ t\geqslant T. \tag{51}$$

Owing to $e = x_1$, which means that the error outpart converges to the designed constrained set after a time interval T. Thus, Property 3 is achieved. This completes the proof. □

Remark 6 Under the proposed finite time prescribed performance control scheme defined by (24), (31), (36), and (41), the error output of the system satisfies $|e(t)| \leqslant c\rho(t)$ after a given finite time T. The speed function $\beta(t)$ restricts the decay rate no less than $(T-t^4/T)\chi^{-1}$ in $[0,T)$ and the smooth function $\rho(t)$ defines a prescribed performance bound for the error output. Compared with the traditional prescribed performance control scheme, this result can enlarge the design freedom for the reason that the error output can converge to the preset constrain bound after a designed time interval at a needed decay rate. Moreover, the initial value of error output need not to be restricted in prescribed performance bound, which releases the conservatism.

4. Simulation results

In this section, a numerical example is given to illustrate the effectiveness of the proposed control scheme. Consider the following nonlinear system in strict feedback form:

$$\begin{cases}\dot{z} = -z + 2\omega_1 v_1\bar{x}_1 + v_2^2 - 2(1+\omega_1)v_1v_2\\ \dot{\bar{x}}_1 = \omega_2 z - v_1\bar{x}_1 + \bar{x}_2 + v_1v_2 - v_1 - \omega_2 v_2^2 - v_2\\ \dot{\bar{x}}_2 = -\bar{x}_2 + u\\ e = \bar{x}_1 - v_2\end{cases} \tag{52}$$

where ω_1 and ω_2 are unknown parameters, and v_1 is a sinusoidal function produced by the exosystem

$$\begin{cases}\dot{v}_1 = v_2\\ \dot{v}_2 = -v_1\end{cases}. \tag{53}$$

By solve the regulator equations, we can obtain $z(v,\omega) = v_2^2$, $\bar{x}_1(v,\omega) = v_2$, $\bar{x}_2(v,\omega) = v_2$, and $u(v,\omega) = v_2 - v_1$. Assumption 3 is satisfied with $\boldsymbol{\pi}_1(v,\omega) = (v_2,\dot{v}_2)^{\mathrm{T}}$, $\boldsymbol{\pi}_2(v,\omega) = (v_2-v_1, -v_1-v_2)^{\mathrm{T}}$, $\boldsymbol{F}_1 = \boldsymbol{F}_2 = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, and $\boldsymbol{\chi}_1 = \boldsymbol{\chi}_2 = [1,0]$. Then $\dot{\pi}_1 = \boldsymbol{F}_1\boldsymbol{\pi}_1$, $\dot{\pi}_2 = \boldsymbol{F}_2\boldsymbol{\pi}_2$, $\bar{x}_2(v,\omega) = \chi_1\boldsymbol{\pi}_1(v,\omega)$, $u(v,\omega) = \chi_2\boldsymbol{\pi}_2(v,\omega)$. Choosing $\boldsymbol{M}_1 = \boldsymbol{M}_2 = \begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}$, $\boldsymbol{N}_1 = \boldsymbol{N}_2 = [2,1]^{\mathrm{T}}$ makes $(\boldsymbol{M}_i,\boldsymbol{N}_i)(i=1,2)$ controllable pairs. By solving the Sylvester equation, we can obtain $\boldsymbol{T}_1 = \boldsymbol{T}_2 = \begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}$. Correspondingly, $\boldsymbol{\gamma}_1(\boldsymbol{\theta}_1) =$

$[1/3, 1/3]\boldsymbol{\theta}_1$, and $\boldsymbol{\gamma}_2(\boldsymbol{\theta}_2) = [1/3, 1/3]\boldsymbol{\theta}_2$.

Using the internal model and by the coordinate transformation, the augmented system in the form of (8) is given as follows:

$$\begin{cases} \dot{z}_0 = -z_0 + 2\omega_1 v_1 x_1 \\ \dot{z}_1 = \boldsymbol{M}_1 z_1 - \boldsymbol{N}_1 \omega_2 z_0 + \boldsymbol{N}_1 v_1 x_1 \\ \dot{z}_2 = \boldsymbol{N}_2 x_2 + \boldsymbol{N}_2 \gamma_1(\eta_1) + \boldsymbol{N}_2 \dot{\gamma}_1(\eta_1) + \boldsymbol{M}_2 Z_2 + \boldsymbol{M}_2 \boldsymbol{N}_2 x_2 \\ \dot{x}_1 = \omega_2 z_0 - v_1 x_1 + x_2 \\ \dot{x}_2 = -x_2 - \boldsymbol{\gamma}_1(\boldsymbol{\eta}_1) - \dot{\boldsymbol{\gamma}}_1(\boldsymbol{\eta}_1) + \boldsymbol{\gamma}_2(\boldsymbol{\eta}_2) + \bar{u} \end{cases}. \tag{54}$$

Performing on the system (54) the following coordinate transformation:

$$\begin{cases} \dfrac{\widetilde{x}_1}{\beta(t)} = x_1 - \alpha_0 \\ \widetilde{x}_2 = x_2 - \alpha_1 \end{cases} \tag{55}$$

where the speed function $\beta(t)$ is as in (16) with $c = 0.1$, the rate function $\chi = \mathrm{e}^t$ and the finite time $T = 2$ s.

Then, the virtual controller and actual controller can be constructed as follows:

$$\begin{cases} \alpha_1(x_1) = -\dfrac{\rho^2(t) - \widetilde{x}_1^2}{2}\widetilde{x}_1^2 \\ u = -2(\widetilde{x}_2^2 + 1)\widetilde{x}_2 + \gamma_2(\eta_2) \\ \dot{\eta}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\eta_1 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}\bar{x}_2 \\ \dot{\eta}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\eta_2 + \begin{bmatrix} 2 \\ 1 \end{bmatrix}u \\ \widetilde{x}_1 = \beta(t)e \\ \widetilde{x}_2 = \bar{x}_2 - \gamma_2(\eta_2) - \alpha_1 \end{cases}. \tag{56}$$

The initial values can be selected as $v_1(0) = 0$, $v_2(0) = 1$, $z(0) = 1.5$, $\bar{x}_1(0) = 1.5$, $\bar{x}_2(0) = 1.5$, and the other initial conditions are chosen as zero. The unknown parameters are $\omega_1 = \omega_2 = 1$. The prescribed performance bound of the tracking error can be chosen as $c\rho(t) = 0.2\mathrm{e}^{-2t} + 0.06$.

The simulation results are shown in Fig. 1 and Fig. 2. According to Remark 6, the finite time prescribed performance for error output can be removed by simply letting $\beta(t) = 1$ and $\rho(t) = \infty$, which yields a traditional control scheme. Fig. 1 compares the trajectories of the error output between the finite time prescribed performance control scheme ((56) with $\chi = \mathrm{e}^t$, $T = 2$ s and $c\rho(t) = 0.2\mathrm{e}^{-2t} + 0.06$) and the traditional control scheme (with $\beta(t) = 1$ and $\rho(t) = \infty$). The dashed line is the prescribed performance bound. It is clear that, under the finite time prescribed performance control scheme, the trajectory of the error output (the solid line) converges to the prescribed set in the given time $T = 2$ s. However, under the traditional control scheme, the Property 3 is not satisfied, as shown by the dotted line in Fig. 1. The plant states $z(t)$, $\bar{x}_1(t)$, $\bar{x}_2(t)$ under the finite time prescribed performance control scheme are pictured in Fig. 2. Apparently, output regulation with prescribe performance in finite time is achieved by using reasonable control efforts.

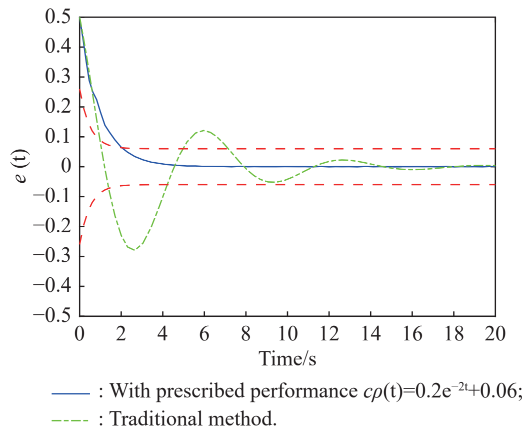

Fig. 1 Tracking error $e(t)$

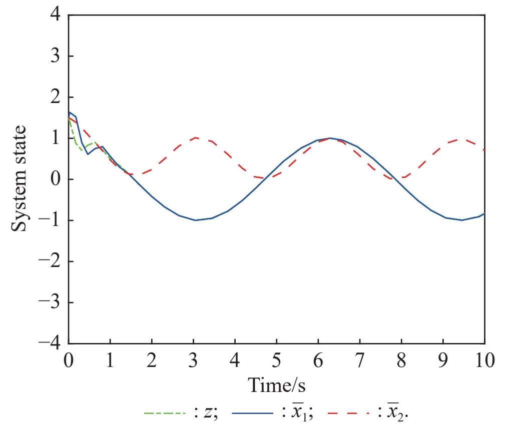

Fig. 2 States of plant

5. Conclusions

The robust output regulation problem with prescribed performance for nonlinear strict feedback system is investigated by speed function technique and BLF technique to guarantee that the output error can be confined in a preset constrained set after a given finite time. On the basis of the normal framework of the general nonlinear robust output regulation problem, a dynamic feedback controller is constructed to deal with the error constrained issue. The design procedure of the finite time prescribed performance control scheme and the transient performance of the resulted closed-loop system are illustrated by a simulation example. In future work, it might be interesting to consider the output regulation with the prospected property problem for the uncertain strict feedback switched systems.

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