Journal of Systems Engineering and Electronics
Vol. 34, No. 1, February 2023, pp.214 – 223

Robust fault detection for delta operator switched fuzzy systems with bilateral packet losses

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Abstract: Considering packet losses, time-varying delay, and parameter uncertainty in the switched fuzzy system, this paper designs a robust fault detection filter at any switching rate and analyzes the H_∞ performance of the system. Firstly, the Takagi-Sugeno (T-S) fuzzy model is used to establish a global fuzzy model for the uncertain nonlinear time-delay switched system, and the packet loss process is modeled as a mathematical model satisfying Bernoulli distribution. Secondly, through the average dwell time method and multiple Lyapunov functions, the exponentially stable condition of the nonlinear network switched system is given. Finally, specific parameters of the robust fault detection filter can be obtained by solving linear matrix inequalities (LMIs). The effectiveness of the method is verified by simulation results.

Keywords: switched fuzzy system, robust fault detection, time-varying delay, bilateral packet losses, uncertainty, average dwell time method.

DOI: 10.23919/JSEE.2023.000025

1. Introduction

The networked control system (NCS) is a closed-loop control system that exchanges data through a public communication network, and it plays an important role in the field of control [1]. Compared with the traditional point-to-point systems, it has obvious advantages, such as simple installation, flexibility, and convenient resource sharing. Therefore, it has received extensive attention from scholars. However, in actual industrial system, the transmission of data through the communication network is not reliable enough. When the communication network exists in the system, it may bring about limited communication, packet losses, disordering, and time delays, which can cause system unstableness [2–4]. Therefore, these factors should be fully considered when studying the NCS to ensure the stability and reliability of the system.

The switched system is a special type of hybrid systems, which is relatively common in industrial production. It usually consists of subsystems and switching rules. In recent years, the switched system has a wealth of theoretical results in robust control, stability analysis, fault detection, etc. And it is widely applied in many engineering fields, such as flight control systems, traffic control, power electronics, communication networks, aerospace industry, NCS, etc. [5−7]. Therefore, the research on the switched system has important scientific significance.

In our daily life, most systems are nonlinear systems, which are difficult to perform modeling and mathematical analysis. The Takagi-Sugeno (T-S) fuzzy model can combine simple linear systems to express complex nonlinear systems through some certain fuzzy rules. It reduces the difficulty of analysis and is an important tool for studying nonlinear systems. In addition, many switched systems also have nonlinear characteristics. Applying the T-S fuzzy model to deal with nonlinear switched system forms the switched fuzzy systems [8−10]. And each subsystem is a fuzzy system constructed by the T-S fuzzy dynamic model. Many references have conducted research on switching fuzzy systems. The problem of dissipative filtering for switched fuzzy nonlinear systems was proposed in [11]. At the same time, random external disturbance and state-dependent switching were considered in the system. Through dissipation analysis, a sufficient condition to ensure the mean square exponential stability of the dynamic error system was obtained. Moreover, the asynchronous H_∞ filtering for discrete-time switched fuzzy systems was discussed in [12], and more consistent Lyapunov functions were constructed for switched systems with asynchronous switching. The stability of a class of discrete-time switched T-S fuzzy systems was analyzed in [13], and improved stability criteria were provided for

Manuscript received May 26, 2021.
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This work was supported by the National Natural Science Foundation of China (61471323).

switched fuzzy systems containing both stable and unstable modes. In [14], robust output feedback control of switched fuzzy systems with both uncertainties and time delays was realized by multiple Lyapunov functions, and sufficient conditions for the asymptotic stability of the system were given.

As we all know, discrete-time systems are more suitable for computer analysis than continuous-time systems. For some continuous-time systems, in order to facilitate mathematical analysis and calculations, they need to be discretized. Common discretization methods mainly include delta operator method and shift operator method. The delta operator has better performance than the shift operator in the case of high-speed sampling, so it has attracted widespread attention from scholars [15]. At present, there have been quite a few research results on the delta operator in filtering, fault detection, stability analysis, etc., which shows that the delta operator has important science significance and research value.

Currently, there are relatively few references about delta operator switched fuzzy systems. The fault-tolerant control of the delta operator switched system was discussed in [16], the full-order dynamic output switching control law and the state switching law were jointly designed for the sensor failure in the system to ensure the asymptotic stability of the closed-loop system. A state feedback controller for the delta operator switched system with time-varying delay and asynchronous actuator failure was designed in [17], so that the system can still maintain exponentially stable with H_∞ performance in the case of actuator failure. However, most of these achievements only consider linear systems. The research on the fault detection for nonlinear switched systems with delta operator has important research significance and application prospects, but related results are still rare. Therefore, this paper combines T-S fuzzy model to deal with the nonlinear system based on the fault detection of the delta operator switched system. And a robust fault detection filter is designed to ensure that the system has exponential mean square stability.

This paper studies the robust fault detection for delta operator switched fuzzy systems with delay, packet losses, and uncertainty. The main contributions are as follows: (i) The nonlinear network switched system is constructed as multiple switched fuzzy systems through T-S fuzzy model, and the corresponding robust fault detection filter is designed; (ii) The exponential mean square stability conditions of the system are given by the average dwell time method; (iii) The specific parameters of the filter can be obtained via solving linear matrix inequalities (LMIs).

This paper is organized as follows. Section 2 establishes the system model for the switched fuzzy systems. The fault detection filter is designed in Section 3. Section 4 verifies the feasibility of the proposed method through simulation results. The conclusions are given in Section 5.

2. Problem statement

The delta operator is defined as follows:

$$\delta x(k) = \frac{x(k+1)-x(k)}{h} \tag{1}$$

where h is the sampling period.

Considering the uncertainty and time-varying delay in the system, the switched fuzzy system with $N_{\sigma(k)}$ fuzzy rules is modeled as follows:

$R^l_{\sigma(k)}$: if ς_1 is $M^l_{\sigma(k)1}$, ς_2 is $M^l_{\sigma(k)2}$, $\cdots$, and ς_p is $M^l_{\sigma(k)p}$, then

$$\begin{cases}\delta \boldsymbol{x}(k)=(\boldsymbol{A}_{\sigma(k)l}+\Delta\boldsymbol{A}_{\sigma(k)l}(k))\boldsymbol{x}(k)+\boldsymbol{B}_{\sigma(k)l}\boldsymbol{u}(k)+\boldsymbol{D}_{\sigma(k)l}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta\sigma(k)l}+\Delta\boldsymbol{A}_{\theta\sigma(k)l}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{\sigma(k)l}\boldsymbol{f}(k)\\ \boldsymbol{y}(k)=\boldsymbol{C}_{\sigma(k)l}\boldsymbol{x}(k)+\boldsymbol{E}_{\sigma(k)l}\boldsymbol{d}(k)\end{cases} \tag{2}$$

where $\sigma(k)\in\overline{M}=\{1,2,\cdots,m\}$ is the switching signal, $\boldsymbol{\varsigma}=[\varsigma_1,\varsigma_2,\cdots,\varsigma_p]$ denotes premise variable, $M^l_{\sigma(k)l}$, $M^l_{\sigma(k)(l+1)}$, $\cdots$, and $M^l_{\sigma(k)p}$ are fuzzy sets, $\boldsymbol{x}(k)\in\mathbf{R}^n$ is state vector, $\boldsymbol{u}(k)\in\mathbf{R}^p$ is control input, $\boldsymbol{d}(k)\in\mathbf{R}^q$ represents interference signals, $\boldsymbol{f}(k)\in\mathbf{R}^m$ is fault signal in the system, $\theta(k)$ denotes time-varying delay, $0<\theta_m\leqslant\theta(k)\leqslant\theta_M$, $\boldsymbol{y}(k)\in\mathbf{R}^t$ is the output signal, $\boldsymbol{A}_{\sigma(k)l},\boldsymbol{A}_{\theta\sigma(k)l},\boldsymbol{B}_{\sigma(k)l},\boldsymbol{D}_{\sigma(k)l}$, $\boldsymbol{C}_{\sigma(k)l},\boldsymbol{E}_{\sigma(k)l},\boldsymbol{F}_{\sigma(k)l}$ are known constant matrices, and $\Delta\boldsymbol{A}_{\sigma(k)l}(k)$ and $\Delta\boldsymbol{A}_{\theta\sigma(k)l}(k)$ are matrices representing the parameters uncertainty in the system.

For convenience, let $\sigma(k)=i$, then the mathematical model of the ith switched fuzzy system is as follows:

R^l_i: if ς_1 is M^l_{i1}, ς_2 is M^l_{i2}, $\cdots$, and ς_p is M^l_{ip}, then

$$\begin{cases}\delta \boldsymbol{x}(k)=(\boldsymbol{A}_{il}+\Delta\boldsymbol{A}_{il}(k))\boldsymbol{x}(k)+\boldsymbol{B}_{il}\boldsymbol{u}(k)+\boldsymbol{D}_{il}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta il}+\Delta\boldsymbol{A}_{\theta il}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{il}\boldsymbol{f}(k)\\ \hat{\boldsymbol{y}}(k)=\alpha(k)\boldsymbol{C}_{il}\boldsymbol{x}(k)+\boldsymbol{E}_{il}\boldsymbol{d}(k)\end{cases} \tag{3}$$

where $l=1,2,\cdots,N_i$, $i=1,2,\cdots,m$, $\hat{\boldsymbol{y}}(k)\in\mathbf{R}^l$ is the measurement output of the system. $\alpha(k)$ is a random variable and satisfies the Bernoulli distribution. When $\alpha(k)=1$, it means that the data of the system can be transmitted successfully. If $\alpha(k)=0$, it indicates the data packet is lost during the transmission process, and the transmission fails.

$$\begin{cases}\text{Prob}\{\alpha(k)=1\}=\alpha_1\\ \text{Prob}\{\alpha(k)=0\}=1-\alpha_1\\ \text{Var}\{\alpha(k)\}=\text{E}\{(\alpha(k)-\alpha_1)^2\}=\sigma_1^2\end{cases}$$

Matrices $\Delta\boldsymbol{A}_{il}$ and $\Delta\boldsymbol{A}_{\theta il}$ are the system uncertainties which satisfy the following assumptions:

$$[\Delta\boldsymbol{A}_{il}(k)\ \ \Delta\boldsymbol{A}_{\theta il}(k)]=\boldsymbol{H}_{il}\boldsymbol{F}_{il}(k)[\boldsymbol{E}_{1il}\ \ \boldsymbol{E}_{2il}]$$

where $\boldsymbol{H}_{il},\boldsymbol{E}_{1il}$, and $\boldsymbol{E}_{2il}$ are constant matrices.

Make the following assumptions for the system in (3):

(i) The data packet is transmitted in a single packet every time;

(ii) The system randomly switches to any subsystem at each switching moment, and the switching signal is variable;

(iii) $\boldsymbol{F}_{il}^{\mathrm{T}}(k)\boldsymbol{F}_{il}(k)\leqslant\boldsymbol{I}$.

According to the fuzzy rules, the global model of the ith sub-fuzzy system is

$$\begin{cases}\delta\boldsymbol{x}(k)=\displaystyle\sum_{l=1}^{N_i}\eta_{il}(\varsigma(k))\,[(\boldsymbol{A}_{il}+\Delta\boldsymbol{A}_{il}(k))\boldsymbol{x}(k)+\boldsymbol{B}_{il}\boldsymbol{u}(k)+\boldsymbol{D}_{il}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta il}+\Delta\boldsymbol{A}_{\theta il}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{il}\boldsymbol{f}(k)]\\ \hat{\boldsymbol{y}}(k)=\displaystyle\sum_{l=1}^{N_i}\eta_{il}(\varsigma(k))\,[\alpha(k)\boldsymbol{C}_{il}\boldsymbol{x}(k)+\boldsymbol{E}_{il}\boldsymbol{d}(k)]\end{cases}\tag{4}$$

where

$$0\leqslant\eta_{il}(\varsigma(k))\leqslant 1,$$

$$\sum_{l=1}^{N_i}\eta_{il}(\varsigma(k))=1,$$

$$\varpi_{il}(\varsigma(k))=\prod_{\upsilon=1}^{p}M_{i\upsilon}^{l}(\varsigma_\upsilon(k)),$$

$$\eta_{il}(\varsigma(k))=\frac{\varpi_{il}(\varsigma(k))}{\displaystyle\sum_{l=1}^{N_i}\varpi_{il}(\varsigma(k))},$$

and $M_{i\upsilon}^{l}(\varsigma_\upsilon(k))$ is the membership degree of $\varsigma_\upsilon(k)$ belonging to the fuzzy set $M_{i\upsilon}^{l}$.

According to the parallel distributed compensation (PDC) algorithm, then the corresponding fuzzy controller is designed as follows:

R_i^l: if ς_1 is M_{i1}^l,ς_2 is $M_{i2}^l,\cdots$, and ς_p is M_{ip}^l, then

$$\hat{\boldsymbol{u}}(k)=\sum_{j=1}^{N_i}\eta_{ij}(\varsigma(\boldsymbol{k}))\boldsymbol{K}_{ij}\hat{\boldsymbol{x}}(k)$$
$$j=1,2,\cdots,N_i,l=1,2,\cdots,m\tag{5}$$

where $\boldsymbol{K}_{ij}\in\mathbf{R}^{p\times n}$ are the controller gain matrices.

In this paper, a fuzzy fault detection filter is designed to detect the faults, which is robust to the interference existing in the system. The mathematical model of the fault detection filter is as follows:

$$\begin{cases}\delta\hat{\boldsymbol{x}}(k)=\displaystyle\sum_{l=1}^{N_i}\eta_{il}(\varsigma(k))\sum_{j=1}^{N_i}\eta_{ij}(\varsigma(k))\,[\boldsymbol{A}_{il}\hat{\boldsymbol{x}}(k)+\boldsymbol{A}_{\theta il}\hat{\boldsymbol{x}}(k-\theta(k))+\\ \qquad\boldsymbol{B}_{il}\bar{\boldsymbol{u}}(k)+\boldsymbol{L}_{il}(\hat{\boldsymbol{y}}(k)-\bar{\alpha}\boldsymbol{C}_{ij}\hat{\boldsymbol{x}}(k))]\\ \boldsymbol{r}(k)=\displaystyle\sum_{l=1}^{N_i}\eta_{il}(\varsigma(k))\sum_{j=1}^{N_i}\eta_{ij}(\varsigma(k))\boldsymbol{Z}_{il}(\hat{\boldsymbol{y}}(k)-\alpha_1\boldsymbol{C}_{ij}\hat{\boldsymbol{x}}(k))\\ \bar{\boldsymbol{u}}(k)=\beta_1\hat{\boldsymbol{u}}(k)\\ \boldsymbol{u}(k)=\beta(k)\hat{\boldsymbol{u}}(k)\end{cases}\tag{6}$$

where $\hat{\boldsymbol{x}}(k)\in\mathbf{R}^n$ is state estimation, and $\bar{\boldsymbol{u}}(k)\in\mathbf{R}^p$ is the control input of the observer. $\boldsymbol{r}(k)\in\mathbf{R}^q$ is the residual signal. $\boldsymbol{L}_{il}\in\mathbf{R}^{n\times l}$ and $\boldsymbol{Z}_{il}\in\mathbf{R}^{m\times l}$ are the observer gain and the residual generator gain respectively. $\beta(k)$ represents the process of packet loss between the controller and the actuator.

$$\begin{cases}\mathrm{Prob}\{\beta(k)=1\}=\beta_1\\ \mathrm{Prob}\{\beta(k)=0\}=1-\beta_1\\ \mathrm{Var}\{\beta(k)\}=\mathrm{E}\{(\beta(k)-\beta_1)^2\}=\sigma_2^2\end{cases}$$

In order to detect the fault more sensitively, the state observation error $\boldsymbol{e}(k)$ and residual error $\tilde{\boldsymbol{r}}(k)$ are defined as follows:

$$\boldsymbol{e}(k)=\boldsymbol{x}(k)-\hat{\boldsymbol{x}}(k),$$
$$\tilde{\boldsymbol{r}}(k)=\boldsymbol{r}(k)-\boldsymbol{f}(k)\ .$$

In addition, construct the residual evaluation function $\boldsymbol{J}(\tilde{\boldsymbol{r}})$ and fault detection threshold $\boldsymbol{J}_{\mathrm{th}}$ as

$$\begin{cases}\boldsymbol{J}(\tilde{\boldsymbol{r}})=\tilde{\boldsymbol{r}}(k)_{2,M}=\left(\displaystyle\sum_{k_0}^{k_0+M}\tilde{\boldsymbol{r}}^{\mathrm{T}}(k)\tilde{\boldsymbol{r}}(k)\right)^{\frac{1}{2}}\\ \boldsymbol{J}_{\mathrm{th}}=\displaystyle\sup_{\omega\in l_2,f=0}\tilde{\boldsymbol{r}}(k)_{2,M}=\left(\displaystyle\sum_{k_0}^{k_0+M}\tilde{\boldsymbol{r}}^{\mathrm{T}}(k)\tilde{\boldsymbol{r}}(k)\right)^{\frac{1}{2}}\end{cases}.\tag{7}$$

Among them, M is the length of evaluation window and k_0 is the initial time. If $\boldsymbol{J}(\tilde{\boldsymbol{r}})>\boldsymbol{J}_{\mathrm{th}}$, it means the system has a fault; if $\boldsymbol{J}(\tilde{\boldsymbol{r}})\leqslant\boldsymbol{J}_{\mathrm{th}}$, it indicates that there is no fault in the system.

Then the following augmented system can be obtained:

$$\begin{cases}\delta\boldsymbol{\phi}(k)=\displaystyle\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\Big[(\bar{\boldsymbol{A}}_{ilj}+\Delta\bar{\boldsymbol{A}}_{il})\boldsymbol{\phi}(k)+\\ \qquad(\alpha(k)-\alpha_1)\bar{\boldsymbol{A}}_{1ilj}\boldsymbol{\phi}(k)+(\beta(k)-\beta_1)\bar{\boldsymbol{A}}_{2ilj}\boldsymbol{\phi}(k)+\\ \qquad((\bar{\boldsymbol{A}}_{\theta il}+\Delta\bar{\boldsymbol{A}}_{\theta il})\boldsymbol{\phi}(k-\theta(k))+\bar{\boldsymbol{B}}_{1ilj}\boldsymbol{\omega}(k)\Big]\\ \tilde{\boldsymbol{r}}(k)=\displaystyle\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\Big[\bar{\boldsymbol{C}}_{ilj}\boldsymbol{\phi}(k)+(\alpha(k)-\alpha_1)\cdot\\ \qquad\bar{\boldsymbol{C}}_{1ilj}\boldsymbol{\phi}(k)+\bar{\boldsymbol{B}}_{2ilj}\boldsymbol{\omega}(k)\Big]\end{cases}\tag{8}$$

where

$$\bar{A}_{ilj}=\begin{bmatrix} A_{il}+\beta_1 B_{il}K_{ij} & -\beta_1 B_{il}K_{ij} \\ 0 & A_{il}-\alpha_1 L_{il}C_{ij}\end{bmatrix},$$

$$\Delta\bar{A}_{il}=\begin{bmatrix} \Delta A_{il} & 0 \\ \Delta A_{il} & 0\end{bmatrix},$$

$$\bar{A}_{1ilj}=\begin{bmatrix} 0 & 0 \\ -L_{il}C_{ij} & 0\end{bmatrix},\bar{A}_{2ilj}=\begin{bmatrix} B_{il}K_{ij} & -B_{il}K_{ij} \\ B_{il}K_{ij} & -B_{il}K_{ij}\end{bmatrix},$$

$$\bar{A}_{\theta il}=\begin{bmatrix} A_{\theta il} & 0 \\ 0 & A_{\theta il}\end{bmatrix},\Delta\bar{A}_{\theta il}=\begin{bmatrix} \Delta A_{\theta il} & 0 \\ \Delta A_{\theta il} & 0\end{bmatrix},$$

$$\bar{B}_{1ilj}=\begin{bmatrix} D_{il} & F_{il} \\ D_{il}-L_{il}E_{ij} & F_{il}\end{bmatrix},\bar{C}_{ilj}=\begin{bmatrix} 0 & \alpha_1 Z_{il}C_{ij}\end{bmatrix},$$

$$\bar{C}_{1ilj}=\begin{bmatrix} Z_{il}C_{ij} & 0\end{bmatrix},\bar{B}_{2ilj}=\begin{bmatrix} Z_{il}E_{ij} & -I\end{bmatrix},$$

$$\phi(k)=\begin{bmatrix} x(k) \\ e(k)\end{bmatrix},\omega(k)=\begin{bmatrix} d(k) \\ f(k)\end{bmatrix}.$$

Remark 1 $\Delta\bar{A}_{il}$ and $\Delta\bar{A}_{\theta il}$ are the parameter uncertainties in the augmented system, the expression is as follows:

$$\begin{bmatrix} \Delta\bar{A}_{il} & \Delta\bar{A}_{\theta il}\end{bmatrix}=\tilde{H}_{il}\tilde{F}_{il}(k)\begin{bmatrix} \tilde{E}_{1il} & \tilde{E}_{2il}\end{bmatrix}$$

where

$$\tilde{F}_{il}^{\mathrm{T}}(k)\tilde{F}_{il}(k)\leqslant I,$$

$$\tilde{H}_{il}=\begin{bmatrix} H_{il} & 0 \\ 0 & H_{il}\end{bmatrix},$$

$$\tilde{E}_{1il}=\begin{bmatrix} E_{1il} & 0 \\ E_{1il} & 0\end{bmatrix},$$

$$\tilde{E}_{2il}=\begin{bmatrix} E_{2il} & 0 \\ E_{2il} & 0\end{bmatrix},$$

$$\tilde{F}_{il}(k)=\begin{bmatrix} F_{il}(k) & 0 \\ 0 & F_{il}(k)\end{bmatrix}.$$

In order to obtain the sufficient conditions for the exponential mean square stability of the system (8) and the design method of the fault detection filter, the following definitions are introduced.

Definition 1 [18] Under zero initial conditions and $0<\lambda<1$, for non-zero external disturbance $\omega(k)\in l_2[0,\infty)$, if the system in (8) is exponentially mean square stable and satisfies:

$$\mathrm{E}\left\{\sum_{s=k_0}^{\infty}(1-\lambda)^s\tilde{r}^{\mathrm{T}}(s)\tilde{r}(s)\right\}\leqslant \mathrm{E}\left\{\sum_{s=k_0}^{\infty}\gamma^2\omega^{\mathrm{T}}(s)\omega(s)\right\},$$

then the system (8) is exponentially mean square stable and has H_∞ performance.

Definition 2 [18] If there exist scalars $\partial>0$ and $0<\chi<1$, when $\omega(k)=0$, $\mathrm{E}\{\|\phi(k_1)\|^2\}<\partial\chi^{(k_1-k_0)}\mathrm{E}\{\|\phi\|_l^{\ 2}\}$, $k_1\geqslant k_0$, the system (8) is exponentially mean square stable, χ is called the decay rate, where

$$\|\phi\|_l=\sup_{k_0-h_M<\theta<k_0}\|\phi_\theta\|$$

Definition 3 [19] For any $k_0\leqslant k\leqslant k_1, N_{\sigma(k)}$ means the switching times of the switching signal $\sigma(k)$ on $[k_0,k_1)$. Given $N_0\geqslant 0$ and $T_a>0$, then

$$N_{\sigma(k)}(k_0,k_1)\leqslant N_0+\frac{k_1-k_0}{T_a}$$

where T_a and N_0 represent the average residence time and jitter bounds respectively.

3. Main results

Theorem 1 Given $0<\lambda<1$, $\mu>1$, for any switching signal $\sigma(k)$ that satisfies $T_a>\overline{T}_a=(-\ln\mu)/\ln(1-\lambda)$, if there exist positive definite matrices P_i,Q_i,R_i,O_i, such that the (9) and (10) hold, then the system (8) is exponentially mean square stable and has H_∞ performance.

$$\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k)).$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12ilj} & \Psi_{13il} \\ * & \Psi_{22} & \Psi_{23il} \\ * & * & \Psi_{33}\end{bmatrix}<0,\ 1\leqslant l\leqslant j\leqslant r \tag{9}$$

$$\begin{cases} P_i\leqslant\mu P_j \\ Q_i\leqslant\mu Q_j \\ R_i\leqslant\mu R_j \\ O_i\leqslant\mu O_j\end{cases} \tag{10}$$

where $*$ denotes symmentric transpose matrix, and

$$\Psi_{12ilj}=\begin{bmatrix} \sqrt{\frac{1}{h}(h\bar{A}_{ilj}+I)^{\mathrm{T}}} & \sqrt{h}\sigma_1\bar{A}_{1ilj}^{\mathrm{T}} & \sqrt{h}\sigma_2\bar{A}_{2ilj}^{\mathrm{T}} & \sqrt{\frac{1}{h}}\bar{C}_{ilj}^{\mathrm{T}} & \sqrt{\frac{1}{h}}\sigma_1\bar{C}_{1ilj} \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{h}\bar{A}_{\theta il}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \sqrt{h}\bar{B}_{1ilj}^{\mathrm{T}} & 0 & 0 & \sqrt{\frac{1}{h}}\bar{B}_{2ilj}^{\mathrm{T}} & 0\end{bmatrix},\Psi_{13il}=\begin{bmatrix} 0 & \varepsilon_i\sqrt{h}\tilde{E}_{1il}^{\mathrm{T}} \\ 0 & 0 \\ 0 & \varepsilon_i\sqrt{h}\tilde{E}_{2il}^{\mathrm{T}} \\ 0 & 0 \\ 0 & 0\end{bmatrix},$$

$$\boldsymbol{\Psi}_{22}=\begin{bmatrix} -\boldsymbol{P}_j^{-1} & 0 & 0 & 0 & 0\\ * & -\boldsymbol{P}_j^{-1} & 0 & 0 & 0\\ * & * & -\boldsymbol{P}_j^{-1} & 0 & 0\\ * & * & * & -\boldsymbol{I} & 0\\ * & * & * & * & -\boldsymbol{I}\end{bmatrix}, \boldsymbol{\Psi}_{23il}=\begin{bmatrix}\tilde{\boldsymbol{H}}_{il}^{\mathrm{T}} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}, \boldsymbol{\Psi}_{33}=\begin{bmatrix}-\varepsilon_i\boldsymbol{I} & 0\\ 0 & -\varepsilon_i\boldsymbol{I}\end{bmatrix},$$

$$\boldsymbol{\Psi}_{11}=\mathrm{diag}\left\{-\frac{1}{h}(1-\lambda)\boldsymbol{P}_i+\frac{1}{h}(\theta_M-\theta_m+1)\boldsymbol{Q}_j+\frac{1}{h}\boldsymbol{R}_j+\frac{1}{h}\boldsymbol{O}_j,-\frac{1}{h}(1-\lambda)^{\theta_m}\boldsymbol{R}_i,-\frac{1}{h}(1-\lambda)^{\theta_m}\boldsymbol{Q}_i,-\frac{1}{h}(1-\lambda)^{\theta_M}\boldsymbol{O}_i,-\frac{1}{h}\gamma^2\boldsymbol{I}\right\}.$$

Proof Choose the following multi-Lyapunov function, denote $\hat{\lambda}=1-\lambda$.

$$\boldsymbol{V}_i(k)=\boldsymbol{\phi}^{\mathrm{T}}(k)\boldsymbol{P}_i\boldsymbol{\phi}(k)+\sum_{s=k-\theta_m}^{k-1}\hat{\lambda}^{k-s-1}\boldsymbol{\phi}^{\mathrm{T}}(s)\boldsymbol{R}_i\boldsymbol{\phi}(s)+$$
$$\sum_{s=k-\theta(k)}^{k-1}\hat{\lambda}^{k-s-1}\boldsymbol{\phi}^{\mathrm{T}}(s)\boldsymbol{Q}_i\boldsymbol{\phi}(s)+\sum_{s=k-\theta_M}^{k-1}\hat{\lambda}^{k-s-1}\boldsymbol{\phi}^{\mathrm{T}}(s)\boldsymbol{O}_i\boldsymbol{\phi}(s)+$$
$$\sum_{j=k-\theta_M+1}^{k-\theta_m}\sum_{s=j}^{k-1}\hat{\lambda}^{k-s-1}\boldsymbol{\phi}^{\mathrm{T}}(k)\boldsymbol{Q}_i\boldsymbol{\phi}(k)$$

Then through calculation, we can get:

$$\frac{1}{h}\mathrm{E}\{h\delta\boldsymbol{V}_i(k)+\lambda\boldsymbol{V}_i(k)+\tilde{\boldsymbol{r}}^{\mathrm{T}}(k)\tilde{\boldsymbol{r}}(k)-\gamma^2\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k)\}\leqslant \boldsymbol{\xi}^{\mathrm{T}}(k)\boldsymbol{\Omega}\boldsymbol{\xi}(k),$$

where

$$\boldsymbol{\xi}^{\mathrm{T}}(k)=\left[\boldsymbol{\phi}^{\mathrm{T}}(k),\boldsymbol{\phi}^{\mathrm{T}}(k-\theta_m),\boldsymbol{\phi}^{\mathrm{T}}(k-\theta(k),\boldsymbol{\phi}^{\mathrm{T}}(k-\theta_M),\boldsymbol{\omega}^{\mathrm{T}}(k)\right],$$
$$\boldsymbol{\Omega}=\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\boldsymbol{\Omega}_{lj},$$

$$\boldsymbol{\Omega}_{lj}=\begin{bmatrix}\boldsymbol{\Omega}_{11ilj} & 0 & \boldsymbol{\Omega}_{13ilj} & 0 & \boldsymbol{\Omega}_{15ilj}\\ * & -\frac{1}{h}(1-\lambda)^{\theta_m}\boldsymbol{R}_{1i} & 0 & 0 & 0\\ * & * & \boldsymbol{\Omega}_{33ilj} & 0 & h(\bar{\boldsymbol{A}}_{\theta il}+\Delta\bar{\boldsymbol{A}}_{\theta il})^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{B}}_{1ilj}\\ * & * & * & -\frac{1}{h}\hat{\lambda}^{\theta_M}\boldsymbol{R}_{2i} & 0\\ * & * & * & * & \boldsymbol{\Omega}_{55ilj}\end{bmatrix},$$

$$\boldsymbol{\Omega}_{11ilj}=\frac{1}{h}(h\bar{\boldsymbol{A}}_{\theta il}+h\Delta\bar{\boldsymbol{A}}_{\theta il}+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}_j(h\bar{\boldsymbol{A}}_{\theta il}+h\Delta\bar{\boldsymbol{A}}_{\theta il}+\boldsymbol{I})+$$
$$h{\sigma_1}^2\bar{\boldsymbol{A}}_{1ilj}^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{A}}_{1ilj}+h{\sigma_2}^2\bar{\boldsymbol{A}}_{2ilj}^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{A}}_{2ilj}-\frac{1}{h}\hat{\lambda}\boldsymbol{P}_i+\frac{1}{h}\bar{\boldsymbol{C}}_{ilj}^{\mathrm{T}}\bar{\boldsymbol{C}}_{ilj}+$$
$$\frac{1}{h}(\theta_M-\theta_m+1)\boldsymbol{Q}_j+\frac{1}{h}\boldsymbol{R}_j+\frac{1}{h}\boldsymbol{O}_j+h{\sigma_1}^2\bar{\boldsymbol{C}}_{1ilj}^{\mathrm{T}}\bar{\boldsymbol{C}}_{1ilj},$$
$$\boldsymbol{\Omega}_{13ilj}=(h\bar{\boldsymbol{A}}_{ilj}+h\Delta\bar{\boldsymbol{A}}_{ilj}+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}_j(\bar{\boldsymbol{A}}_{\theta il}+\Delta\bar{\boldsymbol{A}}_{\theta il}),$$
$$\boldsymbol{\Omega}_{15}=\frac{1}{h}\bar{\boldsymbol{C}}_{ilj}^{\mathrm{T}}\bar{\boldsymbol{B}}_{2ilj}+(h\bar{\boldsymbol{A}}_{ilj}+h\Delta\bar{\boldsymbol{A}}_{ilj}+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{B}}_{1ilj},$$
$$\boldsymbol{\Omega}_{33ilj}=-\frac{1}{h}\hat{\lambda}^{\theta_m}\boldsymbol{Q}_i+h(\bar{\boldsymbol{A}}_{ilj}+\Delta\bar{\boldsymbol{A}}_{ilj})^{\mathrm{T}}\boldsymbol{P}_j(\bar{\boldsymbol{A}}_{\theta il}+\Delta\bar{\boldsymbol{A}}_{\theta il}),$$
$$\boldsymbol{\Omega}_{55ilj}=h\bar{\boldsymbol{B}}_{1ilj}^{\mathrm{T}}\boldsymbol{P}_j\bar{\boldsymbol{B}}_{1ilj}+\frac{1}{h}\bar{\boldsymbol{B}}_{2ilj}^{\mathrm{T}}\bar{\boldsymbol{B}}_{2ilj}-\frac{1}{h}\gamma^2\boldsymbol{I}.$$

According to Schur's supplement lemma [20], we know that $\boldsymbol{\Psi}_{11}-\boldsymbol{\Psi}_{12}\boldsymbol{\Psi}_{22}^{-1}\boldsymbol{\Psi}_{12}^{\mathrm{T}}<0$, then $\mathrm{E}\{\boldsymbol{V}_i(k+1)-\boldsymbol{V}_i(k)+\lambda\boldsymbol{V}_i(k)+\tilde{\boldsymbol{r}}^{\mathrm{T}}(k)\tilde{\boldsymbol{r}}(k)-\gamma^2\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k)\}<0$. When $\boldsymbol{\omega}(k)=0$,

$$\mathrm{E}\left\{\delta V_i(k)+\frac{1}{h}\lambda V_i(k)\right\}<0$$

According to inequality (10): $\mathrm{E}\{\boldsymbol{V}_{\sigma(k)}(k)\}\leqslant(\hat{\lambda}\mu^{\frac{1}{T_a}})^{k_1-k_0}\cdot \boldsymbol{V}_{\sigma(k_0)}(k_0)$. Therefore,

$$\mathrm{E}\left\{\|\boldsymbol{\varphi}(k)\|^2\right\}\leqslant\frac{\kappa_2}{\kappa_1}(\hat{\lambda}\mu^{\frac{1}{T_a}})^{k_1-k_0}\mathrm{E}\left\{\|\boldsymbol{\varphi}(k)\|_l^{\,2}\right\}$$

where

$$\kappa_1=\frac{1}{h}\min_{\forall i\in\mathbf{N}}\lambda_{\min}(\boldsymbol{P}_i),$$
$$\kappa_2=\frac{1}{h}\max_{\forall i\in\mathbf{N}}\lambda_{\max}(\boldsymbol{P}_i)+\frac{1}{h}\max_{\forall i\in\mathbf{N}}\lambda_{\max}(\boldsymbol{R}_i)+\frac{1}{h}(\theta_M-\theta_m+1)\max_{\forall i\in\mathbf{N}}\lambda_{\max}(\boldsymbol{Q}_i)+\frac{1}{h}\max_{\forall i\in\mathbf{N}}\lambda_{\max}(\boldsymbol{O}_i).$$

It can be seen from Definition 2 that the system (8) is exponentially stable. If $\boldsymbol{\omega}(k)\neq 0$, $\boldsymbol{\Gamma}(k)=-\tilde{\boldsymbol{r}}^{\mathrm{T}}(k)\tilde{\boldsymbol{r}}(k)+\gamma^2\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k)$, then

$$\mathrm{E}\{\boldsymbol{V}_i(k)\}<\hat{\lambda}^{k_1-k_0}\boldsymbol{V}_i(k_0)+\mathrm{E}\left\{\sum_{s=k_0}^{k_1-1}\hat{\lambda}^{k_1-s-1}\boldsymbol{\Gamma}(s)\right\}.$$

After derivation, we can get

$$\mathrm{E}\{\boldsymbol{V}_{\sigma(k)}(k)\}\leqslant\hat{\lambda}^{k_1-k_0}\mu^{N(k_0,k_1)}\boldsymbol{V}_{\sigma(k_0)}(k_0)+\mathrm{E}\left\{\sum_{s=k_0}^{k_1-1}\mu^{N(s,k_1)}\hat{\lambda}^{k_1-s-1}\boldsymbol{\Gamma}(s)\right\}.$$

Considering that the initial state is 0, according to Definition 3, we can obtain that

$$\mathrm{E}\left\{\sum_{s=k_0}^{k_1-1}\mu^{\frac{s\ln(1-\alpha)}{\ln\mu}}(1-\lambda)^{k-s-1}\tilde{\boldsymbol{r}}^{\mathrm{T}}(s)\tilde{\boldsymbol{r}}(s)\right\}\leqslant$$
$$\mathrm{E}\left\{\sum_{s=k_0}^{k_1-1}\mu^{-N_\sigma(0,\,s)}(1-\lambda)^{k_1-s-1}\gamma^2\boldsymbol{\omega}^{\mathrm{T}}(k)\boldsymbol{\omega}(k)\right\}.$$

Let $k\to\infty$, according to Definition 1, the system (8) has H_∞ performance.

Remark 2 The inequality (9) is a nonlinear matrix inequality because of the existence of P_j^{-1}, and it cannot be solved using LMI toolbox. Then the Theorem 2 and Theorem 3 introduce matrix variables that make the system exponentially mean square stable and has H_∞ performance γ.

Theorem 2 Given $0<\alpha_1<1$, $0<\beta_1<1, 0<\lambda<1$, and $\mu>1$, if there exist positive definite matrices $\boldsymbol{P}_i,\boldsymbol{Q}_i,\boldsymbol{R}_i,\boldsymbol{O}_i$, and matrices $\boldsymbol{X}_{il},\boldsymbol{K}_{ij},\boldsymbol{L}_{il},\boldsymbol{Z}_{il}$, making

$$\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\begin{bmatrix}\boldsymbol{\varPsi}_{11} & \hat{\boldsymbol{\varPsi}}_{12ilj} & \boldsymbol{\varPsi}_{13il}\\ * & \hat{\boldsymbol{\varPsi}}_{22il} & \hat{\boldsymbol{\varPsi}}_{23il}\\ * & * & \boldsymbol{\varPsi}_{33}\end{bmatrix}<0 \quad (11)$$

holds, then the system in (8) is exponential mean square stable and has H_∞ performance γ $(i,j,l\in\mathbf{N}.)$

In (11),

$$\hat{\boldsymbol{\varPsi}}_{12ilj}=\sqrt{\frac{1}{h}}\begin{bmatrix}\left(h\bar{\boldsymbol{A}}_{ilj}+\boldsymbol{I}\right)^{\mathrm{T}}\boldsymbol{X}_{il} & h\sigma_1\bar{\boldsymbol{A}}_{1ilj}^{\mathrm{T}}\boldsymbol{X}_{il} & h\sigma_2\bar{\boldsymbol{A}}_{2ilj}^{\mathrm{T}}\boldsymbol{X}_{il} & \bar{\boldsymbol{C}}_{ilj}^{\mathrm{T}} & \sigma_1\bar{\boldsymbol{C}}_{1ilj}^{\mathrm{T}}\\ 0&0&0&0&0\\ h\bar{\boldsymbol{A}}_{\theta il}^{\mathrm{T}}\boldsymbol{X}_{il}&0&0&0&0\\ 0&0&0&0&0\\ h\bar{\boldsymbol{B}}_{1il}^{\mathrm{T}}\boldsymbol{X}_{il}&0&0&\bar{\boldsymbol{B}}_{2ilj}^{\mathrm{T}}&0\end{bmatrix},$$

$$\hat{\boldsymbol{\varPsi}}_{23il}=\begin{bmatrix}\boldsymbol{X}_{il}^{\mathrm{T}}\tilde{\boldsymbol{H}}_{il}&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix},$$

$$\hat{\boldsymbol{\varPsi}}_{22il}=\mathrm{diag}\{\boldsymbol{P}_j-(\boldsymbol{X}_{il}+\boldsymbol{X}_{il}^{\mathrm{T}}),\boldsymbol{P}_j-(\boldsymbol{X}_{il}+\boldsymbol{X}_{il}^{\mathrm{T}}),\boldsymbol{P}_j-(\boldsymbol{X}_{il}+\boldsymbol{X}_{il}^{\mathrm{T}}),-\boldsymbol{I},-\boldsymbol{I}\}.$$

Proof As $-\boldsymbol{X}_{il}^{\mathrm{T}}\boldsymbol{P}_j^{-1}\boldsymbol{X}_{il}<\boldsymbol{P}_j-(\boldsymbol{X}_{il}+\boldsymbol{X}_{il}^{\mathrm{T}})$, (11) can be transformed into

$$\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\begin{bmatrix}\boldsymbol{\varPsi}_{11} & \hat{\boldsymbol{\varPsi}}_{12ilj} & \boldsymbol{\varPsi}_{13il}\\ * & \bar{\boldsymbol{\varPsi}}_{22il} & \hat{\boldsymbol{\varPsi}}_{23il}\\ * & * & \boldsymbol{\varPsi}_{33}\end{bmatrix}<0 \quad (12)$$

where

$$\bar{\boldsymbol{\varPsi}}_{22}=\mathrm{diag}\{-\boldsymbol{X}_{il}^{\mathrm{T}}\boldsymbol{P}_j^{-1}\boldsymbol{X}_{il},-\boldsymbol{X}_{il}^{\mathrm{T}}\boldsymbol{P}_j^{-1}\boldsymbol{X}_{il},-\boldsymbol{X}_{il}^{\mathrm{T}}\boldsymbol{P}_j^{-1}\boldsymbol{X}_{il},-\boldsymbol{I},-\boldsymbol{I}\}.$$

Let

$$\boldsymbol{\varDelta}=\mathrm{diag}\{\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{X}_{il}^{-1},\boldsymbol{X}_{il}^{-1},\boldsymbol{X}_{il}^{-1},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I},\boldsymbol{I}\}$$

multiply left and right formulas (12) with matrices $\boldsymbol{\varDelta}^{\mathrm{T}}$ and $\boldsymbol{\varDelta}$, then (9) can be obtained. □

Theorem 3 Given scalars $0<\alpha_1<1$, $0<\beta_1<1$, $0<\lambda<1$, and $\mu>1$, if positive definite matrices $\boldsymbol{P}_i,\boldsymbol{Q}_i,\boldsymbol{R}_i,\boldsymbol{O}_i$ exist, and matrices $\boldsymbol{X}_{il}=\mathrm{diag}\{\boldsymbol{X}_{1il},\boldsymbol{X}_{1il}\}$, $\boldsymbol{K}_{ij},\boldsymbol{L}_{il},\boldsymbol{Z}_{il},\boldsymbol{G}_{ilj},\bar{\boldsymbol{L}}_{il}$ exist such that inequalities (13) and (14) hold, then the system in (8) has H_∞ performance and satisfies exponentially mean square stable $(i,j,l\in\mathbf{N})$. And $\boldsymbol{K}_{ij}=(\boldsymbol{B}_{il}^{\mathrm{T}}\boldsymbol{X}_{il})^{-\mathrm{T}}\boldsymbol{G}_{ilj}{}^{\mathrm{T}}$ is the controller gain matrix, $\boldsymbol{L}_{il}=(\boldsymbol{X}_{1il}^{-1})^{\mathrm{T}}\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}$ is the observer gain matrix.

$$\sum_{l=1}^{N_i}\sum_{j=1}^{N_i}\eta_{il}(\varsigma(k))\eta_{ij}(\varsigma(k))\begin{bmatrix}\boldsymbol{\varPsi}_{11} & \bar{\boldsymbol{\varPsi}}_{12ilj} & \boldsymbol{\varPsi}_{13il}\\ * & \boldsymbol{\varPsi}_{22} & \bar{\boldsymbol{\varPsi}}_{23il}\\ * & * & \boldsymbol{\varPsi}_{33}\end{bmatrix}<0, \quad (13)$$

$$\begin{cases}\boldsymbol{P}_i=\begin{bmatrix}\boldsymbol{P}_{i11}&\boldsymbol{P}_{i12}\\ * & \boldsymbol{P}_{i22}\end{bmatrix}>0\\ \boldsymbol{Q}_i=\begin{bmatrix}\boldsymbol{Q}_{i11}&\boldsymbol{Q}_{i12}\\ * & \boldsymbol{Q}_{i22}\end{bmatrix}>0\\ \boldsymbol{R}_i=\begin{bmatrix}\boldsymbol{R}_{i11}&\boldsymbol{R}_{i12}\\ * & \boldsymbol{R}_{i22}\end{bmatrix}>0\\ \boldsymbol{O}_i=\begin{bmatrix}\boldsymbol{O}_{i11}&\boldsymbol{O}_{i12}\\ * & \boldsymbol{O}_{i22}\end{bmatrix}>0\end{cases}, \quad (14)$$

where

$$\bar{\boldsymbol{\varPsi}}_{12ilj}=\begin{bmatrix}\boldsymbol{\varLambda}_{11ilj}&\boldsymbol{\varLambda}_{12il}&\boldsymbol{\varLambda}_{13ilj}&\sqrt{\frac{1}{h}}\bar{\boldsymbol{C}}_{ilj}^{\mathrm{T}}&\sqrt{\frac{1}{h}}\sigma_1\bar{\boldsymbol{C}}_{1ilj}^{\mathrm{T}}\\ 0&0&0&0&0\\ \boldsymbol{\varLambda}_{31il}&0&0&0&0\\ 0&0&0&0&0\\ \boldsymbol{\varLambda}_{51il}&0&0&\sqrt{\frac{1}{h}}\bar{\boldsymbol{B}}_{2ilj}^{\mathrm{T}}&0\end{bmatrix},$$

$$\bar{\boldsymbol{\varPsi}}_{23il}=\begin{bmatrix}\boldsymbol{\varPi}_{il}&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}^{\mathrm{T}},$$

$$\boldsymbol{\varPi}_{il}=\begin{bmatrix}\boldsymbol{H}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}&0\\0&\boldsymbol{H}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}\end{bmatrix},$$

$$\boldsymbol{\Lambda}_{11ilj}=\begin{bmatrix}\sqrt{h}\boldsymbol{A}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}+\sqrt{h}\beta_1\boldsymbol{G}_{ilj}+\dfrac{1}{\sqrt{h}}\boldsymbol{X}_{1il} & 0\\ -\sqrt{h}\beta_1\boldsymbol{G}_{ilj} & \sqrt{h}\boldsymbol{A}_{ilj}^{\mathrm{T}}\boldsymbol{X}_{1il}-\sqrt{h}\alpha_1\boldsymbol{C}_{il}^{\mathrm{T}}\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}\end{bmatrix},$$

$$\boldsymbol{\Lambda}_{12il}=\begin{bmatrix}0 & -\sqrt{h}\sigma_1\boldsymbol{C}_{il}^{\mathrm{T}}\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}\\ 0 & 0\end{bmatrix},$$

$$\boldsymbol{\Lambda}_{13ilj}=\begin{bmatrix}\sqrt{h}\sigma_2\boldsymbol{G}_{ilj} & \sqrt{h}\sigma_2\boldsymbol{G}_{ilj}\\ -\sqrt{h}\sigma_2\boldsymbol{G}_{ilj} & -\sqrt{h}\sigma_2\boldsymbol{G}_{ilj}\end{bmatrix},$$

$$\boldsymbol{\Lambda}_{31il}=\begin{bmatrix}\sqrt{h}\boldsymbol{A}_{\theta il}^{\mathrm{T}}\boldsymbol{X}_{1il} & 0\\ 0 & \sqrt{h}\boldsymbol{A}_{\theta il}^{\mathrm{T}}\boldsymbol{X}_{1il}\end{bmatrix},$$

$$\boldsymbol{\Lambda}_{51il}=\begin{bmatrix}\sqrt{h}\boldsymbol{D}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il} & \sqrt{h}\boldsymbol{D}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}-\sqrt{h}\boldsymbol{E}_{il}^{\mathrm{T}}\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}\\ \sqrt{h}\boldsymbol{F}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il} & \sqrt{h}\boldsymbol{F}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}\end{bmatrix}.$$

Proof Substitute $\boldsymbol{X}_{il}=\mathrm{diag}\{\boldsymbol{X}_{1il},\boldsymbol{X}_{1il}\}$ into (11), and let $\boldsymbol{G}_{ilj}=\boldsymbol{K}_{ij}^{\mathrm{T}}\boldsymbol{B}_{il}^{\mathrm{T}}\boldsymbol{X}_{il}$, $\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}=\boldsymbol{L}_{il}^{\mathrm{T}}\boldsymbol{X}_{1il}$, so that $\boldsymbol{K}_{ij}=(\boldsymbol{B}_{il}^{\mathrm{T}}\boldsymbol{X}_{il})^{-\mathrm{T}}\boldsymbol{G}_{ilj}{}^{\mathrm{T}}$, $\boldsymbol{L}_{il}=(\boldsymbol{X}_{1il}^{-1})^{\mathrm{T}}\bar{\boldsymbol{L}}_{il}^{\mathrm{T}}$ can be obtained. □

Remark 3 Theorem 2 gives the view that the feasible solution of the described γ can be achieved by solving an optimization problem. Theorem 3 gives the design method of the fault detection filter, specific parameter expression of the designed filter can be obtained by LMIs.

4. Simulation results

This paper constructs a time-delay delta operator switched fuzzy system with uncertainty, and the system switches randomly to any subsystem through switching signals. The mathematical model of the uncertain time-delay switched fuzzy system is as follows:

R_1^1: if $x_1(k)$ is M_{11}^1, then

$$\begin{cases}\delta\boldsymbol{x}(k)=(\boldsymbol{A}_{11}+\Delta\boldsymbol{A}_{11})\boldsymbol{x}(k)+\boldsymbol{B}_{11}\boldsymbol{u}(k)+\boldsymbol{D}_{11}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta11}+\Delta\boldsymbol{A}_{\theta11}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{11}\boldsymbol{f}(k)\\ \hat{\boldsymbol{y}}(k)=\alpha(k)\boldsymbol{C}_{11}\boldsymbol{x}(k)+\boldsymbol{E}_{11}\boldsymbol{d}(k)\end{cases}.$$

R_1^2: if $x_1(k)$ is M_{11}^2, then

$$\begin{cases}\delta\boldsymbol{x}(k)=(\boldsymbol{A}_{12}+\Delta\boldsymbol{A}_{12})\boldsymbol{x}(k)+\boldsymbol{B}_{12}\boldsymbol{u}(k)+\boldsymbol{D}_{12}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta12}+\Delta\boldsymbol{A}_{\theta12}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{12}\boldsymbol{f}(k)\\ \hat{\boldsymbol{y}}(k)=\alpha(k)\boldsymbol{C}_{12}\boldsymbol{x}(k)+\boldsymbol{E}_{12}\boldsymbol{d}(k)\end{cases}.$$

R_2^1: if $x_1(k)$ is M_{21}^1, then

$$\begin{cases}\delta\boldsymbol{x}(k)=(\boldsymbol{A}_{21}+\Delta\boldsymbol{A}_{21})\boldsymbol{x}(k)+\boldsymbol{B}_{21}\boldsymbol{u}(k)+\boldsymbol{D}_{21}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta21}+\Delta\boldsymbol{A}_{\theta21}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{21}\boldsymbol{f}(k)\\ \hat{\boldsymbol{y}}(k)=\alpha(k)\boldsymbol{C}_{21}\boldsymbol{x}(k)+\boldsymbol{E}_{21}\boldsymbol{d}(k)\end{cases}.$$

R_2^2: if $x_1(k)$ is M_{21}^2, then

$$\begin{cases}\delta\boldsymbol{x}(k)=(\boldsymbol{A}_{22}+\Delta\boldsymbol{A}_{22})\boldsymbol{x}(k)+\boldsymbol{B}_{22}\boldsymbol{u}(k)+\boldsymbol{D}_{22}\boldsymbol{d}(k)+\\ \qquad(\boldsymbol{A}_{\theta22}+\Delta\boldsymbol{A}_{\theta22}(k))\boldsymbol{x}(k-\theta(k))+\boldsymbol{F}_{22}\boldsymbol{f}(k)\\ \hat{\boldsymbol{y}}(k)=\alpha(k)\boldsymbol{C}_{22}\boldsymbol{x}(k)+\boldsymbol{E}_{22}\boldsymbol{d}(k)\end{cases}.$$

The parameters selection of the system mainly refers to references [21–22], and some corresponding adjustments have been made. Then the specific parameters are as follows:

$$\boldsymbol{A}_{11}=\begin{bmatrix}-43 & -13\\ -17 & -15\end{bmatrix},\boldsymbol{A}_{12}=\begin{bmatrix}-30 & -15\\ -8 & -20\end{bmatrix},$$

$$\boldsymbol{A}_{21}=\begin{bmatrix}-35 & -21\\ 7 & -15\end{bmatrix},\boldsymbol{A}_{22}=\begin{bmatrix}-40 & -10\\ 6 & -27\end{bmatrix},$$

$$\boldsymbol{A}_{\theta11}=\begin{bmatrix}-10 & 5\\ -8 & 20\end{bmatrix},\boldsymbol{A}_{\theta12}=\begin{bmatrix}10 & 12\\ -10 & -23\end{bmatrix},$$

$$\boldsymbol{A}_{\theta21}=\begin{bmatrix}20 & 5\\ -6 & 15\end{bmatrix},\boldsymbol{A}_{\theta22}=\begin{bmatrix}11 & -8\\ 7 & 16\end{bmatrix},$$

$$\boldsymbol{B}_{11}=\begin{bmatrix}10\\ 30\end{bmatrix},\boldsymbol{B}_{12}=\begin{bmatrix}8\\ 20\end{bmatrix},\boldsymbol{B}_{21}=\begin{bmatrix}-9\\ 15\end{bmatrix},\boldsymbol{B}_{22}=\begin{bmatrix}15\\ 22\end{bmatrix},$$

$$\boldsymbol{D}_{11}=\begin{bmatrix}15\\ 23\end{bmatrix},\boldsymbol{D}_{12}=\begin{bmatrix}-9\\ 21\end{bmatrix},\boldsymbol{D}_{21}=\begin{bmatrix}13\\ 25\end{bmatrix},\boldsymbol{D}_{22}=\begin{bmatrix}-21\\ 31\end{bmatrix}$$

$$\boldsymbol{F}_{11}=\begin{bmatrix}20\\ 18\end{bmatrix},\boldsymbol{F}_{12}=\begin{bmatrix}25\\ 31\end{bmatrix},\boldsymbol{F}_{21}=\begin{bmatrix}32\\ 40\end{bmatrix},\boldsymbol{F}_{22}=\begin{bmatrix}33\\ 21\end{bmatrix},$$

$$\boldsymbol{C}_{11}=\begin{bmatrix}11 & 9\end{bmatrix},\boldsymbol{C}_{12}=\begin{bmatrix}10 & 12\end{bmatrix},$$

$$\boldsymbol{C}_{21}=\begin{bmatrix}20 & 13\end{bmatrix},\boldsymbol{C}_{22}=\begin{bmatrix}15 & 7\end{bmatrix},$$

$$\boldsymbol{E}_{11}=9,\boldsymbol{E}_{12}=12,\boldsymbol{E}_{21}=8,\boldsymbol{E}_{22}=15,$$

$$\boldsymbol{E}_{111}=\begin{bmatrix}-11 & 9\end{bmatrix},\boldsymbol{E}_{112}=\begin{bmatrix}8 & 11\end{bmatrix}.$$

When designing the fault detection filter for a system, robustness is a very important performance indicator. The designed filter needs to be sensitive to fault signals and robust to external interference. There are many factors that affect the robustness of the system. Next, three different delays are selected to analyze the impact of delay on robustness.

4.1 Impact of time delay on system robustness

From Table 1, we notice that as the delay increases, the value of γ keeps increasing. It indicates that the robustness of the system is weakened. Therefore, it is necessary to reduce the time delay in the system in industrial production.

Table 1 Impact of time delay on robustness s

Delay	θ_m	θ_M	$\gamma_{\min}$
$\theta(k)=1+[1+(-1)^k]/2$	1	2	1.135
$\theta(k)=2+\sin(\pi k)$	1	3	1.157
$\theta(k)=3+\sin(\pi k/2)$	2	4	1.180

4.2 Impact of packet loss rate on system robustness

Next, the impact of the packet loss rate on the robustness of the system is discussed. The packet losses occur simultaneously in the two links from the sensor to the controller and the controller to the actuator. Keep the packet loss rate on the sensor-to-controller link unchanged at 0.25, only changes the packet loss rate $1-\beta_1$ from the controller to actuator, and the experimental results are shown in Table 2.

Table 2 Impact of packet loss rate on robustness

Parameter	$1-\beta_1$			
	0.1	0.2	0.3	0.38
$\gamma_{\min}$	1.124 3	1.130 8	1.158 8	No feasible solution

It can be seen from Table 2 that the greater the packet loss rate, the weaker the system robustness. When the packet loss rate exceeds the limit that the system can withstand, the stability of the system will decrease.

Under the above parameters, as the packet loss rate reaches 38%, the system in (8) has no feasible solution. However, when the parameters change, the maximum packet loss rate that the system can endure will also change. For different systems, the maximum packet loss rate that can be tolerated is also different.

4.3 Numerical simulations

Given the following parameters, time delay $\theta(k)=2+[1+(-1)^k]/2$, sampling period $h=0.01$ s, $\varepsilon_1=1.5$, $\varepsilon_2=1.2$, $\mu=2.1$, $\lambda=0.05$, $\alpha_1=0.75$, $\beta_1=0.9$. The switching signal meets the average dwell time that: $T_a \geqslant T_a^*=-\ln\mu/\ln(1-\lambda)$, then we can get $T_a^*=17.890\,6$. For convenience, we select that $T_a=20>T_a^*$. Therefore, the switching law is designed as

$$\sigma=\begin{cases}1, & [k/20]=0,2,4,6,\cdots \\ 2, & [k/20]=1,3,5,7,\cdots\end{cases}.$$

The interference signal and the fault signal are set up as

$$d(k)=0.1\sin(0.1k),\ \ 0\leqslant k\leqslant 200,$$

$$f(k)=\begin{cases}\sin(k), & 30\leqslant k\leqslant 70 \\ 1, & 100\leqslant k\leqslant 145 \\ 0, & \text{others}\end{cases}.$$

In addition, the fuzzy sets are chosen as follows:

$$M_{11}^1(x(k))=1-\frac{1}{1+\mathrm{e}^{-3x(k+1)}},$$
$$M_{11}^2(x(k))=1-M_{11}^1(x(k)),$$
$$M_{21}^1(x(k))=1-\frac{1}{1+\mathrm{e}^{-2.5(x(k)-1)}},$$
$$M_{21}^2(x(k))=1-M_{21}^1(x(k)).$$

By solving the above LMIs, the controller gain $\boldsymbol{K}_{ij}$, observer gain $\boldsymbol{L}_{il}$, residual generator gain $\boldsymbol{Z}_{il}$ and the minimum values of H_∞ performance γ are obtained:

$$\boldsymbol{K}_{11}=[-0.15\ -1.58],\ \boldsymbol{K}_{21}=[0.95\ -2.85],$$
$$\boldsymbol{K}_{12}=[-1.2\ -0.32],\ \boldsymbol{K}_{22}=[-1.91\ -1.58],$$
$$\boldsymbol{L}_{11}=\begin{bmatrix}-1.91\\-0.73\end{bmatrix},\ \boldsymbol{L}_{12}=\begin{bmatrix}-1.35\\-0.96\end{bmatrix},$$
$$\boldsymbol{L}_{21}=\begin{bmatrix}-2.78\\-3.61\end{bmatrix},\ \boldsymbol{L}_{22}=\begin{bmatrix}-1.53\\-0.94\end{bmatrix},$$
$$Z_{11}=0.002\,6, Z_{12}=0.009\,8,$$
$$Z_{21}=0.003\,7, Z_{22}=0.005\,6,$$
$$\gamma=1.135\,4. \tag{15}$$

Next, the simulation results are shown in Fig. 1–Fig. 6.

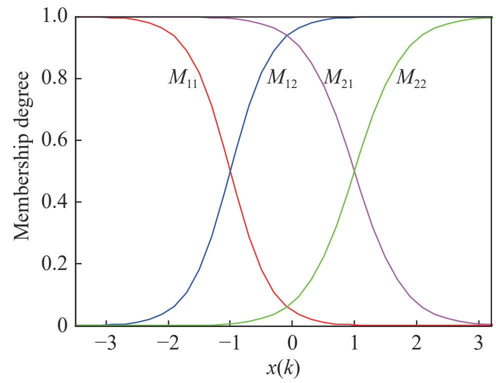

Fig. 1 Membership functions

Fig. 2 is the switching signal of the system; Fig. 3 shows two different types of fault signals.

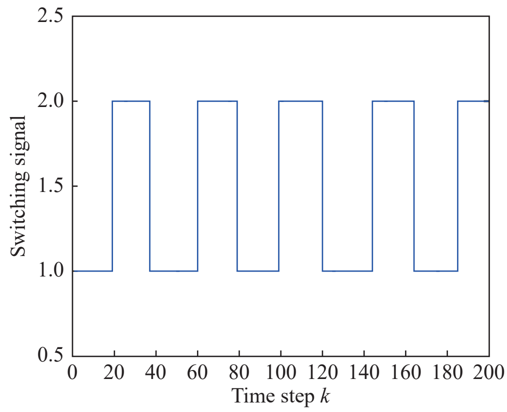

Fig. 2 Switching signal

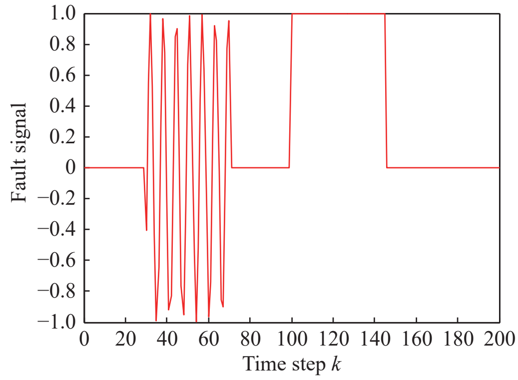

Fig. 3 Fault signal

[Fig. 4](#) represents the state estimation error. From [Fig. 4](#), it can be seen that the residual error will fluctuate greatly when the system has a fault. And if there is no fault, the value of the residual error will oscillate around 0 and tend to be stable.

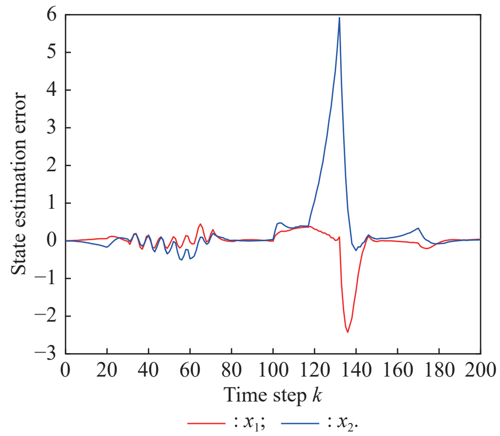

Fig. 4 State estimation error

[Fig. 5](#) shows the residual evaluation function obtained by using the shift operator with the parameters in [21]. As can be seen in [Fig. 5](#), when the shift operator is used for fault detection, the evaluation function detects the fault at the 11th sampling point after the fault occurs. Therefore, this method is not sensitive enough to fault perception. [Fig. 6](#) is the evaluation function using the delta operator. As the picture shows that the fault detection threshold of the system is $J_{\rm th} = 0.083\,1$. When $k = 30$, the system appears a fault. At the same time, the residual evaluation function $J(\tilde{r}) = 0.408\,1 > J_{\rm th}$. It means that the fault can be detected in time. After the first fault is over, the residual evaluation function reaches a new critical value 4.535, which can be used as the threshold for the next fault detection. When $k = 100$, the system fails for the second time. The value of the evaluation function at this time is 4.644, which exceeds the threshold of 4.535. It can be seen that the second fault can still be quickly detected. By comparing [Fig. 5](#) and [Fig. 6](#), it is obvious that the fault detection filter designed in this paper is more effective. And it can quickly detect the fault of the system many times.

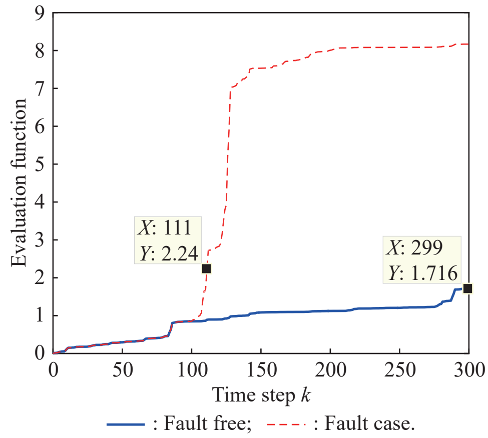

Fig. 5 Evaluation function of shift operator

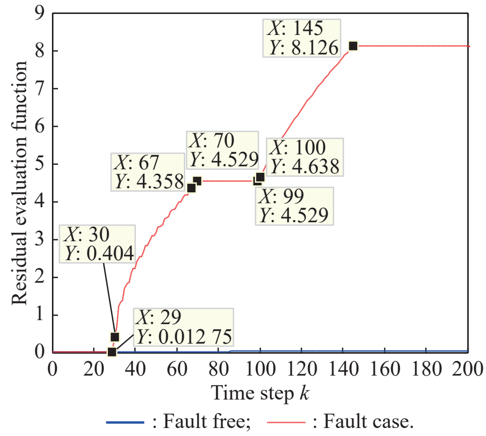

Fig. 6 Evaluation function $\tilde{J}(r)$

5. Conclusions

This paper studies the fault detection problem of the delta operator switched fuzzy system with time-varying delay, bilateral packet losses and uncertainty. A robust fault detection filter under an arbitrary switching law is designed, and the system is proved to have H_∞ performance. By constructing multiple Lyapunov functions, the exponential mean square stability conditions of the system and filter parameters are obtained. Also, the impact of delay and packet loss rate on system performance are analyzed through the numerical simulation results. And the comparison experiment proves that the fault detection filter designed in this paper is faster and more effective. In addition, delta operator switched fuzzy system fault detection under other constraints also has important scientific significance and requires further research.

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