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Cooperative game theory-based steering law design of a CMG system

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Abstract: Spacecraft require a large-angle manoeuvre when performing agile manoeuvring tasks, therefore a control moment gyroscope (CMG) is employed to provide a strong moment. However, the control of the CMG system easily falls into singularity, which renders the actuator unable to output the required moment. To solve the singularity problem of CMGs, the control law design of a CMG system based on a cooperative game is proposed. First, the cooperative game model is constructed according to the quadratic programming problem, and the cooperative strategy is constructed. When the strategy falls into singularity, the weighting coefficient is introduced to carry out the strategy game to achieve the optimal strategy. In theory, it is proven that the cooperative game manipulation law of the CMG system converges, the sum of the CMG frame angular velocities is minimized, the energy consumption is small, and there is no output torque error. Then, the CMG group system is simulated. When the CMG system is near the singular point, it can quickly escape the singularity. When the CMG system falls into the singularity, it can also escape the singularity. Considering the optimization of angular momentum and energy consumption, the feasibility of the CMG system steering law based on a cooperative game is proven.

Keywords: control moment gyroscopes (CMG), cooperative game theory, steering laws.

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1. Introduction

A control moment gyroscope (CMG) is a kind of inertial actuator that is utilized in spacecraft. CMGs are widely utilized in various space missions and can be applied to control the optical platform of spacecraft. A CMG amplifies the torque to obtain a large torque for spacecraft attitude control. The actuators of worldview-1 launched in September 2007 and of worldview-2 launched in October 2009 are equipped with four CMGs with a rotational speed of 6 000 rmp to realize a side swing capacity of ±40°, a manoeuvring angular velocity of 3.5°/s, and an angular acceleration of 1.5°/s . Pleiade series satellites are also equipped with pyramid CMGs, which has simple mechanical structure, low energy consumption, and high reliability.

Presently, the mainstream CMG is a single gimbal CMG (SGCMG). A CMG is mainly composed of a rotor and a single frame, and its output torque linearity is high. A CMG can quickly double the amplification torque and response, so it is widely utilized [1]. A single CMG cannot output a three-axis torque, so it needs multiple CMGs to combine into the CMG group system as the spacecraft actuator, and there are many singular points in the angular momentum envelope of the CMG group system, that is, it cannot output the desired torque in a certain plane [2]. Therefore, scholars have performed much research on how to escape the CMG singularity and design the control law of the CMG group system. The control law of the CMG group system is to calculate the required frame angular velocity based on the current frame angle and command torque [3]. In 2008, the CMG control law based on singular value decomposition was investigated by Zhang et al. [4]. The law can avoid singularity to a certain extent, but the effect of escaping singularity is not ideal. In 2010, Takada et al. [5] proposed the CMG singularity avoidance control law based on a singular surface cost function, which avoids the internal singularity of the system by introducing torque error. In 2011, Nanamori et al. [6] proposed a new optimal initial frame angle control law without knowing the trajectory of the desired torque in advance. In 2011, He [7] proposed an error tolerant control law, which takes into account the rapid departure from singularity but increases the output torque error. In 2013, Wu et al. [8] proposed the CMG compound control law design based on feedforward and feedback, which can avoid the CMG singularity problem but cannot be solved if the CMG initially enters the singular state. In 2014, Geng et al. [9] applied zero motion to

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avoid CMG singularity, making CMG always run outside the dead zone of frame shaft speed. In 2017, Wu et al. [10] proposed the CMG-modified singular direction control law, which is suitable for all CMG configurations, but multilayer correction will lead to a larger torque error. In 2018, Guo et al. [11] proposed a fast singular escape control law in the frame angular space to drive SGCMG to escape from impassable singular surfaces. In 2019, Lei et al. [12] proposed the dynamic adjustment strategy of torque distribution to improve the singularity avoidance ability of the system without considering the energy consumption. In 2020, Guo [13] proposed the singularity avoidance control law of CMGs based on manifold theory and designed the frame angle redirection control law and manifold path selection (MPS) control law without introducing torque error but without considering energy consumption. The CMG control law cannot escape singularity and simultaneously solves the torque error and energy consumption. To solve these problems, this paper proposes a CMG group system manipulation law design based on a cooperative game. Game theory is seldom applied in CMG group systems. In 2005, for the first time, Lee et al. [14] proposed that game theory was applied to CMG singularity avoidance, which can effectively avoid the singularity conditions in the process of large angle attitude manipulation, but the problem of energy consumption is not considered. In 2019, Wu et al. [15] applied game theory to a CMG + RW hybrid actuator but added a momentum wheel, increased weight and required space.

To realize the large angle tracking control problem of the optical platform, this paper examines the CMG group system as the actuator and proposes the CMG group system control law design based on a cooperative game so that the CMG can successfully escape from the singularity near the singularity and when it falls into the singularity, and almost no torque difference is introduced, which can reduce energy consumption and cost.

2. CMG group system configuration

2.1 Pyramidal CMGs system

The system configuration of the CMG group adopts the classical pyramid configuration [16], as shown in Fig. 1.

The frame angle set is $\boldsymbol{\alpha}=[\alpha_1,\alpha_2,\alpha_3,\alpha_4]^{\mathrm{T}}$, the dip angle is β, and the CMG angular momentum is $\boldsymbol{h}_{\mathrm{CMG}}$:

$$\boldsymbol{h}_{\mathrm{CMG}}=h_0\sum_{i=1}^{4}\boldsymbol{A}_{\mathrm{CMG},i}=$$
$$h_0\begin{bmatrix} -c(\beta)s(\alpha_1)-c(\alpha_2)+c(\beta)s(\alpha_3)+c(\alpha_4)\\ c(\alpha_1)-c(\beta)s(\alpha_2)-c(\alpha_3)+c(\beta)s(\alpha_4)\\ s(\beta)(s(\alpha_1)+s(\alpha_2)+s(\alpha_3)+s(\alpha_4))\end{bmatrix} \quad (1)$$

where $c(\beta)=\cos\beta$, $s(\beta)=\sin\beta$, $s(\alpha_i)=\sin\alpha_i$, $c(\alpha_i)=\cos\alpha_i$, and $\boldsymbol{A}_{\mathrm{CMG},i}$ is the angular momentum direction of the ith CMG rotor.

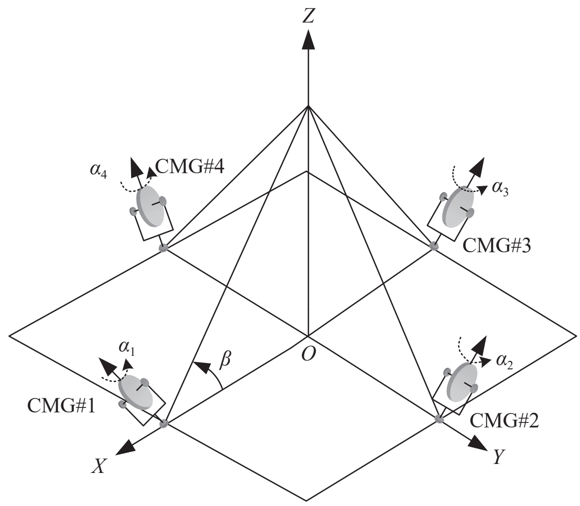


Fig. 1 Pyramidal CMGs system

The time derivative of (1) is

$$\dot{\boldsymbol{h}}=\boldsymbol{u}_{\mathrm{o}}=\boldsymbol{J}\dot{\boldsymbol{\alpha}}=$$
$$h_0\begin{bmatrix} -c(\beta)c(\alpha_1) & s(\alpha_2) & c(\beta)c(\alpha_3) & -s(\alpha_4)\\ -s(\alpha_1) & -c(\beta)c(\alpha_2) & s(\alpha_3) & c(\beta)c(\alpha_4)\\ s(\beta)c(\alpha_1) & s(\beta)c(\alpha_2) & s(\beta)c(\alpha_3) & s(\beta)c(\alpha_4)\end{bmatrix}\begin{bmatrix}\dot{\alpha}_1\\ \dot{\alpha}_2\\ \dot{\alpha}_3\\ \dot{\alpha}_4\end{bmatrix} \quad (2)$$

where $\boldsymbol{J}$ is the Jacobian matrix of the CMG system, $\dot{\boldsymbol{\alpha}}=[\dot{\alpha}_1,\dot{\alpha}_2,\dot{\alpha}_3,\dot{\alpha}_4]^{\mathrm{T}}$ is the frame angular velocity of the CMG system, and $\boldsymbol{u}_{\mathrm{o}}$ is the output torque of the CMG system.

2.2 Singularity analysis of CMGs

When the CMG system falls into singularity, it cannot output the required torque, and its Jacobian matrix is not full of rank:

$$\begin{cases}\mathrm{rank}(\boldsymbol{J})<3\\ \det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})=0\end{cases}. \quad (3)$$

The singularity index S_{CMG} is chosen to determine whether the CMG singular problem occurs [17]. The larger S_{CMG} is, the farther the CMG system is from singularity. If $S_{\mathrm{CMG}}=0$, the CMG comes into singularity.

$$S_{\mathrm{CMG}}=\frac{\det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})}{[\det(\boldsymbol{J}\boldsymbol{J}^{\mathrm{T}})]_{\max}}\in[0,1] \quad (4)$$

where $[\det(\boldsymbol{J}_a\boldsymbol{J}_{\mathrm{a}}^{\mathrm{T}})]_{\max}$ is the maximum of the value that is related to the CMG configuration. In the simulation, $[\det(\boldsymbol{J}_a\boldsymbol{J}_{\mathrm{a}}^{\mathrm{T}})]_{\max}=2.37$ [15].

3. CMGs cooperative game theory steering logic

3.1 Quadratic programming problem of CMGs control

The command torque is $\boldsymbol{u}_c = [u_{c,x}, u_{c,y}, u_{c,z}]^{\mathrm{T}}$. To optimize the angular momentum and energy consumption, the quadratic programming problem is considered:

$$\boldsymbol{u}_{\mathrm{o}} = h_0 \boldsymbol{J}\dot{\boldsymbol{\alpha}} = \boldsymbol{A}\dot{\boldsymbol{\alpha}} = \boldsymbol{u}_c. \tag{5}$$

The problem can be converted to the minimization problem of $L(\dot{\boldsymbol{\alpha}})$:

$$L(\dot{\boldsymbol{\alpha}}) = \sum_{k=1}^{4} \dot{\alpha}_k^2. \tag{6a}$$

Subjected to

$$f(\dot{\boldsymbol{\alpha}}) = \begin{bmatrix} f_1(\dot{\boldsymbol{\alpha}}) \\ f_2(\dot{\boldsymbol{\alpha}}) \\ f_3(\dot{\boldsymbol{\alpha}}) \end{bmatrix} = \boldsymbol{0} \tag{6b}$$

where

$$f_1(\dot{\boldsymbol{\alpha}}) = A_{11}\dot{\alpha}_1 + A_{12}\dot{\alpha}_2 + A_{13}\dot{\alpha}_3 + A_{14}\dot{\alpha}_4 - u_{c,x}, \tag{7a}$$

$$f_2(\dot{\boldsymbol{\alpha}}) = A_{21}\dot{\alpha}_1 + A_{22}\dot{\alpha}_2 + A_{23}\dot{\alpha}_3 + A_{24}\dot{\alpha}_4 - u_{c,y}, \tag{7b}$$

$$f_3(\dot{\boldsymbol{\alpha}}) = A_{31}\dot{\alpha}_1 + A_{32}\dot{\alpha}_2 + A_{33}\dot{\alpha}_3 + A_{34}\dot{\alpha}_4 - u_{c,z}, \tag{7c}$$

and $A_{ij} = h_0 J(i, j)$.

Let

$$H(\dot{\boldsymbol{\alpha}}, \boldsymbol{\lambda}) = L(\dot{\boldsymbol{\alpha}}) + \boldsymbol{\lambda}^{\mathrm{T}} f(\dot{\boldsymbol{\alpha}}) = L(\dot{\boldsymbol{\alpha}}) + \lambda_1 f_1(\dot{\boldsymbol{\alpha}}) + \lambda_2 f_2(\dot{\boldsymbol{\alpha}}) + \lambda_3 f_3(\dot{\boldsymbol{\alpha}}) \tag{8}$$

where the Lagrangian operator is $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^{\mathrm{T}}$.

Thus, we obtain

$$\frac{\partial H(\dot{\alpha}, \lambda)}{\partial \dot{\alpha}_k} = 0, \ k = 1, 2, 3, 4. \tag{9}$$

Substituting (8) into (9), we have

$$\begin{cases} \dfrac{\partial H(\dot{\boldsymbol{\alpha}}, \boldsymbol{\lambda})}{\partial \dot{\alpha}_1} = 2\dot{\alpha}_1 + \lambda_1 A_{11} + \lambda_2 A_{21} + \lambda_3 A_{31} = 0 \\ \dfrac{\partial H(\dot{\boldsymbol{\alpha}}, \boldsymbol{\lambda})}{\partial \dot{\alpha}_2} = 2\dot{\alpha}_2 + \lambda_1 A_{12} + \lambda_2 A_{22} + \lambda_3 A_{32} = 0 \\ \dfrac{\partial H(\dot{\boldsymbol{\alpha}}, \boldsymbol{\lambda})}{\partial \dot{\alpha}_3} = 2\dot{\alpha}_3 + \lambda_1 A_{13} + \lambda_2 A_{23} + \lambda_3 A_{33} = 0 \\ \dfrac{\partial H(\dot{\boldsymbol{\alpha}}, \boldsymbol{\lambda})}{\partial \dot{\alpha}_4} = 2\dot{\alpha}_4 + \lambda_1 A_{14} + \lambda_2 A_{24} + \lambda_3 A_{34} = 0 \end{cases}. \tag{10}$$

Therefore, we obtain

$$\dot{\alpha}_k = -\frac{1}{2}\sum_{i=1}^{3} \lambda_i A_{ik}, \ k = 1, 2, 3, 4. \tag{11}$$

Substituting (11) into (6a), we have

$$L(\dot{\boldsymbol{\alpha}}) = \sum_{k=1}^{4} \dot{\alpha}_k^2 = \frac{1}{4}\left(\sum_{i=1}^{3}\lambda_i A_{i1}\right)^2 + \frac{1}{4}\left(\sum_{i=1}^{3}\lambda_i A_{i2}\right)^2 + \frac{1}{4}\left(\sum_{i=1}^{3}\lambda_i A_{i3}\right)^2 + \frac{1}{4}\left(\sum_{i=1}^{3}\lambda_i A_{i4}\right)^2 = -\frac{1}{2}a_1\lambda_1^2 - \frac{1}{2}a_2\lambda_2^2 - \frac{1}{2}a_3\lambda_3^2 - a_{12}\lambda_1\lambda_2 - a_{13}\lambda_1\lambda_3 - a_{23}\lambda_2\lambda_3, \tag{12a}$$

$$f(\dot{\boldsymbol{\alpha}}) = f(\boldsymbol{A}, \boldsymbol{\lambda}) = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} - \begin{bmatrix} u_{c,x} \\ u_{c,y} \\ u_{c,z} \end{bmatrix}, \tag{12b}$$

where the elements $a_i (i = 1, 2, 3)$ and $a_{ij} (i = 1, 2, 3; j = 1, 2, 3; i \neq j)$ are defined as

$$a_i = -\frac{1}{2}\sum_{k=1}^{4} A_{ik}^2, \quad i = 1, 2, 3, \tag{13a}$$

$$a_{ij} = -\frac{1}{2}\sum_{k=1}^{4} A_{ik}A_{jk}, \ i = 1, 2, 3; j = 1, 2, 3; i < j. \tag{13b}$$

3.2 Design of CMGs cooperative game theory steering logic

3.2.1 Cooperative game model of CMGs

Cooperative game model of CMGs system is $R = \{W, V\}$, where V is the return function and W is the partner, including Partner 1, Partner 2, and Partner 3. When the cooperative game is conducted, the income of at least one partner increases, while the gains of the other partners are not reduced. All cooperative games are included in strategy set D_i, in which the cooperative game situation forms the situation $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]$, and satisfied $|f_j(\boldsymbol{A}, \boldsymbol{\lambda})| \leqslant \varepsilon_D$ $(j = 1, 2, 3)$, where ε_D is a small positive value, D_i includes all strategies λ_i that satisfy $|f_j(\boldsymbol{A}, \boldsymbol{\lambda})| \leqslant \varepsilon_D$. $|f_j(\boldsymbol{A}, \boldsymbol{\lambda})| \leqslant \varepsilon_D$ is the CMG output torque error ε_D, $|u_{o,x} - u_{c,x}| \leqslant \varepsilon_D$, $|u_{o,y} - u_{c,y}| \leqslant \varepsilon_D$ and $|u_{o,z} - u_{c,z}| \leqslant \varepsilon_D$. Therefore, there is a set of optimal policies $\lambda_i^{\#} \in D_i$ such that $|f_j(\boldsymbol{A}, \boldsymbol{\lambda})| \to 0$, $\boldsymbol{u}_o - \boldsymbol{u}_c = \boldsymbol{0}$.

At the beginning of the game, we need to choose a group of strategy initial values $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]$. Each negotiation will produce a group of strategy change values $\delta\boldsymbol{\lambda} = [\delta\lambda_1, \delta\lambda_2, \delta\lambda_3]$, and the new game situation is $\boldsymbol{\lambda} = \boldsymbol{\lambda} + \delta\boldsymbol{\lambda}$. Note that $\delta\boldsymbol{\lambda}$ needs to be designed according to the constraints, and $|f_j(\boldsymbol{A}, \boldsymbol{\lambda})| \to 0$ when $\delta\boldsymbol{\lambda} \to \boldsymbol{0}$. To calculate the strategy change value, we need to design the value function $V(\{i\})$:

$$V(\{i\}) = \begin{cases} \dfrac{1}{3}(1 - \delta\lambda_i), & \delta\lambda_1 < \delta\lambda_1^- \text{ or } \delta\lambda_2 < \delta\lambda_2^- \text{ or } \delta\lambda_3 < \delta\lambda_3^- \\ 0, & \delta\lambda_1 \geqslant \delta\lambda_1^- \text{ and } \delta\lambda_2 \geqslant \delta\lambda_2^- \text{ and } \delta\lambda_3 \geqslant \delta\lambda_3^- \end{cases} \tag{14}$$

where $\delta\lambda_i^-$ is the strategy change value obtained from the last negotiation.

To ensure that at least one partner's value function increases in each negotiation, at least one new λ_i is closer to the optimal strategy $\lambda_i^\#$. If $\delta\lambda_i \geqslant \delta\lambda_i^-$ ($i = 1, 2, 3$), then the negotiation cannot reach an agreement, this plan is not passed, and $V(\{i\}) = 0$; otherwise, this plan is passed. $V(\{i\})$ is applied to judge whether the scheme is passed and whether the strategy change value converges [15].

If $\delta\boldsymbol{\lambda} \to \mathbf{0}$, the strategy converges, the optimal strategy and the maximum value function are obtained, and the cooperative game ends.

3.2.2 Design of CMGs cooperative game theory

The design of the cooperative game manipulation law includes the following steps:

Step 1 Select the policy initial value and value function initial value according to $f(\boldsymbol{A}, \boldsymbol{\lambda}) = 0$,

$$\lambda_3 = \frac{(a_2a_{12}a_{13} - a_{12}^2a_{23})\,u_{c,x}}{\lambda_{3,*}} + \frac{(a_1a_{12}a_{23} - a_{12}^2a_{13})\,u_{c,y}}{\lambda_{3,*}} + \frac{(a_{12}^3 - a_1a_2a_{12})\,u_{c,z}}{\lambda_{3,*}}, \tag{15a}$$

$$\lambda_2 = \frac{a_{12}u_{c,x} - a_1u_{c,y} - (a_{12}a_{13} - a_1a_{23})\,\lambda_3}{\lambda_{2,*}}, \tag{15b}$$

$$\lambda_1 = \frac{u_{c,x} - a_{12}\lambda_2 - a_{13}\lambda_3}{\lambda_{1,*}}, \tag{15c}$$

$$V(\lambda_i) = 0.1,\ \ i = 1, 2, 3, \tag{16}$$

where the singularity indexes of strategy are $\lambda_{1,*}$, $\lambda_{2,*}$, and $\lambda_{3,*}$:

$$\begin{cases} \lambda_{3,*} = a_2a_{12}a_{13}^2 - 2a_{12}^2a_{13}a_{23} + a_1a_{12}a_{23}^2 + \\ \quad a_{12}^3a_3 - a_1a_2a_3a_{12} \\ \lambda_{2,*} = a_{12}^2 - a_1a_2 \\ \lambda_{1,*} = a_1 \end{cases}. \tag{17}$$

Note that CMG singularity leads to Lagrangian singularity. When the CMG system is singular, it cannot output torque.

(i) When the X-channel does not output torque, $A_{11} = A_{12} = A_{13} = A_{14} = 0$,

$$\begin{cases} \lambda_{1,*} = a_1 = -\dfrac{1}{2}\displaystyle\sum_{k=1}^{4} A_{1k}^2 = 0 \\ \lambda_{2,*} = {a_{12}}^2 - a_1a_2 = \dfrac{1}{4}\left(\displaystyle\sum_{k=1}^{4} A_{1k}J_{2k}\right)^2 - \dfrac{1}{4}\displaystyle\sum_{k=1}^{4} A_{1k}^2 \sum_{k=1}^{4} A_{2k}^2 = 0 \\ \lambda_{3,*} = a_2a_{12}a_{13}^2 - 2a_{12}^2a_{13}a_{23} + a_1a_{12}a_{23}^2 + \\ \quad a_{12}^3a_3 - a_1a_2a_3a_{12} = 0 \end{cases}. \tag{18}$$

Thus, λ_1, λ_2 and λ_3 are strange.

(ii) When the Y-channel does not output torque, $A_{21} = A_{22} = A_{23} = A_{24} = 0$,

$$\begin{cases} \lambda_{1,*} = a_1 = -\dfrac{1}{2}\displaystyle\sum_{k=1}^{4} A_{1k}^2 \neq 0 \\ \lambda_{2,*} = {a_{12}}^2 - a_1a_2 = \dfrac{1}{4}\left(\displaystyle\sum_{k=1}^{4} A_{1k}A_{2k}\right)^2 - \dfrac{1}{4}\displaystyle\sum_{k=1}^{4} A_{1k}^2 \sum_{k=1}^{4} A_{2k}^2 = 0. \\ \lambda_{3,*} = a_2a_{12}a_{13}^2 - 2a_{12}^2a_{13}a_{23} + a_1a_{12}a_{23}^2 + \\ \quad a_{12}^3a_3 - a_1a_2a_3a_{12} = 0 \end{cases} \tag{19}$$

Therefore, λ_2 and λ_3 are strange.

(iii) When the Z-channel does not output torque, $A_{31} = A_{32} = A_{33} = A_{34} = 0$,

$$\begin{cases} \lambda_{1,*} = a_1 = -\dfrac{1}{2}\displaystyle\sum_{k=1}^{4} A_{1k}^2 \neq 0 \\ \lambda_{2,*} = {a_{12}}^2 - a_1a_2 = \dfrac{1}{4}\left(\displaystyle\sum_{k=1}^{4} A_{1k}A_{2k}\right)^2 - \dfrac{1}{4}\displaystyle\sum_{k=1}^{4} A_{1k}^2 \sum_{k=1}^{4} A_{2k}^2 \neq 0. \\ \lambda_{3,*} = a_2a_{12}a_{13}^2 - 2a_{12}^2a_{13}a_{23} + a_1a_{12}a_{23}^2 + \\ \quad a_{12}^3a_3 - a_1a_2a_3a_{12} = 0 \end{cases} \tag{20}$$

Thus, λ_3 is strange.

When λ_i is close to singularity, $\lambda_{i,*} \leqslant \varepsilon_\lambda$, where ε^* is a smaller positive number. When the strategy approaches singularity, it can be avoided by modifying (15),

$$\begin{cases} \lambda_3 = \dfrac{(a_2a_{12}a_{13} - a_{12}^2a_{23})\,u_{c,x}}{\lambda_{3,*} + \varepsilon^*} + \dfrac{(a_1a_{12}a_{23} - a_{12}^2a_{13})\,u_{c,y}}{\lambda_{3,*} + \varepsilon^*} + \\ \quad \dfrac{(a_{12}^3 - a_1a_2a_{12})\,u_{c,z}}{\lambda_{3,*} + \varepsilon^*} \\ \lambda_2 = \dfrac{a_{12}u_{c,x} - a_1u_{c,y} - (a_{12}a_{13} - a_1a_{23})\,\lambda_3}{\lambda_{2,*} + \varepsilon^*} \\ \lambda_1 = \dfrac{u_{c,x} - a_{12}\lambda_2 - a_{13}\lambda_3}{\lambda_{1\,*} + \varepsilon^*} \end{cases}. \tag{21}$$

Although introducing ε^* can solve the singular problem of the Lagrange operator, it will increase the output torque error of CMGs, which can be avoided by reasonable design.

Step 2 Calculate the policy change value $\delta\boldsymbol{\lambda} = [\delta\lambda_1, \delta\lambda_2, \delta\lambda_3]$ and the update strategy situation and value function.

If the strategy does not fall into singularity, the change value of the strategy is expressed as follows:

$$\delta\lambda_3 = \frac{(a_2a_{12}a_{13} - a_{12}^2a_{23})\,\Gamma_{31}\,(u_{c,x} - U_1)}{\lambda_{3,*}} + \frac{(a_1a_{12}a_{23} - a_{12}^2a_{13})\,\Gamma_{32}\,(u_{c,y} - U_2)}{\lambda_{3,*}} + \frac{(a_{12}^3 - a_1a_2a_{12})\,\Gamma_{33}\,(u_{c,z} - U_3)}{\lambda_{3,*}}, \tag{22a}$$

$$\delta\lambda_2 = \frac{a_{12}\Gamma_{21}\,(u_{c,x} - U_1)}{\lambda_{2,*}} - \frac{a_1\Gamma_{22}\,(u_{c,y} - U_2)}{\lambda_{2,*}} - \frac{(a_{12}a_{13} - a_1a_{23})\,\delta\lambda_3}{\lambda_{2,*}}, \tag{22b}$$

$$\delta\lambda_1 = \frac{\varGamma_{11}(u_{c,x}-U_1)-a_{12}\delta\lambda_2-a_{13}\delta\lambda_3}{\lambda_{1,*}}, \tag{22c}$$

where the weighting coefficient is $\varGamma_{ij}$ ($i = 1, 2, 3$; $j = 1, 2, 3$; $j \leqslant i$). The weighting coefficient can improve the convergence speed of iteration and the accuracy of CMG output torque, and reasonable design of the weighting coefficient can avoid torque output error. In addition, reasonable design of the weighting coefficient can ensure the convergence of iteration. The iterative moment is $\boldsymbol{U} = [U_1, U_2, U_3]^{\mathrm{T}}$,

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}. \tag{23}$$

If the strategy falls into singularity, the strategy change value is calculated according to the following formula:

$$\delta\lambda_3 = \frac{(a_2a_{12}a_{13}-a_{12}^2a_{23})\varGamma_{31}(u_{c,x}-U_1)}{\lambda_{3,*}+\varepsilon^*}+\frac{(a_1a_{12}a_{23}-a_{12}^2a_{13})\varGamma_{32}(u_{c,y}-U_2)}{\lambda_{3,*}+\varepsilon^*}+\frac{(a_{12}^3-a_1a_2a_{12})\varGamma_{33}(u_{c,z}-U_3)}{\lambda_{3,*}+\varepsilon^*}, \tag{24a}$$

$$\delta\lambda_2 = \frac{a_{12}\varGamma_{21}(u_{c,x}-U_1)}{\lambda_{2,*}+\varepsilon^*}-\frac{a_1\varGamma_{22}(u_{c,y}-U_2)}{\lambda_{2,*}+\varepsilon^*}-\frac{(a_{12}a_{13}-a_1a_{23})\delta\lambda_3}{\lambda_{2,*}+\varepsilon^*}, \tag{24b}$$

$$\delta\lambda_1 = \frac{\varGamma_{11}(u_{c,x}-U_1)-a_{12}\delta\lambda_2-a_{13}\delta\lambda_3}{\lambda_{1,*}+\varepsilon^*}. \tag{24c}$$

The new game situation and value function are expressed as follows:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}+\delta\boldsymbol{\lambda}, \tag{25a}$$

$$V(\{i\})=\begin{cases} \dfrac{1}{3}(1-\delta\lambda_i), & \delta\lambda_1<\delta\lambda_1^- \text{ or } \delta\lambda_2<\delta\lambda_2^- \text{ or } \delta\lambda_3<\delta\lambda_3^- \\ 0, & \delta\lambda_1\geqslant\delta\lambda_1^- \text{ and } \delta\lambda_2\geqslant\delta\lambda_2^- \text{ and } \delta\lambda_3\geqslant\delta\lambda_3^- \end{cases}. \tag{25b}$$

Step 3 If $V(\{i\}) \neq 0$, repeat Steps 2 and 3 until the iteration converges, and obtain the optimal strategy after convergence. The optimal strategy can be obtained after convergence.

Substituting (25a) into (11), we have

$$\boldsymbol{u} = \boldsymbol{J}\dot{\boldsymbol{\alpha}} = -\frac{1}{2}\boldsymbol{J}\begin{bmatrix} \lambda_1J_{11}+\lambda_2J_{21}+\lambda_1J_{31} \\ \lambda_1J_{12}+\lambda_2J_{22}+\lambda_1J_{32} \\ \vdots \\ \lambda_1J_{14}+\lambda_2J_{24}+\lambda_1J_{34} \end{bmatrix}. \tag{26}$$

The energy cost function of CMGs system defined as

$$E_{\text{cost}} = \sum_{i=1}^{4}\frac{1}{2}\boldsymbol{J}_{\text{CMG}i}\dot{\alpha}_i^2 \tag{27}$$

where $\boldsymbol{J}_{\text{CMG}i}$ is the moment of inertia of the ith CMG. Note that each CMG rotor has the same constant speed, so the energy consumption function does not include the energy to maintain the speed of the CMG rotor, that is, it is only used to measure the energy consumption of the CMG frame angle rotation.

3.2.3 Convergence and accuracy analysis of the CMG cooperative game control law

When the CMG cooperative game manipulation law is adopted, the iteration will converge to a certain extent. To prove the convergence of the control law, there are two cases: Case 1, the current strategy has no singularity; Case 2, the current strategy has singularity.

Case 1 There is no singularity in the current strategy, and in the first iteration, we obtain

$$\begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix}\begin{bmatrix} \lambda_{1,1} \\ \lambda_{2,1} \\ \lambda_{3,1} \end{bmatrix} = \begin{bmatrix} u_{c,x} \\ u_{c,y} \\ u_{c,z} \end{bmatrix} \tag{28}$$

where the iterative innings of the first iteration are $\boldsymbol{U}_1 = [U_{1,1}, U_{2,1}, U_{3,1}]^{\mathrm{T}}$ and the policy initial value is $\boldsymbol{\lambda}_1 = [\lambda_{1,1}, \lambda_{2,1}, \lambda_{3,1}]^{\mathrm{T}}$.

Substituting (28) into (22a)−(22c), we have $\delta\lambda_1 = 0$ and policy convergence.

Case 2 The current strategy is singular. In the first iteration, we obtain

$$\delta\boldsymbol{u}_1 = \boldsymbol{u}_c - \begin{bmatrix} U_{1,1} & U_{2,1} & U_{3,1} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} u_{c,x} \\ u_{c,y} \\ u_{c,z} \end{bmatrix} - \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix}\begin{bmatrix} \lambda_{1,1} \\ \lambda_{2,1} \\ \lambda_{3,1} \end{bmatrix} = \begin{bmatrix} \delta u_{1,1} & \delta u_{2,1} & \delta u_{3,1} \end{bmatrix}^{\mathrm{T}} \tag{29}$$

where the iteration (moment) error of the first iteration is $\delta\boldsymbol{u}_1 = [\delta u_{1,1}, \delta u_{2,1}, \delta u_{3,1}]^{\mathrm{T}}$.

(i) When λ_3 is singular,

$$\delta u_{3,1} = u_{c,z} - U_{3,1} = \frac{\varepsilon^*[(a_{13}a_2-a_{12}a_{23})u_{c,x}+(a_1a_{23}-a_{12}a_{13})u_{c,y}+(a_{12}^2-a_1a_2)u_{c,z}]}{(a_{12}^2-a_1a_2)(\lambda_{3,*}+\varepsilon^*)}, \tag{30a}$$

$$\delta u_{2,1} = u_{c,y} - U_{2,1} = 0, \tag{30b}$$

$$\delta u_{1,1} = u_{c,x} - U_{1,1} = 0. \tag{30c}$$

Substituting (29) and (30) into (22a)−(22c), we obtain $\delta\lambda_{1,1} \neq 0$, $\delta\lambda_{2,1} \neq 0$, $\delta\lambda_{3,1} \neq 0$. Therefore, the strategy cannot converge in the first iteration.

Assume that the strategy does not converge in the $(k-1)$th iteration and converges in the kth iteration; we then have

$$\begin{bmatrix} U_{1,k} \\ U_{2,k} \\ U_{3,k} \end{bmatrix} = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \lambda_{1,k} \\ \lambda_{2,k} \\ \lambda_{3,k} \end{bmatrix} =$$

$$\begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \lambda_{1,k-1} \\ \lambda_{2,k-1} \\ \lambda_{3,k-1} \end{bmatrix} + \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \delta\lambda_{1,k-1} \\ \delta\lambda_{2,k-1} \\ \delta\lambda_{3,k-1} \end{bmatrix} =$$

$$\begin{bmatrix} U_{1,k-1} \\ U_{2,k-1} \\ U_{3,k-1} \end{bmatrix} + \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \delta\lambda_{1,k-1} \\ \delta\lambda_{2,k-1} \\ \delta\lambda_{3,k-1} \end{bmatrix} \tag{31}$$

where $\boldsymbol{U}_{1,k} = [U_{1,k}, U_{2,k}, U_{3,k}]^{\mathrm{T}}$ is the iteration moment of the kth iteration.

$$U_{1,k} = U_{1,k-1} + a_1\delta\lambda_{1,k-1} + a_{12}\delta\lambda_{2,k-1} + a_{13}\delta\lambda_{3,k-1} = (1-\varGamma_{11})U_{1,k-1} + \varGamma_{11}u_{c,x}, \tag{32a}$$

$$U_{2,k} = U_{2,k-1} + a_{12}\delta\lambda_{1,k-1} + a_2\delta\lambda_{2,k-1} + a_{23}\delta\lambda_{3,k-1} = (1-\varGamma_{22})U_{2,k-1} + \varGamma_{22}u_{c,y} + \frac{a_{12}}{a_1}(\varGamma_{11} - \varGamma_{21})(u_{c,x} - U_{1,k-1}). \tag{32b}$$

Notably, if $\varGamma_{11} = \varGamma_{22} = 1$, we have

$$U_{1,k} = u_{c,k}, \tag{33a}$$

$$U_{2,k} = u_{c,y}. \tag{33b}$$

Substituting (22a)−(22c) and (33a)−(33b) into (31), we obtain

$$U_{3,k} = U_{3,k-1} + (a_{13}\delta\lambda_{1,k-1} + a_{23}\delta\lambda_{2,k-1} + a_3\delta\lambda_{3,k-1}) = \varGamma_{33}u_{c,z}\frac{\lambda_{3,*}}{\lambda_{3,*}+\varepsilon^*} + U_{3,k-1}\left[1 - \varGamma_{33}\frac{\lambda_{3,*}}{\lambda_{3,*}+\varepsilon^*}\right]. \tag{34}$$

According to (34), let $\varGamma_{33} = \dfrac{\lambda_{3,*}+\varepsilon^*}{\lambda_{3,*}}$. (34) is then expressed as follows:

$$U_{3,k} = u_{c,z}. \tag{35}$$

In conclusion, when λ_3 is singular, let $\varGamma_{11} = 1$, $\varGamma_{22} = 1$, $\varGamma_{33} = \dfrac{\lambda_{3,*}+\varepsilon^*}{\lambda_{3,*}}$, and $\delta\lambda_k = 0$. The iteration converges in the kth cycle.

(ii) When λ_2 and λ_3 are singular,

$$\begin{aligned} \delta u_{3,1} = u_{c,z} - U_{3,1} = {} & \frac{\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)\lambda_{3,*}}\left[a_{12}(a_{12}a_{13} - a_1a_{23})(a_{13}a_{23} - a_3a_{12}) + a_{12}(a_{12}a_{13} - a_1a_{23}) - a_{13}\left(\lambda_{2,*}+\varepsilon^*\right)\right]u_{c,x} + \\ & \frac{\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)\lambda_{3,*}}\left[a_{12}(a_1a_{23} - a_{12}a_{13})\left(a_{13}{}^2 - a_1a_3\right) + a_1(a_1a_{23} - a_{12}a_{13})\right]u_{c,y} + \\ & \frac{\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)\lambda_{3,*}}\left[a_{12}(a_{12}a_{13} - a_1a_{23})^2 + a_1\left(\lambda_{2,*}+\varepsilon^*\right)\right]u_{c,z}, \end{aligned} \tag{36a}$$

$$\begin{aligned} \delta u_{2,1} = u_{c,y} - U_{2,1} = {} & \frac{a_{12}\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left[\left(a_{12}^2 - a_1a_2\right)(a_{13}a_{23} - a_3a_{12}) - \varepsilon^*\right]u_{c,x} + \frac{a_{12}\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left[\left(a_{12}^2 - a_1a_2\right)\left(a_1a_3 - a_{13}{}^2\right) + \varepsilon^*\right]u_{c,y} + \\ & \frac{a_{12}\varepsilon^*}{a_1(\lambda_{2,*}+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}(a_{12}a_{13} - a_1a_{23})\left(a_{12}^2 - a_1a_2\right)u_{c,z}, \end{aligned} \tag{36b}$$

$$\delta u_{1,1} = u_{c,x} - U_{1,1} = 0. \tag{36c}$$

Substituting (29)−(30) into (22a)−(22c), we obtain $\delta\lambda_{1,1} \neq 0$, $\delta\lambda_{2,1} \neq 0$, $\delta\lambda_{3,1} \neq 0$; therefore, the strategy cannot converge in the first iteration.

Assume that the strategy does not converge in the $(k-1)$th iteration and converges in the kth iteration; we then have

$$\begin{cases} \varGamma_{22} = \dfrac{1}{(a_1a_{23} - a_{12}a_{13})^2 - (a_1a_3 - a_{13}{}^2)} \\ \varGamma_{32} = \dfrac{a_1(\lambda_{3,*}+\varepsilon^*)}{a_{12}(a_1a_3 - a_{13}{}^2)(\lambda_{2,*}+\varepsilon^*) + a_{12}(a_1a_{23} - a_{12}a_{13})^2}\varGamma_{22} \\ \varGamma_{33} = \dfrac{a_1(\lambda_{2,*}+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}{a_{12}\lambda_{2,*}\left[(a_1a_{23} - a_{12}a_{13})^2 + (\lambda_{2,*}+\varepsilon^*)(a_1a_3 - a_{13}{}^2)\right]} \end{cases}. \tag{37}$$

In conclusion, if λ_2 and λ_3 are singular, let the weighting coefficient be expressed as follows (37), and the iteration converges in the kth cycle.

(iii) When λ_1, λ_2 and λ_3 are singular,

$$\delta u_{3,1} = u_{c,z} - U_{3,1} =$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2)(\lambda_{3,*}+\varepsilon^*)}\Big[a_{12}a_{13}\left(a_{23}{}^2-a_2a_3\right)\left(a_{12}^2-a_1a_2\right)+a_{12}\left(a_{12}a_{13}-a_{23}\left(\lambda_{1,*}+\varepsilon^*\right)\right)-$$
$$a_{13}\left(a_{12}^2-a_1a_2+\varepsilon^*\right)+\left(a_2a_{12}a_{13}-a_{12}^2a_{23}\right)\left(a_{13}{}^2-a_3\left(\lambda_{1,*}+\varepsilon^*\right)\right)\Big]u_{c,x}+$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2)(\lambda_{3,*}+\varepsilon^*)}\Big[a_{12}a_{13}\left(a_3a_{12}-a_{13}a_{23}\right)+\left(a_{13}{}^2-a_3\left(\lambda_{1,*}+\varepsilon^*\right)\right)\left(a_1a_{12}a_{23}-a_{12}^2a_{13}\right)-$$
$$a_1\left(a_{12}a_{13}-a_{23}\left(\lambda_{1,*}+\varepsilon^*\right)\right)\Big]u_{c,y}+$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2)(\lambda_{3,*}+\varepsilon^*)}\Big[a_{12}\left(a_2a_{13}^2-a_{12}a_{13}a_{23}\right)\left(a_{12}^2-a_1a_2\right)+\left(\lambda_{1,*}+\varepsilon^*\right)\left(a_{12}^2-a_1a_2\right)+$$
$$\left(a_{13}{}^2-a_3\left(\lambda_{1,*}+\varepsilon^*\right)\right)\left(a_{12}^3-a_1a_2a_{12}\right)\Big]u_{c,z}, \tag{38a}$$

$$\delta u_{2,1} = u_{c,y} - U_{2,1} =$$
$$\frac{a_{12}\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left[a_{12}\left(a_{23}{}^2-a_2a_3\right)\lambda_{2,*}+\left(a_{12}a_{13}-a_{23}\left(\lambda_{1,*}+\varepsilon^*\right)\right)(a_2a_{13}-a_{12}a_{23})-\left(\lambda_{3,*}+\varepsilon^*\right)-a_2\varepsilon^*\right]u_{c,x}+$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left[\left(a_1a_{12}a_{23}-a_{12}{}^2a_{13}\right)\left(a_2a_{12}a_{13}-a_{12}^3a_{23}a_{13}-2a_{23}\lambda_{1,*}\right)+\left(a_{12}^2+\lambda_{2,*}+\varepsilon^*\right)\left(\lambda_{3,*}+\varepsilon^*\right)\right]u_{c,y}+$$
$$\frac{1}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left(a_{12}^3-a_1a_2a_{12}\right)\varepsilon^*\left(a_2a_{12}a_{13}-a_{12}^3a_{13}a_{23}-a_{23}\left(\lambda_{1,*}+\varepsilon^*\right)\right)u_{c,z}, \tag{38b}$$

$$\delta u_{1,1} = u_{c,x} - U_{1,1} =$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left(a_1(a_2a_{13}-a_{12}a_{23})\left(a_2a_{12}a_{13}-a_{12}^2a_{23}\right)-a_{13}\varepsilon^{*2}\left(a_2a_{12}a_{13}-a_{12}^2a_{23}\right)-\left(\varepsilon^*-a_1a_2\right)\left(\lambda_{3,*}+\varepsilon^*\right)\right)u_{c,x}+$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left(a_1a_{12}\left(\lambda_{3,*}+\varepsilon^*\right)+\left(a_1(a_2a_{13}-a_{12}a_{23})-a_{13}\varepsilon^*\right)\left(a_1a_{12}a_{23}-a_{12}^2a_{13}\right)\right)u_{c,y}+$$
$$\frac{\varepsilon^*}{(\lambda_{1,*}+\varepsilon^*)(a_{12}^2-a_1a_2+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}\left(a_{12}^3-a_1a_2a_{12}\right)\left(a_1a_2a_{13}-a_1a_{12}a_{23}-a_{13}\varepsilon^*\right)u_{c,z}. \tag{38c}$$

Substituting (29)−(30) into (22a)−(22c), we obtain $\delta\lambda_{1,1}\neq 0$, $\delta\lambda_{2,1}\neq 0$, $\delta\lambda_{3,1}\neq 0$; therefore, the strategy cannot converge in the first iteration.

Assume that the strategy does not converge in the $(k-1)$th iteration but converges in the kth iteration; we then have

$$\begin{cases}A=(a_{12}a_{13}-a_1a_{23})\left[a_{23}\left(a_1+\varepsilon^*\right)-a_{12}a_{13}\right]-\left(a_3\left(a_1+\varepsilon^*\right)-a_{13}{}^2\right)\left(\lambda_{2,*}+\varepsilon^*\right)\\ \Gamma_{11}=\dfrac{[a_{23}(a_1+\varepsilon^*)-a_{12}a_{13}]a_{12}(a_2a_{13}-a_{12}a_{23})-A(\lambda_{3,*}+\varepsilon^*)(a_2(a_1+\varepsilon^*)-a_{12}{}^2)}{(a_{12}{}^2-a_1a_2)(A(\lambda_{3,*}+\varepsilon^*)-a_{13}(a_2a_{13}-a_{12}a_{23}))}\\ \Gamma_{22}=\dfrac{(\lambda_{1,*}+\varepsilon^*)(\lambda_{2,*}+\varepsilon^*)(a_1a_{23}-a_{12}a_{13})\left[(a_{12}a_{23}-a_2a_{13})+a_{13}\varepsilon^{*2}\right]}{a_1(\lambda_{2,*}-a_2\varepsilon^*)(a_1a_{23}-a_{12}a_{13})^2-a_{12}{}^2\varepsilon^*(a_{12}a_{23}-a_2a_{13})^2}\\ \Gamma_{31}=\dfrac{\{\Gamma_{11}(\lambda_{2,*}+\varepsilon^*)+(a_2(a_1+\varepsilon^*)-a_{12}{}^2)\}(\lambda_{3,*}+\varepsilon^*)}{a_{12}\varepsilon^*(a_{12}a_{23}-a_2a_{13})^2}\\ \Gamma_{32}=\dfrac{(\lambda_{1,*}+\varepsilon^*)(\lambda_{2,*}+\varepsilon^*)(\lambda_{3,*}+\varepsilon^*)}{a_1(\lambda_{2,*}-a_2\varepsilon^*)(a_1a_{23}-a_{12}a_{13})^2-a_{12}{}^2\varepsilon^*(a_{12}a_{23}-a_2a_{13})^2}\\ \Gamma_{33}=-\dfrac{1}{(a_{12}^3-a_1a_2a_{12})A}\end{cases}. \tag{39}$$

In conclusion, if λ_1, λ_2, and λ_3 are singular, let the weighting coefficient be expressed as (27), and the iteration converges in the kth cycle.

Theorem 1 When CMG cooperative game theory is employed, if the CMG output torque capacity is not exceeded, the CMG output torque error is 0, that is, $\boldsymbol{u}_{\rm o} = \boldsymbol{u}_{\rm c}$.

Proof When the game situation converges, according to (28), (33a), (33b), and (35), we can obtain

$$\begin{bmatrix} U_{1,k} \\ U_{2,k} \\ U_{3,k} \end{bmatrix} = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \lambda_{1,k} \\ \lambda_{2,k} \\ \lambda_{3,k} \end{bmatrix} = \begin{bmatrix} u_{\rm c,x} \\ u_{\rm c,y} \\ u_{\rm c,z} \end{bmatrix}. \quad (40)$$

Substituting (40) into (12b), we obtain

$$f(\dot{\boldsymbol{\alpha}}) = \begin{bmatrix} a_1 & a_{12} & a_{13} \\ a_{12} & a_2 & a_{23} \\ a_{13} & a_{23} & a_3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} - \begin{bmatrix} u_{c,x} \\ u_{c,y} \\ u_{c,z} \end{bmatrix} = \boldsymbol{J}\dot{\boldsymbol{\alpha}} - \boldsymbol{u}_c = \boldsymbol{0}. \quad (41)$$

The constraint is then satisfied, and $\boldsymbol{u}_o = \boldsymbol{J}\dot{\boldsymbol{\alpha}} = \boldsymbol{u}_c$. Therefore, when the output torque capacity of the CMG system is not exceeded, CMGs can output the torque without error.

4. Numerical simulation and analysis

4.1 Simulation parameter setting

The command torque is $\boldsymbol{u}_c = [0.05\sin(5t); 0.35\cos(2t); 0.1\sin(3t)]$. In this subsection, seven scenarios are considered. In scenario 1, the zero motion occurs at the singularity of the channel Z. In scenarios 2−4, the cooperative game manipulation law is near the singularity of the channels X, Y, and Z. In scenarios 5−7, the cooperative game manipulation law is at the singularity of the channels X, Y, and Z channels. Table 1 shows the design of each simulation scenario.

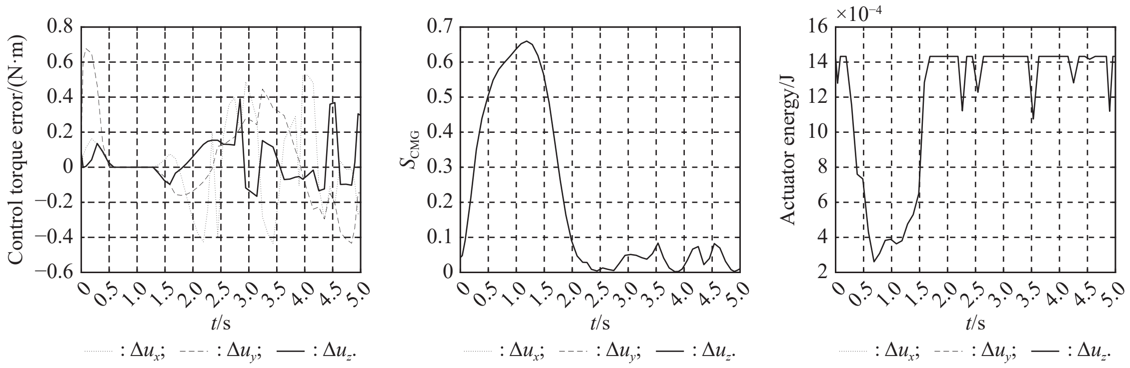
Table 1 Design of each simulation scenario

Scenario	Scenario design	Initial state/(°)	Parameter
1	(Null motion) X-channel singularity (nearby)	$\boldsymbol{\alpha} = [-105, 10, 95, 170]$	$\boldsymbol{W} = \mathrm{diag}[5,5,5,5]$
2	X-channel singularity (nearby)	$\boldsymbol{\alpha} = [-105, 10, 95, 170]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$
3	Y-channel singularity (nearby)	$\boldsymbol{\alpha} = [20, 90, 20, 100]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$
4	Z-channel singularity (nearby)	$\boldsymbol{\alpha} = [105, 115, 110, 105]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$
5	Singularity of X-channel	$\boldsymbol{\alpha} = [90, 0, 90, 0]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$
6	Singularity of Y-channel	$\boldsymbol{\alpha} = [0, 90, 0, 90]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$
7	Singularity of Z-channel	$\boldsymbol{\alpha} = [90, 90, 90, 90]$	$\varepsilon_\lambda = 1\times10^{-12}$, $\varepsilon^* = 1\times10^{-10}$

4.2 Simulation result

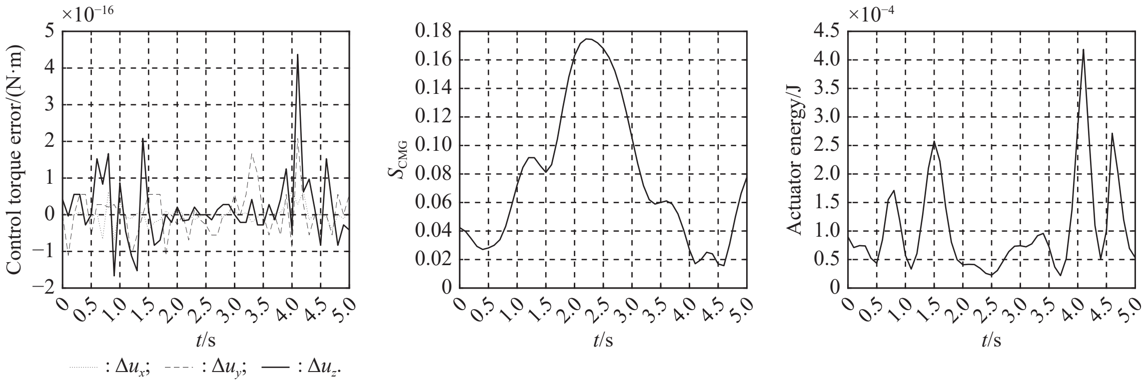
Fig. 2 shows the X-channel singular (nearby) control torque error, CMG singularity index, and actuator energy. As shown in Fig. 2(b), after 2 s, when it falls into singularity due to the expected torque, it cannot escape from the singularity state. It falls into singularity many times from 2.5 s to 5 s, and the output torque error is very large.

Figs. 3−5 show the torque error, singular metric function, and energy consumption when singular near channel X; torque error, singular metric function, and energy consumption when singular near channel Y; torque error, singular metric function, and energy consumption when singular near channel Z. As shown in Fig. 2(b), the initial state is near the singularity, that is, the singularity function is close to 0. The initial state is far from the singularity in less than 0.5 s, and near 4.3 s, it enters the singularity due to the expected torque, but it soon escapes from the singularity.



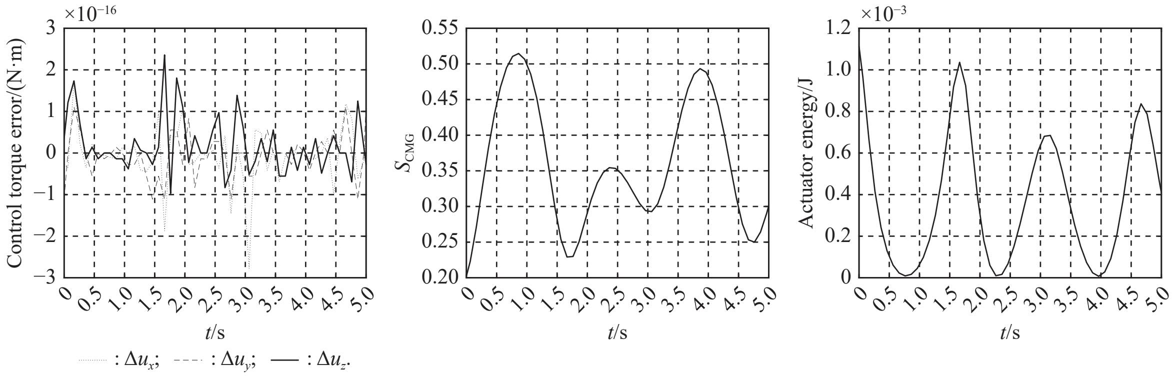
(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 2 Simulation results of Scenario 1



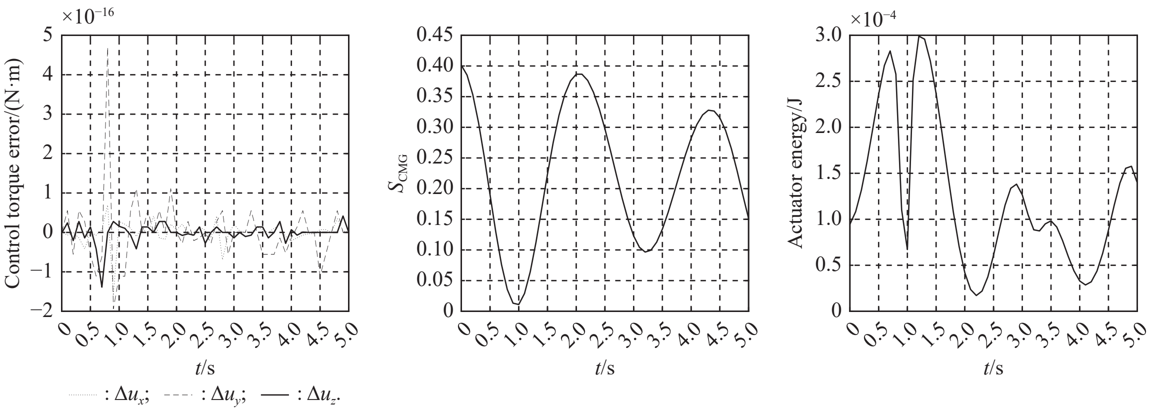
(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 3 Simulation results of Scenario 2



(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 4 Simulation results of Scenario 3

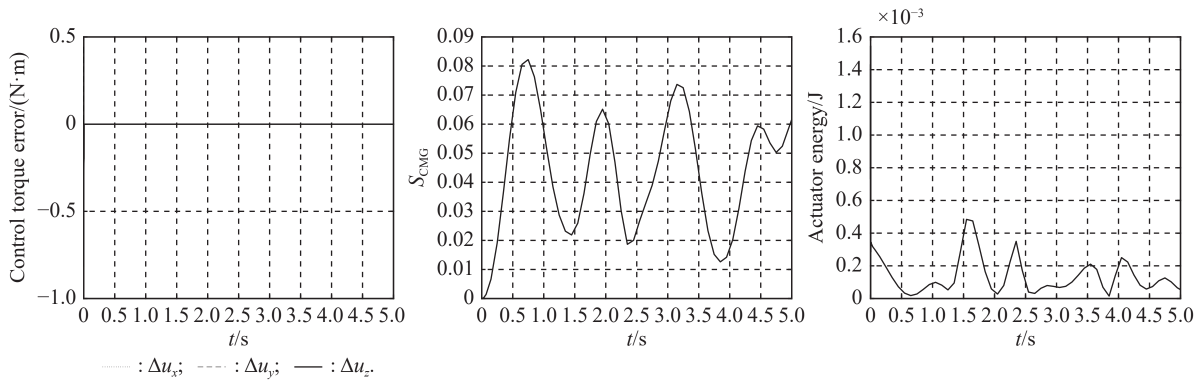


(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 5 Simulation results of Scenario 4

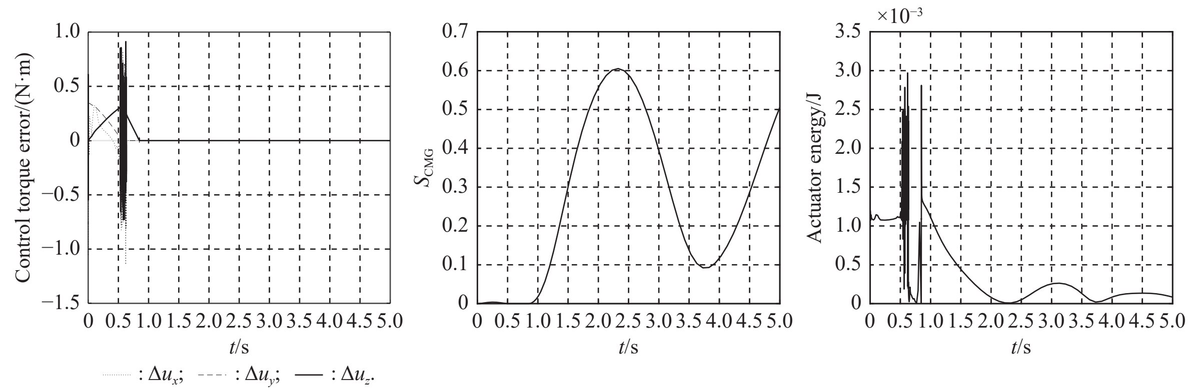
When the CMG works normally, once it is close to the singularity, the cooperative game control law has a role, which makes the CMG group system move quickly away from the singular state and makes the actuator work normally to meet the actual needs. The maximum torque error is 4.5×10^{-16} N·m, which can be disregarded. Simultaneously, the energy consumption is small, which is conducive to practical application.

Figs. 6–8 show the torque error, singular metric function and energy consumption near channel X; torque error, singular metric function and energy consumption when channel Y is singular; and torque error, singular metric function and energy consumption when channel Z is singular. As shown in Fig. 5(b), the initial state is observed at the singular point, that is, the singularity function is 0. When the CMG group system is in this extreme state, it can immediately escape from the singular state and will not re-enter the singular state. The CMG group system will not fall into singularity under the cooperative game manipulation law. When the initial state of the CMG group system is singular, the CMG group system can also escape from the singular state. After escaping from the singular state, the torque error can be disregarded, and the energy consumption is small.



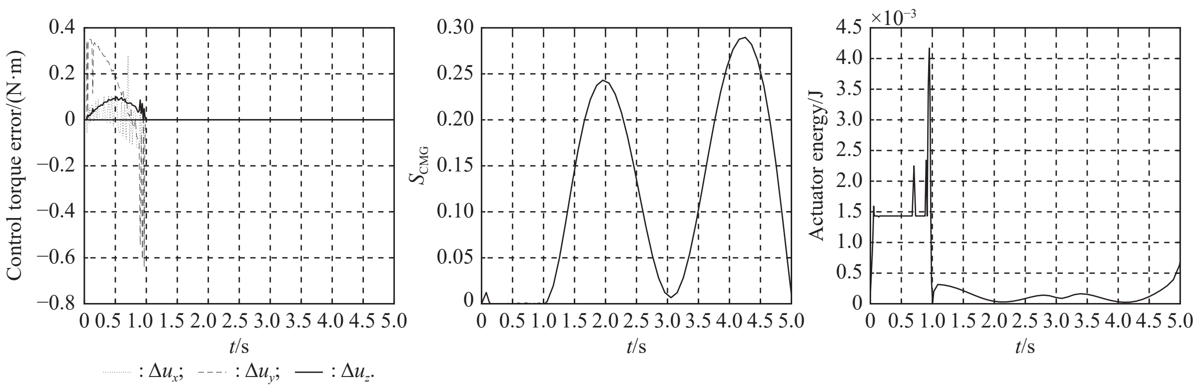
(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 6 Simulation results of scenario 5



(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 7 Simulation results of scenario 6



(a) Control torque error (b) CMG singularity index (c) Actuator energy

Fig. 8 Simulation results of scenario 7

4.3 Simulation conclusions

Table 2 summarizes Scenarios 1−7, the torque difference and the energy consumption. The abovementioned comparison shows that null motion cannot escape singularity, and the torque error is large, and the energy consumption is larger than that of cooperative game control law. When using the cooperative game control law, when the CMG group system works normally, once it is close to the singularity, the cooperative game control law immediately makes the CMG group far from the singularity, that is, the CMGs will not fall into the singularity. Simultaneously, the torque error is approximately 0, and the energy consumption is small, which meets the actual requirements of spacecraft. Once the CMG group system falls into singularity. For example, when the initial frame angle is at the singular point, the CMG group system can be separated from the singular state by using the cooperative game-based manipulation law. After the CMG group system is separated from the singular state, the torque difference can be disregarded, and the energy consumption is small. The availability of the CMG group system manipulation law design is based on a cooperative game.

Table 2 Summary of each simulation scenario

Scenario	Scenario design	Singularity situation	Control torque error/(N·m)	Actuator energy/J
1	(Null motion)Singularity of Z-channel	Singularity	0.6	14×10^{-4}
2	X-channel singularity (nearby)	No singularity	1×10^{-16}	0.42×10^{-3}
3	Y-channel singularity (nearby)	No singularity	1×10^{-16}	1.1×10^{-3}
4	Z-channel singularity (nearby)	No singularity	1×10^{-16}	0.3×10^{-3}
5	Singularity of X-channel	Escape from singularity	After escaping the singularity1×10^{-16}	1.5×10^{-3}
6	Singularity of Y-channel	Escape from singularity	After escaping the singularity1×10^{-16}	3×10^{-3}
7	Singularity of Z-channel	Escape from singularity	After escaping the singularity1×10^{-16}	4×10^{-3}

5. Conclusions

CMGs are widely employed in spacecraft control missions because of their multiple amplified moments and rapid responses. However, the application and development of gyroscopes are limited due to their inherent singularity [11]. In this paper, the pyramid CMG group system is modelled, and the cooperative game model is constructed according to the quadratic programming problem. Considering the angular momentum optimization and management, the angular velocity of the CMG frame is minimized; the energy consumption is small; there is no output torque error; it can effectively move away from the singular state; and it can escape from the singular state in the limit state. Compared with null motion, the energy consumption is lower. Compared with the design of the hybrid actuator, the weight and space of the momentum wheel are saved. The mathematical simulation proves that the control law design of the CMG group system based on a cooperative game enables the CMG group system move effectively away from and out of the singular state, reduces energy consumption, and avoids output torque error and that it can be employed to control spacecraft optical platforms.

References

[1] LIN L C. Research on agile attitude control and full physical simulation technology of small satellite. Beijing: University of Chinese Academy of Sciences (Changchun Institute of Optics, Fine Mechanics and Physics, Chinese Academy of Sciences), 2019. (in Chinese)

[2] ZHI G S. Research on steering law of SGCMGS. Harbin: Harbin Institute of Technology, 2016. (in Chinese)

[3] WU Z, WU H X. Survey of steering laws for single gimbal control moment gyroscope systems. Journal of Astronautics, 2000(4): 140–145. (in Chinese)

[4] ZHANG J R, CHEN L Q. Steering laws analysis of SGCMGs based on singular value decomposition theorys. Applied Mathematics and Mechanics, 2008, 2008(8): 918–926.

[5] TAKADA K, KOJIMA H, MATSUDA N. Control moment gyro singularity-avoidance steering control based on singular-surface cost function. Journal of Guidance, Control, and Dynamics, 2010, 33(5): 1442–1450.

[6] NANAMORI Y, TAKAHASHI M. Steering law of control moment gyros using optimization of initial gimbal angles for satellite attitude control. Journal of System Design and Dynamics, 2011, 5(1): 30–41.

[7] HE Y. Research on attitude maneuvers control of SGCMG based agile small satellite. Harbin: Harbin Institute of Technology, 2011. (in Chinese)

[8] WU Z, DENG S L, WEI K M, et al. Hybrid steering law with feedback and feedforward for single gimbal control moment gyroscopes. Chinese Space Science and Technology, 2013, 33(3): 1–7. (in Chinese)

[9] GENG Y H, HOU Z L. A method of avoiding SGCMG gimbal rate going into dead zone by null-motions. Journal of Astronautics, 2014, 35(4): 418–424. (in Chinese)

[10] WU Y H, HAN F. Modified singular direction steering logic for single gimbal control moment gyroscopes. Proc. of the 36th China Control Conference, 2017: 7–26

[11] GUO J T, WU B L, GENG Y H, et al. Rapid SGCMGs singularity-escape steer-ing law in gimbal angle space. IEEE Trans. on Aerospace and Electronic Systems, 2018, 54(5): 2509–2525.

[12] LEI Y G, YUAN L, WANG S Y, et al. A steering method with torque command adjustment and dynamic distribution for single-gimbal control moment gyro systems. Journal of Astronautics, 2019, 40(7): 794–802. (in Chinese)

[13] GUO J T. Research on attitude control methods for spacecraft using single gimbal control moment gyros. Harbin:

Harbin Institute of Technology, 2020. (in Chinese)

[14] LEE J S, BANG H C, LEE H. Singularity avoidance by game theory for control moment gyros. Proc. of the AIAA Guidance, Navigation, and Control Conference, 2005. DOI: 10.2514/6.2005−5946.

[15] WU Y H, ZHENG M H, HE M J, et al. Cooperative game theory-based optimal angular momentum management of hybrid attitude control actuator. IEEE Access, 2019, 7: 6853–6865.

[16] WIE B. Singularity escape/avoidance steering logic for control moment gyro systems. Journal of Guidance, Control, and Dynamics, 2005, 28(5): 948–956.

[17] WU Y H, HAN F, ZHANG S J, et al. Attitude agile maneuvering control for spacecraft equipped with hybrid actuators. Journal of Guidance, Control, and Dynamics, 2018, 41(3): 809–812.

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